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Ballooning Mode Calculations in Stellarators

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Abstract

An MHD ballooning mode formalism and calculation is developed to show how a field line following code can be used to study MHD stability. The asymptotic analysis of the ballooning equation yields the Mercier condition. It is shown that first order equilibrium effects to the vacuum fields from finite pressure, cancels the intrinsically destabilizing term of the Mercier condition. A ballooning unstable solution is found in a Heliac configuration that has a magnetic well at zero beta and a rotational transform that increases radially outward.

I. Introduction

The general problem of MHD plasma stability in a stellarator configuration is complex owing to the intrinsic three-dimensional equilibrium. However, the analysis simplifies considerably in the ballooning mode limit, where the cross field structure of the mode is assumed localized arbitrarily close to a field line. Typical analysis of such types of modes,[1,2] give stability conditions that are more pessimistic than neighboring modes with larger spatial spread, and hence stability conditions based on ballooning modes should yield a conservative figure of merit for the stability properties of a given stellarator configuration, except perhaps for free-boundary modes.

Ballooning mode equations have been derived rigorously for toroidally symmetric systems, and more recently for a general three-dimensional system.[3] In the limit of extreme localization to a field line, the ballooning mode formalism is readily obtained from the eikonal approximation where a displacement ξ, normal to a flux surface, can be written as $\xi = \xi(s)\exp[iS(\alpha,\beta)]$, where s is the distance along a field line and α and β are two surfaces that label a field line. The perpendicular wave-number is given by $\underset{\sim}{k}_\perp \equiv \underset{\sim}{\nabla} S = \underset{\sim}{\nabla}\alpha \frac{\partial S}{\partial \alpha} + \underset{\sim}{\nabla}\beta \frac{\partial S}{\partial \beta}$. In this work we use Clebsch coordinates for α and β where $\underset{\sim}{B} = \underset{\sim}{\nabla}\alpha \times \underset{\sim}{\nabla}\beta$, and α is taken to label a flux surface while β is an angle-like variable.

In general the ballooning mode equation will be given in terms of $\underset{\sim}{\nabla}\alpha$, which for a given configuration is a complicated geometrical function reflecting the complex undulations of the flux surface. In this work we show how we can construct $\underset{\sim}{\nabla}\alpha$, as well as $\underset{\sim}{\nabla}\beta$, relatively simply, by following the equations of neighboring field lines on the same flux surface. Thus, if one has a mechanism to construct the magnetic fields, one then integrates along the field line and simultaneously solves the ballooning mode equations.

In this note we: (1) write the ballooning equations in a manner that is readily integrated by a field line following code[4]; (2) analyze the asymptotic properties of the ballooning mode equation to rederive the Mercier condition[5] in a form particularly well suited for our computations; and (3) evaluate the low beta stability properties for three typical configurations, Proto-Cleo, Wistor U and a Heliac where the magnetic fields are due to actual vacuum coils plus the currents arising from a localized pressure gradient.

We find if a system has a magnetic well at zero beta, it will be Mercier stable for all beta values consistent with the limitations of the low beta approximation used, a result previously found by Shafranov.[6] However, as ballooning mode stabiity is more pessimistic than the Mercier criterion, additional ballooning mode stability limitations sometimes arise.

II. Ballooning Mode Equation

The Euler equation of the energy principle, using the eikonal approximation for the variation transverse to the magnetic field, is

$$\frac{d}{ds}\frac{k_\perp^2(s)}{B(s)}\frac{d}{ds}\xi(s) + \frac{2}{B^3}(\underset{\sim}{k}\times\underset{\sim}{b}\cdot\underset{\sim}{\nabla}p)(\underset{\sim}{k}\times\underset{\sim}{b}\cdot\underset{\sim}{\kappa})\,\xi(s) = 0 \ , \tag{1}$$

where $\underset{\sim}{b} = \underset{\sim}{B}/|B|$, $\underset{\sim}{\kappa} = (\underset{\sim}{b}\cdot\underset{\sim}{\nabla})\underset{\sim}{b}$, and s is the distance along a field line. Stability of Eq. (1) is determined from the condition $\xi(s)$ is nonzero between $-\infty < s < \infty$. We assume that the lines consist of nested surfaces, labelled by $\alpha \equiv$ const., which can be generated by following a field line. If we choose $\partial S/\partial\alpha = 0$, Eq. (1) then becomes

$$\frac{d}{ds}\frac{|\nabla\beta|^2}{B}\frac{d\xi}{ds} + \frac{2}{B^2}\frac{\partial p}{\partial\alpha}(\underset{\sim}{\nabla}\beta\times\underset{\sim}{b}\cdot\underset{\sim}{\kappa})\xi(s) = 0 \quad . \tag{2}$$

Given the surfaces $\alpha(\underset{\sim}{r})$, the function $\underset{\sim}{\nabla}\beta$ can be determined as follows. From $\underset{\sim}{\nabla}\alpha \times \underset{\sim}{B} = \underset{\sim}{\nabla}\alpha \times (\underset{\sim}{\nabla}\alpha \times \underset{\sim}{\nabla}\beta)$ we have the relation, $\underset{\sim}{\nabla}\beta = \Lambda\underset{\sim}{\nabla}\alpha + \frac{\underset{\sim}{B}\times\underset{\sim}{\nabla}\alpha}{|\nabla\alpha|^2}$ with $\Lambda = \underset{\sim}{\nabla}\alpha\cdot\underset{\sim}{\nabla}\beta/|\nabla\alpha|^2$ and $d\Lambda/ds$ is proportional to the magnetic shear. Then, by applying the operation $\underset{\sim}{B}\times\underset{\sim}{\nabla}\alpha\cdot\underset{\sim}{\nabla}\times$ on the form for $\underset{\sim}{\nabla}\beta$ we obtain the relation

$$\frac{d\Lambda}{ds} = -\frac{B}{|\underset{\sim}{\nabla}\alpha|^4}(\underset{\sim}{b}\times\underset{\sim}{\nabla}\alpha)\cdot\underset{\sim}{\nabla}\times(\underset{\sim}{b}\times\underset{\sim}{\nabla}\alpha) \quad . \tag{3}$$

Thus, with an initial condition $\Lambda(s_0)$, we have determined $\underset{\sim}{\nabla}\beta$ given $\underset{\sim}{\nabla}\alpha$, and Eq. (2) can be integrated from $-\infty < s < \infty$. For simplicity, we shall assume that a point s_0 can be found, where the coefficients of Eq. (2) are even in $s - s_0$, and then the domain of integration may be limited to $0 \leqslant s - s_0 < \infty$, with $d\xi(s_0)/ds = 0$ and $\Lambda(s_0)=0$. We assume that when s_0 is on the outside edge of the torus, we otain the most pessimistic result, although further investigation of this point is needed.

We now need a practical way to calculate $\underset{\sim}{\nabla}\alpha$. This is done by choosing a convenient third coordinate and examining the covariant and contravariant bases sets. For the covariant bases we have $\underset{\sim}{\nabla}\alpha, \underset{\sim}{\nabla}\beta$, and say, $\underset{\sim}{\nabla}\phi$, where ϕ is the usual toroidal angle about a fixed axis so that $\underset{\sim}{\nabla}\phi = \hat{\underset{\sim}{\phi}}/R$, and R is the major radius. The infinitesimal distance, $d\underset{\sim}{r}$ can be expressed in terms of the contravariant basis set $d\underset{\sim}{r}=d\phi\,\underset{\sim}{e}_1+d\alpha\,\underset{\sim}{e}_2+d\beta\,\underset{\sim}{e}_3$, with $\underset{\sim}{e}_1 \equiv \frac{\nabla\alpha\times\nabla\beta}{|\nabla\alpha\times\nabla\beta\cdot\nabla\phi|}$, $\underset{\sim}{e}_2 = \frac{\nabla\beta\times\nabla\phi}{|\nabla\alpha\times\nabla\beta\cdot\nabla\phi|}$, $\underset{\sim}{e}_3 = \frac{\nabla\phi\times\nabla\alpha}{|\nabla\alpha\times\nabla\beta\cdot\nabla\phi|}$ and, similarly, the inverse relations hold, $\nabla\alpha = (\underset{\sim}{e}_3\times\underset{\sim}{e}_1)/|\underset{\sim}{e}_1\times\underset{\sim}{e}_2\cdot\underset{\sim}{e}_3|$,etc. Further, we have $|\underset{\sim}{e}_1 \times \underset{\sim}{e}_2 \cdot \underset{\sim}{e}_3|^{-1} = |\nabla\alpha\times\nabla\beta\cdot\nabla\phi| = \frac{B_\phi}{R}$. It readily follows that $\underset{\sim}{e}_1 = R\underset{\sim}{B}/B_\phi$ and $ds = BRd\phi/B_\phi$. Further, $\underset{\sim}{e}_3$ is perpendicular to $\nabla\phi$ and in the plane of the flux surface. Let us suppose we can find two points, say $\underset{\sim}{r}_1(\phi_0)$ and $\underset{\sim}{r}_2(\phi_0)$, infinitesimally close to each other at the same $\phi = \phi_0$ and on the same flux surface. With these points as initial conditions, we can then generate the curves $\underset{\sim}{r}_1(\phi)$ and $\underset{\sim}{r}_2(\phi)$ each of which satisfy the field line equation $R^{-1}d\underset{\sim}{r}/d\phi=\underset{\sim}{b}B/B_\phi$. We then choose $\underset{\sim}{e}_3$ to satisfy

$$\underset{\sim}{e}_3 = \lim_{\underset{\sim}{r}_2(\phi_0)\to\underset{\sim}{r}_1(\phi_0)} \frac{\underset{\sim}{r}_1(\phi) - \underset{\sim}{r}_2(\phi)}{|\underset{\sim}{r}_1(\phi_0) - \underset{\sim}{r}_2(\phi_0)|} . \tag{4}$$

(We note that the initial points are particularly easy to find for systems for which a plane $\phi = \phi_0$ can be found for which the flux surface has up-down symmetry).

With $\underset{\sim}{e}_3$ defined according to Eq. (4), we then find that $\nabla\alpha = \underset{\sim}{e}_3\times\underset{\sim}{B}$. Hence, in terms of $\underset{\sim}{e}_3$, the ballooning mode equation becomes,

$$\frac{d}{d\phi}\left[\frac{B_\phi}{B^2R}\left(\frac{1}{|\underset{\sim}{e}_3\times\underset{\sim}{b}|^2} + B^2\Lambda^2|\underset{\sim}{e}_3\times\underset{\sim}{b}|^2\right)\right]\frac{d\xi}{d\phi} + 2\frac{\partial p}{\partial\alpha}\frac{R}{B_\phi}\left(\frac{\underset{\sim}{e}_3\times\underset{\sim}{b}\cdot\underset{\sim}{\kappa}}{B|\underset{\sim}{e}_3\times\underset{\sim}{b}|^2} - \Lambda\underset{\sim}{e}_3\cdot\underset{\sim}{\kappa}\right)\xi = 0 \tag{5}$$

with

$$\frac{d\Lambda}{d\phi} = \frac{R}{B_\phi}\left\{\frac{[\underset{\sim}{e}_3 - (\underset{\sim}{e}_3\cdot\underset{\sim}{b})\underset{\sim}{b}]\cdot[(\underset{\sim}{b}\cdot\nabla)(\underset{\sim}{e}_3\times\underset{\sim}{b}) - (\underset{\sim}{e}_3\times\underset{\sim}{b})\cdot\nabla\underset{\sim}{b}]}{|\underset{\sim}{e}_3\times\underset{\sim}{b}|^4}\right\} . \tag{6}$$

The advantage of the ballooning mode equation in this form is that we can determine $\underset{\sim}{e}_3$ and Λ as we generate the equations for a field line. One can show that Eqs. (5) and (6) reduce to the equations derived by Taylor, Hastie and Connor in the symmetric tokamak limit.

III. Mercier Condition and Diamagnetic Shift

We now derive the formal Mercier condition for Eq. (5). We first of all observe that it follows from $\underset{\sim}{j}=\partial p/\partial\alpha[2\lambda\underset{\sim}{B}+(\underset{\sim}{B}\times\underset{\sim}{\nabla}\alpha)/B^2]$, and $\underset{\sim}{\nabla}\cdot\underset{\sim}{j}=0$ that $d\lambda/d\phi=R\,\underset{\sim}{e}_3\cdot\underset{\sim}{\kappa}/B_\phi$. Since λ has to be a single-valued function of space, it follows that the integral $\lambda \equiv \int_0^\phi d\phi\,\frac{R}{B_\phi}\,\underset{\sim}{e}_3\cdot\underset{\sim}{\kappa}$ cannot be secular if a field line is in a bounded region of space.

Now, writing the ballooning mode equation as

$$\frac{d}{d\phi}\left(\frac{1}{|\underset{\sim}{e}_3\times\underset{\sim}{b}|^2}+B^2\Lambda^2|\underset{\sim}{e}_3\times\underset{\sim}{b}|^2\right)\frac{B_\phi}{B^2R}\frac{d\xi}{d\phi}+2\frac{\partial p}{\partial\alpha}\left(\frac{\underset{\sim}{e}_3\times\underset{\sim}{b}\cdot\underset{\sim}{\kappa}R}{|\underset{\sim}{e}_3\times\underset{\sim}{b}|^2B_\phi B}-\Lambda\frac{d\lambda}{d\phi}\right)\xi = 0 \tag{7}$$

we seek a solution, as $\Lambda\to\infty$, of the form, $\xi=\Lambda^\nu\left(\xi_0+\xi_1/\Lambda+\xi_2/\Lambda^2+\ldots\right)$, where ξ_j are non-secular.

Then, with $g^{-1}=|\underset{\sim}{e}_3\times\underset{\sim}{b}|^2B_\phi/R$, we consider the equations

$$\frac{d}{d\phi}\frac{1}{g}\frac{d\xi_0}{d\phi} = 0, \quad \frac{d}{d\phi}\frac{1}{g}\frac{d\xi_1}{d\phi} = -\nu\frac{d}{d\phi}\left(\frac{d\Lambda/d\phi}{g}\right) + 2\frac{\partial p}{\partial\alpha}\frac{d\lambda}{d\phi}$$

$$\frac{d}{d\phi}\frac{1}{g}\frac{d\xi_2}{d\phi} = -\nu\frac{(\nu+1)}{g}\left(\frac{d\Lambda}{d\phi}\right)^2 - (\nu-1)\frac{d}{d\phi}\left[\frac{(d\Lambda/d\phi)\xi_1}{g}\right] - \frac{(\nu+1)}{g}\frac{d\lambda}{d\phi}\frac{d\xi_1}{d\phi}$$

$$- 2\frac{\partial p}{\partial\alpha}\frac{\underset{\sim}{e}_3\times\underset{\sim}{b}\cdot\underset{\sim}{\kappa}g}{B} + 2\frac{\partial p}{\partial\alpha}\xi_1\frac{d\lambda}{d\phi}\,. \tag{8}$$

Hence $\xi_0=1$, and if we introduce the notation, $\langle\lambda\rangle = \int_0^\infty d\phi\lambda g/\int_0^\infty d\phi g$

$G = \int_0^\phi d\phi g$, $\left(\overline{\frac{\Lambda}{G}}\right) = \lim_{\phi\to\infty}\frac{\Lambda}{G}$, the solution for $d\xi_1/d\phi$ with the condition that ξ_1 be bounded, is then found to be

$$\frac{d\xi_1}{d\phi} = -\nu\left[\frac{d\Lambda}{d\phi} - \left(\overline{\frac{\Lambda}{G}}\right)g\right] + 2\frac{\partial p}{\partial\alpha}(\lambda - \langle\lambda\rangle)g\,. \tag{9}$$

If we now integrate Eq. (8) from $-\infty < \phi < \infty$, demanding that $d\xi_2(|\infty|)/d\phi$ be bounded, we obtain the relation,

$$\int_{-\infty}^{\infty} d\phi\,\left\{\frac{\nu(\nu+1)}{g}\left(\frac{d\Lambda}{d\phi}\right)^2 + 2\frac{\partial p}{\partial\alpha}\frac{g(\underset{\sim}{e}_3\times\underset{\sim}{b}\cdot\underset{\sim}{\kappa})}{B} + \left[\frac{(\nu+1)}{g}\frac{d\Lambda}{d\phi}\right.\right.$$

$$\left.\left. + 2\frac{\partial p}{\partial\alpha}(\lambda - \langle\lambda\rangle)\right]\frac{d\xi_1}{d\phi}\right\} = 0\,. \tag{10}$$

Substituting Eq. (9), then leads to the recursion relation $\nu^2+\nu+D = 0$ with

$$D = \lim_{\phi\to\infty} \frac{G}{\Lambda^2(\phi)} \{\int_0^\phi d\phi [\, 2\,\frac{\partial p}{\partial\alpha}\left(g\,\frac{\underset{\sim}{e}_3\times\underset{\sim}{b}\cdot\underset{\sim}{\kappa}}{B} + \frac{d\Lambda}{d\phi}\,(\lambda-\langle\lambda\rangle)\right) + 4\left(\frac{\partial p}{\partial\alpha}\right)^2 g(\lambda^2-\langle\lambda\rangle^2)]\,\} \tag{11}$$

and $D < 1/4$ is required if stability is to be possible.

In general one can apply these Mercier and ballooning equations to a numerically generated finite beta stellarator equilibrium. We confine ourselves here to studies of vacuum fields with a first order pressure modification. To incorporate the flux function shifts due to the self-well we note that $\underset{\sim}{\nabla}\alpha = \underset{\sim}{\nabla}\alpha_0+\underset{\sim}{\nabla}\alpha_1$; $\underset{\sim}{\nabla}\beta = \nabla\beta_0+\underset{\sim}{\nabla}\beta_1$, where the subscript "0" refers to quantities generated soley from vacuum currents and the subscript "1" to perturbed quantities induced by plasma currents. By linearizing in the plasma currents, we have

$$\underset{\sim}{\nabla}\times[\underset{\sim}{\nabla}\alpha_0\times\underset{\sim}{\nabla}\beta_1+\underset{\sim}{\nabla}\alpha_1\times\underset{\sim}{\nabla}\beta_0] = \underset{\sim}{J}= \frac{\partial p}{\partial\alpha}\left(\frac{\underset{\sim}{B}_0\times\underset{\sim}{\nabla}\alpha_0}{B_0^2} + 2\lambda_0\underset{\sim}{B}_0\right) \tag{12}$$

Further, by considering

$$\begin{aligned} \underset{\sim}{\nabla}\times\underset{\sim}{\nabla}\alpha_1 &\equiv \underset{\sim}{\nabla}\times[\gamma_{\alpha\alpha}\underset{\sim}{\nabla}\alpha_0+\gamma_{\alpha\beta}\underset{\sim}{\nabla}\beta_0+\gamma_{\alpha B}\underset{\sim}{B}] = 0 \\ \underset{\sim}{\nabla}\times\underset{\sim}{\nabla}\beta_1 &\equiv \underset{\sim}{\nabla}\times[\gamma_{\beta\alpha}\underset{\sim}{\nabla}\alpha_0+\gamma_{\beta\beta}\underset{\sim}{\nabla}\beta_0+\gamma_{\beta B}\underset{\sim}{B}] = 0 \end{aligned} \tag{13}$$

and ordering, $\underset{\sim}{\nabla}\lambda \equiv \frac{\partial\lambda}{\partial\alpha}\,\underset{\sim}{\nabla}\alpha + O(\varepsilon)$, (i.e. the largest gradients are assumed to be across the magnetic surface) we find on taking the dot product of Eq. (13) with $\underset{\sim}{\nabla}\alpha_0$ and $\underset{\sim}{\nabla}\beta_0$, that $\gamma_{\alpha\beta}\sim\gamma_{\alpha B}\sim\varepsilon\gamma_{\alpha\alpha}$; $\gamma_{\beta\alpha}\sim\gamma_{\beta B}\sim\varepsilon\gamma_{\beta\alpha}$. This model would be exact if the pressure gradient were non-zero only in the neighborhood of the surface under examination. A similar model has been used in Tokamak studies[8] and found to give the same qualitative stability behavior as found from exact global equilibria. Now if we assume $\gamma_{\alpha\alpha} \sim \epsilon \gamma_{\beta\alpha}$, and then substitute for $\underset{\sim}{\nabla}\alpha_1$ and $\underset{\sim}{\nabla}\beta_1$ into Eq. (12) we find, $d\gamma_{\beta\alpha}/ds = - j_{\parallel} B/|\nabla\alpha|^2$, $\gamma_{\alpha\alpha} = - p/B^2$. Using the expression for $j_{\parallel}$ in terms of the pressure gradient we find $\gamma_{\beta\alpha} \sim O(\beta/\epsilon)$ with $\beta \equiv 2p/B^2$, thereby verifying the consistency of our assumption. Thus only the $\nabla\beta_1 = \gamma_{\beta\alpha}\nabla\alpha_0$ correction need be retained and the ballooning mode equation, Eq. (5), can be evaluated with vacuum quantities, except for the equation for Λ which becomes

$$\frac{d\Lambda}{d\phi} = \frac{d\Lambda_0}{d\phi} - 2\,\frac{\partial p}{\partial \alpha}\,\frac{(\lambda - \lambda_0)}{|\underset{\sim}{e}_3 \times \underset{\sim}{b}|^2}\,\frac{R}{B_\phi} \qquad (6')$$

where $d\Lambda_0/d\phi$ represents the right hand side of Eq.(6) evaluated by using vacuum field quantities, and λ_0 is the constant of integration for the parallel current on a flux surface. If we argue that $E_{\parallel} = \eta\, j_{\parallel}$ with η a constant, and that $E_{\parallel} = -\partial\Phi/\partial s$, as there is no inductive field (the "zero current" case) , then the uniqueness of Φ demands $\lambda_0 = \int_{-\infty}^{\infty} ds\, B\lambda / \int_{-\infty}^{\infty} ds\, B$.

If we now define $\delta\Lambda \equiv \Lambda - \Lambda_0 \equiv -2\,\frac{\partial p}{\partial \alpha} \int_0^{\phi} d\phi\ (\lambda - \lambda_0) R / [\,|\underset{\sim}{e}_3 \times \underset{\sim}{b}|^2 B_\phi\,]$, and substitute into Eq. (11) for the Mercier condition, we find that the destabilizing quadratic term is cancelled by the $\delta\Lambda$ term and D becomes

$$D = \lim_{\phi \to \infty} \frac{2\frac{\partial p}{\partial \alpha}\, G}{(\Lambda_0 + \delta\Lambda)^2} \int_0^{\phi} d\phi \left[g\, \frac{\underset{\sim}{e}_3 \times \underset{\sim}{b} \cdot \underset{\sim}{\kappa}}{B} + \frac{d\Lambda_0}{d\phi}\,(\lambda - \langle\lambda\rangle) \right] \qquad (14)$$

and consequently if $D<0$ as $\partial p/\partial\alpha \to 0$, it is negative (and therefore stable to the Mercier criterion) for all beta consistent with our perturbation theory.

IV. NUMERICAL RESULTS

We have solved the ballooning mode equations for a variety of existing and proposed experimental devices. In this study we present the results for Proto-Cleo, Wistor U , and a proposed heliac configuration.

The method of computation is essentially the same for each machine. The fields are determined for a given coil structure by the Biot-Savart law using a code developed by Anderson, et al.[9] One can then generate surfaces, as in Fig. 1, by integrating the equation $d\underset{\sim}{r}/ds = \underset{\sim}{B}/|B|$. Then by recording the "puncture" points on a given ϕ plane, the structure of the surfaces are determined (see Fig. 1). In this way one can obtain the distance of the magnetic axis to the last reasonably defined surface on a given ϕ = const. surface. In practice, a fairly abrupt transition occurs between field lines that generate surfaces, to field lines that are not confined. This transition, though interesting, will not be discussed further in this work.

All the configurations examined here have special planes ϕ = const. that exhibit up-down reflection symmetry and in which $\hat{\underset{\sim}{\phi}}\cdot\underset{\sim}{\nabla}\alpha = 0$. On such planes, points equidistant from the reflection line are on the same flux surface. On such a ϕ = const. plane, we choose two neighboring points, $\underset{\sim}{r}_{10}$ and $\underset{\sim}{r}_{20}$, that straddle and are equidistant to the reflection line, and construct $\underset{\sim}{e}_3$ in accordance with Eq. (4) with the sign chosen so that $\underset{\sim}{e}_3\times\underset{\sim}{b}$ is directed away from the enclosed flux surface. Subsequent values of the vector $\underset{\sim}{e}_3$ are generated according to Eq. (4) by following the equations for the field lines emanating from $\underset{\sim}{r}_{10}$ and $\underset{\sim}{r}_{20}$.

Simultaneously, one can solve the differential equation for Λ and λ. Examples of Λ_0, Λ and λ are shown in Figs. 2 and 3 for a particularly complicated Heliac configuration. We note that Λ and Λ_0 increase secularly, with helical and toroidal modulations and the modification to the secular behavior from the diamagnetic correction is small while the modulation effects for this example is large. As the quantity λ is proportional to the parallel current it needs to be quasi-periodic. This is demonstrated in Fig. 3. We see that λ has rapid helical ripples and large toroidal ripples. However, the quasi-periodicity of λ is confirmed with $\iota \equiv d\theta/d\phi = 2\pi/\bar{\phi}$ the long period in Fig. 3. We choose an integral number of poloidal circuits to integrate Eq. (5) so that averaged quantities are evaluated accurately. We note that $d\alpha = B_0|\underset{\sim}{e}_{30}\times\underset{\sim}{b}|dr_0$ where the subscript "0" refers to the initial ϕ = const. plane. We define the local beta, β_0, of the surface as $\beta_0 = 2(\partial p/\partial\alpha)\, R_{sc}|e_{3_0}\times b_0|/B_0 \equiv R_{sc}(\partial p/\partial r_0)/B_0^2$ where R_{sc} is the distance from the magnetic axis to the outermost flux surface on the ϕ = const. symmetry plane. The total enclosed beta, $\bar{\beta}$, is defined as $\bar{\beta} = \int_{r_a}^{r_s} drB_0^2\beta_0/R_{sc}\, B_{00}^2$ where r_a is the magnetic axis, r_s is the position of the outermost flux surface and B_{00}^2 the magnitude of the magnetic field at the toroidal axis.

To find the marginal stability condition we assume that an eigenfunction centered on the outside of the torus will have the most pessimistic stability properties. If we start at the symmetry plane $\phi=\phi_0$ where quantities in Eq. (5) are even in $(\phi - \phi_0)$, we are allowed to consider initial conditions, $\xi = 1$ and $d\xi/d\phi = 0$. We then evaluate the Mercier condition. If D<0, the configuration has a magnetic well, and we then integrate Eq. (5) in a scan of beta to see if an additional ballooning instability can arise. When ballooning instability is possible, the critical beta is insensitive to the boundary condition $\xi=0$ at $\phi=\phi_{max}$ as long as $\phi_{max} > 2\pi/\iota$. If D>0, we examine ballooning mode stability at the Mercier critical beta β_M, to see if more pessimistic ballooning beta can be found. If the eigenfunction for $\beta_0=\beta_M$ indicates an additional intersection, we then search for the critical beta of the ballooning mode. In Fig. (4) we show an example of eigenfunction in Heliac which was ballooning unstable with D<0.

Our results show that the Proto-Cleo configuration is always stable (D<0 and ballooning modes are not found), whereas Wistor U is Mercier unstable, with no additional ballooning mode limits. In this machine the major radius, R_0, of the axis at the preferred plane is 2.5m and the minor radius from the magnetic axis to the separatrix is .27m. We generate flux surfaces starting at major radii R=2.55m, 2.6m, 2.65m, 2.7m, 2.75m, and find that the respective rotational transforms, initial mod-B, and critical beta, β_0, values are respectively $\iota \equiv .27, .35, .50, .72, 1.2$, mod-B = 3.9T, 4.0T, 4.19T, 4.43T, 4.75T, and $\beta_c \equiv .0031, .0074, .0094, .013, .012$. The mean beta is then found to be $\bar{\beta} = .011$. These results agree with Shafranov's conjecture[7] that if the rotational transform increases outwardly ballooning mode stability is determined solely by the Mercier condition. However, we find a counter example to Shafranov's conjecture in a heliac configuration which is Mercier stable, with radially increasing rotational transform, but ballooning limits are found at some radii, as shown in Fig. (4). We note however, that the critical beta is somewhat high, and we may be over extending the validity of our perturbation method expecially if the pressure gradient is not local to the flux surface. Further study on this aspect is needed.

In conclusion, we have developed a ballooning mode formalism that is amenable to numerical studies. If a magnetic well can be designed at zero beta, there is a strong tendency for the system to be ballooning mode stable for all beta (consistent with the equilibrium beta limitations of our theory) if the rotational transform increases radially, although an exception in a Heliac configuration has been found. Further work is in progress to calculate stellarator equilibrium with global pressure gradients so that ballooning mode stability can be accurately calculated when there are appreciable flux surface shifts due to finite beta.

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