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**CLASSIFICATION, CASIMIR INVARIANTS, AND
STABILITY OF LIE-POISSON SYSTEMS**

by

JEAN-LUC THIFFEAULT, B.S., M.A.

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STABILITY OF LIE-POISSON SYSTEMS**

APPROVED BY
DISSERTATION COMMITTEE:

Supervisor: _____

P. J. Morrison

R. D. Hazeltine

C. W. Horton

R. de la Llave

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A mes parents, Irène et Claude.

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CLASSIFICATION, CASIMIR INVARIANTS, AND
STABILITY OF LIE–POISSON SYSTEMS

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We classify Lie–Poisson brackets that are formed from Lie algebra extensions. The problem is relevant because many physical systems owe their Hamiltonian structure to such brackets. A classification involves reducing all brackets to a set of normal forms, independent under coordinate transformations, and is achieved with the techniques of *Lie algebra cohomology*. For extensions of order less than five, we find that the number of normal forms is small and they involve no free parameters. A special extension, known as the Leibniz extension, is shown to be the unique “maximal” extension.

We derive a general method of finding Casimir invariants of Lie–Poisson bracket extensions. The Casimir invariants of all brackets of order less than five are explicitly computed, using the concept of *coextension*. We obtain the Casimir invariants of Leibniz extensions of arbitrary order. We also offer some physical insight into the nature of the Casimir invariants of compressible reduced magnetohydrodynamics.

We make use of the methods developed to study the stability of extensions for given classes of Hamiltonians. This helps to elucidate the distinction between semidirect extensions and those involving *cocycles*. For compressible reduced magnetohydrodynamics, we find the cocycle has a destabilizing effect on the steady-state solutions.

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Chapter 1

Introduction

The topic of this thesis is the classification and analysis of the properties of Lie–Poisson brackets obtained from extensions of Lie algebras. A large class of finite- and infinite-dimensional dynamical equations admit a Hamiltonian formulation using noncanonical brackets of the Lie–Poisson type. Finite-dimensional examples include the Euler equations for the rigid body [9], the moment reduction of the Kida vortex [65], and a low-order model of atmospheric dynamics [14]. Infinite-dimensional examples include the Euler equation for the ideal fluid [52, 61, 67, 72, 78], the quasigeostrophic equations [39, 94], and the Vlasov equation [60, 66].

In mathematical terms, Lie–Poisson brackets naturally define a Poisson structure (i.e., a symplectic structure [95]) on the dual of a Lie algebra. For the rigid body, the Lie algebra is the one associated with the rotation group, $SO(3)$, while for the Kida vortex moment reduction the underlying group is $SO(2, 1)$. For the two-dimensional ideal fluid, the relevant Lie algebra corresponds to the group of volume-preserving diffeomorphisms of the fluid domain.

Lie–Poisson structures often occur as a result of reduction [59]. Reduction is, in essence, a method of taking advantage of the symmetries of a system to lower its order. However in so doing one perhaps loses the *canonical* nature

of the system: there are no longer any well-defined conjugate positions and momenta. This does not preclude the system from being Hamiltonian, that is these conjugate variables can exist *locally*, up to some possible degeneracy in the system (the *symplectic leaves*). The resulting Hamiltonian system (after reduction) is often of Lie–Poisson type. For example, the reduction of the rigid body in Euler angle coordinates (three angles and three canonical momenta, for a total of six coordinates) gives Euler’s equations (in terms of only the angular momenta, three coordinates), which have a Lie–Poisson structure.

Why seek a bracket formulation of a system at all? If we care about whether a system is Hamiltonian or not, then for noncanonical systems it is a simple way of showing that the equations have such a structure. We are then free to use the powerful machinery of Hamiltonian mechanics. For example, we know that the eigenvalue spectrum of the linearized system has to have four-fold symmetry in the complex plane [6]. If we are concerned with the properties of the truncation of a hydrodynamic system, then knowing the bracket formulation can serve as a guide for finding a finite-dimensional representation of the system which retains the Hamiltonian structure [64, 97]. Also, there exists moment reductions—finite-dimensional subalgebras of infinite-dimensional algebras—that provide exact closures [63, 64, 65, 83].

We will classify low-order bracket extensions and find their Casimir invariants. An extension is simply a new Lie bracket, derived from a base algebra (for example, $SO(3)$), and defined on n -tuples of that algebra. We are ruling out extensions where the individual brackets that appear are not of the same form as that of the base algebra. We are thus omitting some brackets [70, 72, 77], but the brackets we are considering are amenable to a general classification.

The method of extension yields interesting and physically relevant algebras. Using this method we can describe finite-dimensional systems of several variables and infinite-dimensional systems of several fields. For the finite-dimensional case, an example is the two vector model of the heavy top [40], where the two vectors are the angular momentum and the position of the center of mass. For infinite-dimensional systems there are examples of models of two [12, 64, 73], three [32, 51, 73], and four [33, 70] fields. Knowing the bracket allows one to find the Casimir invariants of the system [36, 50, 91]. These are quantities which commute with every functional on the Poisson manifold, and thus are conserved by the dynamics for any Hamiltonian. They are useful for analyzing the constraints in the system [90] and for establishing stability criteria [31, 38, 68, 69, 71].

1.1 Overview

The outline of the thesis is as follows. In Chapter 2, we review the general theory behind Lie–Poisson brackets. We give some examples of physical systems of Lie–Poisson type, both finite- and infinite-dimensional. We introduce the concept of Lie algebra extensions and derive some of their basic properties. Chapter 3 is devoted to the more abstract treatment of extensions through the theory of Lie algebra cohomology [19, 21, 47]. We define some terminology and special extensions such as the semidirect sum and the Leibniz extension. In Chapter 4, we use the cohomology techniques to treat the specific type of extension with which we are concerned, brackets over n -tuples. We give an explicit classification of low-order extensions. By classifying, we mean reducing—through coordinate changes—all possible brackets to independent

normal forms. We find that the normal forms are relatively few and that they involve no free parameters—at least for low-order extensions.

In Chapter 5, we turn to the problem of finding the Casimir invariants of the brackets, those functionals that commute with every other functional in the algebra. We derive some general techniques for doing this that apply to extensions of any order. We treat explicitly some examples, including the Casimir invariants of a particular model of magnetohydrodynamics (MHD), which are also given a physical interpretation. A formula for the invariants of Leibniz extensions of any order is also derived. Then in Section 5.6 we use the classification of Section 4.6 to derive the Casimir invariants for low-order extensions.

We address general stability of Lie–Poisson systems in Chapter 6. We begin by reviewing the concept of stability in Section 6.1, discussing the distinctions between spectral, linearized, formal, and nonlinear stability. We consider the difficulties that arise for infinite-dimensional systems. In Section 6.2 we present a review of the energy-Casimir method for finding equilibria and establishing sufficient conditions for stability. We use the method on compressible reduced MHD. In Section 6.3, we turn to a more general method for stability analysis, that of dynamical accessibility. The method uses variations that have been restricted to symplectic leaves. We then treat several different classes of Hamiltonian and Lie–Poisson brackets and discuss the role of cocycles in equilibria and their stability. Finally, in Chapter 7 we offer some concluding remarks and discuss future directions for research.

Chapter 2

Lie–Poisson Brackets

Lie–Poisson brackets define a natural Poisson structure on duals of Lie algebras. Physically, they often arise in the *reduction* of a system. For our purposes, a reduction is a mapping of the dynamical variables of a system to a smaller set of variables, such that the transformed Hamiltonian and bracket depend only on the smaller set of variables. (For a more detailed mathematical treatment, see for example [1, 10, 28, 57, 58, 59].) The simplest example of a reduction is the case in which a cyclic variable is eliminated, but more generally a reduction exists as a consequence of an underlying symmetry of the system. For instance, the Lie–Poisson bracket for the rigid body is obtained from a reduction of the canonical Euler angle description using the rotational symmetry of the system [40]. The Euler equation for the two-dimensional ideal fluid is obtained from a reduction of the Lagrangian description of the fluid, which has a relabeling symmetry [16, 69, 76, 80].

Here we shall take a more abstract viewpoint: we do not assume that the Lie–Poisson bracket is obtained from a reduction, though it is always possible to do so by the method of Clebsch variables [69]. Rather we proceed directly from a given Lie algebra to build a Lie–Poisson bracket. The choice of algebra can be guided by the symmetries of the system.

In Section 2.1, we give some definitions and review the basic theory behind Lie–Poisson brackets. We then give examples in Section 2.2: the free rigid body, reduced magnetohydrodynamics (RMHD), and compressible reduced magnetohydrodynamics (CRMHD). These last two cases are examples of *Lie algebra extensions*. We describe general Lie algebra extensions in Section 2.3. This introduces the problem, and establishes the framework for the remainder of the thesis.

2.1 Lie–Poisson Brackets on Duals of Lie Algebras

Recall that a *Lie algebra* \mathfrak{g} is a vector space on which is defined a bilinear operation $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$, called the Lie bracket. The Lie bracket is *antisymmetric*,

$$[\alpha, \beta] = -[\beta, \alpha],$$

and satisfies the *Jacobi identity*,

$$[\alpha, [\beta, \gamma]] + [\beta, [\gamma, \alpha]] + [\gamma, [\alpha, \beta]] = 0,$$

for arbitrary elements α, β, γ in \mathfrak{g} . Lie algebras are differentiable manifolds.

A real-valued *functional* defined on a differentiable manifold \mathcal{M} is simply a map from \mathcal{M} to \mathbb{R} . (From now on, when we say functional it will be understood that we mean a real-valued functional.) The vector space of all functionals on \mathcal{M} is denoted by $\mathcal{F}(\mathcal{M})$.

The *dual* \mathfrak{g}^* of \mathfrak{g} is the set of all *linear* functionals on \mathfrak{g} . The elements of \mathfrak{g}^* are denoted by

$$\langle \xi, \cdot \rangle : \mathfrak{g} \rightarrow \mathbb{R}, \quad \langle \xi, \cdot \rangle \in \mathfrak{g}^*,$$

where ξ identifies the elements of \mathfrak{g}^* . It is customary, however, to simply say $\xi \in \mathfrak{g}^*$ and express the *pairing* by $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \rightarrow \mathbb{R}$. This simplifies the procedure of identifying \mathfrak{g} and \mathfrak{g}^* , especially for infinite-dimensional Lie algebras, where the pairing is typically an integral. Note that functionals can be defined on \mathfrak{g}^* , since it is a differentiable manifold. In finite dimensions, \mathfrak{g} and \mathfrak{g}^* are isomorphic as vector spaces (they have the same dimension). However, \mathfrak{g}^* does not naturally inherit a Lie algebra structure from \mathfrak{g} . In infinite dimensions, the two spaces need not be isomorphic.

Let \mathcal{M} be a differentiable manifold. A *Poisson structure* on $\mathcal{F}(\mathcal{M})$ is a Lie algebra on $\mathcal{F}(\mathcal{M})$ with bracket $\{ \cdot, \cdot \}$ that satisfies the derivation property

$$\{F, GH\} = \{F, G\}H + G\{F, H\},$$

where $F, G, H \in \mathcal{F}(\mathcal{M})$. (This property is also called the Leibniz rule.) The manifold \mathcal{M} with the bracket $\{ \cdot, \cdot \}$ is called a *Poisson manifold*.

For the remainder of the thesis, we will be interested in the case where \mathcal{M} is the dual \mathfrak{g}^* of a Lie algebra \mathfrak{g} . The *Lie–Poisson bracket* provides a natural Poisson structure on $\mathcal{F}(\mathfrak{g}^*)$, given the Lie bracket $[\cdot, \cdot]$ in \mathfrak{g} . It is defined as

$$\{F, G\}_{\pm}(\xi) = \pm \left\langle \xi, \left[\frac{\delta F}{\delta \xi}, \frac{\delta G}{\delta \xi} \right] \right\rangle, \quad (2.1)$$

where F and G are real-valued functionals on \mathfrak{g}^* , that is, $F, G : \mathfrak{g}^* \rightarrow \mathbb{R}$, and $\xi \in \mathfrak{g}^*$. The functional derivative $\delta F / \delta \xi \in \mathfrak{g}$ is defined by

$$\delta F[\xi; \delta \xi] := \left. \frac{d}{d\epsilon} F[\xi + \epsilon \delta \xi] \right|_{\epsilon=0} =: \left\langle \delta \xi, \frac{\delta F}{\delta \xi} \right\rangle. \quad (2.2)$$

We shall refer to the bracket $[\cdot, \cdot]$ as the *inner bracket* and to the bracket $\{ \cdot, \cdot \}$ as the Lie–Poisson bracket. The dual \mathfrak{g}^* together with the Lie–Poisson bracket

is a Poisson manifold. The sign choice in (2.1) comes from whether we are considering right invariant (+) or left invariant (−) functions on the cotangent bundle of the Lie group [58, 61], but for our purposes we simply choose the sign as needed.

For finite-dimensional algebras, the Lie–Poisson bracket (2.1) was first written down by Lie [54] and was rediscovered by Berezin [13]; it is also closely related to work of Arnold [5], Kirillov [46], Kostant [48], and Souriau [86].

Before we can describe the dynamics generated by Lie–Poisson brackets, we need a few more definitions. The *adjoint action* of \mathfrak{g} on itself is the same as the bracket in \mathfrak{g} ,

$$\mathrm{ad}_\alpha \beta \equiv [\alpha, \beta],$$

where $\alpha, \beta \in \mathfrak{g}$. From this we define the *coadjoint action* $\mathrm{ad}_\alpha^\dagger$ of \mathfrak{g} on \mathfrak{g}^* by¹

$$\langle \mathrm{ad}_\alpha^\dagger \xi, \beta \rangle := \langle \xi, \mathrm{ad}_\alpha \beta \rangle, \quad (2.3)$$

where $\xi \in \mathfrak{g}^*$. We also define the *coadjoint bracket* $[\cdot, \cdot]^\dagger : \mathfrak{g} \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ to be

$$[\alpha, \xi]^\dagger := \mathrm{ad}_\alpha^\dagger \xi, \quad (2.4)$$

so that

$$\langle [\alpha, \xi]^\dagger, \beta \rangle := \langle \xi, [\alpha, \beta] \rangle; \quad (2.5)$$

the bracket $[\cdot, \cdot]^\dagger$ satisfies the identity

$$\langle [\alpha, \xi]^\dagger, \beta \rangle = -\langle [\beta, \xi]^\dagger, \alpha \rangle.$$

¹We are using the convention of Arnold [9, p. 321], but some authors define ad^\dagger with a minus sign, so that the canonical bracket and its coadjoint bracket have the same sign in (2.12) when \mathfrak{g} and \mathfrak{g}^* are identified.

Since the inner bracket is Lie, it satisfies the Jacobi identity, and consequently the form given by (2.1) for the Lie–Poisson bracket will automatically satisfy the Jacobi identity [2, p. 614]. This is proved in Appendix A.

We are of course interested in generating dynamics from the Lie–Poisson bracket. This is done in the usual manner, by inserting a Hamiltonian functional in the bracket. For any Poisson structure, given a Hamiltonian functional $H : \mathcal{M} \rightarrow \mathbb{R}$, the equation of motion for $\xi \in \mathcal{M}$ is

$$\dot{\xi} = \{\xi, H\},$$

where a dot denotes a time derivative. For a Lie–Poisson bracket, we have $\mathcal{M} = \mathfrak{g}^*$, and we use the definition (2.1) of $\{\cdot, \cdot\}$ to write

$$\dot{\xi} = \left\langle \xi, \left[\Delta, \frac{\delta H}{\delta \xi} \right] \right\rangle,$$

where Δ is a Kronecker or Dirac delta, or a combination of both for an infinite-dimensional system of several fields (that is, ξ can be a vector of field variables).

We then use the definition of the coadjoint bracket (2.5),

$$\dot{\xi} = - \left\langle \left[\frac{\delta H}{\delta \xi}, \xi \right]^\dagger, \Delta \right\rangle,$$

and finally use the property of the delta function to identify $\dot{\xi}$ with the left slot of the pairing,

$$\dot{\xi} = - \left[\frac{\delta H}{\delta \xi}, \xi \right]^\dagger. \quad (2.6)$$

Thus, for Lie–Poisson brackets the dynamical evolution of ξ is generated by the coadjoint bracket.

We close this section by commenting on the nature of the dynamics generated by Lie–Poisson brackets. The elements of a Lie algebra \mathfrak{g} are usually

regarded as infinitesimal generators of the elements of a *Lie group* G near the identity. (We also say that the Lie algebra is the tangent space of the Lie group at the identity.) The *coadjoint orbit* through $\xi \in \mathfrak{g}^*$ is defined as

$$\text{Orb}(\xi) := \{\text{Ad}_a^\dagger \xi \mid a \in G\}.$$

(We will not rigorously define it here, but simply think of $\text{Ad}_a^\dagger : \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ as a finite version of the infinitesimal coadjoint action $\text{ad}_\xi^\dagger : \mathfrak{g}^* \rightarrow \mathfrak{g}^*$. See for example Arnold [9, pp. 319–321].) The coadjoint orbits tell us what parts of \mathfrak{g}^* can be reached from a given element ξ^* by acting with the group elements. For example, the coadjoint orbits for the rotation group $SO(3)$ are spheres [58, p. 400], so two elements of \mathfrak{g}^* belong to the same coadjoint orbit if they lie on the same sphere (the elements can be mapped onto each other by a rotation).

The infinitesimal generator at ξ of the coadjoint action is

$$\eta_{\mathfrak{g}^*}(\xi) := \text{ad}_\eta^\dagger \xi \tag{2.7}$$

Comparing this to the equation of motion (2.6), and recalling the definition of the coadjoint bracket (2.4), we see that $\dot{\xi}$ lies along the direction of the infinitesimal generator $\text{ad}_{\delta H/\delta \xi}^\dagger$ at ξ .

What does this all mean? The time-evolved trajectory $\{\xi(t) \mid t \geq 0\}$ of ξ must go through points in \mathfrak{g}^* that can be reached by $\text{Ad}_a^\dagger \xi(0)$, where $\xi(0)$ is an initial condition, for some $a \in G$. To put it more succinctly,

$$\{\xi(t) \mid t \geq 0\} \subseteq \text{Orb}(\xi(0)). \tag{2.8}$$

For $G = SO(3)$, since the coadjoint orbits are spheres then the only trajectories allowed must lie on spheres. This makes $SO(3)$ the natural group to describe

the motion of the rigid body, as we will see in Section 2.2.1. Note that equality in (2.8) does not usually hold, since the trajectory is one-dimensional, whereas the coadjoint orbits are usually of higher dimension.

2.2 Examples of Lie–Poisson Systems

We will say that a physical systems can be described by a given Lie–Poisson bracket and Hamiltonian if its equations of motion can be written as (2.6) for some H ; the system is then said to be Hamiltonian of the Lie–Poisson type. We give four examples: the first is finite-dimensional (the free rigid body, Section 2.2.1) and the second infinite-dimensional (Euler’s equation for the ideal fluid, Section 2.2.2). The third and fourth examples are also infinite-dimensional and serve to introduce the concept of extension. They are low-beta reduced magnetohydrodynamics (MHD) in Section 2.2.3 and compressible reduced MHD in Section 2.2.4. These last two examples are meant to illustrate the physical relevance of Lie algebra extensions.

2.2.1 The Free Rigid Body

The classic example of a Lie–Poisson bracket is obtained by taking for \mathfrak{g} the Lie algebra of the rotation group $SO(3)$. If the $\hat{\mathbf{e}}_{(i)}$ denote a basis of $\mathfrak{g} = so(3)$, the Lie bracket is given by

$$[\hat{\mathbf{e}}_{(i)}, \hat{\mathbf{e}}_{(j)}] = c_{ij}^k \hat{\mathbf{e}}_{(k)},$$

where the $c_{ij}^k = \varepsilon_{ijk}$ are the structure constants of the algebra, in this case the totally antisymmetric symbol. Using as a pairing the usual contraction between

upper and lower indices, with (2.1) we are led to the Lie–Poisson bracket

$$\{f, g\} = -c_{ij}^k \ell_k \frac{\partial f}{\partial \ell_i} \frac{\partial g}{\partial \ell_j},$$

where the three-vector ℓ is in \mathfrak{g}^* , and we have chosen the minus sign in (2.1).

The coadjoint bracket is obtained using (2.3),

$$[\beta, \ell]_i^\dagger = -c_{ij}^k \beta^j \ell_k. \quad (2.9)$$

If we use this coadjoint bracket and insert the Hamiltonian

$$H = \frac{1}{2}(I^{-1})^{ij} \ell_i \ell_j \quad (2.10)$$

in (2.6) we obtain

$$\dot{\ell}_m = \{\ell_m, H\} = c_{mj}^k (I^{-1})^{jp} \ell_k \ell_p.$$

Notice how the moment of inertia tensor I plays the role of a metric—it allows us to build a quadratic form (the Hamiltonian) from two elements of \mathfrak{g}^* . If we take $I = \text{diag}(I_1, I_2, I_3)$, we recover Euler’s equations for the motion of the free rigid body

$$\dot{\ell}_1 = \left(\frac{1}{I_2} - \frac{1}{I_3} \right) \ell_2 \ell_3,$$

and cyclic permutations of 1,2,3. The ℓ_i are the angular momenta about the axes and the I_i are the principal moments of inertia. This result is naturally appealing because we expect the rigid body equations to be invariant under the rotation group, hence the choice of $SO(3)$ for G .

2.2.2 The Two-dimensional Ideal Fluid

Consider now an ideal fluid with the flow taking place over a two-dimensional domain Ω . Let \mathfrak{g} be the infinite-dimensional Lie algebra associated with the Lie

group of volume-preserving diffeomorphisms of Ω . In two spatial dimensions this is the same as the group of canonical transformations on Ω . The bracket in \mathfrak{g} is the canonical bracket

$$[a, b] = \frac{\partial a}{\partial x} \frac{\partial b}{\partial y} - \frac{\partial b}{\partial x} \frac{\partial a}{\partial y}. \quad (2.11)$$

We formally identify \mathfrak{g} and \mathfrak{g}^* by using as the pairing $\langle \cdot, \cdot \rangle$ the usual integral over the fluid domain,

$$\langle F, G \rangle = \int_{\Omega} F(\mathbf{x}) G(\mathbf{x}) d^2x,$$

where $\mathbf{x} := (x, y)$. For infinite-dimensional spaces, there are functional analytic issues about whether we can make this identification, and take $\mathfrak{g}^{**} = \mathfrak{g}$. We will assume here that these relationships hold formally. See Marsden and Weinstein [57] for references on this subject and Audin [10] for a treatment of the identification of \mathfrak{g} and \mathfrak{g}^* .

For simplicity, we assume that the boundary conditions are such that surface terms vanish, and we get

$$[\cdot, \cdot]^{\dagger} = -[\cdot, \cdot] \quad (2.12)$$

from (2.5). (Without this assumption the coadjoint bracket would involve extra boundary terms.) We take the vorticity ω as the field variable ξ and write for the Hamiltonian

$$H[\omega] = -\frac{1}{2} \langle \omega, \nabla^{-2} \omega \rangle,$$

where

$$(\nabla^{-2} \omega)(\mathbf{x}) := \int_{\Omega} K(\mathbf{x}|\mathbf{x}') \omega(\mathbf{x}') d^2x',$$

and K is Green's function for the Laplacian. The Green's function plays the role of a metric since it maps an element of \mathfrak{g}^* (the vorticity ω) into an element of \mathfrak{g} to be used in the right slot of the pairing. This relationship is only weak: the mapping K is not surjective, and thus the metric cannot formally be inverted (it is called *weakly nondegenerate*). When we have identified \mathfrak{g} and \mathfrak{g}^* we shall often drop the comma in the pairing and write

$$H[\omega] = -\frac{1}{2} \langle \omega, \phi \rangle = \frac{1}{2} \langle |\nabla \phi|^2 \rangle,$$

where $\omega = \nabla^2 \phi$ defines the streamfunction ϕ . We work out the evolution equation for ω explicitly:

$$\begin{aligned} \dot{\omega}(\mathbf{x}) &= \{\omega, H\} = \int_{\Omega} \omega(\mathbf{x}') \left[\frac{\delta \omega(\mathbf{x})}{\delta \omega(\mathbf{x}')} , \frac{\delta H}{\delta \omega(\mathbf{x}')} \right] d^2 x' \\ &= \int_{\Omega} \omega(\mathbf{x}') [\delta(\mathbf{x} - \mathbf{x}'), -\phi(\mathbf{x}')] d^2 x' \\ &= \int_{\Omega} \delta(\mathbf{x} - \mathbf{x}') [\omega(\mathbf{x}'), \phi(\mathbf{x}')] d^2 x' \\ &= [\omega(\mathbf{x}), \phi(\mathbf{x})]. \end{aligned}$$

This is Euler's equation for a two-dimensional ideal fluid. We could also have written this result down directly from (2.6) using $[\cdot, \cdot]^\dagger = -[\cdot, \cdot]$.

2.2.3 Low-beta Reduced MHD

This example will illustrate the concept of a Lie algebra extension, the central topic of this thesis. Essentially, the idea is to use an algebra of n -tuples, which we call an extension, to describe a physical system with more than one dynamical variable. As in Section 2.2.2 we consider a flow taking place over a two-dimensional domain Ω . The Lie algebra \mathfrak{g} is again taken to be that of volume preserving diffeomorphisms on Ω , but now we consider also the vector

space V of real-valued functions on Ω (an Abelian Lie algebra under addition). The *semidirect sum* of \mathfrak{g} and V is a new Lie algebra whose elements are two-tuples (α, v) with a bracket defined by

$$[(\alpha, v), (\beta, w)] := ([\alpha, \beta], [\alpha, w] - [\beta, v]), \quad (2.13)$$

where α and $\beta \in \mathfrak{g}$, v and $w \in V$. This is a Lie algebra, so we can use the prescription of Section 2.1 to build a Lie–Poisson bracket,

$$\{F, G\} = \int_{\Omega} \left(\omega \left[\frac{\delta F}{\delta \omega}, \frac{\delta G}{\delta \omega} \right] + \psi \left(\left[\frac{\delta F}{\delta \omega}, \frac{\delta G}{\delta \psi} \right] - \left[\frac{\delta G}{\delta \omega}, \frac{\delta F}{\delta \psi} \right] \right) \right) d^2x.$$

Let $\omega = \nabla^2 \phi$ be the (scalar) parallel vorticity, where ϕ is the electric potential, ψ is the poloidal magnetic flux, and $J = \nabla^2 \psi$ is the poloidal current. (We use the same symbol for the electric field as for the streamfunction in Section 2.2.2 since they play a similar role.) The pairing used is a dot product of the vectors followed by an integral over the fluid domain (again identifying \mathfrak{g} and \mathfrak{g}^* as in Section 2.2.2). The Hamiltonian

$$H[\omega; \psi] = \frac{1}{2} \int_{\Omega} \left(|\nabla \phi|^2 + |\nabla \psi|^2 \right) d^2x$$

with the above bracket leads to the equations of motion

$$\begin{aligned} \dot{\omega} &= [\omega, \phi] + [\psi, J], \\ \dot{\psi} &= [\psi, \phi]. \end{aligned} \quad (2.14)$$

This is a model for low-beta reduced MHD [73, 87, 98], obtained by an expansion in the inverse aspect ratio ϵ of a tokamak, with ϵ small. With a strong toroidal magnetic field, the dynamics are then approximately two-dimensional. The model is referred to as low-beta because the electron beta (the ratio of electron pressure to magnetic pressure, see (2.16)) is of order ϵ^2 .

For high-beta reduced MHD, the electron beta is taken to be of order ϵ . There is then an additional advected pressure variable, which couples to the vorticity equation, and the system still has a semidirect sum structure [35, 88].

Benjamin [12] used a system with a similar Lie–Poisson structure, but for waves in a density-stratified fluid. Semidirect sum structures are ubiquitous in advective systems: one variable (in this example, ϕ) “drags” the others along [90].

2.2.4 Compressible Reduced MHD

In general there are other, more general ways to extend Lie algebras besides the semidirect sum. The model derived by Hazeltine *et al.* [33, 34, 35] for two-dimensional compressible reduced MHD (CRMHD) is an example. This model has four fields, and as for the low-beta reduced MHD system in Section 2.2.3 it is obtained from an expansion in the inverse aspect ratio of a tokamak. It includes compressibility and finite ion Larmor radius effects. The Hamiltonian is

$$H[\omega, v, p, \psi] = \frac{1}{2} \int_{\Omega} \left(|\nabla\phi|^2 + v^2 + \frac{(p - 2\beta_e x)^2}{\beta_e} + |\nabla\psi|^2 \right) d^2x, \quad (2.15)$$

where v is the ion parallel (toroidal) velocity, p is the electron pressure,² β_e is the electron beta,

$$\beta_e := \frac{2T_e}{v_A^2}, \quad (2.16)$$

a parameter that measures compressibility, v_A is the Alfvén speed, and T_e is the electron temperature. The other variables are as in Section 2.2.3. The

²The variable p is actually a deviation of the pressure from a linear gradient. The total pressure is $\bar{p} = p - 2\beta_e x$.

coordinate x points outward from the center of the tokamak in the horizontal plane and y is the vertical coordinate. The motion is made two-dimensional by the strong toroidal magnetic field. The bracket we will use is

$$\begin{aligned} \{F, G\} = & \int_{\Omega} \left(\omega \left[\frac{\delta F}{\delta \omega}, \frac{\delta G}{\delta \omega} \right] + v \left(\left[\frac{\delta F}{\delta \omega}, \frac{\delta G}{\delta v} \right] + \left[\frac{\delta F}{\delta v}, \frac{\delta G}{\delta \omega} \right] \right) \right. \\ & + p \left(\left[\frac{\delta F}{\delta \omega}, \frac{\delta G}{\delta p} \right] + \left[\frac{\delta F}{\delta p}, \frac{\delta G}{\delta \omega} \right] \right) + \psi \left(\left[\frac{\delta F}{\delta \omega}, \frac{\delta G}{\delta \psi} \right] + \left[\frac{\delta F}{\delta \psi}, \frac{\delta G}{\delta \omega} \right] \right) \\ & \left. - \beta_e \psi \left(\left[\frac{\delta F}{\delta p}, \frac{\delta G}{\delta v} \right] + \left[\frac{\delta F}{\delta v}, \frac{\delta G}{\delta p} \right] \right) \right) d^2x. \quad (2.17) \end{aligned}$$

Together this bracket and the Hamiltonian (2.15) lead to the equations

$$\begin{aligned} \dot{\omega} &= [\omega, \phi] + [\psi, J] + 2[p, x] \\ \dot{v} &= [v, \phi] + [\psi, p] + 2\beta_e [x, \psi] \\ \dot{p} &= [p, \phi] + \beta_e [\psi, v] \\ \dot{\psi} &= [\psi, \phi], \end{aligned}$$

which reduce to the example of Section 2.2.3 in the limit of $v = p = \beta_e = 0$ (when compressibility effects are unimportant). In the limit of $\beta_e = 0$, the parallel velocity decouples from the other equations, and we recover the three equations of high-beta reduced MHD for ω , ψ , and p [35].

It is far from clear that the Jacobi identity is satisfied for (2.17). A direct verification is straightforward (if tedious), but we shall see in Section 2.3 that there is an easier way.

2.3 General Lie Algebra Extensions

We wish to generalize the types of bracket used in Sections 2.2.3 and 2.2.4. We build an algebra extension by forming an n -tuple of elements of a single Lie

algebra \mathfrak{g} ,

$$\alpha := (\alpha_1, \dots, \alpha_n), \quad (2.18)$$

where $\alpha_i \in \mathfrak{g}$. The most general bracket on this n -tuple space obtained from a linear combination of the one in \mathfrak{g} has components

$$[\alpha, \beta]_\lambda = \sum_{\mu, \nu=1}^n W_\lambda^{\mu\nu} [\alpha_\mu, \beta_\nu], \quad \lambda = 1, \dots, n, \quad (2.19)$$

where the $W_\lambda^{\mu\nu}$ are constants. (From now on we will assume that repeated indices are summed unless otherwise noted.) Since the bracket in \mathfrak{g} is antisymmetric the W 's must be symmetric in their upper indices,

$$W_\lambda^{\mu\nu} = W_\lambda^{\nu\mu}. \quad (2.20)$$

This bracket must also satisfy the Jacobi identity

$$[\alpha, [\beta, \gamma]]_\lambda + [\beta, [\gamma, \alpha]]_\lambda + [\gamma, [\alpha, \beta]]_\lambda = 0, \quad \lambda = 1, \dots, n.$$

The first term can be written

$$[\alpha, [\beta, \gamma]]_\lambda = W_\lambda^{\sigma\tau} W_\sigma^{\mu\nu} [\alpha_\tau, [\beta_\mu, \gamma_\nu]],$$

which when added to the other two gives

$$W_\lambda^{\sigma\tau} W_\sigma^{\mu\nu} ([\alpha_\tau, [\beta_\mu, \gamma_\nu]] + [\beta_\tau, [\gamma_\mu, \alpha_\nu]] + [\gamma_\tau, [\alpha_\mu, \beta_\nu]]) = 0.$$

We cannot yet make use of the Jacobi identity in \mathfrak{g} : the subscripts of α , β , and γ are different in each term so they represent different elements of \mathfrak{g} . We first relabel the sums and then make use of the Jacobi identity in \mathfrak{g} to obtain

$$\begin{aligned} & (W_\lambda^{\sigma\tau} W_\sigma^{\mu\nu} - W_\lambda^{\sigma\nu} W_\sigma^{\tau\mu}) [\alpha_\tau, [\beta_\mu, \gamma_\nu]] \\ & + (W_\lambda^{\sigma\mu} W_\sigma^{\nu\tau} - W_\lambda^{\sigma\nu} W_\sigma^{\tau\mu}) [\beta_\mu, [\gamma_\nu, \alpha_\tau]] = 0. \end{aligned}$$

This identity is satisfied if and only if

$$W_\lambda^{\sigma\tau} W_\sigma^{\mu\nu} = W_\lambda^{\sigma\nu} W_\sigma^{\tau\mu}, \quad (2.21)$$

which together with (2.20) implies that the quantity $W_\lambda^{\sigma\tau} W_\sigma^{\mu\nu}$ is symmetric in all three free upper indices. If we write the W 's as n matrices $W^{(\nu)}$ with rows labeled by λ and columns by μ ,

$$[W^{(\nu)}]_\lambda^\mu := W_\lambda^{\mu\nu}, \quad (2.22)$$

then (2.21) says that those matrices pairwise commute:

$$W^{(\nu)} W^{(\sigma)} = W^{(\sigma)} W^{(\nu)}. \quad (2.23)$$

Equations (2.20) and (2.23) form a necessary and sufficient condition: a set of n commuting matrices of size $n \times n$ satisfying the symmetry given by (2.20) can be used to make a good Lie algebra bracket. From this Lie bracket we can build a Lie–Poisson bracket using the prescription of (2.1) to obtain

$$\{F, G\}_\pm(\xi) = \pm \sum_{\lambda, \mu, \nu=1}^n W_\lambda^{\mu\nu} \left\langle \xi^\lambda, \left[\frac{\delta F}{\delta \xi^\mu}, \frac{\delta G}{\delta \xi^\nu} \right] \right\rangle.$$

We now return to the two extension examples of Sections 2.2.3 and 2.2.4 and examine them in light of the general extension concept introduced here.

2.3.1 Low-beta Reduced MHD

For this example we have $(\xi^0, \xi^1) = (\omega, \psi)$, with

$$W^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad W^{(1)} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The reason why we start labeling at 0 will become clearer in Section 4.4. The two $W^{(\mu)}$ must commute since $W^{(0)} = I$, the identity. The tensor W also satisfies the symmetry property (2.20). Hence, the bracket is a good Lie algebra bracket.

2.3.2 Compressible Reduced MHD

We have $n = 4$ and take $(\xi^0, \xi^1, \xi^2, \xi^3) = (\omega, v, p, \psi)$, so the tensor W is given by

$$\begin{aligned} W^{(0)} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & W^{(1)} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\beta_e & 0 \end{pmatrix}, \\ W^{(2)} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -\beta_e & 0 & 0 \end{pmatrix}, & W^{(3)} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (2.24)$$

It is easy to verify that these matrices commute and that the tensor W satisfies the symmetry property, so that the Lie–Poisson bracket given by (2.17) satisfies the Jacobi identity. (See Section 4.4 for an explanation of why the labeling is chosen to begin at zero.)

The 3-tensor W can be represented as a cubical array of numbers, in the same way a matrix is a square array. In Figure 2.1 we show a schematic representation of W for CRMHD. The blocks represent nonzero elements.

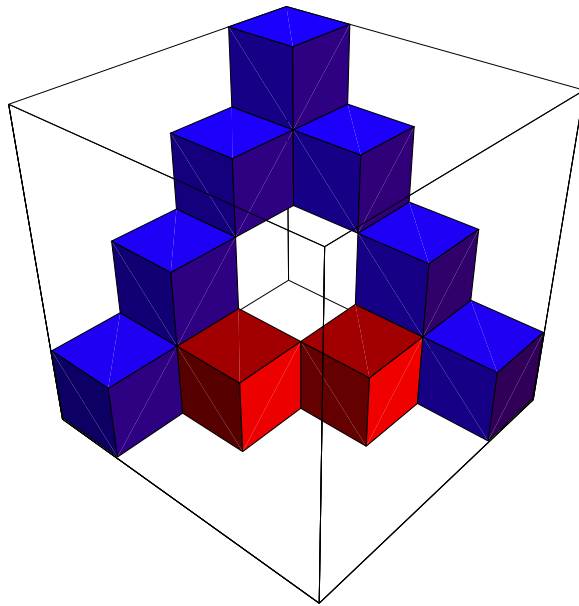


Figure 2.1: Schematic representation of the 3-tensor W for compressible reduced MHD. The blue cubes represent unit entries, the red cubes are equal to $-\beta_e$, and all other entries vanish. The vertical axis is the lower index λ of $W_\lambda^{\mu\nu}$, and the two horizontal axes are the symmetric upper indices μ and ν . The origin is at the top-rear.

Chapter 3

Extension of a Lie Algebra

In this chapter we review the theory of Lie algebra cohomology and its application to extensions. This is useful for shedding light on the methods used in Chapter 4 for classifying the extensions. However, the mathematical details presented in this chapter can be skipped without seriously compromising the flavor of the classification scheme of Chapter 4. Most necessary mathematical concepts will be defined as needed, but the reader wishing more extensive definitions may want to consult books such as Azcárraga and Izquierdo [21] or Choquet-Bruhat and DeWitt-Morette [20].

3.1 Cohomology of Lie Algebras

We now introduce the abstract formalism of Lie algebra cohomology. Historically there were two different reasons for the development of this theory. One, known as the Chevalley–Eilenberg formulation [19], was developed from de Rham cohomology. de Rham cohomology concerns the relationship between exact and closed differential forms, which is determined by the global properties (topology) of a differentiable manifold. A Lie group is a differentiable manifold and so has an associated de Rham cohomology. If invariant differential forms are used in the computation, one is led to the cohomology of Lie

algebras presented in this section [21, 20, 47]. The second motivation is the one that concerns us: we will show in Section 3.2 that the extension problem—the problem of enumerating extensions of a Lie algebra—can be related to the cohomology of Lie algebras.

Let \mathfrak{g} be a Lie algebra, and let the vector space V over the field K (which we take to be the real numbers later) be a left \mathfrak{g} -module,¹ that is, there is an operator $\rho : \mathfrak{g} \times V \rightarrow V$ such that

$$\begin{aligned}\rho_\alpha(v + v') &= \rho_\alpha v + \rho_\alpha v', \\ \rho_{\alpha+\alpha'} v &= \rho_\alpha v + \rho_{\alpha'} v, \\ \rho_{[\alpha, \alpha']} v &= [\rho_\alpha, \rho_{\alpha'}] v,\end{aligned}\tag{3.1}$$

for $\alpha, \alpha' \in \mathfrak{g}$ and $v, v' \in V$. The operator ρ is known as a left *action*. A \mathfrak{g} -module gives a representation of \mathfrak{g} on V . The action ρ defines a Lie algebra *homomorphism* from \mathfrak{g} to the algebra of linear transformations on V . A Lie algebra homomorphism $f : \mathfrak{g} \rightarrow \mathfrak{a}$ is a linear mapping between two Lie algebras \mathfrak{g} and \mathfrak{a} which preserves the Lie algebra structure, that is

$$f([\alpha, \beta]_{\mathfrak{g}}) = [f(\alpha), f(\beta)]_{\mathfrak{a}}, \quad \alpha, \beta \in \mathfrak{g}.$$

An n -dimensional V -valued *cochain* ω_n for \mathfrak{g} , or just n -cochain for short, is a skew-symmetric n -linear mapping

$$\omega_n : \mathfrak{g} \times \overset{\leftarrow n \rightarrow}{\mathfrak{g} \times \cdots \times \mathfrak{g}} \longrightarrow V.$$

¹When V is a right \mathfrak{g} -module, we have $\rho_{[\alpha, \alpha']} = -[\rho_\alpha, \rho_{\alpha'}]$. The results of this section can be adapted to a right action by changing the sign every time a commutator appears. This sign choice is for similar reasons as that of (2.1).

Cochains are Lie algebra cohomology analogues of differential forms on a manifold. Addition and scalar multiplication of n -cochains are defined in the obvious manner by

$$\begin{aligned} (\omega_n + \omega'_n)(\alpha_1, \dots, \alpha_n) &:= \omega_n(\alpha_1, \dots, \alpha_n) + \omega'_n(\alpha_1, \dots, \alpha_n), \\ (a\omega_n)(\alpha_1, \dots, \alpha_n) &:= a\omega_n(\alpha_1, \dots, \alpha_n), \end{aligned}$$

where $\alpha_1, \dots, \alpha_n \in \mathfrak{g}$ and $a \in K$. The set of all n -cochains thus forms a vector space over the field K and is denoted by $C^n(\mathfrak{g}, V)$. The 0-cochains are defined to be just elements of V , so that $C^0(\mathfrak{g}, V) = V$.

The *coboundary operator* is the map between cochains,

$$s_n : C^n(\mathfrak{g}, V) \longrightarrow C^{n+1}(\mathfrak{g}, V),$$

defined by

$$\begin{aligned} (s_n \omega_n)(\alpha_1, \dots, \alpha_{n+1}) &:= \sum_{i=1}^{n+1} (-)^{i+1} \rho_{\alpha_i} \omega_n(\alpha_1, \dots, \hat{\alpha}_i, \dots, \alpha_{n+1}) \\ &\quad + \sum_{\substack{j,k=1 \\ j < k}}^{n+1} (-)^{j+k} \omega_n([\alpha_j, \alpha_k], \alpha_1, \dots, \hat{\alpha}_j, \dots, \hat{\alpha}_k, \dots, \alpha_{n+1}), \end{aligned}$$

where the caret means an argument is omitted. We shall often drop the n subscript on s_n , deducing it from the dimension of the cochain on which s acts.

We shall make use mostly of the first few cases,

$$(s\omega_0)(\alpha_1) = \rho_{\alpha_1} \omega_0, \tag{3.2}$$

$$(s\omega_1)(\alpha_1, \alpha_2) = \rho_{\alpha_1} \omega_1(\alpha_2) - \rho_{\alpha_2} \omega_1(\alpha_1) - \omega_1([\alpha_1, \alpha_2]), \tag{3.3}$$

$$\begin{aligned} (s\omega_2)(\alpha_1, \alpha_2, \alpha_3) &= \rho_{\alpha_1} \omega_2(\alpha_2, \alpha_3) + \rho_{\alpha_2} \omega_2(\alpha_3, \alpha_1) + \rho_{\alpha_3} \omega_2(\alpha_1, \alpha_2) \\ &\quad - \omega_2([\alpha_1, \alpha_2], \alpha_3) - \omega_2([\alpha_2, \alpha_3], \alpha_1) - \omega_2([\alpha_3, \alpha_1], \alpha_2). \end{aligned} \tag{3.4}$$

It is easy to verify that $s\omega_n$ defines an $(n+1)$ -cochain, and it is straightforward (if tedious) to show that $s_{n+1}s_n = s^2 = 0$. For this to be true, the homomorphism property (3.1) of ρ is crucial.

An *n-cocycle* is an element ω_n of $C^n(\mathfrak{g}, V)$ such that $s_n\omega_n = 0$. An *n-coboundary* ω_{cob} is an element of $C^n(\mathfrak{g}, V)$ for which there exists an element ω_{n-1} of $C^{n-1}(\mathfrak{g}, V)$ such that $\omega_{\text{cob}} = s\omega_{n-1}$. Note that all coboundaries are cocycles, but not vice-versa.

Let

$$Z_\rho^n(\mathfrak{g}, V) = \ker s_n$$

be the vector subspace of all *n-cocycles*, $Z_\rho^n(\mathfrak{g}, V) \subset C^n(\mathfrak{g}, V)$, and let

$$B_\rho^n(\mathfrak{g}, V) = \text{range } s_{n-1}$$

be the vector subspace of all *n-coboundaries*, $B_\rho^n(\mathfrak{g}, V) \subset C^n(\mathfrak{g}, V)$. The *nth cohomology group* of \mathfrak{g} with coefficients in V is defined to be the quotient vector space

$$H_\rho^n(\mathfrak{g}, V) := Z_\rho^n(\mathfrak{g}, V) / B_\rho^n(\mathfrak{g}, V). \quad (3.5)$$

Note that for $n > \dim \mathfrak{g}$, we have $H_\rho^n(\mathfrak{g}, V) = Z_\rho^n(\mathfrak{g}, V) = B_\rho^n(\mathfrak{g}, V) = 0$. This is because one cannot build a nonvanishing antisymmetric quantity with more indices than the dimension of the space (at least two of the indices would always be equal, which implies that the quantity is zero).

3.2 Application of Cohomology to Extensions

In Section 2.3 we gave a definition of extension that is specific to our problem, in terms of the tensors W . We will now define extensions in a more abstract

manner. We then show how the cohomology of Lie algebras of Section 3.1 is related to the problem of classifying extensions. In Chapter 4 we will return to the more concrete concept of extension, of the form given in Section 2.3.

Let $f_i : \mathfrak{g}_i \rightarrow \mathfrak{g}_{i+1}$ be a collection of Lie algebra homomorphisms,

$$\cdots \longrightarrow \mathfrak{g}_i \xrightarrow{f_i} \mathfrak{g}_{i+1} \xrightarrow{f_{i+1}} \mathfrak{g}_{i+2} \longrightarrow \cdots .$$

By the homomorphism property of f_i , we have

$$f_i([\alpha, \beta]_{\mathfrak{g}_i}) = [f_i(\alpha), f_i(\beta)]_{\mathfrak{g}_{i+1}}, \quad \alpha, \beta \in \mathfrak{g}_i.$$

The subscript on the brackets denotes the algebra to which it belongs.

The sequence f_i is called an *exact sequence* of Lie algebra homomorphisms if

$$\text{range } f_i = \ker f_{i+1}.$$

Let \mathfrak{g} , \mathfrak{h} , and \mathfrak{a} be Lie algebras. The algebra \mathfrak{h} is said to be an *extension* of \mathfrak{g} by \mathfrak{a} if there is a short exact sequence of Lie algebra homomorphisms

$$0 \longrightarrow \mathfrak{a} \xrightarrow{i} \mathfrak{h} \xrightleftharpoons[\pi]{\tau} \mathfrak{g} \longrightarrow 0. \quad (3.6)$$

The homomorphism i is an insertion (injection), and π is a projection (surjection). We shall distinguish brackets in the different algebras by appropriate subscripts. We also define $\tau : \mathfrak{g} \rightarrow \mathfrak{h}$ to be a linear mapping such that $\pi \circ \tau = 1|_{\mathfrak{g}}$ (the identity mapping in \mathfrak{g}). Note that τ is not unique, since the kernel of π is not trivial. Let $\beta \in \mathfrak{h}$, $\eta \in \mathfrak{a}$; then

$$\pi[\beta, i\eta]_{\mathfrak{h}} = [\pi\beta, \pi i\eta]_{\mathfrak{g}} = 0,$$

using the homomorphism property of π and $\pi \circ i = 0$, a consequence of the exactness of the sequence. Thus $[\beta, i\eta]_{\mathfrak{h}} \in \ker \pi = \text{range } i$, and $i\mathfrak{a}$ is an ideal

in \mathfrak{h} since $[\beta, i\eta] \in i\mathfrak{a}$. Hence, we can form the quotient algebra $\mathfrak{h}/\mathfrak{a}$, with equivalence classes denoted by $\beta + \mathfrak{a}$. By exactness $\pi(\beta + \mathfrak{a}) = \pi\beta$, so \mathfrak{g} is isomorphic to $\mathfrak{h}/\mathfrak{a}$ and we write $\mathfrak{g} = \mathfrak{h}/\mathfrak{a}$.

Though $i\mathfrak{a}$ is a subalgebra of \mathfrak{h} , $\tau\mathfrak{g}$ is not necessarily a subalgebra of \mathfrak{h} , for in general

$$[\tau\alpha, \tau\beta]_{\mathfrak{h}} \neq \tau[\alpha, \beta]_{\mathfrak{g}},$$

for $\alpha, \beta \in \mathfrak{g}$; that is, τ is not necessarily a homomorphism. The classification problem essentially resides in the determination of how much τ differs from a homomorphism. The cohomology machinery of Section 3.1 is the key to quantifying this difference, and we proceed to show this.

To this end, we use the algebra \mathfrak{a} as the vector space V of Section 3.1, so that \mathfrak{a} will be a left \mathfrak{g} -module. We define the left action as

$$\rho_{\alpha}\eta := i^{-1}[\tau\alpha, i\eta]_{\mathfrak{h}} \quad (3.7)$$

for $\alpha \in \mathfrak{g}$ and $\eta \in \mathfrak{a}$. For \mathfrak{a} to be a left \mathfrak{g} -module, we need ρ to be a homomorphism, i.e., ρ must satisfy (3.1). Therefore consider

$$\begin{aligned} [\rho_{\alpha}, \rho_{\beta}]\eta &= (\rho_{\alpha}\rho_{\beta} - \rho_{\beta}\rho_{\alpha})\eta \\ &= \rho_{\alpha}i^{-1}[\tau\beta, i\eta]_{\mathfrak{h}} - \rho_{\beta}i^{-1}[\tau\alpha, i\eta]_{\mathfrak{h}} \\ &= i^{-1}\left[\tau\alpha, [\tau\beta, i\eta]_{\mathfrak{h}}\right]_{\mathfrak{h}} - i^{-1}\left[\tau\beta, [\tau\alpha, i\eta]_{\mathfrak{h}}\right]_{\mathfrak{h}}, \end{aligned}$$

which upon using the Jacobi identity in \mathfrak{h} becomes

$$\begin{aligned} [\rho_{\alpha}, \rho_{\beta}]\eta &= i^{-1}\left[[\tau\alpha, \tau\beta]_{\mathfrak{h}}, i\eta\right]_{\mathfrak{h}} \\ &= i^{-1}\left[\tau[\alpha, \beta]_{\mathfrak{g}}, i\eta\right]_{\mathfrak{h}} + i^{-1}\left[\left([\tau\alpha, \tau\beta]_{\mathfrak{h}} - \tau[\alpha, \beta]_{\mathfrak{g}}\right), i\eta\right]_{\mathfrak{h}} \\ &= \rho_{[\alpha, \beta]_{\mathfrak{g}}}\eta + i^{-1}\left[\left([\tau\alpha, \tau\beta]_{\mathfrak{h}} - \tau[\alpha, \beta]_{\mathfrak{g}}\right), i\eta\right]_{\mathfrak{h}}. \end{aligned} \quad (3.8)$$

By applying π on the expression in parentheses of the last term of (3.8), we see that it vanishes and so is in $\ker \pi$, and by exactness it is also in $i \mathfrak{a}$. Thus the \mathfrak{h} commutator above involves two elements of $i \mathfrak{a}$. We define $\omega : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{a}$ by

$$\omega(\alpha, \beta) := i^{-1} \left([\tau \alpha, \tau \beta]_{\mathfrak{h}} - \tau [\alpha, \beta]_{\mathfrak{g}} \right). \quad (3.9)$$

The mapping i^{-1} is well defined on $i \mathfrak{a}$. Equation (3.8) becomes

$$[\rho_{\alpha}, \rho_{\beta}] \eta = \rho_{[\alpha, \beta]_{\mathfrak{g}}} \eta + [\omega(\alpha, \beta), \eta]_{\mathfrak{a}}. \quad (3.10)$$

Therefore, ρ satisfies the homomorphism property if either of the following is true:

- (i) \mathfrak{a} is Abelian,
- (ii) τ is a homomorphism,

Condition (i) implies $[\ ,]_{\mathfrak{a}} = 0$, while condition (ii) means

$$[\tau \alpha, \tau \beta]_{\mathfrak{h}} = \tau [\alpha, \beta]_{\mathfrak{g}},$$

which implies $\omega \equiv 0$. If either of these conditions is satisfied, \mathfrak{a} with the action ρ is a left \mathfrak{g} -module. We treat these two cases separately in Sections 3.3 and 3.4, respectively.

3.3 Extension by an Abelian Lie Algebra

In this section we assume that the homomorphism condition (i) at the end of Section 3.2 is met. Therefore \mathfrak{a} is a left \mathfrak{g} -module, and we can define \mathfrak{a} -valued cochains on \mathfrak{g} . In particular, ω defined by (3.9) is a 2-cochain, $\omega \in C^2(\mathfrak{g}, \mathfrak{a})$, that

measures the “failure” of τ to be a homomorphism. We now show, moreover, that ω is a 2-cocycle, $\omega \in Z_\rho^2(\mathfrak{g}, \mathfrak{a})$. By using (3.4),

$$\begin{aligned} (s\omega)(\alpha, \beta, \gamma) &= \rho_\alpha \omega(\beta, \gamma) + \rho_\beta \omega(\gamma, \alpha) + \rho_\gamma \omega(\alpha, \beta) \\ &\quad - \omega([\alpha, \beta]_{\mathfrak{g}}, \gamma) - \omega([\beta, \gamma]_{\mathfrak{g}}, \alpha) - \omega([\gamma, \alpha]_{\mathfrak{g}}, \beta), \\ &= i^{-1}[\tau\alpha, i\omega(\beta, \gamma)]_{\mathfrak{h}} - \omega([\alpha, \beta]_{\mathfrak{g}}, \gamma) + \text{cyc. perm.}, \end{aligned}$$

where we have written “cyc. perm.” to mean cyclic permutations of α, β , and γ .

Using the definition (3.9) of ω , we have

$$\begin{aligned} (s\omega)(\alpha, \beta, \gamma) &= i^{-1} \left[\tau\alpha, [\tau\beta, \tau\gamma]_{\mathfrak{h}} - \tau[\beta, \gamma]_{\mathfrak{g}} \right]_{\mathfrak{h}} \\ &\quad - i^{-1} \left(\left[\tau[\alpha, \beta]_{\mathfrak{g}}, \tau\gamma \right]_{\mathfrak{h}} - \tau \left[[\alpha, \beta]_{\mathfrak{g}}, \gamma \right]_{\mathfrak{g}} \right) + \text{cyc. perm.}, \\ &= i^{-1} \left(\left[\tau\alpha, [\tau\beta, \tau\gamma]_{\mathfrak{h}} \right]_{\mathfrak{h}} + \text{cyc. perm.} \right) \\ &\quad + i^{-1} \tau \left(\left[[\alpha, \beta]_{\mathfrak{g}}, \gamma \right]_{\mathfrak{g}} + \text{cyc. perm.} \right) = 0. \end{aligned}$$

The first parenthesis vanishes by the Jacobi identity in \mathfrak{h} , the second by the Jacobi identity in \mathfrak{g} , and the other terms were canceled in pairs. Hence ω is a 2-cocycle.

Two extensions \mathfrak{h} and \mathfrak{h}' are equivalent if there exists a Lie algebra isomorphism σ such that the diagram

$$\begin{array}{ccccc} & & \mathfrak{h} & & \\ & i \nearrow & \downarrow \sigma & \searrow \pi & \\ 0 & \longrightarrow & \mathfrak{a} & & \mathfrak{g} \longrightarrow 0 \\ & i' \searrow & \downarrow \sigma & \nearrow \pi' & \\ & & \mathfrak{h}' & & \end{array} \quad (3.11)$$

is commutative, that is if $\sigma \circ i = i'$ and $\pi = \pi' \circ \sigma$.

There will be an injection τ associated with π and a τ' associated with π' , such that $\pi \circ \tau = 1|_{\mathfrak{g}} = \pi' \circ \tau'$. The linear map $\nu = \sigma^{-1}\tau' - \tau$ must be from \mathfrak{g} to $i\mathfrak{a}$, so $i^{-1}\nu \in C^1(\mathfrak{g}, \mathfrak{a})$. Consider ρ and ρ' respectively defined using τ, i and τ', i' by (3.7). Then

$$\begin{aligned}
(\rho_\alpha - \rho'_\alpha)\eta &= i^{-1}[\tau\alpha, i\eta]_{\mathfrak{h}} - i'^{-1}[\tau'\alpha, i'\eta]_{\mathfrak{h}'} \\
&= i^{-1}[\tau\alpha, i\eta]_{\mathfrak{h}} - i^{-1}\sigma^{-1}[\sigma(\nu + \tau)\alpha, \sigma i\eta]_{\mathfrak{h}'} \\
&= i^{-1}[\tau\alpha, i\eta]_{\mathfrak{h}} - i^{-1}[(\nu + \tau)\alpha, i\eta]_{\mathfrak{h}} \\
&= -i^{-1}[\nu\alpha, i\eta]_{\mathfrak{h}} = 0,
\end{aligned} \tag{3.12}$$

since \mathfrak{a} is Abelian. Hence, τ and τ' define the same ρ . Now consider the 2-cocycles ω and ω' defined from τ and τ' by (3.9). We have

$$\begin{aligned}
\omega'(\alpha, \beta) - \omega(\alpha, \beta) &= i'^{-1} \left([\tau'\alpha, \tau'\beta]_{\mathfrak{h}'} - \tau'[\alpha, \beta]_{\mathfrak{g}} \right) \\
&\quad - i^{-1} \left([\tau\alpha, \tau\beta]_{\mathfrak{h}} - \tau[\alpha, \beta]_{\mathfrak{g}} \right) \\
&= i^{-1}\sigma^{-1} \left([\sigma(\nu + \tau)\alpha, \sigma(\nu + \tau)\beta]_{\mathfrak{h}'} - \sigma(\nu + \tau)[\alpha, \beta]_{\mathfrak{g}} \right) \\
&\quad - i^{-1} \left([\tau\alpha, \tau\beta]_{\mathfrak{h}} - \tau[\alpha, \beta]_{\mathfrak{g}} \right) \\
&= i^{-1} \left([(\nu + \tau)\alpha, (\nu + \tau)\beta]_{\mathfrak{h}} - \nu[\alpha, \beta]_{\mathfrak{g}} - [\tau\alpha, \tau\beta]_{\mathfrak{h}} \right) \\
&= i^{-1} \left([\tau\alpha, \nu\beta]_{\mathfrak{h}} + [\nu\alpha, \tau\beta]_{\mathfrak{h}} - \nu[\alpha, \beta]_{\mathfrak{g}} \right) \\
&= \rho_\alpha(i^{-1}\nu\beta) - \rho_\beta(i^{-1}\nu\alpha) - i^{-1}\nu[\alpha, \beta]_{\mathfrak{g}}.
\end{aligned}$$

Comparing this with (3.3), we see that

$$\omega' - \omega = s(i^{-1}\nu), \tag{3.13}$$

so ω and ω' differ by a coboundary. Hence, they represent the same element in $H_\rho^2(\mathfrak{g}, \mathfrak{a})$. Equivalent extensions uniquely define an element of the second cohomology group $H_\rho^2(\mathfrak{g}, \mathfrak{a})$. Note that this is true in particular for $\mathfrak{h} = \mathfrak{h}'$, $\sigma = 1$, so that the element of $H_\rho^2(\mathfrak{g}, \mathfrak{a})$ is independent of the choice of τ . equivalent

We are now ready to write down explicitly the bracket in \mathfrak{h} . We can represent an element $\alpha \in \mathfrak{h}$ as a two-tuple: $\alpha = (\alpha_1, \alpha_2)$ where $\alpha_1 \in \mathfrak{g}$ and $\alpha_2 \in \mathfrak{a}$ ($\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{a}$ as a vector space). The injection i is then $i \alpha_2 = (0, \alpha_2)$, the projection π is $\pi(\alpha_1, \alpha_2) = \alpha_1$, and since the extension is independent of the choice of τ we take $\tau \alpha_1 = (\alpha_1, 0)$. By linearity,

$$\begin{aligned} [\alpha, \beta]_{\mathfrak{h}} &= [(\alpha_1, 0), (\beta_1, 0)]_{\mathfrak{h}} + [(0, \alpha_2), (0, \beta_2)]_{\mathfrak{h}} \\ &\quad + [(\alpha_1, 0), (0, \beta_2)]_{\mathfrak{h}} + [(0, \alpha_2), (\beta_1, 0)]_{\mathfrak{h}}. \end{aligned}$$

We know that $[(0, \alpha_2), (0, \beta_2)]_{\mathfrak{h}} = 0$ since \mathfrak{a} is Abelian. By definition of the cocycle ω , Eq. (3.9), we have

$$\begin{aligned} [(\alpha_1, 0), (\beta_1, 0)]_{\mathfrak{h}} &= [\tau \alpha_1, \tau \beta_1]_{\mathfrak{h}} \\ &= i \omega(\alpha_1, \beta_1) + \tau [\alpha_1, \beta_1]_{\mathfrak{g}} \\ &= ([\alpha_1, \beta_1]_{\mathfrak{g}}, \omega(\alpha_1, \beta_1)). \end{aligned}$$

Finally, by the definition of ρ , Eq. (3.7),

$$[(\alpha_1, 0), (0, \beta_2)]_{\mathfrak{h}} = [\tau \alpha_1, i \beta_2]_{\mathfrak{h}} = \rho_{\alpha_1} \beta_2, \quad (3.14)$$

and similarly for $[(0, \alpha_2), (\beta_1, 0)]_{\mathfrak{h}}$, with opposite sign. So the bracket is

$$[\alpha, \beta]_{\mathfrak{h}} = \left([\alpha_1, \beta_1]_{\mathfrak{g}}, \rho_{\alpha_1} \beta_2 - \rho_{\beta_1} \alpha_2 + \omega(\alpha_1, \beta_1) \right). \quad (3.15)$$

As a check we work out the Jacobi identity in \mathfrak{h} :

$$\begin{aligned} \left[\alpha, [\beta, \gamma]_{\mathfrak{h}} \right]_{\mathfrak{h}} &= \left([\alpha_1, [\beta, \gamma]_{\mathfrak{h}}]_{\mathfrak{g}}, \rho_{\alpha_1} [\beta, \gamma]_2 - \rho_{[\beta, \gamma]_1} \alpha_2 + \omega(\alpha_1, [\beta, \gamma]_1) \right) \\ &= \left(\left[\alpha_1, [\beta_1, \gamma_1]_{\mathfrak{g}} \right]_{\mathfrak{g}}, \rho_{\alpha_1} (\rho_{\beta_1} \gamma_2 - \rho_{\gamma_1} \beta_2 + \omega(\beta_1, \gamma_1)) \right. \\ &\quad \left. - \rho_{[\beta_1, \gamma_1]_{\mathfrak{g}}} \alpha_2 + \omega(\alpha_1, [\beta_1, \gamma_1]_{\mathfrak{g}}) \right). \end{aligned}$$

Upon adding permutations, the first component will vanish by the Jacobi identity in \mathfrak{g} . We are left with

$$\begin{aligned} \left[\alpha, [\beta, \gamma]_{\mathfrak{h}} \right]_{\mathfrak{h}} + \text{cyc. perm.} &= \left(0, \left(\rho_{\alpha_1} \rho_{\beta_1} - \rho_{\beta_1} \rho_{\alpha_1} - \rho_{[\alpha_1, \beta_1]_{\mathfrak{g}}} \right) \gamma_2 \right. \\ &\quad \left. + \rho_{\alpha_1} \omega(\beta_1, \gamma_1) - \omega([\alpha_1, \beta_1]_{\mathfrak{g}}, \gamma_1) \right) + \text{cyc. perm.}, \end{aligned}$$

which vanishes by the the homomorphism property of ρ and the fact that ω is a 2-cocycle, Eq. (3.4).

Equation (3.15) is the most general form of the Lie bracket for extension by an Abelian Lie algebra. It turns out that the theory of extension by a non-Abelian algebra can be reduced to the study of extension by the center of \mathfrak{a} , which is Abelian [21]. We will not need this fact here, as the only extensions by non-Abelian algebras we will deal with are of the simpler type of Section 3.4.

We have thus shown that equivalent extensions are enumerated by the second cohomology group $H_{\rho}^2(\mathfrak{g}, \mathfrak{a})$. The coordinate transformation σ used in (3.11) to define equivalence of extensions preserves the form of \mathfrak{g} and \mathfrak{a} as subsets of \mathfrak{h} . However, we have the freedom to choose coordinate transformations which do transform these subsets. All we require is that the isomorphism σ between \mathfrak{h} and \mathfrak{h}' be a Lie algebra homomorphism. We can represent this by the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathfrak{a} & \xrightarrow{i} & \mathfrak{h} & \xrightarrow{\pi} & \mathfrak{g} \longrightarrow 0 \\ & & & & \downarrow \sigma & & \\ 0 & \longrightarrow & \mathfrak{a}' & \xrightarrow{i} & \mathfrak{h}' & \xrightarrow{\pi} & \mathfrak{g}' \longrightarrow 0. \end{array} \quad (3.16)$$

The primed and the unprimed extensions are not equivalent, but they are isomorphic [96, p. 199]. Cohomology for us is not the whole story, since we are

interested in isomorphic extensions, but it will guide our classification scheme. We discuss this point further in Section 4.3.

Diagrams (3.11) and (3.16) are related to the “Short Five Lemma,” which states that if the diagram of Lie algebra homomorphisms

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & \mathfrak{a} & \xrightarrow{i} & \mathfrak{h} & \xrightarrow{\pi} & \mathfrak{g} & \longrightarrow & 0 \\
 & & \downarrow \gamma & & \downarrow \sigma & & \downarrow \delta & & \\
 0 & \longrightarrow & \mathfrak{a}' & \xrightarrow{i} & \mathfrak{h}' & \xrightarrow{\pi} & \mathfrak{g}' & \longrightarrow & 0
 \end{array}$$

is commutative, with the top and bottom rows exact, then

- (i) γ, δ monomorphisms $\implies \sigma$ monomorphism;
- (ii) γ, δ epimorphisms $\implies \sigma$ epimorphism;
- (iii) γ, δ isomorphisms $\implies \sigma$ isomorphism.

A monomorphism is injective, an epimorphism is surjective, and an isomorphism is bijective. The important point is that the converse of the Lemma is not true: if σ is an isomorphism then it says nothing about the properties (or even the existence) of γ and δ . Note that (iii) follows immediately from (i) and (ii). The proof can be found in Mac Lane and Birkhoff [55] or Hungerford [42], for example.

3.4 Semidirect and Direct Extensions

Assume now that ω defined by (3.9) is a coboundary. By (3.13) there exists an equivalent extension with $\omega \equiv 0$. For that equivalent extension τ is a homomorphism and condition (ii) at the end of Section 3.2 is satisfied. Thus the sequence

$$\mathfrak{h} \xleftarrow{\tau} \mathfrak{g} \longleftarrow 0 \tag{3.17}$$

is an exact sequence of Lie algebra homomorphisms, as well as the sequence given by (3.6). We then say that the extension is a semidirect extension (or a semidirect sum of algebras) by analogy with the group case. More generally, we say that \mathfrak{h} splits if it is isomorphic to a semidirect sum, which corresponds to ω being a coboundary, not necessarily zero. If \mathfrak{a} is not Abelian, then (3.12) is not satisfied and two equivalent extensions (or two different choices of τ) do not necessarily lead to the same ρ .

Representing elements of \mathfrak{h} as 2-tuples, as in Section 3.3, we can derive the bracket in \mathfrak{h} for a semidirect sum. The difference is that τ is a homomorphism so that

$$[(\alpha_1, 0), (\beta_1, 0)]_{\mathfrak{h}} = [\tau \alpha_1, \tau \beta_1]_{\mathfrak{h}} = \tau [\alpha_1, \beta_1]_{\mathfrak{g}} = ([\alpha_1, \beta_1]_{\mathfrak{g}}, 0),$$

and \mathfrak{a} is not assumed Abelian,

$$[(0, \alpha_2), (0, \beta_2)]_{\mathfrak{h}} = [i \alpha_2, i \beta_2]_{\mathfrak{h}} = i [\alpha_2, \beta_2]_{\mathfrak{a}} = (0, [\alpha_2, \beta_2]_{\mathfrak{a}}),$$

which together with (3.14) gives

$$[\alpha, \beta]_{\mathfrak{h}} = \left([\alpha_1, \beta_1]_{\mathfrak{g}}, \rho_{\alpha_1} \beta_2 - \rho_{\beta_1} \alpha_2 + [\alpha_2, \beta_2]_{\mathfrak{a}} \right), \quad (3.18)$$

Verifying Jacobi for (3.18) we find the ρ must also satisfy

$$\rho_{\alpha_1} [\beta_2, \gamma_2]_{\mathfrak{a}} = [\rho_{\alpha_1} \beta_2, \gamma_2]_{\mathfrak{a}} + [\beta_2, \rho_{\alpha_1} \gamma_2]_{\mathfrak{a}},$$

which is trivially satisfied if \mathfrak{a} is Abelian, but in general this condition states that ρ_{α} is a derivation on \mathfrak{a} .

Now consider the case where i^{-1} is a homomorphism and

$$\ker i^{-1} = \text{range } \tau.$$

Then the sequence

$$0 \rightleftarrows \mathfrak{a} \begin{array}{c} \xleftarrow{i^{-1}} \\ \xrightarrow{i} \end{array} \mathfrak{h} \begin{array}{c} \xleftarrow{\tau} \\ \xrightarrow{\pi} \end{array} \mathfrak{g} \rightleftarrows 0$$

is exact in both directions and, hence, both i and $\pi = \tau^{-1}$ are bijections. The action of \mathfrak{g} on \mathfrak{a} is

$$\rho_\alpha \eta = i^{-1}[\tau \alpha, i\eta]_{\mathfrak{h}} = [i^{-1}\tau \alpha, \eta]_{\mathfrak{a}} = 0$$

since by exactness $i^{-1} \circ \tau = 0$. This is called a direct sum. Note that in this case the role of \mathfrak{g} and \mathfrak{a} is interchangeable and they are both ideals in \mathfrak{h} . The bracket in \mathfrak{h} is easily obtained from (3.18) by letting $\rho = 0$,

$$[\alpha, \beta]_{\mathfrak{h}} = \left([\alpha_1, \beta_1]_{\mathfrak{g}}, [\alpha_2, \beta_2]_{\mathfrak{a}} \right). \quad (3.19)$$

Semidirect and direct extensions play an important role in physics. A simple example of a semidirect sum structure is when \mathfrak{g} is the Lie algebra $so(3)$ associated with the rotation group $SO(3)$ and \mathfrak{a} is \mathbb{R}^3 . Their semidirect sum is the algebra of the six parameter Euclidean group of rotations and translations. This algebra can be used in a Lie–Poisson bracket to describe the dynamics of the heavy top (see for example [40, 56, 92]). We have already discussed the semidirect sum in Section 2.2.3. The bracket (2.13) is a semidirect sum, with \mathfrak{g} the algebra of the group of volume-preserving diffeomorphisms and \mathfrak{a} the Abelian Lie algebra of functions on \mathbb{R}^2 . The action is just the adjoint action $\rho_\alpha v := [\alpha, v]$ obtained by identifying \mathfrak{g} and \mathfrak{a} .

In general, semidirect Lie–Poisson structures appear in systems where the field variables are in some sense “slaved” to the base variable (the one associated with \mathfrak{h}) [57, 90]. Here, the advected quantities are forced to move

on the coadjoint orbits of the Lie group G . This is seen directly from the equations of motion (2.6), since, for a semidirect sum,

$$\dot{\xi}^\mu = - \left[\frac{\delta H}{\delta \xi^0}, \xi^\mu \right]^\dagger = -\text{ad}_{\delta H / \delta \xi^0}^\dagger \xi^\mu,$$

which is by definition the infinitesimal generator of the coadjoint orbits of the Lie group [58] (see Section 2.1). For example, the coadjoint orbits of $SO(3)$ are spheres, so the semidirect product² of $SO(3)$ and \mathbb{R}^3 leads to a physical system where the dynamics are confined to spheres, which naturally describes rigid body motion. In other words, the coadjoint orbits of the semidirect product of G and \mathbb{R}^3 are isomorphic to the coadjoint orbits of G . We shall have more to say on this in Section 6.3.4.

A Lie–Poisson bracket built from a direct sum is just a sum of the separate brackets. The dynamical interaction between the variables can only come from the Hamiltonian or from constitutive equations. For example, in the baroclinic instability model of two superimposed two fluid layers with different potential vorticities, the two layers are coupled through the potential vorticity relation [64]. A very similar model with a direct sum structure exists in MHD for studying magnetic reconnection [17].

3.4.1 Classification of Splitting Extensions

We now briefly mention the connection between the first cohomology group and splitting extensions. This will not be used directly in the classification scheme of Chapter 4, but we include it for completeness. We assume in this section that \mathfrak{a} is Abelian. In (3.17) we had chosen the canonical τ , $\tau(\alpha) = (\alpha, 0)$. Now

²Semidirect *product* is the term used for groups, semidirect *sum* for algebras.

suppose we use instead

$$\tau'(\alpha) = (\alpha, \nu(\alpha)). \quad (3.20)$$

Here ν is a linear map from \mathfrak{g} to \mathfrak{a} and is thus an element of $C^1(\mathfrak{g}, \mathfrak{a})$, a 1-cochain. If τ' is a Lie algebra homomorphism,

$$\tau'([\alpha, \beta]_{\mathfrak{g}}) = \left([\alpha, \beta]_{\mathfrak{g}}, \nu([\alpha, \beta]_{\mathfrak{g}}) \right) \quad (3.21)$$

must be equal to

$$[\tau'(\alpha), \tau'(\beta)]_{\mathfrak{h}} = \left([\alpha, \beta]_{\mathfrak{g}}, \rho_{\alpha} \nu(\beta) - \rho_{\beta} \nu(\alpha) \right) \quad (3.22)$$

subtracting (3.21) and (3.22) gives

$$\rho_{\alpha} \nu(\beta) - \rho_{\beta} \nu(\alpha) - \nu([\alpha, \beta]_{\mathfrak{g}}) = s \nu(\alpha, \beta) = 0, \quad (3.23)$$

from (3.3). Hence ν is a cocycle, with coboundaries given by

$$\nu(\alpha) = \rho_{\alpha} \eta_0, \quad \eta_0 \in \mathfrak{a}, \quad (3.24)$$

The first cohomology group $H_{\rho}^1(\mathfrak{g}, \mathfrak{a})$ classifies splitting extensions of \mathfrak{h} by \mathfrak{a} modulo those given in terms of the coboundaries (3.24).

Chapter 4

Classification of Extensions of a Lie Algebra

In this chapter we return to the main problem introduced in Section 2.3: the classification of algebra extensions built by forming n -tuples of elements of a single Lie algebra \mathfrak{g} . The elements of this Lie algebra \mathfrak{h} are written as $\alpha := (\alpha_1, \dots, \alpha_n)$, $\alpha_i \in \mathfrak{g}$, with a bracket defined by

$$[\alpha, \beta]_\lambda = W_\lambda^{\mu\nu} [\alpha_\mu, \beta_\nu], \quad (2.19)$$

where $W_\lambda^{\mu\nu}$ are constants. We will call n the *order* of the extension. Recall (see Section 2.3) that the W 's are symmetric in their upper indices,

$$W_\lambda^{\mu\nu} = W_\lambda^{\nu\mu}, \quad (2.20)$$

and commute,

$$W^{(\nu)} W^{(\sigma)} = W^{(\sigma)} W^{(\nu)}, \quad (2.23)$$

where the $n \times n$ matrices $W^{(\nu)}$ are defined by $[W^{(\nu)}]_\lambda^\mu := W_\lambda^{\nu\mu}$. Since the W 's are 3-tensors we can also represent their elements by matrices obtained by fixing the lower index,

$$W_{(\lambda)} : [W_{(\lambda)}]^{\mu\nu} := W_\lambda^{\mu\nu}, \quad (4.1)$$

which are symmetric but do not commute. Either collection of matrices, (2.22) or (4.1), completely describes the Lie bracket, and which one we use will be understood by whether the parenthesized index is up or down.

What do we mean by a classification? A classification is achieved if we obtain a set of normal forms for the extensions which are independent, that is not related by linear transformations. We use linear transformations because they preserve the Lie–Poisson structure—they amount to transformations of the W tensor. We thus begin by assuming the most general W possible.

We first show in Section 4.1 how an extension can be broken down into a direct sum of degenerate subblocks (degenerate in the sense that the eigenvalues have multiplicity greater than unity). The classification scheme is thus reduced to the study of a single degenerate subblock. In Section 4.2 we couch our particular extension problem in terms of the Lie algebra cohomology language of Section 3.2 and apply the techniques therein. The limitations of this cohomology approach are investigated in Section 4.3, and we look at other coordinate transformations that do not necessarily preserve the extension structure of the algebra, as expressed in diagram (3.16). In Section 4.5 we introduce a particular type of extension, called the Leibniz extension, that is in a sense the “maximal” extension. Finally, in Section 4.6 we give an explicit classification of solvable extensions up to order four.

4.1 Direct Sum Structure

A set of commuting matrices can be put into simultaneous block-diagonal form by a coordinate transformation, each block corresponding to a degenerate eigenvalue [89]. Let us denote the change of basis by a matrix $M_{\beta}^{\bar{\alpha}}$, with inverse $(M^{-1})_{\bar{\alpha}}^{\beta}$, such that the matrix $\widetilde{W}^{(\nu)}$, whose components are given by

$$\widetilde{W}_{\beta}^{\bar{\alpha}\nu} = (M^{-1})_{\beta}^{\lambda} W_{\lambda}^{\mu\nu} M_{\mu}^{\bar{\alpha}} ,$$

is in block-diagonal form for all ν [89]. However, $W_\lambda^{\mu\nu}$ is a 3-tensor and so the third index is also subject to the coordinate change:

$$\overline{W}_\beta^{\alpha\tilde{\gamma}} = \widetilde{W}_\beta^{\alpha\nu} M_\nu^{\tilde{\gamma}} .$$

This last step only adds linear combinations of the $\widetilde{W}^{(\nu)}$'s together, so the $\widetilde{W}^{(\nu)}$'s and the $\overline{W}^{(\tilde{\gamma})}$'s have the same block-diagonal structure. Note that the $\overline{W}_\beta^{\alpha\tilde{\gamma}}$ are still symmetric in their upper indices, since this property is preserved by a change of basis:

$$\begin{aligned} \overline{W}_\beta^{\alpha\tilde{\gamma}} &= (M^{-1})_{\tilde{\beta}}^\lambda W_\lambda^{\mu\nu} M_\mu^{\tilde{\alpha}} M_\nu^{\tilde{\gamma}} \\ &= (M^{-1})_{\tilde{\beta}}^\lambda W_\lambda^{\nu\mu} M_\nu^{\tilde{\alpha}} M_\mu^{\tilde{\gamma}} \quad (\text{Relabeling } \mu \text{ and } \nu) \\ &= (M^{-1})_{\tilde{\beta}}^\lambda W_\lambda^{\mu\nu} M_\mu^{\tilde{\gamma}} M_\nu^{\tilde{\alpha}} \\ &= \overline{W}_\beta^{\tilde{\gamma}\tilde{\alpha}} . \end{aligned}$$

So from now on we just assume that we are working in a basis where the $W^{(\nu)}$'s are block-diagonal and symmetric in their upper indices; this symmetry means that if we look at a W as a cube, then in the block-diagonal basis it consists of smaller cubes along the main diagonal. This is the 3-tensor equivalent of a block-diagonal matrix, as illustrated in Figure 4.1, a pictorial representation of a direct sum of extensions.

4.1.1 Example: three-field model of MHD

We consider as an example of a direct sum structure a three-field model of MHD due to Hazeltine [30, 32]. In addition to the vorticity ω and the magnetic flux ψ (see Section 2.2.3), the model also includes a field χ which measures plasma density perturbations. The model includes as limits the RMHD system

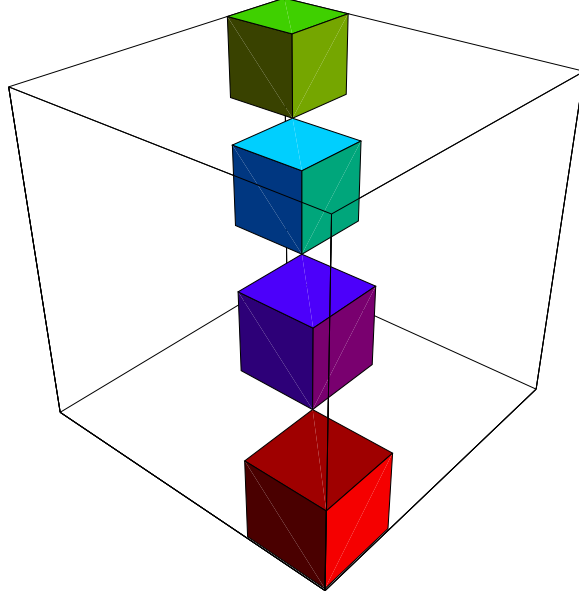


Figure 4.1: Schematic representation of the 3-tensor W for a direct sum of extensions. The cubes represent potentially nonzero elements.

of Section 2.2.3 and the Charney–Hasegawa–Mima equation [41]. We thus have $\xi = (\omega, \psi, \chi)$, with the Hamiltonian

$$H = \frac{1}{2} \langle |\nabla\phi|^2 + |\nabla\psi|^2 + \alpha\chi^2 \rangle, \quad (4.2)$$

and bracket represented by the matrices

$$W^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad W^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad W^{(3)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

The matrices commute and obey the symmetry (2.20), so they form a good bracket. As in Section 2.2.3, the electric potential is denoted by ϕ and the electric current by J . The equations of motion are given by

$$\begin{aligned} \dot{\omega} &= [\omega, \phi] + [\psi, J], \\ \dot{\psi} &= [\psi, \phi] + \alpha[\chi, \psi], \\ \dot{\chi} &= [\chi, \phi] + [\psi, J]. \end{aligned} \quad (4.3)$$

The $W^{(\mu)}$'s are not in block triangular form, and since $W^{(3)}$ has eigenvalues which are not threefold degenerate we know the extension can be blocked-up further. Indeed, the coordinate transformation $\eta^{\bar{\mu}} = \xi^\nu M_\nu^{\bar{\mu}}$, with

$$M = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \quad (4.4)$$

will transform the extension to

$$\overline{W}^{(1)} = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right), \quad \overline{W}^{(2)} = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), \quad \overline{W}^{(3)} = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right),$$

where we have explicitly indicated the blocks. The extension is also block-diagonal in the alternate, lower-indexed representation,

$$\overline{W}_{(1)} = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), \quad \overline{W}_{(2)} = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), \quad \overline{W}_{(3)} = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right).$$

This is what was meant by ‘‘cubes’’ at the end of the previous section.

At the bracket level the variables $\eta^{\bar{1}}$ and $\eta^{\bar{2}}$ are decoupled from $\eta^{\bar{3}}$. But under the transformation (4.4) the Hamiltonian (4.2) becomes

$$\bar{H} = \frac{1}{2} \left\langle |\nabla(\eta^{\bar{1}} + \eta^{\bar{3}})|^2 + |\nabla\eta^{\bar{2}}|^2 + \alpha |\eta^{\bar{1}}|^2 \right\rangle.$$

The new equations of motion are thus

$$\begin{aligned} \dot{\eta}^{\bar{1}} &= [\eta^{\bar{1}}, \bar{\phi}] + [\eta^{\bar{2}}, \bar{J}], \\ \dot{\eta}^{\bar{2}} &= [\eta^{\bar{2}}, \bar{\phi} - \alpha \eta^{\bar{1}}], \\ \dot{\eta}^{\bar{3}} &= [\eta^{\bar{3}}, \bar{\phi}]. \end{aligned}$$

with $\nabla^2 \bar{\phi} := \eta^{\bar{1}} + \eta^{\bar{3}}$ and $\bar{J} := \nabla^2 \eta^{\bar{2}}$. The variable $\eta^{\bar{3}}$ is still coupled to the other variables through the defining relation for $\bar{\phi}$.

4.1.2 Lower-triangular Structure

Block-diagonalization is the first step in the classification: each subblock of W is associated with an ideal (hence, a subalgebra) in the full n -tuple algebra \mathfrak{g} . (A subset $\mathfrak{a} \subseteq \mathfrak{h}$ is an ideal in the Lie algebra \mathfrak{h} if $[\mathfrak{h}, \mathfrak{a}] \subseteq \mathfrak{a}$. Ideals are subalgebras.) Hence, by the definition of Section 3.4, the algebra \mathfrak{g} is a direct sum of the algebra denoted by each subblock. Each of these algebras can be studied independently, that is we can focus our attention on a *single subblock*. So from now on we assume that we have n commuting matrices, each with n -fold degenerate eigenvalues. The eigenvalues can, however, be different for each matrix.

Such a set of commuting matrices can be put into lower-triangular form by a coordinate change, and again the transformation of the third index preserves this structure (though it can change the eigenvalue of each matrix). The eigenvalue of each matrix lies on the diagonal; we denote the eigenvalue of $W^{(\mu)}$ by $\Lambda^{(\mu)}$. We write the quantity $W_1^{\mu\nu}$ as the matrix

$$W_{(1)} = \begin{pmatrix} \Lambda^{(1)} & 0 & 0 & \dots & 0 \\ \Lambda^{(2)} & 0 & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ \Lambda^{(n)} & 0 & 0 & \dots & 0 \end{pmatrix},$$

which consists of the first row of the lower-triangular matrices $W^{(\mu)}$ as prescribed by (4.1). Evidently, the symmetry of $W_{(1)}$ requires

$$\Lambda^{(\nu)} = \theta \delta_1^\nu,$$

that is, all the matrices $W^{(\mu)}$ are nilpotent (their eigenvalues vanish) except for $W^{(1)}$ when $\theta \neq 0$. If this first eigenvalue is nonzero then it can be scaled to $\theta = 1$ by the coordinate transformation $M_\nu^{\bar{\alpha}} = \theta^{-1} \delta_\nu^{\bar{\alpha}}$. We will use the

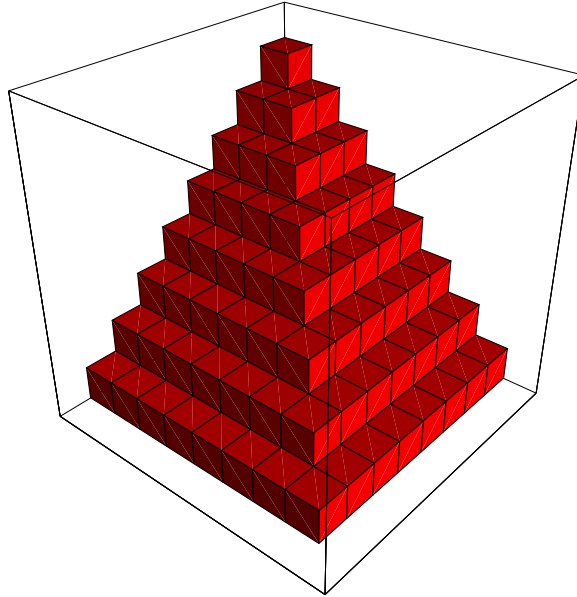


Figure 4.2: Schematic representation of the 3-tensor W for a solvable extension. The cubes represent potentially nonzero elements. The vertical axis is the lower index λ of $W_\lambda^{\mu\nu}$, and the two horizontal axes are the symmetric upper indices μ and ν . The origin is at the top-rear. The pyramid-like structure is a consequence of the symmetry of W and of its lower-triangular structure in this basis.

symbol θ to mean a variable which can take the value 0 or 1. Figure 4.2 shows the structure, with $\theta = 0$, of a degenerate extension, after lower-triangularity and symmetry of the upper indices of W are taken into account.

4.2 Connection to Cohomology

We now bring together the abstract notions of Chapter 3 with the n -tuple extensions of Section 2.3. It is shown in Section 4.2.1 that we need only classify the case of $\theta = 0$. This case will be seen to correspond to solvable extensions, which we classify in Section 4.2.2.

4.2.1 Preliminary Splitting

Assume we are in the basis described at the end of Section 4.1 and, for now, suppose $\theta = 1$. To place the structure of W in the context of Lie algebras, we first give some definitions. The *derived series* $\mathfrak{g}^{(k)}$ of \mathfrak{g} has terms

$$\begin{aligned}\mathfrak{g}^{(0)} &= \mathfrak{g} \\ \mathfrak{g}^{(1)} &= [\mathfrak{g}, \mathfrak{g}] \\ \mathfrak{g}^{(2)} &= [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] \\ &\vdots \\ \mathfrak{g}^{(k)} &= [\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}],\end{aligned}\tag{4.5}$$

where by $[\mathfrak{g}, \mathfrak{g}]$ we mean the set obtained by taking all the possible Lie brackets of elements of \mathfrak{g} . The *lower central series* \mathfrak{g}^k has terms defined by

$$\begin{aligned}\mathfrak{g}^0 &= \mathfrak{g} \\ \mathfrak{g}^1 &= [\mathfrak{g}, \mathfrak{g}] \\ \mathfrak{g}^2 &= [\mathfrak{g}, \mathfrak{g}^1] \\ &\vdots \\ \mathfrak{g}^k &= [\mathfrak{g}, \mathfrak{g}^{(k-1)}].\end{aligned}\tag{4.6}$$

An algebra \mathfrak{g} is said to be *solvable* if its derived series terminates, $\mathfrak{g}^{(k)} = 0$, for some k . An algebra \mathfrak{g} is said to be *nilpotent* if its lower central series terminates, $\mathfrak{g}^k = 0$, for some k . Note that a nilpotent algebra is solvable, but not vice-versa [44].

The set of elements of the form $\beta = (0, \beta_2, \dots, \beta_n)$ is a nilpotent ideal in \mathfrak{h} that we denote by \mathfrak{a} (\mathfrak{a} is thus a solvable subalgebra). To see this, observe

that (4.6) involves nested brackets, so that the elements \mathfrak{a}^k of the lower central series will involve k th powers of the $W^{(\mu)}$. But since the $W^{(\mu)}$ with $\mu > 1$ are lower-triangular with zeros along the diagonal, we have $(W^{(\mu)})^{n-1} = 0$, and the lower central series must eventually vanish.

Because \mathfrak{a} is an ideal, we can construct the algebra $\mathfrak{g} = \mathfrak{h}/\mathfrak{a}$, so that \mathfrak{h} is an extension of \mathfrak{g} by \mathfrak{a} . If \mathfrak{g} is semisimple, then \mathfrak{a} is the radical of \mathfrak{h} (the maximal solvable ideal). It is easy to see that the elements of \mathfrak{g} embedded in \mathfrak{h} are of the form $\alpha = (\alpha_1, 0, \dots, 0)$. We will now show that \mathfrak{h} splits; that is, there exist coordinates in which \mathfrak{h} is manifestly the semidirect sum of \mathfrak{g} and the (in general non-Abelian) algebra \mathfrak{a} .

In Appendix B we give a lower-triangular coordinate transformation that makes $W^{(1)} = I$, the identity matrix. Assuming we have effected this transformation, the mappings i , π , and τ of Section 3.2 are given by

$$\begin{aligned} i : \mathfrak{a} &\longrightarrow \mathfrak{h}, & i(\alpha_2, \dots, \alpha_n) &= (0, \alpha_2, \dots, \alpha_n), \\ \pi : \mathfrak{h} &\longrightarrow \mathfrak{g}, & \pi(\alpha_1, \alpha_2, \dots, \alpha_n) &= \alpha_1, \\ \tau : \mathfrak{g} &\longrightarrow \mathfrak{h}, & \tau(\alpha_1) &= (\alpha_1, 0, \dots, 0), \end{aligned}$$

and the cocycle of Eq. (3.9) is

$$\begin{aligned} i\omega(\alpha, \beta) &= [\tau\alpha, \tau\beta]_{\mathfrak{h}} - \tau[\alpha, \beta]_{\mathfrak{g}} \\ &= [(\alpha_1, 0, \dots, 0), (\beta_1, 0, \dots, 0)]_{\mathfrak{h}} - ([\alpha_1, \beta_1], 0, \dots, 0) \\ &= (W_1^{11} [\alpha_1, \beta_1], 0, \dots, 0) - ([\alpha_1, \beta_1], 0, \dots, 0) \\ &= 0, \end{aligned}$$

since $W_1^{11} = 1$. Hence, the extension is a semidirect sum. The coordinate transformation that made $W^{(1)} = I$ removed a coboundary, making the above cocy-

cle vanish identically. For the case where \mathfrak{g} is finite-dimensional and semisimple, we have an explicit demonstration of the Levi decomposition theorem: any finite-dimensional¹ Lie algebra \mathfrak{h} (of characteristic zero) with radical \mathfrak{a} is the semidirect sum of a semisimple Lie algebra \mathfrak{g} and \mathfrak{a} [44].

4.2.2 Solvable Extensions

Above we assumed the eigenvalue θ of the first matrix was unity; however, if this eigenvalue vanishes, then we have a solvable algebra of n -tuples to begin with. Since n is arbitrary we can study these two solvable cases together.

Thus, we now suppose \mathfrak{h} is a solvable Lie algebra of n -tuples (we reuse the symbols \mathfrak{h} , \mathfrak{g} , and \mathfrak{a} to parallel the notation of Section 3.1), where all of the the $W^{(\mu)}$'s are lower-triangular with zeros along the diagonal. Note that $W^{(n)} = 0$, so the set of elements of the form $\alpha = (0, \dots, 0, \alpha_n)$ forms an Abelian subalgebra of \mathfrak{h} . In fact, this subalgebra is an ideal. Now assume \mathfrak{h} contains an Abelian ideal of order $n - m$ (the order of this ideal is at least 1), which we denote by \mathfrak{a} . The elements of \mathfrak{a} can always be cast in the form

$$\alpha = (0, \dots, 0, \alpha_{m+1}, \dots, \alpha_n)$$

via a coordinate transformation that preserves the lower-triangular, nilpotent form of the $W^{(\mu)}$.

We also denote by \mathfrak{g} the algebra of m -tuples with bracket

$$[(\alpha_1, \dots, \alpha_m), (\beta_1, \dots, \beta_m)]_{\mathfrak{g}\lambda} = \sum_{\mu, \nu=1}^m W_{\lambda}^{\mu\nu} [\alpha_{\mu}, \beta_{\nu}], \quad \lambda = 1, \dots, m.$$

¹The inner bracket can be infinite dimensional, but the order of the extension is finite.

It is trivial to show that $\mathfrak{g} = \mathfrak{h}/\mathfrak{a}$, so that \mathfrak{h} is an extension of \mathfrak{g} by \mathfrak{a} . Since \mathfrak{a} is Abelian we can use the formalism of Section 3.1 (the other case we used above was for \mathfrak{a} non-Abelian but where the extension was semidirect). The injection and projection maps are given by

$$\begin{aligned} i : \mathfrak{a} &\longrightarrow \mathfrak{h}, & i(\alpha_{m+1}, \dots, \alpha_n) &= (0, \dots, 0, \alpha_{m+1}, \dots, \alpha_n), \\ \pi : \mathfrak{h} &\longrightarrow \mathfrak{g}, & \pi(\alpha_1, \alpha_2, \dots, \alpha_n) &= (\alpha_1, \dots, \alpha_m), \\ \tau : \mathfrak{g} &\longrightarrow \mathfrak{h}, & \tau(\alpha_1, \dots, \alpha_m) &= (\alpha_1, \dots, \alpha_m, 0, \dots, 0). \end{aligned}$$

From the definition of the action, Eq. (3.7), we have for $\alpha \in \mathfrak{g}$ and $\eta \in \mathfrak{a}$,

$$\begin{aligned} i \rho_\alpha \eta &= [\tau \alpha, i \eta]_{\mathfrak{h}} \\ &= [(\alpha_1, \dots, \alpha_m, 0, \dots, 0), (0, \dots, 0, \eta_{m+1}, \dots, \eta_n)]_{\mathfrak{h}} \\ &= \sum_{\mu=1}^m \sum_{\nu=m+1}^{n-1} (0, \dots, 0, W_{m+2}^{\mu\nu}[\alpha_\mu, \eta_\nu], \dots, W_n^{\mu\nu}[\alpha_\mu, \eta_\nu]). \end{aligned} \quad (4.7)$$

In addition to the action, the solvable extension is also characterized by the cocycle defined in Eq. (3.9),

$$\begin{aligned} i \omega(\alpha, \beta) &= [\tau \alpha, \tau \beta]_{\mathfrak{h}} - \tau [\alpha, \beta]_{\mathfrak{g}} \\ &= [(\alpha_1, \dots, \alpha_m, 0, \dots, 0), (\beta_1, \dots, \beta_m, 0, \dots, 0)]_{\mathfrak{h}} \\ &\quad - \tau [(\alpha_1, \dots, \alpha_m), (\beta_1, \dots, \beta_m)]_{\mathfrak{g}} \\ &= \sum_{\mu, \nu=1}^m (0, \dots, 0, W_{m+1}^{\mu\nu}[\alpha_\mu, \beta_\nu], \dots, W_n^{\mu\nu}[\alpha_\mu, \beta_\nu]). \end{aligned} \quad (4.8)$$

We can illustrate which parts of the W 's contribute to the action and which to the cocycle by writing

$$W_{(\lambda)} = \left(\begin{array}{c|c} \mathbf{w}_\lambda & \mathbf{r}_\lambda \\ \mathbf{r}_\lambda^T & \mathbf{0} \end{array} \right), \quad \lambda = m+1, \dots, n, \quad (4.9)$$

where the \mathbf{w}_λ 's are $m \times m$ symmetric matrices that determine the cocycle ω and the \mathbf{r}_λ 's are $m \times (n - m)$ matrices that determine the action ρ . The zero matrix of size $(n - m) \times (n - m)$ on the bottom right of the $W_{(\lambda)}$'s appears as a consequence of \mathfrak{a} being Abelian.

The algebra \mathfrak{g} is completely characterized by the $W_{(\lambda)}$, $\lambda = 1, \dots, m$. Hence, we can look for the maximal Abelian ideal of \mathfrak{g} and repeat the procedure we used for the full \mathfrak{h} . It is straightforward to show that although coordinate transformations of \mathfrak{g} might change the cocycle ω and the action ρ , they will not alter the *form* of (4.9) .

Recall that in Section 3.1 we defined 2-coboundaries as 2-cocycles obtained from 1-cochains by the coboundary operator, s . The 2-coboundaries turned out to be removable obstructions to a semidirect sum structure. Here the coboundaries are associated with the parts of the $W_{(\lambda)}$ that can be removed by (a restricted class of) coordinate transformations, as shown below.

Let us explore the connection between 1-cochains and coboundaries in the present context. Since a 1-cochain is just a linear mapping from \mathfrak{g} to \mathfrak{a} , for $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathfrak{g}$ we can write this as

$$\omega_\mu^{(1)}(\alpha) = - \sum_{\lambda=1}^m k_\mu^\lambda \alpha_\lambda, \quad \mu = m + 1, \dots, n, \quad (4.10)$$

where the k_μ^λ are arbitrary constants. To find the form of a 2-coboundary we act on the 1-cochain (4.10) with the coboundary operator; using (3.3) and (4.7)

we obtain

$$\begin{aligned}
\omega_\lambda^{\text{cob}}(\alpha, \beta) &= (s\omega^{(1)})(\alpha, \beta) \\
&= \rho_\alpha \omega^{(1)}(\beta) + \rho_\beta \omega^{(1)}(\alpha) - \omega^{(1)}([\alpha, \beta]_{\mathfrak{g}}) \\
&= \sum_{\mu=1}^m \sum_{\nu=m+1}^n W_\lambda^{\mu\nu} [\alpha_\mu, \omega_\nu^{(1)}(\beta)] - \sum_{\mu=1}^m \sum_{\nu=m+1}^n W_\lambda^{\mu\nu} [\beta_\mu, \omega_\nu^{(1)}(\alpha)] \\
&\quad + \sum_{\mu, \nu, \sigma=1}^m k_\lambda^\sigma W_\sigma^{\mu\nu} [\alpha_\mu, \beta_\nu]. \tag{4.11}
\end{aligned}$$

After inserting (4.10) into (4.11) and relabeling, we obtain the general form of a 2-coboundary

$$\omega_\lambda^{\text{cob}}(\alpha, \beta) = \sum_{\mu, \nu=1}^m V_\lambda^{\mu\nu} [\alpha_\mu, \beta_\nu], \quad \lambda = m+1, \dots, n,$$

where

$$V_\lambda^{\mu\nu} := \sum_{\tau=1}^m k_\lambda^\tau W_\tau^{\mu\nu} - \sum_{\sigma=m+1}^n (k_\sigma^\mu W_\lambda^{\nu\sigma} + k_\sigma^\nu W_\lambda^{\mu\sigma}). \tag{4.12}$$

To see how coboundaries are removed, consider the lower-triangular coordinate transformation

$$[M_\sigma^{\bar{\tau}}] = \left(\begin{array}{c|c} \mathbf{I} & \mathbf{0} \\ \mathbf{k} & c\mathbf{I} \end{array} \right),$$

where σ labels rows. This transformation subtracts $V_{(\lambda)}$ from $W_{(\lambda)}$ for $\lambda > m$ and leaves the first m of the $W_{(\lambda)}$'s unchanged. In other words, if \overline{W} is the transformed W ,

$$\overline{W}_{(\lambda)} = \begin{cases} W_{(\lambda)} & \lambda = 1, \dots, m; \\ \left(\begin{array}{c|c} c^{-1}(\mathbf{w}_\lambda - \mathbf{V}_\lambda) & \mathbf{r}_\lambda \\ \mathbf{r}_\lambda^T & \mathbf{0} \end{array} \right) & \lambda = m+1, \dots, n. \end{cases} \tag{4.13}$$

We have also included in this transformation an arbitrary scale factor c . Since by (4.8) the block in the upper-left characterizes the cocycle, we see that the

transformed cocycle is the cocycle characterized by \mathbf{w}_λ minus the coboundary characterized by \mathbf{V}_λ .

The special case we will encounter most often is when the maximal Abelian ideal of \mathfrak{h} simply consists of elements of the form $(0, \dots, 0, \alpha_n)$. For this case $m = n - 1$, and the action vanishes since $W_n^{\mu n} = 0$ (the extension is central). The cocycle ω is entirely determined by $W_{(n)}$. The form of the coboundary is reduced to

$$V_n^{\mu\nu} = \sum_{\tau=1}^{n-1} k_n^\tau W_\tau^{\mu\nu}, \quad (4.14)$$

that is, a linear combinations of the first $(n - 1)$ matrices. Thus it is easy to see at a glance which parts of the cocycle characterized $W_{(n)}$ can be removed by lower-triangular coordinate transformations.

4.3 Further Coordinate Transformations

In the previous section we restricted ourselves to lower-triangular coordinate transformations, which in general preserve the lower-triangular structure of the $W^{(\mu)}$. But when the $W^{(\mu)}$ matrices are relatively sparse, there exist non-lower-triangular coordinate transformations that nonetheless preserve the lower-triangular structure. As alluded to in Section 3.3, these transformations are outside the scope of cohomology theory, which is restricted to transformations that preserve the exact form of the action and the algebras \mathfrak{g} and \mathfrak{a} , as shown by (4.13). In other words, cohomology theory classifies extensions *given* \mathfrak{g} , \mathfrak{a} , and ρ . We need not obey this restriction. We can allow non-lower-triangular coordinate transformations as long as they preserve the lower-triangular structure of the $W^{(\mu)}$'s.

We now discuss a particular class of such transformations that will be useful in Section 4.6. Consider the case where both the algebra of $(n - 1)$ -tuples \mathfrak{g} and that of 1-tuples \mathfrak{a} are Abelian. Then the possible (solvable) extensions, in lower triangular form, are characterized by $W_{(\lambda)} = 0$, $\lambda = 1, \dots, n - 1$, with $W_{(n)}$ arbitrary (except for $W_n^{\mu n} = 0$). Let us apply a coordinate change of the form

$$M = \left(\begin{array}{c|c} \mathbf{m} & \mathbf{0} \\ \hline \mathbf{0} & c \end{array} \right),$$

where \mathbf{m} is an $(n - 1) \times (n - 1)$ nonsingular matrix and c is again a nonzero scale factor. Denoting by \overline{W} the transformed W , we have

$$\overline{W}_{(\lambda)} = \begin{cases} 0 & \lambda = 1, \dots, n - 1; \\ \left(\begin{array}{c|c} c^{-1} \mathbf{m}^T \mathbf{w}_\lambda \mathbf{m} & \mathbf{0} \\ \hline \mathbf{0} & 0 \end{array} \right) & \lambda = n. \end{cases} \quad (4.15)$$

This transformation does not change the lower-triangular form of the extension, even if \mathbf{m} is not lower-triangular. The manner in which \mathbf{w}_n is transformed by M is very similar to that of a (possibly singular) metric tensor: it can be diagonalized and rescaled such that all its eigenvalues are 0 or ± 1 . We can also change the overall sign of the eigenvalues using c (something that cannot be done for a metric tensor). Hence, we shall order the eigenvalues such that the $+1$'s come first, followed by the -1 's, and finally by the 0 's. We will show in Section 4.6 how the negative eigenvalues can be eliminated to harmonize the notation.

4.4 Appending a Semisimple Part

In Section 4.2 we showed that because of the Levi decomposition theorem we only needed to classify the solvable part of the extension for a given de-

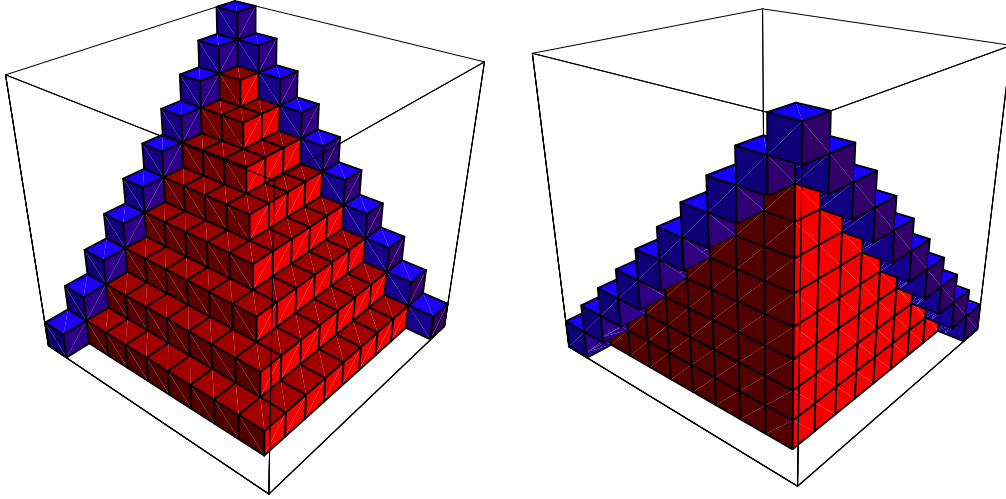


Figure 4.3: Front and rear views of a schematic representation of the 3-tensor W for an arbitrary semidirect extension with cocycle. The solvable part is in red. The semisimple part is in blue and consists of unit entries. The axes are as in Figure 4.2. An extension with *all* these elements nonzero cannot actually occur.

generate block. Most physical applications have a semisimple part ($\theta = 1$); when this is so, we shall label the matrices by $W^{(0)}, W^{(1)}, \dots, W^{(n)}$, where they are now of size $n + 1$ and $W^{(0)}$ is the identity.² Thus the matrices labeled by $W^{(1)}, \dots, W^{(n)}$ will always form a solvable subalgebra. This explains the labeling in Sections 2.3.1 and 2.3.2.

If the extension has a semisimple part ($\theta = 1$, or equivalently $W^{(0)} = I$), we shall refer to it as *semidirect*. This was the case treated in Section 4.2.1. A pictorial representation of an arbitrary semidirect extension with nonvanishing cocycle is shown in Figure 4.3. If the extension is not semidirect, then it is solvable (and contains n matrices instead of $n + 1$). This is the extension

²The term semisimple is not quite precise: if the base algebra is not semisimple then neither is the extension. However we will use the term to distinguish the different cases.

represented in Figure 4.2.

Given a solvable algebra of n -tuples we can carry out in some sense the inverse of the Levi decomposition and append a semisimple part to the extension. Effectively, this means that the $n \times n$ matrices $W^{(1)}, \dots, W^{(n)}$ are made $n + 1 \times n + 1$ by adding a row and column of zeros. Then we simply append the matrix $W^{(0)} = I$ to the extension. In this manner we construct a semisimple extension from a solvable one. This is useful since we will be classifying solvable extensions, and afterwards we will want to recover their semidirect counterpart.

The extension obtained by appending a semisimple part to the completely Abelian algebra of n -tuples will be called *pure semidirect*. It is characterized by $W^{(0)} = I$, and $W_\lambda^{\mu\nu} = 0$ for $\mu, \nu > 0$. This is shown schematically in Figure 4.4.

4.5 Leibniz Extension

A particular extension that we shall consider is called the Leibniz extension [81].

For the solvable case this extension has the form

$$W^{(1)} =: N = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & & \ddots & \\ & & & & 1 & 0 \end{pmatrix} \quad (4.16)$$

or $W_\lambda^{\mu 1} = \delta_{\lambda-1}^\mu$, $\lambda > 1$. The first matrix is an $n \times n$ Jordan block. In this case the other matrices, in order to commute with $W^{(1)}$, must be in striped

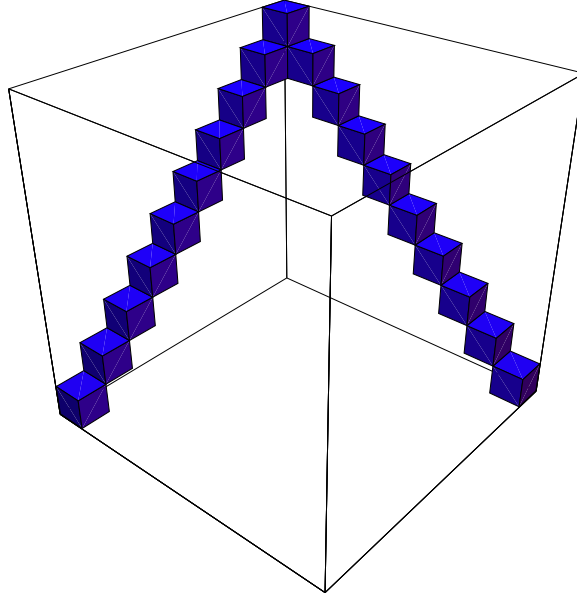


Figure 4.4: Schematic representation of the 3-tensor W for a pure semidirect extension. The axes are as in Figure 4.2.

lower-triangular form [89],

$$W^{(\nu)} = \begin{pmatrix} 0 & & & & & & & & \\ a & 0 & & & & & & & \\ b & a & 0 & & & & & & \\ c & b & a & 0 & & & & & \\ d & c & b & a & 0 & & & & \\ \vdots & & & & & \ddots & & & \end{pmatrix}. \quad (4.17)$$

But by symmetry of the upper indices the first column of matrix $W^{(\nu)}$ must be $W_{\lambda}^{1(\nu)} = \delta_{\lambda}^{\nu}$, so that

$$W^{(\nu)} = (N)^{\nu}, \quad (4.18)$$

where on the right-hand side the ν denotes an exponent, not a superscript. An equivalent way of characterizing the Leibniz extension is

$$W_{\lambda}^{\mu\nu} = \delta_{\lambda}^{\mu+\nu}, \quad \mu, \nu, \lambda = 1, \dots, n. \quad (4.19)$$

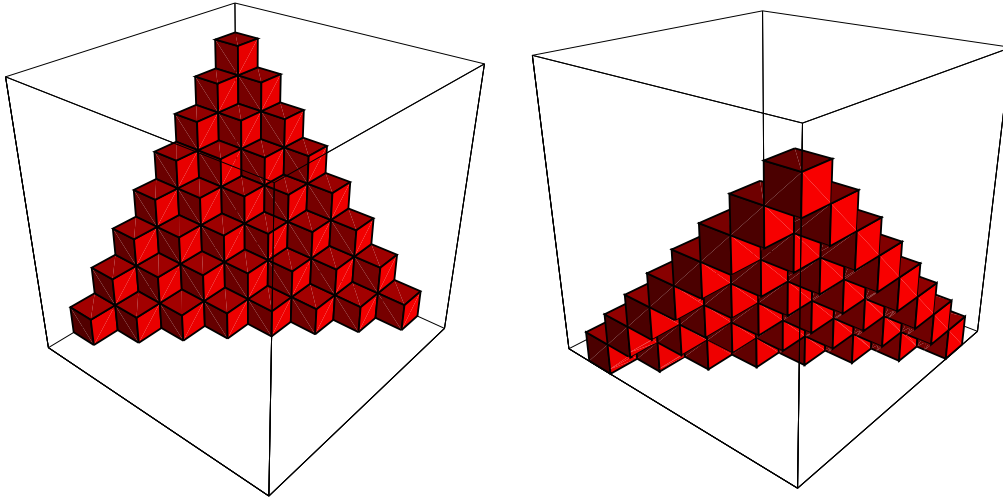


Figure 4.5: Two views of the 3-tensor W for a solvable Leibniz extension, where each cube denotes a 1. The axes are as in Figure 4.2. The Leibniz extension is “hollow.”

The tensor δ is an ordinary Kronecker delta. Note that neither (4.18) nor (4.19) are covariant expressions, reflecting the coordinate-dependent nature of the Leibniz extension.

The Leibniz extension is in some sense a “maximal” extension: it is the only extension that has $W_{(\lambda)} \neq 0$ for *all* $\lambda > 1$ (up to coordinate transformations). Its uniqueness will become clear in Section 4.6, and is proved in Section 4.7. We show two schematic views of the extension in Figure 4.5. Fans of 1980’s arcade games will understand why the author is suggesting the alternate name Q*Bert extension,³ since Leibniz has no dearth of things named after him (see Figure 4.6).

To construct the semidirect Leibniz extension, we append $W^{(0)} = I$, a square matrix of size $n+1$, to the solvable Leibniz extension above, as described

³Q*Bert™ is a trademark of the Sony Corporation.



Figure 4.6: Screenshot of the Q*Bert game. Compare with Figure 4.5!

in Section 4.4. The characterization given by Eq. (4.19) can be used for the semidirect Leibniz extension by simply letting the indices run from 0 to n .

4.6 Low-order Extensions

We now classify algebra extensions of low order. As demonstrated in Section 4.2 we only need to classify solvable algebras, which means that $W^{(n)} = 0$ for all cases. We will do the classification up to order $n = 4$. For each case we first write down the most general set of lower-triangular matrices $W^{(\nu)}$ (we have already used the fact that a set of commuting matrices can be lower-triangularized) with the symmetry $W_{\lambda}^{\mu\nu} = W_{\lambda}^{\nu\mu}$ built in. Then we look at what sort of restrictions the commutativity of the matrices places on the elements. Finally, we eliminate coboundaries for each case by the methods of Sections 4.2 and 4.3. This requires coordinate transformations, but we usually will not bother using new symbols and just assume the transformation were

effected.

Note that, due to the lower-triangular structure of the extensions, the classification found for an m -tuple algebra applies to the first m elements of an n -tuple algebra, $n > m$. Thus, $W_{(n)}$ is the cocycle that contains all of the new information not included in the previous $m = n - 1$ classification. These comments will become clearer as we proceed.

There are three generic cases that we will encounter for any order:

1. The Leibniz extension, discussed in Section 4.5.
2. An extension with $W_{(\lambda)} \equiv 0$, $\lambda = 1, \dots, n - 1$, and $W_{(n)}$ arbitrary (and symmetric). This extension automatically satisfies the commutativity requirement, because the product of any two $W^{(\mu)}$ vanishes. It can be further classified by the methods of Section 4.3. Later we will refer to this case as having a *vanishing coextension* (see Section 5.4 and Figure 6.3).
3. The Abelian extension, which vanishes identically: $W_{(\lambda)} \equiv 0$, $\lambda = 1, \dots, n$. This is a special case of 2, above. When appended to a semidirect part (as explained in Section 4.4), the Abelian extension generates the pure semidirect extension.

We shall call an order n extension *trivial* if $W_{(n)} \equiv 0$, so that the cocycle appended to the order $n - 1$ extension contributes nothing to the bracket.

We now proceed with the classification for orders $n = 1$ to 4.

4.6.1 $n=1$

This case is Abelian, with the only possible element $W_1^{11} = 0$.

4.6.2 $\mathfrak{n}=2$

The most general lower-triangular form for the matrices is

$$W^{(1)} = \begin{pmatrix} 0 & 0 \\ W_2^{11} & 0 \end{pmatrix}, \quad W^{(2)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

If $W_2^{11} \neq 0$, then we can rescale it to unity. Hence, we let $W_2^{11} := \theta_1$, where $\theta_1 = 0$ or 1 . The case $\theta_1 = 0$ is the Abelian case, while for $\theta_1 = 1$ we have the $n = 2$ Leibniz extension (Section 4.5). Thus for $n = 2$ there are only two possible algebras. The cocycle which we have added at this stage is characterized by θ_1 .

4.6.3 $\mathfrak{n}=3$

Using the result of Section 4.6.2, the most general lower-triangular form is

$$W^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ \theta_1 & 0 & 0 \\ W_3^{11} & W_3^{21} & 0 \end{pmatrix}, \quad W^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ W_3^{21} & W_3^{22} & 0 \end{pmatrix},$$

and $W^{(3)} = 0$. These satisfy the symmetry condition (2.20), and the requirement that the matrices commute leads to the condition

$$\theta_1 W_3^{22} = 0.$$

The symmetric matrix representing the cocycle is

$$W_{(3)} = \begin{pmatrix} W_3^{11} & W_3^{21} & 0 \\ W_3^{21} & W_3^{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (4.20)$$

If $\theta_1 = 1$, then W_3^{22} must vanish. Then, by (4.14) we can remove from $W_{(3)}$ a multiple of $W_{(2)}$, and therefore we may assume W_3^{11} vanishes. A suitable rescaling allows us to write $W_3^{21} = \theta_2$, where $\theta_2 = 0$ or 1 . The cocycle for the

case $\theta_1 = 1$ is thus

$$W_{(3)} = \begin{pmatrix} 0 & \theta_2 & 0 \\ \theta_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For $\theta_2 = 1$ we have the Leibniz extension (Section 4.5).

If $\theta_1 = 0$, we have the case discussed in Section 4.3. For this case we can diagonalize and rescale $W_{(3)}$ such that

$$W_{(3)} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where (λ_1, λ_2) can be $(1, 1)$, $(1, 0)$, $(0, 0)$, or $(1, -1)$. This last case, as alluded to at the end of Section 4.3, can be transformed so that it corresponds to $\theta_1 = 0$, $\theta_2 = 1$. The choice $(1, 0)$ can be transformed to the $\theta_1 = 1$, $\theta_2 = 0$ case. Finally for $(\lambda_1, \lambda_2) = (1, 1)$ we can use the complex transformation

$$\xi^1 \rightarrow \frac{1}{\sqrt{2}}(\xi^1 + \xi^2), \quad \xi^2 \rightarrow -\frac{i}{\sqrt{2}}(\xi^1 - \xi^2), \quad \xi^3 \rightarrow \xi^3,$$

to transform to the $\theta_1 = 0$, $\theta_2 = 1$ case.

We allow complex transformations in our classification because we are chiefly interested in finding Casimir invariants for Lie–Poisson brackets. If we disallowed complex transformations, the final classification would contain a few more members. The use of complex transformations will be noted as we proceed.

There are thus four independent extensions for $n = 3$, corresponding to

$$(\theta_1, \theta_2) \in \{(0, 0), (0, 1), (1, 0), (1, 1)\}.$$

These will be referred to as Cases 1–4, respectively. Cases 1 and 3 have $\theta_2 = 0$, and so are trivial ($W_{(3)} = 0$). Case 2 is the solvable part of the compressible reduced MHD bracket (Section 2.3.2). Case 4 is the solvable Leibniz extension.

4.6.4 n=4

Proceeding as before and using the result of Sections 4.6.2 and 4.6.3, we now know that we need only write

$$W_{(4)} = \begin{pmatrix} W_4^{11} & W_4^{21} & W_4^{31} & 0 \\ W_4^{21} & W_4^{22} & W_4^{32} & 0 \\ W_4^{31} & W_4^{32} & W_4^{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (4.21)$$

The matrices $W_{(1)}$, $W_{(2)}$, and $W_{(3)}$ are given by their $n = 3$ analogues padded with an extra row and column of zeros (owing to the lower-triangular form of the matrices). The requirement that the matrices $W^{(1)} \dots W^{(4)}$ commute leads to the conditions

$$\begin{aligned} \theta_2 W_4^{33} &= 0, \\ \theta_2 W_4^{31} &= \theta_1 W_4^{22}, \\ \theta_2 W_4^{32} &= 0, \\ \theta_1 W_4^{32} &= 0. \end{aligned} \quad (4.22)$$

There are four cases to look at, corresponding to the possible values of θ_1 and θ_2 .

Case 1 $\theta_1 = 0$, $\theta_2 = 0$.

This is the unconstrained case discussed in Section 4.3, that is, all the commutation relations (4.22) are automatically satisfied. We can diagonalize to give

$$W_{(4)} = \begin{pmatrix} \lambda'_1 & 0 & 0 & 0 \\ 0 & \lambda'_2 & 0 & 0 \\ 0 & 0 & \lambda'_3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where

$$(\lambda'_1, \lambda'_2, \lambda'_3) \in \{(1, 1, 1), (1, 1, 0), (1, 0, 0), (0, 0, 0), (1, 1, -1), (1, -1, 0)\},$$

so there are six distinct cases. The exact form of the transformation is unimportant, but the $(1, 1, 0)$ extension can be mapped to Case 2 (the transformation is complex), $(1, 0, 0)$ can be mapped to Case 3a, and $(1, -1, 0)$ can be mapped to Case 2. Finally the $(1, 1, 1)$ extension can be mapped to the $(1, 1, -1)$ case by a complex transformation.

After transforming that $(1, 1, -1)$ case, we are left with

$$W_{(4)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

These will be called Cases 1a and 1b.

Case 2 $\theta_1 = 0, \theta_2 = 1$.

The commutation relations (4.22) reduce to $W_4^{31} = W_4^{32} = W_4^{33} = 0$, and we have

$$W_{(4)} = \begin{pmatrix} W_4^{11} & W_4^{21} & 0 & 0 \\ W_4^{21} & W_4^{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We can remove W_4^{21} because it is a coboundary (in this case a multiple of $W_{(3)}$).

We can also rescale appropriately to obtain the four possible extensions

$$W_{(4)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Again, the form of the transformation is unimportant, but it turns out that the second extension can be mapped to Case 3c, and the third and fourth to Case 3b. This last transformation is complex. Thus there is only one independent possibility, the trivial extension $W_{(4)} = 0$.

Case 3 $\theta_1 = 1, \theta_2 = 0$.

We can remove W_4^{11} using a coordinate transformation. From the commutation requirement (4.22) we obtain $W_4^{22} = W_4^{32} = 0$. We are left with $W_{(3)} = 0$ and

$$W_{(4)} = \begin{pmatrix} 0 & W_4^{21} & W_4^{31} & 0 \\ W_4^{21} & 0 & 0 & 0 \\ W_4^{31} & 0 & W_4^{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using the fact that elements of the form $(0, \alpha_2, 0, \alpha_4)$ are an Abelian ideal of this bracket, we find that $W_4^{33}W_4^{31} = 0$. Using an upper-triangular transformation we can also make $W_4^{21}W_4^{31} = 0$. After suitable rescaling we find there are five cases: the trivial extension $W_{(4)} = 0$, and

$$W_{(4)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

However the last of these may be mapped to Case 4 (below) with $\theta_3 = 0$. We will refer to the trivial extension as Case 3a and to the remaining three extensions as Cases 3b–d, respectively.

Case 4 $\theta_1 = 1, \theta_2 = 1$.

The elements W_4^{11} and W_4^{21} are coboundaries that can be removed by a coordinate transformation. From (4.22) we have $W_4^{33} = W_4^{32} = 0, W_4^{22} = W_4^{31} =: \theta_3$, so that

$$W_{(4)} = \begin{pmatrix} 0 & 0 & \theta_3 & 0 \\ 0 & \theta_3 & 0 & 0 \\ \theta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Case	$W_{(2)}$	$W_{(3)}$	$W_{(4)}$			
			a	b	c	d
1	(0)	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$		
2	(0)	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			
3	(1)	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
4	(1)	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$		

Table 4.1: Enumeration of the independent extensions up to $n = 4$. We have $W_{(1)} = 0$ for all the cases, and we have left out a row and a column of zeros at the end of each matrix.

For $\theta_3 = 1$ we have the Leibniz extension. The two cases will be referred to as Case 4a for $\theta_3 = 0$ and 4b for $\theta_3 = 1$.

Table 4.1 summarizes the results. There are a total of nine independent $n = 4$ extensions, four of which are trivial ($W_{(4)} = 0$). As noted in Section 4.5 only the Leibniz extension, Case 4b, has nonvanishing $W_{(i)}$ for all $1 < i \leq n$.

The surprising fact is that even to order four the normal forms of the extensions involve no free parameters: all entries in the coefficients of the bracket are either zero or one. There is no obvious reason this should hold true if we try

to classify extensions of order $n > 4$. It would be interesting to find out, but the classification scheme used here becomes prohibitive at such high order. The problem is that some of the transformations used to relate extensions cannot be systematically derived and were obtained by educated guessing.

4.7 Leibniz as the Maximal Extension

We mentioned in Section 4.5 that the Leibniz extension is maximal: it is the only extension that has $W_{(\lambda)} \neq 0$ for *all* $\lambda > 1$. Having seen the classification process at work in Section 4.6, we are now in a position to show why the Leibniz extension has this property. We will demonstrate that the only way to extend a Leibniz extension nontrivially (i.e., with a nonvanishing cocycle) is to append a cocycle such that the new extension is again Leibniz.

Consider a solvable Leibniz extension of order $n - 1$, denoted by the 3-tensor \widetilde{W} . We increase the order of \widetilde{W} by one by appending the most general cocycle possible (as was done in Section 4.6) to obtain an extension of order n denoted by the tensor W . The form of the matrices $W^{(\mu)}$ of the new extension is

$$W^{(\mu)} = \left(\begin{array}{c|c} \widetilde{W}^{(\mu)} & \\ \hline W_n^{(\mu)} & 0 \end{array} \right), \quad \mu = 1, \dots, n - 1, \quad (4.23)$$

and $W^{(n)} \equiv 0$. The quantity $W_n^{(\mu)}$ is a row vector defined in the obvious manner as $[W_n^{(\mu)}]^\nu = W_n^{\mu\nu}$.

In particular, the first matrix of the n th order extension is

$$W^{(1)} = \left(\begin{array}{cccccc|c} 0 & & & & & & \\ 1 & 0 & & & & & \\ & 1 & 0 & & & & \\ & & \dots & \dots & & & \\ & & & & 1 & 0 & \\ \hline W_n^{11} & W_n^{12} & \dots & W_n^{1,n-2} & W_n^{1,n-1} & & 0 \end{array} \right), \quad (4.24)$$

where the $W_n^{\mu\nu}$ represent the appended cocycle, and we have explicitly delimited the order $n - 1$ Leibniz extension. It is not difficult to show that the $W_n^{1\nu}$, $\nu = 1, \dots, n - 2$, are coboundaries and so can be removed by a coordinate transformation. We thus assume that $W_n^{1\nu} = 0$, $\nu = 1, \dots, n - 2$. The only potentially nonzero element of that row is $W_n^{1,n-1}$.

Taking the commutator of two matrices of the form (4.23) gives the conditions

$$\sum_{\sigma=1}^{n-1} W_n^{\mu\sigma} \widetilde{W}_\sigma^{\nu\tau} = \sum_{\sigma=1}^{n-1} W_n^{\nu\sigma} \widetilde{W}_\sigma^{\mu\tau}, \quad \mu, \nu, \tau = 1, \dots, n - 1.$$

Substituting the form of the Leibniz extension (4.19) for \widetilde{W} , this becomes

$$\sum_{\sigma=1}^{n-1} W_n^{\mu\sigma} \delta_\sigma^{\nu+\tau} = \sum_{\sigma=1}^{n-1} W_n^{\nu\sigma} \delta_\sigma^{\mu+\tau},$$

or

$$W_n^{\mu,\nu+\tau} = \begin{cases} W_n^{\mu+\tau,\nu} & \mu + \tau < n, \\ 0 & \mu + \tau \geq n, \end{cases}$$

where $\nu + \tau < n$. For $\tau = 1$, this is

$$W_n^{\mu,\nu+1} = \begin{cases} W_n^{\mu+1,\nu} & \mu < n - 1, \\ 0 & \mu = n - 1, \end{cases}$$

for $\nu = 1, \dots, n - 2$, which says that $W_{(n)}$ has a banded structure. Because we have that $W_n^{1\nu} = 0$, $\nu = 1, \dots, n - 2$, it must be that

$$W_{(n)} = \begin{pmatrix} 0 & 0 & \dots & 0 & W_n^{1,n-1} & 0 \\ 0 & 0 & \dots & W_n^{1,n-1} & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & W_n^{1,n-1} & \dots & 0 & 0 & 0 \\ W_n^{1,n-1} & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}$$

So either $W_n^{1,n-1} = 0$ (the extension is trivial), or $W_n^{1,n-1}$ can be rescaled to unity (the extension is of the Leibniz type).

Thus, if one has a Leibniz extension of size $n - 1$ then the only way to nontrivially extend it is to make it the Leibniz extension of size n . But since the Leibniz extension is the only nontrivial extension of order 2 (see Section 4.6.2), we have shown the uniqueness of the maximal extension, up to a change of coordinates.

Chapter 5

Casimir Invariants for Extensions

In this chapter we will use the bracket extensions of Chapter 4 to make Lie–Poisson brackets, following the prescription of Chapter 2. In Section 5.1 we write down the general form of the Casimir condition (the condition under which a functional is a Casimir invariant) for a general class of inner brackets. Then in Section 5.2 we see how the Casimirs separate for a direct sum of algebras, the case discussed in Section 4.1. Section 5.3 discusses the particular properties of Casimirs of solvable extensions. In Section 5.4 we give a general solution to the Casimir problem and introduce the concept of *coextension*. Finally, in Section 5.5 we work out the Casimir invariants for some specific examples, including CRMHD and the Leibniz extension.

5.1 Casimir Condition

A generalized Casimir invariant (or Casimir for short) is a function $C : \mathfrak{g}^* \rightarrow \mathbb{R}$ for which

$$\{F, C\} \equiv 0,$$

for all $F : \mathfrak{g}^* \rightarrow \mathbb{R}$. Using (2.1) and (2.5), we can write this as

$$\left\langle \xi, \left[\frac{\delta F}{\delta \xi}, \frac{\delta C}{\delta \xi} \right] \right\rangle = - \left\langle \left[\frac{\delta C}{\delta \xi}, \xi \right]^\dagger, \frac{\delta F}{\delta \xi} \right\rangle.$$

Since this vanishes for all F we conclude

$$\left[\frac{\delta C}{\delta \xi}, \xi \right]^\dagger = 0. \quad (5.1)$$

To figure out the coadjoint bracket corresponding to (2.19), we write

$$\langle \xi, [\alpha, \beta] \rangle = \langle \xi^\lambda, W_\lambda{}^{\mu\nu} [\alpha_\mu, \beta_\nu] \rangle,$$

which after using the coadjoint bracket in \mathfrak{g} becomes

$$\langle [\beta, \xi]^\dagger, \alpha \rangle = \langle W_\lambda{}^{\mu\nu} [\beta_\nu, \xi^\lambda]^\dagger, \alpha_\mu \rangle$$

so that

$$[\beta, \xi]^\dagger{}^\nu = W_\lambda{}^{\mu\nu} [\beta_\mu, \xi^\lambda]^\dagger. \quad (5.2)$$

We can now write the Casimir condition (5.1) for the bracket extension as

$$W_\lambda{}^{\mu\nu} \left[\frac{\delta C}{\delta \xi^\mu}, \xi^\lambda \right]^\dagger = 0, \quad \nu = 0, \dots, n. \quad (5.3)$$

We now specialize the bracket to the case of most interested to us, where the inner bracket is of canonical form (2.11). (We will touch briefly on the finite-dimensional case in Section 5.1.1, but the remainder of the thesis will deal with a canonical inner bracket unless otherwise noted.) As we saw in Chapter 2, this is the bracket for 2-D fluid flows. Further, we assume that the form of the Casimir invariants is

$$C[\xi] = \int_\Omega \mathcal{C}(\xi(\mathbf{x})) \, d^2x, \quad (5.4)$$

and thus, since \mathcal{C} does not contain derivatives of ξ , functional derivatives of C can be written as ordinary partial derivatives of \mathcal{C} . We can then rewrite (5.3) as

$$W_\lambda{}^{\mu\nu} \frac{\partial^2 \mathcal{C}}{\partial \xi^\mu \partial \xi^\sigma} [\xi^\sigma, \xi^\lambda] = 0, \quad \nu = 0, \dots, n. \quad (5.5)$$

In the canonical case where the inner bracket is like (2.11) the $[\xi^\sigma, \xi^\lambda]$ are independent and antisymmetric in λ and σ . Thus a necessary and sufficient condition for the Casimir condition to be satisfied is

$$W_\lambda{}^{\mu\nu} \frac{\partial^2 \mathcal{C}}{\partial \xi^\mu \partial \xi^\sigma} = W_\sigma{}^{\mu\nu} \frac{\partial^2 \mathcal{C}}{\partial \xi^\mu \partial \xi^\lambda} , \quad (5.6)$$

for $\lambda, \sigma, \nu = 0, \dots, n$. Sometimes we shall abbreviate this as

$$W_\lambda{}^{\mu\nu} \mathcal{C}_{,\mu\sigma} = W_\sigma{}^{\mu\nu} \mathcal{C}_{,\mu\lambda} , \quad (5.7)$$

that is, any subscript μ on \mathcal{C} following a comma indicates differentiation with respect to ξ^μ . Equation (5.7) is trivially satisfied when \mathcal{C} is a linear function of the ξ 's. That solution usually follows from special cases of more general solutions, and we shall only mention it in Section 5.4.2 where it is the only solution.

An important result is immediate from (5.7) for a semidirect extension. Whenever the extension is semidirect we shall label the variables $\xi^0, \xi^1, \dots, \xi^n$, because the subset ξ^1, \dots, ξ^n then forms a solvable subalgebra (see Section 4.4 for terminology). For a semidirect extension, $W^{(0)}$ is the identity matrix, and thus (5.7) gives

$$\begin{aligned} \delta_\lambda{}^\mu \mathcal{C}_{,\mu\sigma} &= \delta_\sigma{}^\mu \mathcal{C}_{,\mu\lambda} , \\ \mathcal{C}_{,\lambda\sigma} &= \mathcal{C}_{,\sigma\lambda} , \end{aligned}$$

which is satisfied because we can interchange the order of differentiation. Hence, $\nu = 0$ does not lead to any conditions on the Casimir. However, the variables μ, λ, σ still take values from 0 to n in (5.7).

5.1.1 Finite-dimensional Casimirs

For completeness, we briefly outline the derivation of condition (5.7) for a finite-dimensional algebra, though we shall be concerned with the canonical inner bracket for the remainder of the thesis. The Lie–Poisson bracket can be written

$$\{f, g\} = W_\lambda{}^{\mu\nu} c_{ij}^k \xi_k^\lambda \frac{\partial f}{\partial \xi_i^\mu} \frac{\partial g}{\partial \xi_j^\nu}, \quad (5.8)$$

where the c_{ij}^k are the structure constants of the algebra \mathfrak{g} . The roman indices denote the components of each ξ^μ , in the same manner as the rigid body example of Section 2.2.1, and f and g are ordinary functions of the ξ_i^μ . The Casimir condition (5.3) is thus

$$W_\lambda{}^{\mu\nu} c_{ij}^k \frac{\partial C}{\partial \xi_i^\mu} \xi_k^\lambda = 0, \quad (5.9)$$

where both ν and j are free indices. From the structure constants we can construct the *Cartan–Killing form* [29, 44],

$$\mathfrak{K}_{ij} := c_{is}^t c_{jt}^s. \quad (5.10)$$

The Cartan–Killing form is symmetric, and is nondegenerate for a semisimple algebra. We assume this is the case for \mathfrak{g} , and denote the inverse of \mathfrak{K}_{ij} by \mathfrak{K}^{ij} .

For definiteness we take a Casimir of the form

$$C = \frac{1}{2} \mathfrak{K}^{ij} C_{\mu\nu} \xi_i^\mu \xi_j^\nu, \quad (5.11)$$

where $C_{\mu\nu}$ is a symmetric tensor. Inserting this into (5.9), we get

$$W_\lambda{}^{\mu\nu} c_{ij}^k \mathfrak{K}^{is} C_{\mu\sigma} \xi_s^\sigma \xi_k^\lambda = 0. \quad (5.12)$$

The symbol $c_j^{sk} := \mathfrak{K}^{si} c_{ij}^k$ can be shown to be antisymmetric in its upper indices. (We use the Cartan–Killing form as a metric to raise and lower indices.) We

can then define the bracket $[,]^* : \mathfrak{g}^* \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ by

$$[\xi, \eta]_k^* := c_k^{ij} \xi_i \eta_j, \quad (5.13)$$

which is a Lie bracket on \mathfrak{g}^* induced by the Cartan–Killing form \mathfrak{K} . The Casimir condition (5.12) can be rewritten neatly in terms of the bracket $[,]^*$ as

$$W_\lambda^{\mu\nu} C_{\mu\sigma} [\xi^\sigma, \xi^\lambda]^* = 0. \quad (5.14)$$

This should be compared with condition (5.5), for the infinite-dimensional case, where the bracket $[,]^*$ is obtained from the identification of \mathfrak{g} and \mathfrak{g}^* . The Casimir (5.11) is thus the finite-dimensional analogue of (5.4). Since condition (5.12) has to be true for any value of the ξ , it follows that we must have

$$W_\lambda^{\mu\nu} C_{\mu\sigma} = W_\sigma^{\mu\nu} C_{\mu\lambda}, \quad (5.15)$$

the same condition as (5.7). We conclude that, even though we shall be concerned with the canonical bracket case, many of the subsequent results of this chapter apply to finite-dimensional brackets.

5.2 Direct Sum

For the direct sum we found in Section 4.1 that if we look at the 3-tensor W as a cube, then it “blocks out” into smaller cubes, or subblocks, along its main diagonal, each subblock representing a subalgebra. We denote each subblock of $W_\lambda^{\mu\nu}$ by $W_{i\lambda}^{\mu\nu}$, $i = 1, \dots, r$, where r is the number of subblocks. We can rewrite (2.1) as

$$\begin{aligned} \{A, B\} &= \sum_{i=1}^r \left\langle \xi_i^\lambda, W_{i\lambda}^{\mu\nu} \left[\frac{\delta A}{\delta \xi_i^\mu}, \frac{\delta B}{\delta \xi_i^\nu} \right] \right\rangle \\ &=: \sum_{i=1}^r \{A, B\}_i, \end{aligned}$$

where i labels the different subblocks and the greek indices run over the size of the i th subblock. Each of the subbrackets $\{, \}_i$ depends on different fields. In particular, if the functional C is a Casimir, then, for any functional F

$$\{F, C\} = \sum_{i=1}^r \{F, C\}_i = 0 \quad \implies \quad \{F, C\}_i = 0, \quad i = 1, \dots, r.$$

The solution for this is

$$C[\xi] = C_1[\xi_1] + \dots + C_r[\xi_r], \quad \text{where } \{F, C_i\}_i = 0, \quad i = 1, \dots, r,$$

that is, the Casimir is just the sum of the Casimir for each subbracket. Hence, the question of finding the Casimirs can be treated separately for each component of the direct sum. We thus assume we are working on a single degenerate subblock, as we did for the classification in Chapter 4, and henceforth we drop the subscript i .

There is a complication when a single (degenerate) subblock has more than one simultaneous eigenvector. By this we mean k vectors $u^{(a)}$, $a = 1, \dots, k$, such that

$$W_\lambda^{\mu(\nu)} u_\mu^{(a)} = \Lambda^{(\nu)} u_\lambda^{(a)}.$$

Note that lower-triangular matrices always have at least the eigenvector given by $u_\mu = \delta_\mu^n$. Let $\eta^{(a)} := u_\rho^{(a)} \xi^\rho$, and consider a function $\mathcal{C}(\eta^{(1)}, \dots, \eta^{(k)})$. Then

$$\begin{aligned} W_\lambda^{\mu(\nu)} \frac{\partial^2 \mathcal{C}}{\partial \xi^\mu \partial \xi^\sigma} &= W_\lambda^{\mu(\nu)} \sum_{a,b=1}^k u_\mu^{(a)} u_\sigma^{(b)} \frac{\partial^2 \mathcal{C}}{\partial \eta^{(a)} \partial \eta^{(b)}}, \\ &= \Lambda^{(\nu)} \sum_{a,b=1}^k u_\lambda^{(a)} u_\sigma^{(b)} \frac{\partial^2 \mathcal{C}}{\partial \eta^{(a)} \partial \eta^{(b)}}. \end{aligned}$$

Because the eigenvalue $\Lambda^{(\nu)}$ does not depend on a (the block was assumed to have degenerate eigenvalues), the above expression is symmetric in λ and σ . Hence, the Casimir condition (5.6) is satisfied.

The reason this is introduced here is that if a degenerate block splits into a direct sum, then it will have several simultaneous eigenvectors. The Casimir invariants $\mathcal{C}^{(a)}(\eta^{(a)})$ and $\mathcal{C}^{(b)}(\eta^{(b)})$ corresponding to each eigenvector, instead of adding as $\mathcal{C}^{(a)}(\eta^{(a)}) + \mathcal{C}^{(b)}(\eta^{(b)})$, will combine into one function to give $\mathcal{C}(\eta^{(a)}, \eta^{(b)})$, a more general functional dependence. However, these situations with more than one eigenvector are not limited to direct sums. For instance, they occur in semidirect sums. In Section 5.6 we will see examples of both cases.

5.3 Local Casimirs for Solvable Extensions

In the solvable case, when all the $W^{(\mu)}$'s are lower-triangular with vanishing eigenvalues, a special situation occurs. If we consider the Casimir condition (5.5), we notice that derivatives with respect to ξ^n do not occur at all, since $W^{(n)} = 0$. Hence, the functional

$$C[\xi] = \int_{\Omega} \xi^n(\mathbf{x}') \delta(\mathbf{x} - \mathbf{x}') d^2x' = \xi^n(\mathbf{x})$$

is conserved. The variable $\xi^n(\mathbf{x})$ is *locally* conserved. It cannot have any dynamics associated with it. This holds true for any other simultaneous null eigenvectors the extension happens to have, but for the solvable case ξ^n is always such a vector (provided the matrices have been put in lower-triangular form, of course).

Hence, there are at most $n - 1$ dynamical variables in an order n solvable extension. An interesting special case occurs when the only nonvanishing $W_{(\mu)}$ is for $\mu = n$. Then the Lie–Poisson bracket is

$$\{F, G\} = \sum_{\mu, \nu=1}^{n-1} W_n^{\mu\nu} \int_{\Omega} \xi^n(\mathbf{x}) \left[\frac{\delta F}{\delta \xi^\mu(\mathbf{x})}, \frac{\delta G}{\delta \xi^\nu(\mathbf{x})} \right] d^2x,$$

where $\xi^n(\mathbf{x})$ is some function of our choosing. This bracket is not what we would normally call Lie–Poisson because $\xi^n(\mathbf{x})$ is not dynamical. It gives equations of motion of the form

$$\dot{\xi}^\nu = W_n^{\nu\mu} \left[\frac{\delta H}{\delta \xi^\mu}, \xi^n \right],$$

which can be used to model, for example, advection of scalars in a specified flow given by $\xi^n(\mathbf{x})$. This bracket occurs naturally when a Lie–Poisson bracket is linearized [58, 69].

5.4 Solution of the Casimir Problem

We now proceed to find the solution to (5.5). We assume that all the $W^{(\mu)}$, $\mu = 0, \dots, n$, are in lower-triangular form, and that the matrix $W^{(0)}$ is the identity matrix (which we see saw can always be done). Though this is the semidirect form of the extension, we will see that we can also recover the Casimir invariants of the solvable part. We assume $\nu > 0$ in (5.5), since $\nu = 0$ does not lead to a condition on the Casimir (Section 5.1). Therefore $W_\lambda^{n\nu} = 0$. Thus, we separate the Casimir condition into a part involving indices ranging from $0, \dots, n - 1$ and a part that involves only n . The condition

$$\sum_{\mu, \sigma, \lambda=0}^n W_\lambda^{\mu\nu} \mathcal{C}_{,\mu\sigma} [\xi^\lambda, \xi^\sigma] = 0, \quad \nu > 0,$$

becomes

$$\sum_{\lambda=0}^n \left(\sum_{\mu, \sigma=0}^{n-1} W_\lambda^{\mu\nu} \mathcal{C}_{,\mu\sigma} [\xi^\lambda, \xi^\sigma] + \sum_{\mu=0}^{n-1} W_\lambda^{\mu\nu} \mathcal{C}_{,\mu n} [\xi^\lambda, \xi^n] \right) = 0,$$

where we have used $W_\lambda^{n\nu} = 0$ to limit the sum on μ . Separating the sum in λ ,

$$\begin{aligned} \sum_{\lambda=0}^{n-1} \left(\sum_{\mu,\sigma=0}^{n-1} W_\lambda^{\mu\nu} \mathcal{C}_{,\mu\sigma} [\xi^\lambda, \xi^\sigma] + \sum_{\mu=0}^{n-1} W_\lambda^{\mu\nu} \mathcal{C}_{,\mu n} [\xi^\lambda, \xi^n] \right) \\ + \sum_{\mu,\sigma=0}^{n-1} W_n^{\mu\nu} \mathcal{C}_{,\mu\sigma} [\xi^n, \xi^\sigma] + \sum_{\mu=0}^{n-1} W_n^{\mu\nu} \mathcal{C}_{,\mu n} [\xi^n, \xi^n] = 0. \end{aligned}$$

The last sum vanishes because $[\xi^n, \xi^n] = 0$. Now we separate the condition into semisimple and solvable parts,

$$\begin{aligned} \sum_{\mu=1}^{n-1} \left(\sum_{\lambda,\sigma=0}^{n-1} W_\lambda^{\mu\nu} \mathcal{C}_{,\mu\sigma} [\xi^\lambda, \xi^\sigma] - \sum_{\sigma=0}^{n-1} W_\sigma^{\mu\nu} \mathcal{C}_{,\mu n} [\xi^n, \xi^\sigma] \right. \\ \left. + \sum_{\sigma=0}^{n-1} W_n^{\mu\nu} \mathcal{C}_{,\mu\sigma} [\xi^n, \xi^\sigma] \right) + \sum_{\lambda,\sigma=0}^{n-1} W_\lambda^{0\nu} \mathcal{C}_{,0\sigma} [\xi^\lambda, \xi^\sigma] \\ - \sum_{\sigma=0}^{n-1} W_\sigma^{0\nu} \mathcal{C}_{,0n} [\xi^n, \xi^\sigma] + \sum_{\sigma=0}^{n-1} W_n^{0\nu} \mathcal{C}_{,0\sigma} [\xi^n, \xi^\sigma] = 0. \end{aligned}$$

Using $W_\sigma^{0\nu} = \delta_\sigma^\nu$, we can separate the conditions into a part for $\nu = n$ and one for $0 < \nu < n$. For $\nu = n$, the only term that survives is the last sum

$$\sum_{\sigma=0}^{n-1} \mathcal{C}_{,0\sigma} [\xi^n, \xi^\sigma] = 0.$$

Since the commutators are independent, we have the conditions,

$$\mathcal{C}_{,0\sigma} = 0, \quad \sigma = 0, \dots, n-1. \quad (5.16)$$

and for $0 < \nu < n$,

$$\begin{aligned} \sum_{\mu=1}^{n-1} \left(\sum_{\lambda,\sigma=1}^{n-1} W_\lambda^{\mu\nu} \mathcal{C}_{,\mu\sigma} [\xi^\lambda, \xi^\sigma] - \sum_{\sigma=1}^{n-1} W_\sigma^{\mu\nu} \mathcal{C}_{,\mu n} [\xi^n, \xi^\sigma] \right. \\ \left. + \sum_{\sigma=1}^{n-1} W_n^{\mu\nu} \mathcal{C}_{,\mu\sigma} [\xi^n, \xi^\sigma] \right) - \mathcal{C}_{,0n} [\xi^n, \xi^\nu] = 0, \end{aligned}$$

where we have used (5.16). Using independence of the inner brackets gives

$$\widetilde{W}_\lambda{}^{\mu\nu}\mathcal{C}_{,\mu\sigma} = \widetilde{W}_\sigma{}^{\mu\nu}\mathcal{C}_{,\mu\lambda}, \quad (5.17)$$

$$g^{\nu\mu}\mathcal{C}_{,\mu\sigma} = \widetilde{W}_\sigma{}^{\nu\mu}\mathcal{C}_{,\mu n} + \delta^\nu{}_\sigma\mathcal{C}_{,0n}, \quad (5.18)$$

for $0 < \sigma, \lambda, \nu, \mu < n$. From now on in this section repeated indices are summed, and all greek indices run from 1 to $n-1$ unless otherwise noted. We have written a tilde over the W 's to stress the fact that the indices run from 1 to $n-1$, so that the \widetilde{W} represent a solvable order $(n-1)$ subextension of W . This subextension does not include $W_{(n)}$. We have also made the definition

$$g^{\mu\nu} := W_n{}^{\mu\nu}. \quad (5.19)$$

Equation (5.17) is a Casimir condition: it says that \mathcal{C} is also a Casimir of \widetilde{W} . We now proceed to solve (5.18) for the case where g is nonsingular. In Section 5.4.2 we will solve the singular g case. We will see that in both cases (5.17) follows from (5.18).

5.4.1 Nonsingular g

The simplest case occurs when g has an inverse, which we will call $\bar{g}_{\mu\nu}$. Then Eq. (5.18) has solution

$$\mathcal{C}_{,\tau\sigma} = A_{\tau\sigma}^\mu \mathcal{C}_{,\mu n} + \bar{g}_{\tau\sigma} \mathcal{C}_{,0n}, \quad (5.20)$$

where

$$A_{\tau\sigma}^\mu := \bar{g}_{\tau\nu} \widetilde{W}_\sigma{}^{\nu\mu}. \quad (5.21)$$

We now verify that $A_{\tau\sigma}^\mu = A_{\sigma\tau}^\mu$, as required by the symmetry of the left-hand side of (5.20).

$$\begin{aligned} A_{\tau\sigma}^\mu &= \bar{g}_{\tau\nu} \widetilde{W}_\kappa^{\nu\mu} \delta_\sigma^\kappa \\ &= \bar{g}_{\tau\nu} \widetilde{W}_\kappa^{\nu\mu} g^{\rho\kappa} \bar{g}_{\sigma\rho} \\ &= \bar{g}_{\tau\nu} \left(\sum_{\kappa=1}^n \widetilde{W}_\kappa^{\nu\mu} W_n^{\rho\kappa} \right) \bar{g}_{\sigma\rho}, \end{aligned}$$

where we used the fact that $W_n^{\rho n} = 0$ to extend the sum. Then we can use the commutativity property (2.21) to interchange ρ and ν ,

$$\begin{aligned} A_{\tau\sigma}^\mu &= \bar{g}_{\tau\nu} \left(\sum_{\kappa=1}^n \widetilde{W}_\kappa^{\rho\mu} W_n^{\nu\kappa} \right) \bar{g}_{\sigma\rho} \\ &= \bar{g}_{\tau\nu} W_n^{\nu\kappa} \bar{g}_{\sigma\rho} \widetilde{W}_\kappa^{\rho\mu} \\ &= \delta_\tau^\kappa A_{\sigma\kappa}^\mu \\ &= A_{\sigma\tau}^\mu, \end{aligned}$$

which shows that A is symmetric in its lower indices.

In (5.20), it is clear that the n th variable is “special”; this suggests that we try the following form for the Casimir:

$$\mathcal{C}(\xi^0, \xi^1, \dots, \xi^n) = \sum_{i \geq 0} \mathcal{D}^{(i)}(\xi^0, \xi^1, \dots, \xi^{n-1}) f_i(\xi^n), \quad (5.22)$$

where f is arbitrary and f_i is the i th derivative of f with respect to its argument. One immediate advantage of this form is that (5.17) follows from (5.18). Indeed, taking a derivative of (5.18) with respect to ξ^λ , inserting (5.22), and equating derivatives of f leads to

$$g^{\nu\mu} \mathcal{D}_{,\mu\sigma\lambda}^{(i)} = \widetilde{W}_\sigma^{\nu\mu} \mathcal{D}_{,\mu\lambda}^{(i+1)},$$

where we have used (5.16). Since the left-hand side is symmetric in λ and σ then so is the right-hand side, and (5.17) is satisfied.

Now, inserting the form of the Casimir (5.22) into the solution (5.20), we can equate derivatives of f to obtain for $\tau, \sigma = 1, \dots, n-1$,

$$\begin{aligned} \mathcal{D}_{,\tau\sigma}^{(0)} &= 0, \\ \mathcal{D}_{,\tau\sigma}^{(i)} &= A_{\tau\sigma}^{\mu} \mathcal{D}_{,\mu}^{(i-1)} + \bar{g}_{\tau\sigma} \mathcal{D}_{,0}^{(i-1)}, \quad i \geq 1. \end{aligned} \quad (5.23)$$

The first condition, together with (5.16), says that $\mathcal{D}^{(0)}$ is linear in ξ^0, \dots, ξ^{n-1} . There are no other conditions on $\mathcal{D}^{(0)}$, so we can obtain n independent solutions by choosing

$$\mathcal{D}^{(0)\nu} = \xi^{\nu}, \quad \nu = 0, \dots, n-1. \quad (5.24)$$

The equation for $\mathcal{D}^{(1)\nu}$ is

$$\mathcal{D}_{,\tau\sigma}^{(1)\nu} = \begin{cases} \bar{g}_{\tau\sigma} & \nu = 0, \\ A_{\tau\sigma}^{\nu} & \nu = 1, \dots, n-1. \end{cases} \quad (5.25)$$

Thus $\mathcal{D}^{(1)\nu}$ is a quadratic polynomial (the arbitrary linear part does not yield an independent Casimir, so we set it to zero). Note that $\mathcal{D}^{(1)\nu}$ does not depend on ξ^0 since $\tau, \sigma = 1, \dots, n-1$. Hence, for $i > 1$ we can drop the $\mathcal{D}_{,0}^{(i-1)}$ term in (5.23). Taking derivatives of (5.23), we obtain

$$\mathcal{D}_{,\tau_1\tau_2\dots\tau_{(i+1)}}^{(i)\nu} = A_{\tau_1\tau_2}^{\mu_1} A_{\mu_1\tau_3}^{\mu_2} \cdots A_{\mu_{(i-2)}\tau_i}^{\mu_{(i-1)}} \mathcal{D}_{,\mu_{(i-1)}\tau_{(i+1)}}^{(1)\nu}. \quad (5.26)$$

We know the series will terminate because the $\widetilde{W}^{(\mu)}$, and hence the $A_{(\mu)}$, are nilpotent. The solution to (5.26) is

$$\mathcal{D}^{(i)\nu} = \frac{1}{(i+1)!} D_{\tau_1\tau_2\dots\tau_{(i+1)}}^{(i)\nu} \xi^{\tau_1} \xi^{\tau_2} \cdots \xi^{\tau_{(i+1)}}, \quad i > 1, \quad (5.27)$$

where the constants D are defined by

$$D_{\tau_1\tau_2\dots\tau_{(i+1)}}^{(i)\nu} := A_{\tau_1\tau_2}^{\mu_1} A_{\mu_1\tau_3}^{\mu_2} \cdots A_{\mu_{(i-2)}\tau_i}^{\mu_{(i-1)}} \mathcal{D}_{,\mu_{(i-1)}\tau_{(i+1)}}^{(1)\nu}. \quad (5.28)$$

In summary, the $\mathcal{D}^{(i)}$'s of (5.22) are given by (5.24), (5.25), and (5.27).

Because the left-hand side of (5.26) is symmetric in all its indices, we require

$$A_{\tau\sigma}^{\mu} A_{\mu\lambda}^{\nu} = A_{\tau\lambda}^{\mu} A_{\mu\sigma}^{\nu}, \quad i > 1. \quad (5.29)$$

This is straightforward to show, using (2.21) and the symmetry of A :

$$\begin{aligned} A_{\tau\sigma}^{\mu} A_{\mu\lambda}^{\nu} &= A_{\sigma\tau}^{\mu} A_{\lambda\mu}^{\nu} \\ &= (\bar{g}_{\sigma\kappa} \widetilde{W}_{\tau}{}^{\kappa\mu}) (\bar{g}_{\lambda\rho} \widetilde{W}_{\mu}{}^{\rho\nu}) \\ &= \bar{g}_{\sigma\kappa} \bar{g}_{\lambda\rho} \widetilde{W}_{\tau}{}^{\kappa\mu} \widetilde{W}_{\mu}{}^{\rho\nu} \\ &= \bar{g}_{\sigma\kappa} \bar{g}_{\lambda\rho} \widetilde{W}_{\tau}{}^{\rho\mu} \widetilde{W}_{\mu}{}^{\kappa\nu} \\ &= A_{\tau\lambda}^{\mu} A_{\mu\sigma}^{\nu} \end{aligned}$$

If we compare this to (2.21), we see that A satisfies all the properties of an extension, except with the dual indices. Thus we will call A the *coextension* of \widetilde{W} with respect to g . Essentially, g serves the role of a metric that allows us to raise and lower indices. The formulation presented here is, however, not covariant. We have not been able to find a covariant formulation of the coextension, which is especially problematic for the singular g case (Section 5.4.2). Since the coextension depends strongly on the lower-triangular form of the $W^{(\mu)}$'s, it may well be that a covariant formulation does not exist.

For a solvable extension we simply restrict $\nu > 0$ and the above treatment still holds. We conclude that the Casimirs of the solvable part of a semidirect extension are Casimirs of the full extension. We have also shown, for the case of nonsingular g , that the number of independent Casimirs is equal to the order of the extension.

5.4.2 Singular g

In general, g is singular and thus has no inverse. However, it always has a (symmetric and unique) pseudoinverse $\bar{g}_{\mu\nu}$ such that

$$\bar{g}_{\mu\sigma} g^{\sigma\tau} \bar{g}_{\tau\nu} = \bar{g}_{\mu\nu}, \quad (5.30)$$

$$g^{\mu\sigma} \bar{g}_{\sigma\tau} g^{\tau\nu} = g^{\mu\nu}. \quad (5.31)$$

The pseudoinverse is also known as the strong generalized inverse or the Moore–Penrose inverse [79]. It follows from (5.30) and (5.31) that the matrix operator

$$P^\nu{}_\tau := g^{\nu\kappa} \bar{g}_{\kappa\tau}$$

projects onto the range of g . The system (5.18) only has a solution if the following solvability condition is satisfied:

$$P^\nu{}_\tau (\widetilde{W}_\sigma{}^{\tau\mu} \mathcal{C}_{,\mu n} + \delta^\tau{}_\sigma \mathcal{C}_{,0n}) = \widetilde{W}_\sigma{}^{\nu\mu} \mathcal{C}_{,\mu n} + \delta^\nu{}_\sigma \mathcal{C}_{,0n}; \quad (5.32)$$

that is, the right-hand side of (5.18) must live in the range of g .

If $\mathcal{C}_{,0n} \neq 0$, the quantity $\widetilde{W}_\sigma{}^{\nu\mu} \mathcal{C}_{,\mu n} + \delta^\nu{}_\sigma \mathcal{C}_{,0n}$ has rank equal to n , because the quantity $\widetilde{W}_\sigma{}^{\nu\mu} \mathcal{C}_{,\mu n}$ is lower-triangular (it is a linear combination of lower-triangular matrices). Thus, the projection operator must also have rank n . But then this implies that g has rank n and so is nonsingular, which contradicts the hypothesis of this section. Hence, $\mathcal{C}_{,0n} = 0$ for the singular g case, which together with (5.16) means that a Casimir that depends on ξ^0 can only be of the form $\mathcal{C} = f(\xi^0)$. However, since ξ^0 is not an eigenvector of the $W^{(\mu)}$'s, the only possibility is $\mathcal{C} = \xi^0$, the trivial linear case mentioned in Section 5.1.

The solvability condition (5.32) can thus be rewritten as

$$(P^\nu{}_\tau \widetilde{W}_\sigma{}^{\tau\mu} - \widetilde{W}_\sigma{}^{\nu\mu}) \mathcal{C}_{,\mu n} = 0. \quad (5.33)$$

An obvious choice would be to require $P^\nu{}_\tau \widetilde{W}_\sigma{}^{\tau\mu} = \widetilde{W}_\sigma{}^{\nu\mu}$, but this is too strong. We will derive a weaker requirement shortly.

By an argument similar to that of Section 5.4.1, we now assume \mathcal{C} is of the form

$$\mathcal{C}(\xi^1, \dots, \xi^n) = \sum_{i \geq 0} \mathcal{D}^{(i)}(\xi^1, \dots, \xi^{n-1}) f_i(\xi^n), \quad (5.34)$$

where again f_i is the i th derivative of f with respect to its argument. As in Section 5.4.1, we only need to show (5.18), and (5.17) will follow. The number of independent solutions of (5.18) is equal of the rank of g . The choice

$$\mathcal{D}^{(0)\nu} = P^\nu{}_\rho \xi^\rho, \quad \nu = 1, \dots, n-1, \quad (5.35)$$

provides the right number of solutions because the rank of P is equal to the rank of g . It also properly specializes to (5.24) when g is nonsingular, for then $P^\nu{}_\rho = \delta^\nu{}_\rho$.

The solvability condition (5.33) with this form for the Casimir becomes

$$(P^\nu{}_\tau \widetilde{W}_\sigma{}^{\tau\mu} - \widetilde{W}_\sigma{}^{\nu\mu}) \mathcal{D}_{,\mu}^{(i)\nu} = 0, \quad i \geq 0. \quad (5.36)$$

For $i = 0$ the condition can be shown to simplify to

$$P^\nu{}_\tau \widetilde{W}_\sigma{}^{\tau\mu} = \widetilde{W}_\sigma{}^{\nu\tau} P^\mu{}_\tau,$$

or to the equivalent matrix form

$$P \widetilde{W}_{(\sigma)} = \widetilde{W}_{(\sigma)} P, \quad (5.37)$$

since P is symmetric [79].

Equation (5.18) becomes

$$\begin{aligned} g^{\kappa\mu} \mathcal{D}_{,\mu\sigma}^{(0)\nu} &= 0, \\ g^{\kappa\mu} \mathcal{D}_{,\mu\sigma}^{(i)\nu} &= \widetilde{W}_\sigma{}^{\kappa\mu} \mathcal{D}_{,\mu}^{(i-1)\nu}, \quad i > 0. \end{aligned}$$

If (5.33) is satisfied, we know this has a solution given by

$$\mathcal{D}_{,\lambda\sigma}^{(i)\nu} = \bar{g}_{\lambda\rho} \widetilde{W}_{\sigma}{}^{\rho\mu} \mathcal{D}_{,\mu}^{(i-1)\nu} + (\delta_{\lambda}{}^{\mu} - \bar{g}_{\lambda\rho} g^{\rho\mu}) \mathcal{E}_{\mu\sigma}^{(i-1)\nu}, \quad i > 0,$$

where \mathcal{E} is arbitrary, and $(\delta_{\lambda}{}^{\mu} - \bar{g}_{\lambda\rho} g^{\rho\mu})$ projects onto the null space of g . The left-hand side is symmetric in λ and σ , but not the right-hand side. We can symmetrize the right-hand side by an appropriate choice of the null eigenvector,

$$\mathcal{E}_{\lambda\sigma}^{(i)\nu} := \bar{g}_{\sigma\rho} \widetilde{W}_{\lambda}{}^{\rho\mu} \mathcal{D}_{,\mu}^{(i)\nu}, \quad i \geq 0,$$

in which case

$$\mathcal{D}_{,\lambda\sigma}^{(i)\nu} = A_{\lambda\sigma}^{\mu} \mathcal{D}_{,\mu}^{(i-1)\nu}, \quad i > 0,$$

where

$$A_{\lambda\sigma}^{\nu} := \bar{g}_{\sigma\rho} \widetilde{W}_{\lambda}{}^{\rho\nu} + \bar{g}_{\lambda\rho} \widetilde{W}_{\sigma}{}^{\rho\nu} - \bar{g}_{\lambda\rho} \bar{g}_{\sigma\kappa} g^{\rho\mu} \widetilde{W}_{\mu}{}^{\kappa\nu}, \quad (5.38)$$

which is symmetric in λ and σ . Equation (5.38) also reduces to (5.21) when g is nonsingular, for then the null eigenvector vanishes. The full solution is thus given in the same manner as (5.26) by

$$\mathcal{D}^{(i)\nu} = \frac{1}{(i+1)!} D_{\tau_1\tau_2\dots\tau_{(i+1)}}^{(i)\nu} \xi^{\tau_1} \xi^{\tau_2} \dots \xi^{\tau_{(i+1)}}, \quad i > 0, \quad (5.39)$$

where the constants D are defined by

$$D_{\tau_1\tau_2\dots\tau_{(i+1)}}^{(i)\nu} := A_{\tau_1\tau_2}^{\mu_1} A_{\mu_1\tau_3}^{\mu_2} \dots A_{\mu_{(i-2)}\tau_i}^{\mu_{(i-1)}} A_{\mu_{(i-1)}\tau_{(i+1)}}^{\mu_i} P^{\nu}{}_{\mu_i}, \quad (5.40)$$

and $\mathcal{D}^{(0)}$ is given by (5.35).

The A 's must still satisfy the coextension condition (5.29). Unlike the nonsingular case this condition does not follow directly and is an extra requirement in addition to the solvability condition (5.36). Note that only the $i = 0$ case, Eq. (5.37), needs to be satisfied, for then (5.36) follows. Both these

conditions are coordinate-dependent, and this is a drawback. Nevertheless, we have found in obtaining the Casimir invariants for the low-order brackets that if these conditions are not satisfied, then the extension is a direct sum and the Casimirs can be found by the method of Section 5.2. However, this has not been proved rigorously.

5.5 Examples

We now illustrate the methods developed for finding Casimirs with a few examples. First we treat our prototypical case of CRMHD, and give a physical interpretation of invariants. Then, we derive the Casimir invariants for Leibniz extensions of arbitrary order. Finally, we give an example involving a singular g .

5.5.1 Compressible Reduced MHD

The W tensors representing the bracket for CRMHD (see Section 2.2.4) were given in Section 2.3.2. We have $n = 3$, so from (5.19) we get

$$g = \begin{pmatrix} 0 & -\beta_e \\ -\beta_e & 0 \end{pmatrix}, \quad \bar{g} = g^{-1} = \begin{pmatrix} 0 & -\beta_e^{-1} \\ -\beta_e^{-1} & 0 \end{pmatrix}. \quad (5.41)$$

In this case, the coextension is trivial: all three matrices $A^{(\nu)}$ defined by (5.21) vanish. Using (5.22) and (5.24), with $\nu = 1$ and 2, the Casimirs for the solvable part are

$$\mathcal{C}^1 = \xi^1 g(\xi^3) = v g(\psi), \quad \mathcal{C}^2 = \xi^2 h(\xi^3) = p h(\psi),$$

and the Casimir associated with the eigenvector ξ^3 is

$$\mathcal{C}^3 = k(\xi^3) = k(\psi).$$

Since g is nonsingular we also get another Casimir from the semidirect sum part,

$$\mathcal{C}^0 = \xi^0 f(\xi^3) - \frac{1}{\beta_e} \xi^1 \xi^2 f'(\xi^3) = \omega f(\psi) - \frac{1}{\beta_e} p v f'(\psi).$$

The physical interpretation of the invariant \mathcal{C}^3 is given in Morrison [68] and Thiffeault and Morrison [90]. This invariant implies the preservation of contours of ψ , so that the value ψ_0 on a contour labels that contour for all times. This is a consequence of the lack of dissipation and the divergence-free nature of the velocity. Substituting $\mathcal{C}^3(\psi) = \psi^k$ we also see that all the moments of the magnetic flux are conserved. By choosing $\mathcal{C}^3(\psi) = \Theta(\psi(\mathbf{x}) - \psi_0)$, a heavyside function, and inserting into (5.4), it follows that the area inside of any ψ -contour is conserved.

To understand the Casimirs \mathcal{C}^1 and \mathcal{C}^2 , we also let $g(\psi) = \Theta(\psi - \psi_0)$ in \mathcal{C}^1 . In this case we have

$$\mathcal{C}^1[v; \psi] = \int_{\Omega} v g(\psi) d^2x = \int_{\Psi_0} v(\mathbf{x}) d^2x,$$

where Ψ_0 represents the (not necessarily connected) region of Ω enclosed by the contour $\psi = \psi_0$ and $\partial\Psi_0$ is its boundary. By the interpretation we gave of \mathcal{C}^3 , the contour $\partial\Psi_0$ moves with the fluid. So the total value of v inside of a ψ -contour is conserved by the flow. The same is true of the pressure p . (See Thiffeault and Morrison [90] for an interpretation of these invariants in terms of relabeling symmetries, and a comparison with the rigid body.)

The total pressure and parallel velocity inside of any ψ -contour are preserved. To understand \mathcal{C}^4 , we use the fact that $\omega = \nabla^2\phi$ and integrate by parts to obtain

$$\mathcal{C}^4[\omega, v, p, \psi] = - \int_{\Omega} \left(\nabla\phi \cdot \nabla\psi + \frac{vp}{\beta_e} \right) f'(\psi) d^2x.$$

The quantity in parentheses is thus invariant inside of any ψ -contour. It can be shown that this is a remnant of the conservation by the full MHD model of the cross helicity,

$$V = \int_{\Omega} \mathbf{v} \cdot \mathbf{B} \, d^2x,$$

at second order in the inverse aspect ratio, while the conservation of $C^1[v; \psi]$ is a consequence of preservation of this quantity at first order. Here \mathbf{B} is the magnetic field. The quantities $C^3[\psi]$ and $C^2[p; \psi]$ they are, respectively, the first and second order remnants of the preservation of helicity,

$$W = \int_{\Omega} \mathbf{A} \cdot \mathbf{B} \, d^2x,$$

where \mathbf{A} is the magnetic vector potential.

5.5.2 Leibniz Extension

We first treat the nilpotent case. The Leibniz extension of Section 4.5 can be characterized by

$$W_{\lambda}^{\mu\nu} = \delta_{\lambda}^{\mu+\nu}, \quad \mu, \nu, \lambda = 1, \dots, n, \quad (4.19)$$

where the tensor δ is an ordinary Kronecker delta. Upon restricting the indices to run from 1 to $n-1$ (the tilde notation of Section 5.4), we have

$$g^{\mu\nu} = \widetilde{W}_n^{\mu\nu} = \delta_n^{\mu+\nu}, \quad \mu, \nu = 1, \dots, n-1.$$

The matrix g is nonsingular with inverse equal to itself: $\bar{g}_{\mu\nu} = \delta_{\mu+\nu}^n$. The coextension of \widetilde{W} is thus

$$A_{\tau\sigma}^{\mu} = \sum_{\nu=1}^{n-1} \bar{g}_{\tau\nu} \widetilde{W}_{\sigma}^{\nu\mu} = \sum_{\nu=1}^{n-1} \delta_{\tau+\nu}^n \delta_{\sigma}^{\nu+\mu} = \delta_{\tau+\sigma}^{\mu+n}.$$

Equation (5.28) becomes

$$\begin{aligned}
D_{\tau_1 \tau_2 \dots \tau_{(i+1)}}^{(i)\nu} &= A_{\tau_1 \tau_2}^{\mu_1} A_{\mu_1 \tau_3}^{\mu_2} \cdots A_{\mu_{(i-2)} \tau_i}^{\mu_{(i-1)}} A_{\mu_{(i-1)} \tau_{(i+1)}}^{\nu} \\
&= \delta_{\tau_1 + \tau_2}^{\mu_1 + n} \delta_{\mu_1 + \tau_3}^{\mu_2 + n} \cdots \delta_{\mu_{(i-2)} + \tau_i}^{\mu_{(i-1)} + n} \delta_{\mu_{(i-1)} + \tau_{(i+1)}}^{\nu + n} \\
&= \delta_{\tau_1 + \tau_2 + \dots + \tau_{(i+1)}}^{\nu + in}, \quad \nu = 1, \dots, n-1,
\end{aligned}$$

which, as required, this is symmetric under interchange of the τ_i . Using (5.22), (5.24), (5.25), and (5.27) we obtain the $n-1$ Casimir invariants

$$\mathcal{C}^\nu(\xi^1, \dots, \xi^n) = \sum_{i \geq 0} \frac{1}{(i+1)!} \delta_{\tau_1 + \tau_2 + \dots + \tau_{(i+1)}}^{\nu + in} \xi^{\tau_1} \cdots \xi^{\tau_{(i+1)}} f_i^\nu(\xi^n), \quad (5.42)$$

for $\nu = 1, \dots, n-1$. The superscript ν on f indicates that the arbitrary function is different for each Casimir, and recall the subscript i denotes the i th derivative with respect to ξ^n . The n th invariant is simply $\mathcal{C}^\nu(\xi^n) = f^n(\xi^n)$, corresponding to the null eigenvector in the system. Thus there are n independent Casimirs, as stated in Section 5.4.1.

For the Leibniz semidirect sum case, since g is nonsingular, there will be an extra Casimir given by (5.42) with $\nu = 0$, and the τ_i sums run from 0 to $n-1$. This is the same form as the $\nu = 1$ Casimir of the order $(n+1)$ nilpotent extension.

For the i th term in (5.42), the maximal value of any τ_j is achieved when all but one (say, τ_1) of the τ_j are equal to $n-1$, their maximum value. In this case we have

$$\tau_1 + \tau_2 + \cdots + \tau_{i+1} = \tau_1 + i(n-1) = \nu + in,$$

so that $\tau_1 = i + \nu$. Hence, the i th term depends only on $(\xi^{\nu+i}, \dots, \xi^n)$, and the ν th Casimir depends on (ξ^ν, \dots, ξ^n) . Also,

$$\max(\tau_1 + \cdots + \tau_{i+1}) = (i+1)(n-1) = \nu + in,$$

n	Invariant
1	$f(\xi^1)$
2	$\xi^1 f(\xi^2)$
3	$\xi^1 f(\xi^3) + \frac{1}{2}(\xi^2)^2 f'(\xi^3)$
4	$\xi^1 f(\xi^4) + \xi^2 \xi^3 f'(\xi^4) + \frac{1}{3!}(\xi^3)^3 f''(\xi^4)$
5	$\xi^1 f(\xi^5) + (\xi^2 \xi^4 + \frac{1}{2}(\xi^3)^2) f'(\xi^5) + \frac{1}{2} \xi^3 (\xi^4)^2 f''(\xi^5) + \frac{1}{4!} (\xi^4)^4 f'''(\xi^5)$

Table 5.1: Casimir invariants for Leibniz extensions up to order $n = 5$ ($\nu = 1$). The primes denote derivatives.

which leads to $\max i = n - \nu - 1$. Thus the sum (5.42) terminates, as claimed in Section 5.4.1. We rewrite (5.42) in the more complete form

$$\mathcal{C}^\nu(\xi^\nu, \dots, \xi^n) = \sum_{k=1}^{n-\nu} \frac{1}{k!} \delta_{\tau_1+\tau_2+\dots+\tau_k}^{\nu+(k-1)n} \xi^{\tau_1} \dots \xi^{\tau_k} f_{k-1}^\nu(\xi^n),$$

for $\nu = 0, \dots, n$. Table 5.1 gives the $\nu = 1$ Casimirs up to order $n = 5$.

5.5.3 Singular g

Now consider the $n = 4$ extension from Section 4.6.4, Case 3c. We have

$$\widetilde{W}_{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad g = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

with $\widetilde{W}_{(1)} = \widetilde{W}_{(3)} = 0$. The pseudoinverse of g is $\bar{g} = g$ and the projection operator is

$$P^\nu_\tau := g^{\nu\kappa} \bar{g}_{\kappa\tau} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The solvability condition (5.37) is obviously satisfied. We build the coextension given by (5.38), which in matrix form is

$$A^{(\nu)} = \widetilde{W}^{(\nu)} \bar{g} + (\widetilde{W}^{(\nu)} \bar{g})^T - \bar{g} g \widetilde{W}^{(\nu)} \bar{g},$$

to obtain

$$A^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A^{(2)} = A^{(3)} = 0.$$

These are symmetric and obviously satisfy (5.29), so we have a good coextension. Using (5.34), (5.35), (5.39), and (5.40) we can write, for $\nu = 1$ and 3,

$$\mathcal{C}^1 = \xi^1 f(\xi^4) + \xi^2 \xi^3 f'(\xi^4),$$

$$\mathcal{C}^3 = \xi^3 g(\xi^4).$$

This extension has two null eigenvectors, so from Section 5.2 we also have the Casimir $h(\xi^2, \xi^4)$. The functions f , g , and h are arbitrary, and the prime denotes differentiation with respect to argument.

5.6 Casimir Invariants for Low-order Extensions

Using the techniques developed so far, we now find the Casimir invariants for the low-order extensions classified in Section 4.6. We first find the Casimir invariants for the solvable extensions, since these are also invariants for the semidirect sum case. Then, we obtain the extra Casimir invariants for the semidirect case, when they exist.

5.6.1 Solvable Extensions

Now we look for the Casimirs of solvable extensions. As mentioned in Section 5.3, the Casimirs associated with null eigenvectors (the only kind of eigenvector for solvable extensions) are actually conserved locally. We shall still

write them in the form $\mathcal{C} = f(\xi^n)$, where \mathcal{C} is as in (5.4), so they have the correct form as invariants for the semidirect case of Section 5.6.2 (for which they are no longer locally conserved).

$n=1$

Since the bracket is Abelian, any function $\mathcal{C} = \mathcal{C}(\xi^1)$ is a Casimir.

$n=2$

For the Abelian case we have $\mathcal{C} = \mathcal{C}(\xi^1, \xi^2)$. The only other case is the Leibniz extension,

$$\mathcal{C}(\xi^1, \xi^2) = \xi^1 f(\xi^2) + g(\xi^2).$$

$n=3$

As shown in Section 4.6.3, there are four cases. Case 1 is the Abelian case, for which any function $\mathcal{C} = \mathcal{C}(\xi^1, \xi^2, \xi^3)$ is a Casimir. Case 2 is essentially the solvable part of the CRMHD bracket, which we treated in Section 5.5.1. Case 3 is a direct sum of the Leibniz extension for $n = 2$, which has the bracket

$$[(\alpha_1, \alpha_2), (\beta_1, \beta_2)] = (0, [\alpha_1, \beta_1]),$$

with the Abelian algebra $[\alpha_3, \beta_3] = 0$. Hence, the Casimir invariant is the same as for the $n = 2$ Leibniz extension with the extra ξ^3 dependence of the arbitrary function (see Section 5.2). Finally, Case 4 is the Leibniz Casimir. These results are summarized in Table 5.2.

Cases 1 and 3 are trivial extensions, that is, the cocycle appended to the $n = 2$ case vanishes. The procedure of then adding ξ^n dependence to the arbitrary function works in general.

Case	Invariant
1	$\mathcal{C}(\xi^1, \xi^2, \xi^3)$
2	$\xi^1 f(\xi^3) + \xi^2 g(\xi^3) + h(\xi^3)$
3	$\xi^1 f(\xi^2) + g(\xi^2, \xi^3)$
4	$\xi^1 f(\xi^3) + \frac{1}{2}(\xi^2)^2 f'(\xi^3) + \xi^2 g(\xi^3) + h(\xi^3)$

Table 5.2: Casimir invariants for solvable extensions of order $n = 3$.

$n=4$

As shown in Section 4.6.4, there are nine cases to consider. We shall proceed out of order, to group together similar Casimir invariants.

Cases 1a, 2, 3a, and 4a are trivial extensions, and as mentioned in Section 5.6.1 they involve only addition of ξ^4 dependence to their $n = 3$ equivalents. Case 3b is a direct sum of two $n = 2$ Leibniz extensions, so the Casimirs add.

Case 3c is the semidirect sum of the $n = 2$ Leibniz extension with an Abelian algebra defined by $[(\alpha_3, \alpha_4), (\beta_3, \beta_4)] = (0, 0)$, with action given by

$$\rho_{(\alpha_1, \alpha_2)}(\beta_3, \beta_4) = (0, [\alpha_1, \beta_3]).$$

The Casimir invariants for this extension were derived in Section 5.5.3.

Case 3d has a nonsingular g , so the techniques of Section 5.4.1 can be applied directly.

Finally, Case 4b is the $n = 4$ Leibniz extension, the Casimir invariants of which were derived in Section 5.5.2. The invariants are all summarized in Table 5.3.

Case	Invariant
1a	$\mathcal{C}(\xi^1, \xi^2, \xi^3, \xi^4)$
1b	$\xi^1 f(\xi^4) + \xi^2 g(\xi^4) + \xi^3 h(\xi^4) + k(\xi^4)$
2	$\xi^1 f(\xi^3) + \xi^2 g(\xi^3) + h(\xi^3, \xi^4)$
3a	$\xi^1 f(\xi^2) + g(\xi^2, \xi^3, \xi^4)$
3b	$\xi^1 f(\xi^2) + \xi^3 g(\xi^4) + h(\xi^2, \xi^4)$
3c	$\xi^1 f(\xi^4) + \xi^2 \xi^3 f'(\xi^4) + \xi^3 g(\xi^4) + h(\xi^2, \xi^4)$
3d	$\xi^1 f(\xi^4) + \frac{1}{2}(\xi^2)^2 f'(\xi^4) + \xi^3 g(\xi^4) + \xi^2 h(\xi^4) + k(\xi^4)$
4a	$\xi^1 f(\xi^3) + \frac{1}{2}(\xi^2)^2 f'(\xi^3) + \xi^2 g(\xi^3) + h(\xi^3, \xi^4)$
4b	$\xi^1 f(\xi^4) + \xi^2 \xi^3 f'(\xi^4) + \frac{1}{3!}(\xi^3)^3 f''(\xi^4)$ $+ \xi^2 g(\xi^4) + \frac{1}{2}(\xi^3)^2 g'(\xi^4) + \xi^3 h(\xi^4) + k(\xi^4)$

Table 5.3: Casimir invariants for solvable extensions of order $n = 4$.

5.6.2 Semidirect Extensions

Now that we have derived the Casimir invariants for solvable extensions, we look at extensions involving the semidirect sum of an algebra with these solvable extensions. We label the new variable (the one which acts on the solvable part) by ξ^0 . In Section 5.4.1 we showed that the Casimirs of the solvable part were also Casimirs of the full extension. We also concluded that a necessary condition for obtaining a new Casimir (other than the linear case $\mathcal{C}(\xi^0) = \xi^0$) from the semidirect sum was that $\det W_{(n)} \neq 0$. We go through the solvable cases and determine the Casimirs associated with the semidirect extension, if any exist.

n=1

There is only one solvable extension, so upon appending a semidirect part we have

$$W_{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad W_{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since $\det W_{(1)} \neq 0$, we expect another Casimir. In fact this extension is of the semidirect Leibniz type and has the same Casimir form as the $n = 2$ solvable Leibniz (Section 5.5.2) extension. Thus, the new Casimir is just $\xi^0 f(\xi^1)$.

n=2

Of the two possible extensions only the Leibniz one satisfies $\det W_{(2)} \neq 0$. The Casimir is thus

$$\mathcal{C}_{\text{sd}} = \xi^0 f(\xi^2) + \frac{1}{2}(\xi^1)^2 f'(\xi^2).$$

Case	Invariant
1b	$\xi^0 f(\xi^4) + (\xi^1 \xi^3 + \frac{1}{2}(\xi^2)^2) f'(\xi^4)$
3d	$\xi^0 f(\xi^4) + (\xi^1 \xi^2 + \frac{1}{2}(\xi^3)^2) f'(\xi^4) + \frac{1}{3!}(\xi^2)^3 f''(\xi^4)$
4b	$\xi^0 f(\xi^4) + (\xi^1 \xi^3 + \frac{1}{2}(\xi^2)^2) f'(\xi^4) + \frac{1}{2}\xi^2(\xi^3)^2 f''(\xi^4) + \frac{1}{4!}(\xi^3)^4 f'''(\xi^4)$

Table 5.4: Casimir invariants for semidirect extensions of order $n = 5$. These extensions also possess the corresponding Casimir invariants in Table 5.3.

$n=3$

Cases 2 and 4 have a nonsingular $W_{(3)}$. The Casimir for Case 2 is

$$\mathcal{C}_{\text{sd}} = \xi^0 f(\xi^3) + \xi^1 \xi^2 f'(\xi^3),$$

and for Case 4 it is of the Leibniz form

$$\mathcal{C}_{\text{sd}} = \xi^0 f(\xi^3) + \xi^1 \xi^2 f'(\xi^3) + \frac{1}{3!}(\xi^2)^3 f''(\xi^3).$$

$n=4$

Cases 1b, 3d, and 4b have a nonsingular $W_{(4)}$. The Casimirs are shown in Table 5.4.

Chapter 6

Stability

In this chapter we discuss the general problem of stability of steady solutions of Lie–Poisson systems, for different classes of Hamiltonians. We first define, in Section 6.1, what we mean by a steady solution being stable. We review the different types of stability and discuss how they are related. In Section 6.2 we discuss the energy-Casimir method for finding sufficient conditions for stability, and demonstrate its use by a few examples. The energy-Casimir method for fluids uses an infinite-dimensional analogue of Lagrange multipliers to find constrained extrema of the Hamiltonian (extrema of the free energy).

In Section 6.3 we turn to a different method of establishing stability, that of dynamical accessibility. The technique involves restricting the variations of the energy to lie on the symplectic leaves of the system. It is more general than the energy-Casimir method since it yields all equilibria of the equations of motion. The dynamical accessibility method is closely related to the energy-Casimir method, which we will see is reflected in the fact that the concept of coextension of Chapter 5 is used in the solution.

For the different types of extensions, we derive as general a result as possible, and then specialize to particular forms of the bracket and Hamiltonian, until usable stability conditions are obtained. We will treat CRMHD in detail,

using both the energy-Casimir and dynamical accessibility methods.

6.1 The Many Faces of Stability

A somewhat universally accepted definition of stability is as follows: Let ξ_e be an equilibrium solution of the (not necessarily Hamiltonian) system

$$\dot{\xi} = \mathcal{F}(\xi), \quad (6.1)$$

i.e., $\mathcal{F}(\xi_e) = 0$. The system is said to be *nonlinearly stable*, or simply *stable*, if for every neighborhood U of ξ_e there is a neighborhood V of ξ_e such that trajectories $\xi(t)$ initially in V never leave U (in finite time).

In terms of a norm $\|\cdot\|$, this definition is equivalent to demanding that for every $\epsilon > 0$, there is a $\delta > 0$ such that if $\|\xi(0) - \xi_e\| < \delta$, then $\|\xi(t) - \xi_e\| < \epsilon$ for all $t > 0$.

We also consider the linearized system,

$$\delta\dot{\xi} = \left\langle \delta\xi, \frac{\delta\mathcal{F}}{\delta\xi} \right\rangle \Big|_{\xi=\xi_e}, \quad (6.2)$$

where $\delta\xi$ is an infinitesimal perturbation. From this we define the formally self-adjoint linear operator \mathfrak{F} by

$$\langle \delta\eta, \mathfrak{F} \delta\zeta \rangle := \left\langle \delta\eta, \frac{\delta^2\mathcal{F}}{\delta\eta\delta\zeta} \delta\zeta \right\rangle \Big|_{\xi=\xi_e}. \quad (6.3)$$

From this definition we distinguish four basic types of stability:

- *Spectral stability.* The linearized system (6.2) is spectrally stable if the spectrum of the linear operator \mathfrak{F} defined by (6.3) has no eigenvalue with a positive real part. A special case is *neutral stability*, for which the

spectrum is purely imaginary. Hamiltonian systems are neutrally stable if they are spectrally stable.

- *Linear stability.* If the linearized system (6.2) is stable according to the above definition, then the system (6.1) is said to be linearly stable (or linearized stable). This implies spectral stability.
- *Formal stability* (Dirichlet criterion). The equilibrium is formally stable if we can find a conserved quantity whose first variation vanishes when evaluated at the equilibrium, and whose second variation is positive (or negative) definite when evaluated at the same equilibrium. In finite dimensions, this implies nonlinear stability. When the system is Hamiltonian and separable (i.e., it can be written as a sum of kinetic and potential energy), this criterion becomes Lagrange's theorem.
- *Nonlinear stability.* This is just the nonlinear stability of the full system as defined above. Note that this only implies that there exists a sufficiently small neighborhood V such that trajectories never leave U . It does not imply absence of *finite-amplitude instability*, called nonlinear instability by some authors, which says that the system is unstable for large enough perturbations.

Figure 6.1 summarizes the relationships between the various types of stability. See Siegel and Moser [85], Holm et al. [38], or Morrison [69] for examples and counterexamples of these relationships.

We have stated that formal stability implies nonlinear stability for finite-dimensional systems. Before discussing this point, we prove a stability theorem for finite-dimensional systems that has its origins with Lagrange. It was

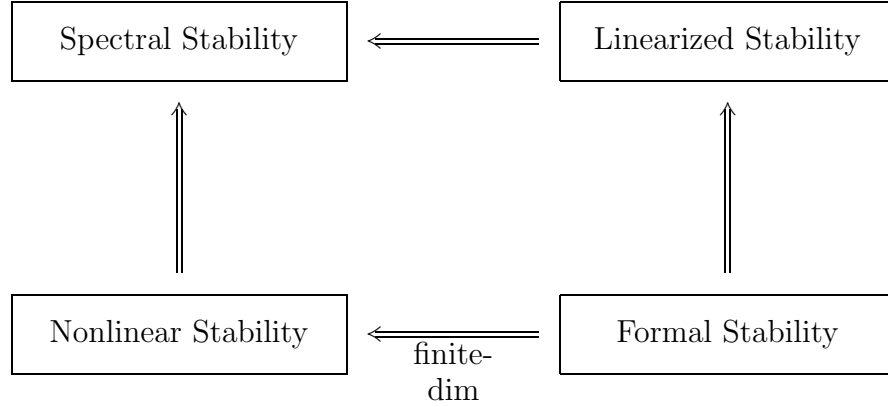


Figure 6.1: Relationship between the different types of stability.

proved in a less general form than presented here by Dirichlet [53], and was subsequently generalized by Liapunov.

The theorem is as follows. If the system (6.1), in finite dimensions, has a constant of the motion \mathfrak{K} that has a relative extremum in the strong sense [27, p. 13] at the equilibrium point $\xi = \xi_e$, then the equilibrium solution is stable.¹

We follow the proof of Siegel and Moser [85, p. 208]. See also Hirsch and Smale for a thorough treatment [37]. Since \mathfrak{K} can be replaced by $-\mathfrak{K}$, we can assume it has a minimum without loss of generality. By the strong minimum hypothesis, there exists a $\varrho > 0$ such that

$$\mathfrak{K}(\xi_e) < \mathfrak{K}(\xi) \text{ whenever } \|\xi - \xi_e\| < \varrho, \quad (6.4)$$

for some norm $\|\cdot\|$. Now, let

$$\mathfrak{M}_\varepsilon := \{\xi \mid \|\xi - \xi_e\| < \varepsilon\}, \quad 0 < \varepsilon < \varrho,$$

¹In finite dimensions a strong minimum is just a minimum with respect to the usual Euclidean norm, $\|\xi\| = |\xi|$.

be a ball of radius ε around the equilibrium point. Let $\mu(\varepsilon)$ be the minimum value of \mathfrak{K} on the surface of the ball \mathfrak{M}_ε ,

$$\mu(\varepsilon) := \min_{\|\xi - \xi_e\| = \varepsilon} \mathfrak{K}(\xi).$$

Using the strong minimum hypothesis, (6.4), we have

$$\mathfrak{K}(\xi) < \mu(\varepsilon), \quad \text{for } \xi \in \mathfrak{M}_\varepsilon.$$

Now consider a trajectory with initial conditions $\xi(0)$ in \mathfrak{M}_ε . Then

$$\mathfrak{K}(\xi(t)) = \mathfrak{K}(\xi(0)) < \mu(\varepsilon).$$

But by continuity this implies $\xi(t) \in \mathfrak{M}_\varepsilon$ since otherwise we would have had $\mathfrak{K}(\xi(t)) \geq \mu(\varepsilon)$ at some point in the trajectory. Thus, $\xi(t)$ lies in \mathfrak{M}_ε whenever $\xi(0)$ does. We then have stability, because \mathfrak{M}_ε is a neighborhood of ξ_e and we can make ε as small as we want.

In finite dimensions, positive or negative definiteness of the second variation of \mathfrak{K} is sufficient for the strong minimum requirement (6.4). In infinite dimensions this is not the case [8, 11, 23, 27, 38, 62, 84]. Further convexity arguments must be made, as done for several physical systems in Holm et al. [38]. Another crucial requirement, which is immediate in finite dimensions, is that the invariant \mathfrak{K} be *continuous* in the norm $\|\cdot\|$. In general an infinite-dimensional minimum will not necessarily satisfy this condition [27, 38].

Ball and Marsden [11] give an example from elasticity theory of a system that is formally stable but is nonlinearly unstable. Finn and Sun [23] discuss additional requirements for nonlinear stability of an ideal fluid in a gravitational field (for an exponential atmosphere), which is formally stable. One does not

know how stringent these requirements are—they could be far from the actual instability threshold. We take the viewpoint here that establishing definiteness of the second variation—showing formal stability—is a good indicator of stability. Indeed, formal stability is often used to mean stability, as is the case with δW stability criteria in MHD, which are actually second-order variations of the potential energy. For the Grad–Shafranov equilibria of reduced MHD (no flow), the sufficient conditions for formal stability are the same as for nonlinear stability [38, pp. 41–43].

It will be the topic of future work to try and make these general stability conditions more rigorous by making more stringent convexity arguments. Certainly formal stability implies linearized stability, since the second variation of the constant of motion provides a norm (conserved by the linearized dynamics) that can be used to establish stability of the linearized system.

Finally, note that Dirichlet’s theorem does *not* imply that if F does not have an extremum at ξ_e , then the system is unstable. It gives a sufficient, but not necessary, condition for stability of an equilibrium.

6.2 The Energy-Casimir Method

The energy-Casimir method has a long history which dates back to Fjortoft [24], Newcomb [93], Kruskal and Oberman [49], Fowler [25], and Gardner [26], but is usually called “Arnold’s method” or “Arnold’s theorem” [3, 4, 5, 7, 8]. We illustrate the method for a Lie–Poisson system. The equations of motion for the field variables ξ in terms of a given Hamiltonian H are

$$\dot{\xi} = - \left[\frac{\delta H}{\delta \xi}, \xi \right]^\dagger. \quad (2.6)$$

This can be rewritten

$$\dot{\xi} = - \left[\frac{\delta H}{\delta \xi} + \frac{\delta C}{\delta \xi}, \xi \right]^\dagger,$$

where C is any function of the Casimirs. It follows that if

$$\delta(H + C)[\xi_e] =: \delta F[\xi_e] = 0,$$

then ξ_e is an equilibrium of the system. We call F the free energy. The free energy F is a constant of the motion whose first variation vanishes at an equilibrium point. Therefore, if we can show it also has a strong extremum at that point then we have proved stability, by the theorem of Dirichlet. Showing that $\delta^2 F$ is definite (that is, showing formal stability) is almost sufficient to show stability, in the sense discussed at the end of Section 6.1.

We now apply the energy-Casimir method to compressible reduced MHD. We will give more examples in Section 6.3 when we introduce the method of dynamical accessibility, which is more general and includes the energy-Casimir result as a special case.

6.2.1 CRMHD Stability

The free energy functional F is built from the Hamiltonian (2.15) and the Casimir invariants found in Section 5.5.1,

$$F[\omega, v, p, \psi] := H + C,$$

where

$$C = \langle f(\psi) + v g(\psi) + p h(\psi) + (\omega k(\psi) - \beta_e^{-1} p v k'(\psi)) \rangle$$

is a combination of the Casimirs of the system. We use the same angle brackets as for the pairing, without the comma, to denote an integral over the fluid domain (we assume that we have identified \mathfrak{g} and \mathfrak{g}^*).

Equilibrium Solutions

We seek equilibria of the system that extremize the free energy. The first variation of F yields

$$\begin{aligned} \delta F = & \left\langle (-\phi + k(\psi)) \delta\omega + \left(v + g(\psi) - \beta_e^{-1} p k'(\psi) \right) \delta v \right. \\ & + \left(\beta_e^{-1} (p - 2\beta_e x) + h(\psi) - \beta_e^{-1} v k'(\psi) \right) \delta p \\ & \left. + \left(-J + f'(\psi) + v g'(\psi) + p h'(\psi) + (\omega k'(\psi) - \beta_e^{-1} p v k''(\psi)) \right) \delta\psi \right\rangle. \end{aligned}$$

An equilibrium solution $(\omega_e, v_e, p_e, \psi_e)$ for which $\delta F = 0$ must therefore satisfy

$$\phi_e = \Phi(\psi_e), \quad (6.5)$$

$$v_e = \beta_e^{-1} p_e \Phi'(\psi_e) - g(\psi_e), \quad (6.6)$$

$$p_e = v_e \Phi'(\psi_e) + \beta_e (2x - h(\psi_e)), \quad (6.7)$$

$$J_e = f'(\psi_e) + v_e g'(\psi_e) + p_e h'(\psi_e) + \omega_e \Phi'(\psi_e) - \beta_e^{-1} p_e v_e \Phi''(\psi_e), \quad (6.8)$$

where we have defined $\Phi(\psi) := k(\psi)$.

Since $\phi_e = \Phi(\psi_e)$, we have $\nabla\phi_e = \Phi'(\psi_e) \nabla\psi_e$. Hence, $\mathbf{v}_{e\perp} = \Phi'(\psi_e) \mathbf{B}_{e\perp}$, so the perpendicular (poloidal) velocity and magnetic field are collinear at an equilibrium.

We can use (6.6) and (6.7) to solve for v_e and p_e ,

$$\begin{pmatrix} v_e \\ p_e \end{pmatrix} = \left(\frac{|\Phi'(\psi_e)|^2}{\beta_e} - 1 \right)^{-1} \begin{pmatrix} g(\psi_e) + (h(\psi_e) - 2x) \Phi'(\psi_e) \\ g(\psi_e) \Phi'(\psi_e) + \beta_e (h(\psi_e) - 2x) \end{pmatrix}, \quad (6.9)$$

except where $|\Phi'(\psi_e)|^2 = \beta_e$. This singularity represents a resonance in the system, about which we will say more later. Equation (6.9) implies

$$\begin{aligned} \left(\nabla v_e - 2\Phi'(\psi_e) (1 - \beta_e^{-1} |\Phi'(\psi_e)|^2)^{-1} \hat{\mathbf{x}} \right) \times \nabla\psi_e &= 0, \\ \left(\nabla p_e - 2\beta_e (1 - \beta_e^{-1} |\Phi'(\psi_e)|^2)^{-1} \hat{\mathbf{x}} \right) \times \nabla\psi_e &= 0. \end{aligned}$$

An important class of equilibria are given by

$$\Phi(\psi_e) = c^{-1} \psi_e(x, y),$$

where c is a constant. We call those *Alfvénic* solutions. (The true Alfvén solutions are the particular case with $c = \pm 1$.) We then have $\omega_e \Phi'(\psi_e) = J_e/c^2$, and so from (6.8)

$$\left(1 - \frac{1}{c^2}\right) J_e = f'(\psi_e) + v_e g'(\psi_e) + p_e h'(\psi_e). \quad (6.10)$$

Note that, because of (6.9), the right-hand side of (6.10) depends explicitly on x , unless we have

$$g'(\psi_e) = -\beta_e c h'(\psi_e), \quad (6.11)$$

in which case (6.10) simplifies to

$$\left(1 - \frac{1}{c^2}\right) J_e(\psi_e) = f'(\psi_e) - g(\psi_e) g'(\psi_e). \quad (6.12)$$

Such an equation, with no explicit independence on x , has an analogue in low-beta reduced MHD, but cannot occur for a system like high-beta reduced MHD [35, p. 59] without a vanishing pressure gradient. Here, with CRMHD, we can eliminate the x dependence because we can set up an equilibrium gradient in the parallel velocity which cancels the pressure gradient.

If in (6.10) we let

$$f'(\psi_e) - g(\psi_e) g'(\psi_e) = \left(1 - \frac{1}{c^2}\right) \exp(-2\psi_e),$$

then we have the particular solution

$$\psi_e(x, y) = \ln(a \cosh y + \sqrt{a^2 - 1} \cos x). \quad (6.13)$$

This solution, the Kelvin–Stuart cat’s eye formula [15, 22, 82], is plotted in Figure 6.2.

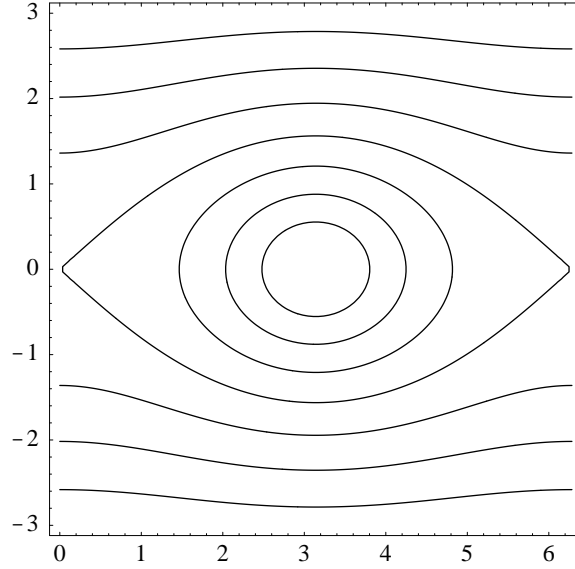


Figure 6.2: Contour plot of the magnetic flux $\psi_e(x, y)$ for the cat's eye solution (6.13), with $a = 1.5$.

Formal Stability

The second variation of F is given by

$$\begin{aligned} \delta^2 F = & \left\langle -\delta\omega (\nabla^2)^{-1} \delta\omega + |\delta v|^2 + \frac{1}{\beta_e} |\delta p|^2 - \delta\psi (\nabla^2)^{-1} \delta\psi + 2k'(\psi) \delta\omega \delta\psi \right. \\ & + \left(f''(\psi) + v g''(\psi) + p h''(\psi) + \omega k''(\psi) - \beta_e^{-1} p v k'''(\psi) \right) |\delta\psi|^2 \\ & + 2 \left(g'(\psi) - \beta_e^{-1} p k''(\psi) \right) \delta\psi \delta v + 2 \left(h'(\psi) - \beta_e^{-1} v k''(\psi) \right) \delta\psi \delta p \\ & \left. - 2\beta_e^{-1} k'(\psi) \delta v \delta p \right\rangle. \end{aligned}$$

We want to determine when this is non-negative. Using $\delta\omega = \nabla^2 \delta\phi$, we can write

$$\begin{aligned} & \langle |\nabla \delta\phi|^2 + |\nabla \delta\psi|^2 + 2k'(\psi) (\nabla^2 \delta\phi) \delta\psi \rangle \\ & = \langle |\nabla \delta\phi|^2 + |\nabla \delta\psi|^2 - 2\nabla(k'(\psi) \delta\psi) \cdot \nabla \delta\phi \rangle \\ & = \langle |\nabla \delta\phi - \nabla(k'(\psi) \delta\psi)|^2 - |\nabla(k'(\psi) \delta\psi)|^2 + |\nabla(\delta\psi)|^2 \rangle, \end{aligned}$$

which, after expanding the $|\nabla(k'(\psi) \delta\psi)|^2$ term, becomes

$$\begin{aligned} \langle |\nabla\delta\phi|^2 + |\nabla\delta\psi|^2 + 2k'(\psi) (\nabla^2\delta\phi) \delta\psi \rangle = \\ \langle |\nabla\delta\phi - \nabla(k'(\psi) \delta\psi)|^2 + (1 - |k'(\psi)|^2) |\nabla\delta\psi|^2 \\ + k'(\psi) \nabla^2 k'(\psi) |\delta\psi|^2 \rangle, \end{aligned} \quad (6.14)$$

so that the second variation, evaluated at the equilibrium solution (6.5)–(6.8), is now

$$\begin{aligned} \delta^2 F_e = \left\langle |\nabla\delta\phi - \nabla(\Phi'(\psi_e) \delta\psi)|^2 + (1 - |\Phi'(\psi_e)|^2) |\nabla\delta\psi|^2 + |\delta v|^2 + \frac{1}{\beta_e} |\delta p|^2 \right. \\ \left. + 2 \left(g'(\psi_e) - \beta_e^{-1} p_e \Phi''(\psi_e) \right) \delta\psi \delta v + 2 \left(h'(\psi_e) - \beta_e^{-1} v_e \Phi''(\psi_e) \right) \delta\psi \delta p \right. \\ \left. + \Theta(x, y) |\delta\psi|^2 - 2\beta_e^{-1} \Phi'(\psi_e) \delta v \delta p \right\rangle, \end{aligned} \quad (6.15)$$

where

$$\begin{aligned} \Theta(x, y) := f''(\psi_e) + v_e g''(\psi_e) + p_e h''(\psi_e) \\ + \omega_e \Phi''(\psi_e) - \beta_e^{-1} p_e v_e \Phi'''(\psi_e) + \Phi'(\psi_e) \nabla^2 \Phi'(\psi_e). \end{aligned}$$

For positive-definiteness of (6.15), we require

$$|\Phi'(\psi_e)| \leq 1. \quad (6.16)$$

If we have equality in (6.16), then we obtain a *family* of marginally stable equilibria, the Alfvén solutions.

Assuming (6.16) is satisfied, a sufficient condition for stability is to show that the $(\delta v, \delta p, \delta\psi)$ part of the second variation is non-negative. We thus demand the quadratic form represented by the symmetric matrix

$$\begin{pmatrix} 1 & -\beta_e^{-1} \Phi'(\psi_e) & g'(\psi_e) - \beta_e^{-1} p_e \Phi''(\psi_e) \\ -\beta_e^{-1} \Phi'(\psi_e) & \beta_e^{-1} & h'(\psi_e) - \beta_e^{-1} v_e \Phi''(\psi_e) \\ g'(\psi_e) - \beta_e^{-1} p_e \Phi''(\psi_e) & h'(\psi_e) - \beta_e^{-1} v_e \Phi''(\psi_e) & \Theta(x, y) \end{pmatrix}$$

be non-negative. A necessary and sufficient condition for this is that the *principal minors* of the matrix be non-negative. The principal minors are simply the determinants of the submatrices of increasing size along the diagonal. Thus, the first two principal minors are

$$\begin{aligned}\mu_1 &= |1| > 0, \\ \mu_2 &= \begin{vmatrix} 1 & -\beta_e^{-1} \Phi'(\psi_e) \\ -\beta_e^{-1} \Phi'(\psi_e) & \beta_e^{-1} \end{vmatrix} = \beta_e^{-1} \left(1 - \frac{|\Phi'(\psi_e)|^2}{\beta_e} \right) \geq 0,\end{aligned}$$

and the third is just the determinant of the matrix,

$$\begin{aligned}\mu_3 &= \mu_2 \left(\Theta(x, y) - [g'(\psi_e) - \beta_e^{-1} p_e \Phi''(\psi_e)]^2 \right) \\ &\quad - [h'(\psi_e) + \beta_e^{-1} g'(\psi_e) \Phi'(\psi_e) - \beta_e^{-1} (v_e + \beta_e^{-1} p_e \Phi'(\psi_e))]^2 \geq 0.\end{aligned}$$

Combining (6.16) with the requirement $\mu_2 \geq 0$, we have

$$|\Phi'(\psi_e)|^2 \leq \min(1, \beta_e). \quad (6.17)$$

According to this condition, for $\beta_e < 1$ CRMHD is *less* stable than the RMHD case. This is a direct manifestation of the nontrivial cocycle in the bracket: there is a new resonance, associated with the *acoustic* resonance, so-named because at that point the flow velocity equals the ion-acoustic speed (proportional to $2T_e$). We will see in Section 6.3.6 that new resonances are a generic feature of Lie–Poisson systems with cocycles.

The condition that μ_3 be non-negative is of a more complicated form. For the Alfvénic case, with $\Phi(\psi_e) = c^{-1} \psi_e(x, y)$, and assuming condition (6.11), so that $J_e = J_e(\psi_e)$, the condition $\mu_3 \geq 0$ simplifies to

$$\mu_2 \left(1 - \frac{1}{c^2} \right) J_e'(\psi_e) \geq 0.$$

Since $\mu_2 \geq 0$ and, by (6.17), $1/c^2 \leq \min(1, \beta_e)$, we can simply write

$$J_e'(\psi_e) \geq 0. \quad (6.18)$$

Hence, for $\beta_e \geq 1$, Alfvénic solutions have the same stability characteristics as for RMHD.

6.3 Dynamical Accessibility

We turn now to a different method of finding equilibria and ascertaining their stability. Finding the solutions for which the first variation of the free energy vanishes yields some, but not all of the equilibria of the equations of motion. For example, this method fails to detect the static equilibrium of the heavy top [69]. For the 2-D Euler system, the equilibria it yields are those for which the streamfunction is a monotonic function of the vorticity, but there are equilibria which do not have this form. This is tied to the rank-changing of the cosymplectic form: there are equilibria that arise because the bracket itself vanishes [69]. The method of dynamical accessibility was used by Morrison and Pfirsch to examine the stability of the Vlasov–Maxwell system [74, 75]. Isichenko [43] made use of a similar method to study hydrodynamic stability, based on ideas of Arnold [6].

We first explain the method of dynamically accessible variations, and then apply it to extensions. We derive general results for pure semidirect extensions and extensions with a nonsingular g . For both cases, we examine several different types of Hamiltonians.

6.3.1 The Method

Consider a perturbation defined as

$$\delta\xi_{\text{da}} := \{\mathcal{G}, \xi\}, \quad (6.19)$$

with the perturbation given in terms of the generating function χ by

$$\mathcal{G} := \langle \xi, \chi \rangle.$$

The χ are arbitrary “constant” functions (i.e., they do not depend on ξ , but do depend on \mathbf{x}). We call (6.19) a *dynamically accessible* perturbation. The first-order variation of the Casimir invariant of the bracket is given by

$$\delta C_{\text{da}} = \left\langle \delta\xi_{\text{da}}, \frac{\delta C}{\delta \xi} \right\rangle = \left\langle \{\mathcal{G}, \xi\}, \frac{\delta C}{\delta \xi} \right\rangle. \quad (6.20)$$

If we now assume that the bracket $\{\cdot, \cdot\}$ is of the Lie–Poisson type (Eq. (2.1)), we have

$$\delta C_{\text{da}} = \left\langle [\chi, \xi]^\dagger, \frac{\delta C}{\delta \xi} \right\rangle = \left\langle \xi, \left[\chi, \frac{\delta C}{\delta \xi} \right] \right\rangle = \{\mathcal{G}, C\} = 0.$$

Hence, to first order, *Casimirs are unchanged by a dynamically accessible perturbation*. The first-order variation of the Hamiltonian is

$$\delta H_{\text{da}} = \delta F_{\text{da}} = \left\langle [\chi, \xi]^\dagger, \frac{\delta H}{\delta \xi} \right\rangle = - \left\langle \left[\frac{\delta H}{\delta \xi}, \xi \right]^\dagger, \chi \right\rangle.$$

The variation of the Hamiltonian and of the free energy are the same because they differ only by Casimirs. If we look for equilibrium solutions by requiring that $\delta H_{\text{da}} = 0$ for all χ , we obtain

$$\left[\frac{\delta H}{\delta \xi}(\xi_e), \xi_e \right]^\dagger = 0,$$

which is equivalent to looking for steady solutions of the equation of motion (2.6).

Because we want to establish formal stability, we have to take second-order dynamically accessible variations that preserve the Casimirs. If we denote the second-order part of the dynamically accessible variation by $\delta^2\xi_{\text{da}}$, and the first and second order generating functions by $\chi^{(1)}$ and $\chi^{(2)}$, we have

$$\begin{aligned}
\delta^2 C_{\text{da}} &= \frac{1}{2} \left\langle \delta\xi_{\text{da}}, \frac{\delta^2 C}{\delta\xi \delta\xi} \delta\xi_{\text{da}} \right\rangle + \left\langle \delta^2\xi_{\text{da}}, \frac{\delta C}{\delta\xi} \right\rangle \\
&= \frac{1}{2} \left\langle \{\mathcal{G}^{(1)}, \xi\}, \frac{\delta^2 C}{\delta\xi \delta\xi} \{\mathcal{G}^{(1)}, \xi\} \right\rangle + \left\langle \delta^2\xi_{\text{da}}, \frac{\delta C}{\delta\xi} \right\rangle \\
&= \frac{1}{2} \left\langle \{\mathcal{G}^{(1)}, \xi\}, \frac{\delta}{\delta\xi} \left\langle \{\mathcal{G}^{(1)}, \xi\}, \frac{\delta C}{\delta\xi} \right\rangle - \left[\chi^{(1)}, \frac{\delta C}{\delta\xi} \right] \right\rangle + \left\langle \delta^2\xi_{\text{da}}, \frac{\delta C}{\delta\xi} \right\rangle \\
&= -\frac{1}{2} \left\langle [\chi^{(1)}, \{\mathcal{G}^{(1)}, \xi\}]^\dagger, \frac{\delta C}{\delta\xi} \right\rangle + \left\langle \delta^2\xi_{\text{da}}, \frac{\delta C}{\delta\xi} \right\rangle \\
&= \left\langle \delta^2\xi_{\text{da}} - \frac{1}{2} \{\mathcal{G}^{(1)}, \{\mathcal{G}^{(1)}, \xi\}\}, \frac{\delta C}{\delta\xi} \right\rangle.
\end{aligned}$$

We made use of the fact that (6.20) vanishes identically. In order for $\delta^2 C_{\text{da}}$ to be zero, we can set

$$\begin{aligned}
\delta^2\xi_{\text{da}} &= \{\mathcal{G}^{(2)}, \xi\} + \frac{1}{2} \{\mathcal{G}^{(1)}, \{\mathcal{G}^{(1)}, \xi\}\} \\
&= [\chi^{(2)}, \xi]^\dagger + \frac{1}{2} [\chi^{(1)}, [\chi^{(1)}, \xi]^\dagger]^\dagger.
\end{aligned} \tag{6.21}$$

The second-order dynamically accessible variation of H is

$$\begin{aligned}
\delta^2 H_{\text{da}} &= \frac{1}{2} \left\langle \delta\xi_{\text{da}}, \frac{\delta^2 H}{\delta\xi \delta\xi} \delta\xi_{\text{da}} \right\rangle + \left\langle \delta^2\xi_{\text{da}}, \frac{\delta H}{\delta\xi} \right\rangle \\
&= \frac{1}{2} \left\langle \{\mathcal{G}^{(1)}, \xi\}, \frac{\delta^2 H}{\delta\xi \delta\xi} \{\mathcal{G}^{(1)}, \xi\} \right\rangle \\
&\quad + \left\langle \{\mathcal{G}^{(2)}, \xi\} + \frac{1}{2} \{\mathcal{G}^{(1)}, \{\mathcal{G}^{(1)}, \xi\}\}, \frac{\delta H}{\delta\xi} \right\rangle,
\end{aligned}$$

which upon using (6.21) becomes

$$\begin{aligned} \delta^2 H_{\text{da}} &= \frac{1}{2} \left\langle [\chi^{(1)}, \xi]^\dagger, \frac{\delta^2 H}{\delta \xi \delta \xi} [\chi^{(1)}, \xi]^\dagger \right\rangle \\ &\quad + \left\langle [\chi^{(2)}, \xi]^\dagger + \frac{1}{2} [\chi^{(1)}, [\chi^{(1)}, \xi]^\dagger]^\dagger, \frac{\delta H}{\delta \xi} \right\rangle \end{aligned}$$

The piece involving $\chi^{(2)}$ can be written as

$$\left\langle [\chi^{(2)}, \xi]^\dagger, \frac{\delta H}{\delta \xi} \right\rangle = - \left\langle \left[\frac{\delta H}{\delta \xi}, \xi \right]^\dagger, \chi^{(2)} \right\rangle,$$

which vanishes when evaluated at an equilibrium of the equations of motion (6.3.1). Hence, for purposes of testing stability we may neglect the second-order generating function entirely. We therefore drop the superscripts on \mathcal{G} and χ , and write

$$\begin{aligned} \delta^2 H_{\text{da}} &= \frac{1}{2} \left\langle [\chi, \xi]^\dagger, \frac{\delta^2 H}{\delta \xi \delta \xi} [\chi, \xi]^\dagger \right\rangle + \frac{1}{2} \left\langle [\chi, [\chi, \xi]^\dagger]^\dagger, \frac{\delta H}{\delta \xi} \right\rangle \\ &= \frac{1}{2} \left\langle [\chi, \xi]^\dagger, \frac{\delta^2 H}{\delta \xi \delta \xi} [\chi, \xi]^\dagger + \left[\chi, \frac{\delta H}{\delta \xi} \right] \right\rangle \end{aligned} \quad (6.22)$$

To more easily determine sufficient stability conditions, we want to write (6.22) as a function of $\delta \xi_{\text{da}}$. (Then (6.22) will be a quadratic form in $\delta \xi_{\text{da}}$.) We now show that this is always possible. This is a generalization of a proof by Arnold [5] for 2-D Euler.

Assume that we have a dynamically accessible variation given in terms of a second generating function χ' ,

$$\delta' \xi_{\text{da}} = [\chi', \xi]^\dagger, \quad \delta'^2 \xi_{\text{da}} = \frac{1}{2} [\chi', [\chi', \xi]^\dagger]^\dagger,$$

such that $\delta \xi_{\text{da}} = \delta' \xi_{\text{da}}$. Then the difference in the second order variation of the

energy is

$$\begin{aligned}
2\delta^2 H_{\text{da}} - 2\delta'^2 H_{\text{da}} &= \left\langle [\chi, \xi]^\dagger, \frac{\delta^2 H}{\delta \xi \delta \xi} [\chi, \xi]^\dagger + \left[\chi, \frac{\delta H}{\delta \xi} \right] \right\rangle \\
&\quad - \left\langle [\chi', \xi]^\dagger, \frac{\delta^2 H}{\delta \xi \delta \xi} [\chi', \xi]^\dagger + \left[\chi', \frac{\delta H}{\delta \xi} \right] \right\rangle \\
&= \left\langle [\chi, \xi]^\dagger, \left[\chi, \frac{\delta H}{\delta \xi} \right] \right\rangle - \left\langle [\chi, \xi]^\dagger, \left[\chi', \frac{\delta H}{\delta \xi} \right] \right\rangle. \quad (6.23)
\end{aligned}$$

Using (2.5) and the Jacobi identity in \mathfrak{g} , we have that for any $\alpha, \beta, \gamma \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$,

$$\begin{aligned}
\langle [\alpha, \xi]^\dagger, [\beta, \gamma] \rangle &= \langle \xi, [\alpha, [\beta, \gamma]] \rangle \\
&= -\langle \xi, ([\beta, [\gamma, \alpha]] + [\gamma, [\alpha, \beta]]) \rangle \\
&= \langle [\beta, \xi]^\dagger, [\alpha, \gamma] \rangle - \langle [\gamma, \xi]^\dagger, [\alpha, \beta] \rangle.
\end{aligned}$$

Making use of this identity in the last term of (6.23), we get

$$\begin{aligned}
2\delta^2 H_{\text{da}} - 2\delta'^2 H_{\text{da}} &= \left\langle [\chi, \xi]^\dagger, \left[\chi, \frac{\delta H}{\delta \xi} \right] \right\rangle - \left\langle [\chi', \xi]^\dagger, \left[\chi, \frac{\delta H}{\delta \xi} \right] \right\rangle \\
&\quad + \left\langle \left[\frac{\delta H}{\delta \xi}, \xi \right]^\dagger, [\chi, \chi'] \right\rangle.
\end{aligned}$$

The first two terms cancel, and from (2.6) we are left with

$$2\delta^2 H_{\text{da}} - 2\delta'^2 H_{\text{da}} = -\langle \dot{\xi}, [\chi, \chi'] \rangle,$$

which vanishes at an equilibrium of the equations of motion, for any χ, χ' . We conclude that $\delta^2 H_{\text{da}}$ depends on χ only through $\delta \xi_{\text{da}}$. Thus, it is always possible to rewrite $\delta^2 H_{\text{da}}$ in terms of only the dynamically accessible perturbations χ .

6.3.2 2-D Euler

An equilibrium of the equation of motion for 2-D Euler (see Section 2.2.2) satisfies $[\phi_e, \omega_e] = 0$. The most general equilibrium solution can thus be written

$$\phi_e = \Phi(u(\mathbf{x})); \quad \omega_e = \Omega(u(\mathbf{x})),$$

where $u(\mathbf{x})$ is an arbitrary function. Contrary to the energy-Casimir result, neither the function Φ or Ω need be invertible (i.e., monotonic in their argument).

We can then examine stability by taking the dynamically accessible second variation of the energy. This is given by (6.22) with $[\chi, \xi_e]^\dagger = -[\chi, \omega_e]$,

$$\begin{aligned} \delta^2 H_{\text{da}}[\omega_e] &= \frac{1}{2} \left\langle [\chi, \omega_e]^\dagger, (-\nabla^{-2}) [\chi, \omega_e]^\dagger - [\chi, \phi_e] \right\rangle \\ &= \frac{1}{2} \left\langle [\chi, \omega_e], (-\nabla^{-2}) [\chi, \omega_e] + [\chi, \phi_e] \right\rangle \\ &= \frac{1}{2} \left\langle |\nabla \delta \phi_{\text{da}}|^2 + [\chi, \omega_e][\chi, \phi_e] \right\rangle \\ &= \frac{1}{2} \left\langle |\nabla \delta \phi_{\text{da}}|^2 + \Phi'(u) \Omega'(u) [\chi, u]^2 \right\rangle, \end{aligned}$$

where $\nabla^2 \delta \phi_{\text{da}} := \delta \omega_{\text{da}}$. A sufficient condition for $\delta^2 H_{\text{da}}[\omega_e]$ to be non-negative is

$$\Phi'(u) \Omega'(u) \geq 0, \quad (6.24)$$

that is, the derivatives of Φ and Ω must have opposite signs. The energy-Casimir result is recovered by letting $\Phi(u) = u$, for then we have $\omega_e = \Omega(\phi_e)$ and the stability condition is the usual Rayleigh criterion, $\Omega'(\phi_e) \geq 0$. The stability result (6.24) obtained using the dynamical accessibility method is more general.

6.3.3 Reduced MHD

The equations of motion and bracket for RMHD are described in Section 2.2.3.

The dynamical variables are $(\xi^0, \xi^1) = (\omega, \psi)$.

Equilibrium Solutions

We must first determine equilibrium solutions (ω_e, ψ_e) of the equations of motion (2.14), which must satisfy

$$[\omega_e, \phi_e] + [\psi_e, J_e] = 0,$$

$$[\psi_e, \phi_e] = 0.$$

To satisfy the second of these conditions we must have $\phi_e = \Phi(u)$, $\psi_e = \Psi(u)$, with $u = u(\mathbf{x})$. Using the fact that, for any $g(\mathbf{x})$ and $f(u(\mathbf{x}))$,

$$[g(\mathbf{x}), f(u)] = f'(u) [g(\mathbf{x}), u] = [f'(u) g(\mathbf{x}), u], \quad (6.25)$$

the first equilibrium condition can be written as

$$[\Phi'(u) \omega_e - \Psi'(u) J_e, u] = 0.$$

This is solved by

$$J_e = \frac{\Upsilon'(u) + \Phi'(u) \omega_e}{\Psi'(u)}, \quad (6.26)$$

where $\Upsilon(u)$ is an arbitrary function. Note that this does *not* necessarily imply that ω_e or J_e are functions of u only.

Formal Stability

Using the coadjoint bracket for extensions (5.2), the dynamically accessible perturbations are given by

$$\delta\omega_{\text{da}} = [\chi_0, \omega]^\dagger + [\chi_1, \psi]^\dagger = -[\chi_0, \omega] - [\chi_1, \psi],$$

$$\delta\psi_{\text{da}} = [\chi_0, \psi]^\dagger = -[\chi_0, \psi].$$

The second-order variation of the Hamiltonian, (6.22), is

$$\begin{aligned}
\delta^2 H_{\text{da}}[\omega_e; \psi_e] &= \frac{1}{2} \langle \delta\omega_{\text{da}}, (-\nabla^{-2}) \delta\omega_{\text{da}} - [\chi_0, \phi_e] \rangle \\
&\quad + \frac{1}{2} \langle \delta\psi_{\text{da}}, (-\nabla^2) \delta\psi_{\text{da}} - [\chi_0, J_e] - [\chi_1, \phi_e] \rangle \\
&= \frac{1}{2} \langle |\nabla \delta\phi_{\text{da}}|^2 + |\nabla \delta\psi_{\text{da}}|^2 \rangle \\
&\quad - \frac{1}{2} \langle [\chi_0, \phi_e] \delta\omega_{\text{da}} + [\chi_0, J_e] \delta\psi_{\text{da}} + [\chi_1, \phi_e] \delta\psi_{\text{da}} \rangle \\
&= \frac{1}{2} \left\langle |\nabla \delta\phi_{\text{da}}|^2 + |\nabla \delta\psi_{\text{da}}|^2 + \frac{\Phi'}{\Psi'} \delta\psi_{\text{da}} \delta\omega_{\text{da}} \right\rangle \\
&\quad - \frac{1}{2} \langle [\chi_0, J_e] \delta\psi_{\text{da}} + \Phi' [\chi_1, u] \delta\psi_{\text{da}} \rangle, \tag{6.27}
\end{aligned}$$

where we have defined $\nabla^2 \delta\phi_{\text{da}} := \delta\omega_{\text{da}}$. Now we use

$$\begin{aligned}
[\chi_1, u] &= \frac{1}{\Psi'} (\Psi' [\chi_1, u] + [\chi_0, \omega_e]) - \frac{1}{\Psi'} [\chi_0, \omega_e] \\
&= -\frac{1}{\Psi'} (\delta\omega_{\text{da}} + [\chi_0, \omega_e]),
\end{aligned}$$

to get

$$\begin{aligned}
\delta^2 H_{\text{da}}[\omega_e; \psi_e] &= \frac{1}{2} \left\langle |\nabla \delta\phi_{\text{da}}|^2 + |\nabla \delta\psi_{\text{da}}|^2 + 2 \frac{\Phi'}{\Psi'} \delta\psi_{\text{da}} \delta\omega_{\text{da}} \right\rangle \\
&\quad + \frac{1}{2} \left\langle \left(\frac{\Phi'}{\Psi'} [\chi_0, \omega_e] - [\chi_0, J_e] \right) \delta\psi_{\text{da}} \right\rangle \\
&= \frac{1}{2} \left\langle |\nabla \delta\phi_{\text{da}}|^2 + |\nabla \delta\psi_{\text{da}}|^2 + 2 \frac{\Phi'}{\Psi'} \delta\psi_{\text{da}} \delta\omega_{\text{da}} \right\rangle \\
&\quad - \frac{1}{2} \left\langle \left(\omega_e \left[\chi_0, \frac{\Phi'}{\Psi'} \right] + \left[\chi_0, \frac{\Upsilon'}{\Psi'} \right] \right) \delta\psi_{\text{da}} \right\rangle,
\end{aligned}$$

where we substituted (6.26) to eliminate J_e . To simplify the notation, we define the differential operator \mathbb{D} by

$$\mathbb{D}f(u) := \frac{1}{\Psi'(u)} \frac{d}{du} f(u), \tag{6.28}$$

so that

$$\begin{aligned}
\delta^2 H_{\text{da}}[\omega_e; \psi_e] &= \frac{1}{2} \langle |\nabla \delta\phi_{\text{da}}|^2 + |\nabla \delta\psi_{\text{da}}|^2 + 2 \mathbb{D}\Phi \delta\psi_{\text{da}} \nabla^2 \delta\phi_{\text{da}} \rangle \\
&\quad - \frac{1}{2} \langle (\omega_e [\chi_0, \mathbb{D}\Phi] + [\chi_0, \mathbb{D}\Upsilon]) \delta\psi_{\text{da}} \rangle. \tag{6.29}
\end{aligned}$$

Note that the first angle bracket in (6.29) is the same as (6.14), with k' replaced by $\mathbb{D}\Phi$. Hence, we can use identity (6.14) to obtain

$$\begin{aligned} \delta^2 H_{\text{da}} = & \frac{1}{2} \langle |\nabla \delta \phi_{\text{da}} - \nabla(\mathbb{D}\Phi \delta \psi_{\text{da}})|^2 + (1 - |\mathbb{D}\Phi|^2) |\nabla \delta \psi_{\text{da}}|^2 \rangle \\ & + \frac{1}{2} \langle (\mathbb{D}\Phi \nabla^2(\mathbb{D}\Phi) + \omega_e \mathbb{D}^2\Phi + \mathbb{D}^2\Upsilon) |\delta \psi_{\text{da}}|^2 \rangle. \end{aligned} \quad (6.30)$$

Sufficient conditions for the perturbation energy (6.30) to be non-negative are [31]

$$|\mathbb{D}\Phi| \leq 1, \quad (6.31)$$

$$\mathbb{D}\Phi \nabla^2(\mathbb{D}\Phi) + \nabla^2\Phi \mathbb{D}^2\Phi + \mathbb{D}^2\Upsilon \geq 0. \quad (6.32)$$

In the second expression we have substituted $\omega_e = \nabla^2\Phi$. The first condition says that $|\Phi'(u)| \leq |\Psi'(u)|$, that is, the gradient of the magnetic flux is greater or equal to the gradient of the electric potential. This is a similar condition to (6.16), and says that the flow needs to be *sub-Alfvénic* to be formally stable [45]. This is due to the well-known fact that the magnetic field provides a restoring force for perturbations of the flow, so that a large enough magnetic field can potentially stabilize the system (but not necessarily so, because the magnetic field can also have a destabilizing effect [18]). Indeed, condition (6.31) is actually *necessary* for positive-definiteness of $\delta^2 H_{\text{da}}$. If we choose $\delta \phi_{\text{da}} = \mathbb{D}\Phi \delta \psi_{\text{da}}$ in (6.30), then the first term vanishes. We can then pick a variation of $\delta \psi_{\text{da}}$ with as steep a gradient as we want, while maintaining the value of $\delta \psi_{\text{da}}$ bounded [27, p. 103]. This means that the $|\nabla \delta \psi_{\text{da}}|^2$ term can always be made to dominate, so that we *require* $|\mathbb{D}\Phi| \leq 1$ for positive-definiteness of $\delta^2 H_{\text{da}}$.

This places a limitation on the method of dynamical accessibility: if we want to satisfy $|\mathbb{D}\Phi| = \Phi'/\Psi' \leq 1$ everywhere, then on their domain of defini-

tion *the zeros of Ψ must also be zeros of Φ* with equal or higher multiplicity. (However, the function Φ could potentially have *more* zeros than Ψ .)

The simplest case is when Ψ has no zeros, but then $\Psi(u)$ is invertible, and we can recover the energy-Casimir result by solving for $u = u(\Psi)$. In practice, this inversion may be difficult, and using the dynamical accessibility method is often easier.

As an example, we will derive equilibria for *magnetic islands with flow*. Consider the RMHD equilibrium relation (6.26), multiplied by $\Psi'(u)$,

$$\Psi'(u)J_e - \Phi'(u)\omega_e = \Upsilon'(u). \quad (6.33)$$

where $J_e = \nabla^2\Psi(u)$. Using the fact that

$$\omega_e = \nabla^2\Phi(u) = \Phi'(u)\nabla^2u + \Phi''(u)|\nabla u|^2, \quad (6.34)$$

and the analogous relation for J_e , we can rewrite (6.33) as

$$((\Psi')^2 - (\Phi')^2)\nabla^2u + (\Psi'\Psi'' - \Phi'\Phi'')|\nabla u|^2 = \Upsilon'(u),$$

or equivalently

$$((\Psi')^2 - (\Phi')^2)\nabla^2u + \frac{1}{2}((\Psi')^2 - (\Phi')^2)'|\nabla u|^2 = \Upsilon'(u). \quad (6.35)$$

We can get rid of the $|\nabla u|^2$ term, and make the equation easier to solve, by choosing

$$(\Psi')^2 - (\Phi')^2 = \kappa^2.$$

(Choosing a different sign for the right-hand side would lead to solutions with $\mathbb{D}\Phi > 1$.) An obvious solution is

$$\Psi'(u) = \kappa \cosh(\nu u), \quad (6.36)$$

$$\Phi'(u) = \kappa \sinh(\nu u). \quad (6.37)$$

These satisfy $|\mathbb{D}\Phi| = |\tanh(\nu u)| < 1$, condition (6.31).

Equation (6.35) becomes

$$\nabla^2 u = \kappa^{-2} \Upsilon'(u), \quad (6.38)$$

to be solved for $u(\mathbf{x})$. This equation has the same form as (6.12), which was an equation for $\psi_e(\mathbf{x})$, so it has the same Kelvin–Stuart cat’s eye solution,

$$u(x, y) = \ln(a \cosh y + \sqrt{a^2 - 1} \cos x),$$

with $\Upsilon'(u) = \kappa^2 \exp(-2u)$. The difference is that now the physical variables are given in terms of u by (6.36) and (6.37), so that the electric potential (and so the flow velocity) does not necessarily vanish, as opposed to the usual magnetic island solutions, which are recovered in the limit $\nu = 0$. The stability of the islands with flow could be very different, since now $\Phi' \neq 0$ in (6.29). However, as for the usual magnetic islands, the sufficient condition (6.32) is not satisfied, so that stability must be determined by test perturbations, or by direct numerical simulation [15, 22, 38, 82].

6.3.4 Pure Semidirect Sum

We now treat the general stability of the pure semidirect sum structure, with no cocycles (see Section 4.4). This structure is given simply by the $n + 1 \times n + 1$ matrices $W^{(0)} = I$, and $\widetilde{W}^{(\mu)} = 0$, $\mu = 1, \dots, n$. We denote the 0th field variable by $\xi^0 = \varpi$, and the remaining n variables by ξ^1, \dots, ξ^n .

Equilibrium Solutions

An equilibrium $(\varpi_e, \{\xi_e^\mu\})$ of the equations of motion for a pure semidirect extension satisfies

$$\dot{\varpi}_e = -[H_{,0}, \varpi_e]^\dagger - \sum_{\mu=1}^n [H_{,\mu}, \xi_e^\mu]^\dagger = 0, \quad (6.39)$$

$$\dot{\xi}_e^\mu = -[H_{,0}, \xi_e^\mu]^\dagger = 0, \quad \mu = 1, \dots, n. \quad (6.40)$$

To unclutter the notation, we assume that the first and second derivatives of the Hamiltonian H are evaluated at the equilibrium $(\varpi_e, \{\xi_e^\mu\})$, unless otherwise noted.

We now specialize the bracket to the 2-D canonical one, Eq. (2.11), so that $[,]^\dagger = -[,]$. To satisfy condition (6.40), we require

$$H_{,0} = -\Phi(u), \quad \xi_e^\mu = \Xi^\mu(u), \quad \mu = 1, \dots, n, \quad (6.41)$$

for arbitrary functions Φ , Ξ^μ , and $u = u(\mathbf{x})$. (The choice of the minus sign for the definition of Φ is purely a convention to agree with the sign of the streamfunction in 2-D Euler, for which $H_{,0} = \delta H / \delta \omega = -\phi$.) Condition (6.39) is then

$$-[\Phi(u), \varpi_e] + \sum_{\mu=1}^n [H_{,\mu}, \Xi^\mu(u)] = 0,$$

or, using (6.25),

$$\left[u, \Phi'(u) \varpi_e + \sum_{\mu=1}^n H_{,\mu} \Xi^{\mu'}(u) \right] = 0,$$

which has solution

$$\Phi'(u) \varpi_e + \sum_{\mu=1}^n H_{,\mu} \Xi^{\mu'}(u) = \Upsilon'(u). \quad (6.42)$$

Equation (6.42) should be compared with (6.26), the equivalent solution for reduced MHD, for which $n = 1$ and $H_{,1} = \delta H / \delta \psi = -J$.

Now that we have the equilibria, using (5.2) we write down the dynamically accessible perturbations

$$\delta \varpi_{\text{da}} = [\chi_0, \varpi]^\dagger + \sum_{\nu=1}^n [\chi_\nu, \xi^\nu]^\dagger, \quad (6.43)$$

$$\delta \xi_{\text{da}}^\mu = [\chi_0, \xi^\mu]^\dagger, \quad \mu = 1, \dots, n, \quad (6.44)$$

and from (6.22) we get the second-order dynamically accessible variation of the Hamiltonian,

$$\begin{aligned} \delta^2 H_{\text{da}} &= \frac{1}{2} \left\langle \delta \varpi_{\text{da}}, H_{,00} \delta \varpi_{\text{da}} + \sum_{\mu=1}^n H_{,0\mu} \delta \xi_{\text{da}}^\mu + [\chi_0, H_{,0}] \right\rangle \\ &+ \sum_{\mu=1}^n \frac{1}{2} \left\langle \delta \xi_{\text{da}}^\mu, \sum_{\nu=1}^n H_{,\mu\nu} \delta \xi_{\text{da}}^\nu + H_{,\mu 0} \delta \varpi_{\text{da}} + [\chi_0, H_{,\mu}] + [\chi_\mu, H_{,0}] \right\rangle. \end{aligned}$$

Because the second-order functional derivative is formally a self-adjoint operator, we have the identity

$$\langle \delta \varpi_{\text{da}}, H_{,0\mu} \delta \xi_{\text{da}}^\mu \rangle = \langle \delta \xi_{\text{da}}^\mu, H_{,\mu 0} \delta \varpi_{\text{da}} \rangle,$$

which we use in $\delta^2 H_{\text{da}}$ to combine two terms and obtain

$$\begin{aligned} \delta^2 H_{\text{da}} &= \frac{1}{2} \left\langle \delta \varpi_{\text{da}}, H_{,00} \delta \varpi_{\text{da}} + 2 \sum_{\mu=1}^n H_{,0\mu} \delta \xi_{\text{da}}^\mu + [\chi_0, H_{,0}] \right\rangle \\ &+ \sum_{\mu=1}^n \frac{1}{2} \left\langle \delta \xi_{\text{da}}^\mu, \sum_{\nu=1}^n H_{,\mu\nu} \delta \xi_{\text{da}}^\nu + [\chi_0, H_{,\mu}] + [\chi_\mu, H_{,0}] \right\rangle. \quad (6.45) \end{aligned}$$

Using the equilibrium solution (6.41), the dynamically accessible variations given by (6.44) can be rewritten

$$\delta \xi_{\text{da}}^\mu = -\Xi^{\mu'}(u) [\chi_0, u]$$

Observe that the perturbations of the ξ^μ are not independent: they all depend on a single generating function, χ_0 . We choose to write all the variations in terms of $\delta\Xi_{\text{da}}^n$. We define

$$\psi(\mathbf{x}) := \xi^n(\mathbf{x}), \quad \Psi(u) := \Xi^n(u),$$

to explicitly show the special role of ξ^n . Then we have

$$\delta\xi_{\text{da}}^\mu = \frac{\Xi^{\mu'}(u)}{\Psi'(u)} \delta\psi_{\text{da}} = \mathbb{D}\Xi^\mu \delta\psi_{\text{da}}, \quad (6.46)$$

where we have used the previous definition of the operator \mathbb{D} ,

$$\mathbb{D}f(u) := \frac{1}{\Psi'(u)} \frac{d}{du} f(u). \quad (6.28)$$

Note that $\mathbb{D}\Xi^n = \mathbb{D}\Psi = 1$. We could have chosen any field instead of ξ^n , but in Section 6.3.5 this particular choice will prove advantageous due to the lower-triangular structure of our extensions.

The dynamically accessible variations must obey the constraints of the system, that is they must lie on the coadjoint orbits. We have already discussed briefly this property of the semidirect sum in Section 3.4.

To illustrate the situation we consider the equations of motion for a finite-dimensional semidirect sum, specifically a semidirect sum of the rotation group $SO(3)$ (associated with our old friend the rigid body) with \mathbb{R}^3 (see Section 2.2). We take ϖ to be ℓ , the angular momentum vector, with Hamiltonian H given by the usual kinetic energy, Eq. (2.10). The variables ξ^μ are three-vectors, and their equations of motion are given in terms of the bracket (2.9) by

$$\begin{aligned} \dot{\xi}^\mu &= -[H_{,0}, \xi^\mu]^\dagger \\ &= (I^{-1}\ell) \times \xi^\mu. \end{aligned} \quad (6.47)$$

Note the angular momentum ℓ is analogous to the vorticity ω , and $I^{-1}\ell$ is analogous to the streamfunction $\phi = \nabla^{-2}\omega$. Equation (6.47) says that the vector ξ^μ is rotating with the rigid body, keeping its length constant (the length of ξ^μ is a Casimir). Thus, each ξ^μ can be used to describe a point in the rigid body, such as the center of gravity. Adding a coupling term to the Hamiltonian can provide us with, for instance, a description of the heavy top in a gravitational field, but this would not change the form of (6.47). The point is that the ξ^μ are constrained to rotate rigidly, and the dynamically accessible perturbations must obey the same constraint—they must depend on the perturbation applied to ℓ , but by themselves there are no dynamically accessible perturbations that allow the ξ^μ to change length or rotate independently. Physically, this makes sense, because we are not allowing the rigid body to have other degrees of freedom than the rotational ones. If we did, we would have to rethink our description, which would lead to different dynamically accessible perturbations; but within the confines of rigidity those perturbations make sense.

The situation in infinite dimensions is analogous to the rigid body. Here the typical case is an ideal fluid with passive scalars: we take $\varpi = \omega(\mathbf{x})$, the vorticity, and a Hamiltonian of the form $H[\omega] = -\frac{1}{2} \langle \phi, \omega \rangle$. The equations of motion for the $\xi^\mu(\mathbf{x})$ are given by

$$\dot{\xi}^\mu = -[H, \xi^\mu]^\dagger = -[\phi, \xi^\mu]. \quad (6.48)$$

Thus, the $\xi^\mu(\mathbf{x})$ are advected along by the fluid. The $\xi^\mu(\mathbf{x})$ can be used to describe passive scalars, since they do not enter the Hamiltonian (they do not affect the flow itself). An interaction term in H could describe, for example, the effect of temperature on the flow in the Boussinesq approximation, but

this would not modify (6.48): only the equation for $\dot{\omega}$ would change. Much like for the rigid body, the quantities ξ^μ are constrained to move with the fluid, regardless of the form of the Hamiltonian. This is also true for the dynamically accessible perturbations of the ξ^μ , which must then be induced by the perturbation on ω .

Formal Stability

We now try to rewrite the second-order variation of the Hamiltonian (6.45) only in terms of dynamically accessible variations. We have, from (6.41),

$$\begin{aligned} [\chi_0, H_{,0}] &= -\Phi'(u) [\chi_0, u] \\ &= \mathbb{D}\Phi(u) \delta\psi_{\text{da}}. \end{aligned}$$

From the second line of (6.45), we can write

$$\begin{aligned} \sum_{\mu=1}^n \left\langle \delta\xi_{\text{da}}^\mu, [\chi_\mu, H_{,0}] \right\rangle &= - \sum_{\mu=1}^n \left\langle \delta\psi_{\text{da}} \mathbb{D}\Xi^\mu \Phi' [\chi_\mu, u] \right\rangle \\ &= - \sum_{\mu=1}^n \left\langle \delta\psi_{\text{da}} \mathbb{D}\Phi [\chi_\mu, \Xi^\mu] \right\rangle \quad (6.49) \\ &= \left\langle \mathbb{D}\Phi (\delta\varpi_{\text{da}} + [\chi_0, \varpi_e]) \delta\psi_{\text{da}} \right\rangle, \end{aligned}$$

where we have made use of (6.43) and (6.46). Finally, we have

$$\begin{aligned} \sum_{\mu=1}^n \left\langle \delta\xi_{\text{da}}^\mu, [\chi_0, H_{,\mu}] \right\rangle &= \sum_{\mu=1}^n \left\langle \delta\psi_{\text{da}} \mathbb{D}\Xi^\mu [\chi_0, H_{,\mu}] \right\rangle \\ &= \sum_{\mu=1}^n \left\langle \delta\psi_{\text{da}} [\chi_0, H_{,\mu} \mathbb{D}\Xi^\mu] - \delta\psi_{\text{da}} H_{,\mu} [\chi_0, \mathbb{D}\Xi^\mu] \right\rangle, \end{aligned}$$

in which we make use of (6.42) to obtain

$$\begin{aligned}
\sum_{\mu=1}^n \left\langle \delta \xi_{\text{da}}^\mu, [\chi_0, H_{,\mu}] \right\rangle &= \left\langle \delta \psi_{\text{da}} \left([\chi_0, \mathbb{D}\Upsilon - \mathbb{D}\Phi \varpi_e] - \sum_{\mu=1}^n H_{,\mu} (\mathbb{D}\Xi^\mu)' [\chi_0, u] \right) \right\rangle \\
&= \left\langle (\mathbb{D}^2\Phi \varpi_e + \sum_{\mu=1}^n H_{,\mu} \mathbb{D}^2\Xi^\mu - \mathbb{D}^2\Upsilon) |\delta \psi_{\text{da}}|^2 \right\rangle \\
&\quad - \left\langle \mathbb{D}\Phi [\chi_0, \varpi_e] \delta \psi_{\text{da}} \right\rangle,
\end{aligned} \tag{6.50}$$

The last term in (6.50) cancels part of (6.49), and we get

$$\begin{aligned}
\delta^2 H_{\text{da}} &= \frac{1}{2} \left\langle \delta \varpi_{\text{da}} H_{,00} \delta \varpi_{\text{da}} + 2 \sum_{\mu=1}^n \delta \varpi_{\text{da}} H_{,0\mu} \delta \xi_{\text{da}}^\mu + 2 \mathbb{D}\Phi \delta \varpi_{\text{da}} \delta \psi_{\text{da}} \right. \\
&\quad \left. + \sum_{\mu,\nu=1}^n \delta \xi_{\text{da}}^\mu H_{,\mu\nu} \delta \xi_{\text{da}}^\nu + (\mathbb{D}^2\Phi \varpi_e + \sum_{\mu=1}^n H_{,\mu} \mathbb{D}^2\Xi^\mu - \mathbb{D}^2\Upsilon) |\delta \psi_{\text{da}}|^2 \right\rangle.
\end{aligned} \tag{6.51}$$

Further progress cannot be made without assuming some particular form for the second-order functional derivative operator of H .

Hamiltonian without operators

The simplest case we can study is when H contains no differential or integral operators. Then $H_{,\mu\nu}$ is just a symmetric matrix. Using (6.46), we can simplify (6.51) to

$$\begin{aligned}
\delta^2 H_{\text{da}} &= \frac{1}{2} \left\langle H_{,00} |\delta \varpi_{\text{da}}|^2 + 2 \left(\sum_{\mu=1}^n H_{,0\mu} \mathbb{D}\Xi^\mu + \mathbb{D}\Phi \right) \delta \varpi_{\text{da}} \delta \psi_{\text{da}} \right. \\
&\quad \left. + \left(\sum_{\mu,\nu=1}^n \mathbb{D}\Xi^\mu H_{,\mu\nu} \mathbb{D}\Xi^\nu + \mathbb{D}^2\Phi \varpi_e + \sum_{\mu=1}^n H_{,\mu} \mathbb{D}^2\Xi^\mu - \mathbb{D}^2\Upsilon \right) |\delta \psi_{\text{da}}|^2 \right\rangle.
\end{aligned}$$

This can be rewritten as a quadratic form,

$$\delta^2 H_{\text{da}} = \frac{1}{2} (\delta \varpi_{\text{da}} \quad \delta \psi_{\text{da}}) \mathcal{Q} \begin{pmatrix} \delta \varpi_{\text{da}} \\ \delta \psi_{\text{da}} \end{pmatrix},$$

where \mathcal{Q} is the 2×2 matrix

$$\mathcal{Q} := \left(\begin{array}{c|c} H_{,00} & H_{,0\mu} \mathbb{D}\Xi^\mu + \mathbb{D}\Phi \\ \hline H_{,0\mu} \mathbb{D}\Xi^\mu + \mathbb{D}\Phi & \mathbb{D}\Xi^\mu H_{,\mu\nu} \mathbb{D}\Xi^\nu + \mathbb{D}^2\Phi \varpi_e + H_{,\mu} \mathbb{D}^2\Xi^\mu - \mathbb{D}^2\Upsilon \end{array} \right).$$

We assume repeated indices are summed from 1 to n . The matrix \mathcal{Q} is non-negative if and only if its principal minors are non-negative, i.e.,

$$H_{,00} \geq 0, \quad (6.52)$$

$$\det \mathcal{Q} \geq 0. \quad (6.53)$$

Hence, to have formal stability it is imperative to have that the energy associated with the perturbation of ϖ be non-negative. Also note that the contribution of $(H_{,0\mu} \mathbb{D}\Xi^\mu + \mathbb{D}\Phi)$ is always destabilizing. For an equilibrium without flow ($\mathbb{D}\Phi \equiv 0$) and with $H_{,0\nu} = 0$, condition (6.53) reduces to

$$\mathbb{D}\Xi^\mu H_{,\mu\nu} \mathbb{D}\Xi^\nu + H_{,\mu} \mathbb{D}^2\Xi^\mu - \mathbb{D}^2\Upsilon \geq 0.$$

Advection Scalars

We now treat the problem of advection of scalars. We shall not restrict ourselves to passive advection, and the form we choose for H is general enough to encompass systems with *generalized vorticities*,² such as the quasigeostrophic equations [39, 94].

Let q denote the generalized vorticity, related to the stream function ϕ by

$$q = \nabla^2 \phi - \mathcal{F} \phi + f, \quad (6.54)$$

²Also called the *potential vorticity*.

for some given functions $\mathcal{F}(\mathbf{x})$ and $f(\mathbf{x})$. Taking $\xi^0 = \varpi = q$, we consider a Hamiltonian

$$\begin{aligned} H &= \left\langle \frac{1}{2} (|\nabla\phi|^2 + \mathcal{F}\phi^2) + \mathcal{V}(\mathbf{x}, q, \xi^1, \dots, \xi^n) \right\rangle \\ &= \left\langle \frac{1}{2} (q - f)(\mathcal{F} - \nabla^2)^{-1}(q - f) + \mathcal{V}(\mathbf{x}, q, \xi^1, \dots, \xi^n) \right\rangle, \end{aligned}$$

where \mathcal{V} does not contain any operators. We have the first derivatives

$$H_{,0} = -\phi + \mathcal{V}_{,0}, \quad H_{,\mu} = \mathcal{V}_{,\mu},$$

and the second derivative operators

$$H_{,00} = (\mathcal{F} - \nabla^2)^{-1} + \mathcal{V}_{,00},$$

$$H_{,\mu\nu} = \mathcal{V}_{,\mu\nu},$$

$$H_{,0\nu} = \mathcal{V}_{,0\nu}.$$

Using identity (6.14), we can rewrite the first line of the second dynamically accessible variation of the energy (6.51) as

$$\begin{aligned} & \frac{1}{2} \left\langle \delta q_{\text{da}} ((\mathcal{F} - \nabla^2)^{-1} + \mathcal{V}_{,00}) \delta q_{\text{da}} + 2 (\mathcal{V}_{,0\mu} \mathbb{D}\Xi^\mu + \mathbb{D}\Phi) \delta q_{\text{da}} \delta \psi_{\text{da}} \right\rangle \\ &= \frac{1}{2} \left\langle \delta \phi_{\text{da}} (\mathcal{F} - \nabla^2) \delta \phi_{\text{da}} + \mathcal{V}_{,00} |\delta q_{\text{da}}|^2 - 2\mathcal{K}(u) \delta \psi_{\text{da}} (\mathcal{F} - \nabla^2) \delta \phi_{\text{da}} \right\rangle \\ &= \frac{1}{2} \left\langle |\nabla \delta \phi_{\text{da}} - \nabla (\mathcal{K} \delta \psi_{\text{da}})|^2 - \mathcal{K}^2 |\nabla \delta \psi_{\text{da}}|^2 + \mathcal{V}_{,00} |\delta q_{\text{da}}|^2 \right. \\ & \quad \left. + \mathcal{F} |\delta \phi_{\text{da}} - \mathcal{K} \delta \psi_{\text{da}}|^2 + \mathcal{K} (\nabla^2 \mathcal{K} - \mathcal{F} \mathcal{K}) |\delta \psi_{\text{da}}|^2 \right\rangle \end{aligned} \quad (6.55)$$

where

$$\mathcal{K}(u) := \mathcal{V}_{,0\mu} \mathbb{D}\Xi^\mu(u) + \mathbb{D}\Phi(u). \quad (6.56)$$

The term proportional to $|\nabla \delta \psi_{\text{da}}|^2$ in (6.55) is negative definite unless we require an equilibrium with $\mathcal{K}(u) \equiv 0$, that is

$$\mathcal{V}_{,0\mu} \mathbb{D}\Xi^\mu(u) + \mathbb{D}\Phi(u) = 0.$$

Using $\mathcal{K} \equiv 0$ in (6.55) and writing out the rest of (6.51), we obtain

$$\begin{aligned} \delta^2 H_{\text{da}} = & \frac{1}{2} \left\langle |\nabla \delta \phi_{\text{da}}|^2 + \mathcal{F} |\delta \phi_{\text{da}}|^2 + \mathcal{V}_{,00} |\delta q_{\text{da}}|^2 \right. \\ & \left. + \left(\mathbb{D}^2 \Phi q_e + \mathbb{D} \Xi^\mu \mathcal{V}_{,\mu\nu} \mathbb{D} \Xi^\nu + \mathcal{V}_{,\mu} \mathbb{D}^2 \Xi^\mu - \mathbb{D}^2 \Upsilon \right) |\delta \psi_{\text{da}}|^2 \right\rangle. \end{aligned}$$

For Hamiltonians with $\mathcal{V}_{,0\mu} = 0$, the only equilibria for which we can demonstrate formal stability are ones without flow. If we assume this is the case, then from (6.42) equilibria satisfy $\mathcal{V}_{,\mu} \mathbb{D} \Xi^\mu(u) = \mathbb{D} \Upsilon(u)$. Note that f in (6.54) enters the stability expression through $q_e = \nabla^2 \Phi - \mathcal{F} \Phi + f$.

Combining (6.55) with the rest of (6.51) we have the the sufficient conditions for stability

$$\begin{aligned} \mathcal{F} & \geq 0, \\ \mathcal{V}_{,00} & \geq 0, \\ \mathbb{D} \Xi^\mu \mathcal{V}_{,\mu\nu} \mathbb{D} \Xi^\nu + \mathcal{V}_{,\mu} \mathbb{D}^2 \Xi^\mu - \mathbb{D}^2 \Upsilon & \geq 0, \end{aligned}$$

where we have assumed $\mathbb{D} \Phi \equiv 0$ so that $\mathcal{K}(u) = \mathcal{V}_{,0\mu} \mathbb{D} \Xi^\mu(u)$. This is the same stability condition as for a Hamiltonian without operators, (6.3.4), because we have chose a form of the Hamiltonian which decouples the operator part (kinetic energy) and the potential, so we get a Lagrange-theorem-like condition on the potential.

RMHD-like System

Another case of interest, a generalization of the RMHD system of Sections 2.2.3 and 6.3.3, involves a Hamiltonian of the form

$$H = \frac{1}{2} \left\langle (|\nabla \phi|^2 + \mathcal{F} \phi^2) + 2\mathcal{V}(\mathbf{x}, q, \xi^1, \dots, \xi^{n-1}, \psi) + |\nabla \psi|^2 \right\rangle. \quad (6.57)$$

Here q , \mathcal{F} , and ϕ are as in the previous section in (6.54). As before, we have labeled ξ^n by ψ as a reminder of its distinguished role: it enters the Hamiltonian as a gradient. (In this section greek indices run from 1 to $n - 1$.) The first derivatives of H are given by

$$H_{,0} = -\phi + \mathcal{V}_{,0}, \quad H_{,\mu} = \mathcal{V}_{,\mu}, \quad H_{,n} = -J + \mathcal{V}_{,n}, \quad (6.58)$$

and the second derivative operators are

$$\begin{aligned} H_{,00} &= (\mathcal{F} - \nabla^2)^{-1} + \mathcal{V}_{,00} & H_{,0n} &= \mathcal{V}_{,0n} \\ H_{,\mu\nu} &= \mathcal{V}_{,\mu\nu} & H_{,\mu n} &= \mathcal{V}_{,\mu n} \\ H_{,0\nu} &= \mathcal{V}_{,0\nu} & H_{,nn} &= -\nabla^2 + \mathcal{V}_{,nn}. \end{aligned} \quad (6.59)$$

The quantity $J := \nabla^2 \psi$ is analogous to the electric current in RMHD. As before, we use $\Xi^\mu(u)$ to denote the equilibrium solution of ξ^μ for $\mu = 1, \dots, n$, and the equilibrium solution of ξ^n is written $\xi_e^n = \Psi(u)$. Also as done previously, we use the relation

$$\delta \xi_{\text{da}}^\mu = \mathbb{D} \Xi^\mu \delta \psi_{\text{da}}, \quad (6.46)$$

where \mathbb{D} is defined by (6.28). Adding the $|\nabla \delta \psi_{\text{da}}|^2$ contribution to (6.55), we obtain

$$\begin{aligned} & \frac{1}{2} \left\langle \delta q_{\text{da}} \left((\mathcal{F} - \nabla^2)^{-1} + \mathcal{V}_{,00} \right) \delta q_{\text{da}} + 2\mathcal{K} \delta q_{\text{da}} \delta \psi_{\text{da}} + |\nabla \delta \psi_{\text{da}}|^2 \right\rangle \\ &= \frac{1}{2} \left\langle \delta \phi_{\text{da}} (\mathcal{F} - \nabla^2) \delta \phi_{\text{da}} + \mathcal{V}_{,00} |\delta q_{\text{da}}|^2 - 2\mathcal{K} \delta \psi_{\text{da}} (\mathcal{F} - \nabla^2) \delta \phi_{\text{da}} + |\nabla \delta \psi_{\text{da}}|^2 \right\rangle \\ &= \frac{1}{2} \left\langle |\nabla \delta \phi_{\text{da}} - \nabla (\mathcal{K} \delta \psi_{\text{da}})|^2 + (1 - \mathcal{K}^2) |\nabla \delta \psi_{\text{da}}|^2 + \mathcal{V}_{,00} |\delta q_{\text{da}}|^2 \right. \\ & \quad \left. + \mathcal{F} |\delta \phi_{\text{da}} - \mathcal{K} \delta \psi_{\text{da}}|^2 + \mathcal{K} (\nabla^2 \mathcal{K} - \mathcal{F} \mathcal{K}) |\delta \psi_{\text{da}}|^2 \right\rangle \end{aligned} \quad (6.60)$$

where \mathcal{K} is defined by (6.56). The energy provided by the new $|\nabla \delta \psi_{\text{da}}|^2$ term in the Hamiltonian (magnetic line-bending energy in MHD) allows us to have

formally stable equilibria provided $\mathcal{K}^2 \leq 1$. Thus, in contrast to the system in the previous section, there exist formally stable equilibria with flow even for a potential with $\mathcal{V}_{,0\mu} = 0$.

6.3.5 Nonsingular g

Now that we have demonstrated the procedure for obtaining equilibria and determining their stability for brackets with no cocycles (Section 6.3.4), we are in a position to deal with the more complicated case of an arbitrary semidirect-type extensions with a nonsingular $W_{(n)} = g$. We shall make heavy use of the concept of coextension introduced in Section 5.4.1.

Equilibrium Solutions

First we must look for equilibria of the equations of motion, which from (2.6) and (5.2) are

$$\dot{\varpi}_e = 0 = -[H_{,0}, \varpi_e]^\dagger - [H_{,\mu}, \xi_e^\mu]^\dagger - [H_{,n}, \psi_e]^\dagger, \quad (6.61)$$

$$\dot{\xi}_e^\mu = 0 = -[H_{,0}, \xi_e^\mu]^\dagger - \widetilde{W}_\lambda{}^{\mu\nu} [H_{,\nu}, \xi_e^\lambda]^\dagger - g^{\mu\nu} [H_{,\nu}, \psi_e]^\dagger, \quad (6.62)$$

$$\dot{\psi}_e = 0 = -[H_{,0}, \psi_e]^\dagger. \quad (6.63)$$

Unless otherwise noted, in this section all greek indices take values from 1 to $n - 1$, and repeated indices are summed. The tensors \widetilde{W} were defined in Section 5.4: they are the subtensors of W with indices restricted from 1 to $n - 1$. They form a solvable extension. We have also made use of the definition $g^{\mu\nu} := W_{(n)}{}^{\mu\nu}$. As in Section 6.3.4, we have set the variable ξ^n apart and labeled it by ψ , but now it does actually play a distinguished role in the solution of the problem, as it did in Section 5.4. Also note that the derivatives of the Hamiltonian are implicitly evaluated at the equilibrium $(\varpi_e, \{\xi_e^\mu\}, \psi_e)$.

We now specialize the bracket to the 2-D canonical one, given by (2.12). Equation (6.63) is satisfied if

$$H_{,0} = -\Phi(u), \quad \psi_e = \Psi(u), \quad (6.64)$$

for functions Φ and Ψ , and some $u = u(\mathbf{x})$. Equation (6.62) is quite a bit dicier to solve. The trick is to use the lower-triangular form of the $\widetilde{W}^{(\mu)}$ to solve for the $H_{,\nu}$. We multiply (6.62) by $\bar{g} := g^{-1}$, and use (6.64), to obtain

$$-\left[\Phi(u), \bar{g}_{\tau\mu} \xi_e^\mu \right] + \bar{g}_{\tau\mu} \widetilde{W}_\lambda^{\mu\nu} \left[H_{,\nu}, \xi_e^\lambda \right] + \bar{g}_{\tau\mu} g^{\mu\nu} \left[H_{,\nu}, \Psi(u) \right] = 0,$$

or, using the definition (5.21) of the coextension, $A_{\tau\lambda}^\nu := \widetilde{W}_\tau^{\nu\mu} \bar{g}_{\mu\lambda}$,

$$\left[H_{,\tau} \Psi'(u) + \Phi'(u) \bar{g}_{\tau\mu} \xi_e^\mu, u \right] + A_{\tau\lambda}^\nu \left[H_{,\nu}, \xi_e^\lambda \right] = 0. \quad (6.65)$$

Since the $\widetilde{W}^{(\mu)}$'s are lower-triangular, the $A^{(\nu)}$'s have the form

$$A^{(\nu)} = \begin{pmatrix} 0 & 0 \\ 0 & \boxed{} \end{pmatrix}, \quad \nu = 1, \dots, n-1,$$

where the box represents a square $(n - \nu - 1)$ -dimensional symmetric matrix of possibly nonzero elements. There are never any nonvanishing elements in the first row of $A^{(\nu)}$, so setting $\tau = 1$ in (6.65) gives

$$\left[H_{,1} \Psi'(u) + \Phi'(u) \bar{g}_{1\mu} \xi_e^\mu, u \right] = 0. \quad (6.66)$$

We write the solution as

$$H_{,1} = k_1(u) - \mathbb{D}\Phi(u) \bar{g}_{1\mu} \xi_e^\mu,$$

where $k_1(u)$ is an arbitrary function and the operator \mathbb{D} is defined by (6.28).

Equation (6.65) with $\tau = 2$ is

$$\left[H_{,2} \Psi'(u) + \Phi'(u) \bar{g}_{2\mu} \xi_e^\mu, u \right] + A_{2\lambda}^1 \left[H_{,1}, \xi_e^\lambda \right] = 0.$$

If we then substitute in the solution for $H_{,1}$, Eq. (6.66), we have

$$\left[H_{,2} \Psi'(u) + \Phi'(u) \bar{g}_{2\mu} \xi_e^\mu, u \right] + A_{2\lambda}^1 \left[k_1(u) - \mathbb{D}\Phi(u) \bar{g}_{1\mu} \xi_e^\mu, \xi^\lambda \right] = 0. \quad (6.67)$$

Note that $A_{2\lambda}^1 \bar{g}_{1\mu} = A_{2\lambda}^\nu \bar{g}_{\nu\mu} = \bar{g}_{2\kappa} W_\lambda^{\kappa\nu} \bar{g}_{\nu\mu} = \bar{g}_{2\kappa} A_{\lambda\mu}^\kappa$ is symmetric in λ and μ .

Hence,

$$\begin{aligned} A_{2\lambda}^1 \left[\mathbb{D}\Phi(u) \bar{g}_{1\mu} \xi_e^\mu, \xi_e^\lambda \right] &= \bar{g}_{2\kappa} A_{\lambda\mu}^\kappa \left[\mathbb{D}\Phi(u) \xi_e^\mu, \xi_e^\lambda \right] \\ &= \mathbb{D}\Phi(u) \bar{g}_{2\kappa} A_{\lambda\mu}^\kappa \left[\xi_e^\mu, \xi_e^\lambda \right] + \bar{g}_{2\kappa} A_{\lambda\mu}^\kappa \xi_e^\mu \left[\mathbb{D}\Phi(u), \xi_e^\lambda \right] \\ &= \frac{1}{2} \mathbb{D}\Phi'(u) \bar{g}_{2\kappa} A_{\lambda\mu}^\kappa \left(\xi_e^\mu \left[u, \xi_e^\lambda \right] + \xi_e^\lambda \left[u, \xi_e^\mu \right] \right) \\ &= \frac{1}{2} \mathbb{D}\Phi'(u) \bar{g}_{2\kappa} A_{\lambda\mu}^\kappa \left[u, \xi_e^\lambda \xi_e^\mu \right] \end{aligned}$$

We can now solve (6.67) for $H_{,2}$, resulting in

$$H_{,2} = k_2(u) - \mathbb{D}\Phi(u) \bar{g}_{2\mu} \xi_e^\mu + A_{2\lambda}^\kappa \mathbb{D}k_\kappa(u) \xi_e^\lambda - \frac{1}{2} \mathbb{D}^2\Phi(u) \bar{g}_{2\kappa} A_{\lambda\mu}^\kappa \xi_e^\lambda \xi_e^\mu,$$

where $k_2(u)$ is another arbitrary function. The procedure carries on in the same manner for $\tau > 2$, and in general we have

$$H_{,\tau} = k_\tau(u) + \sum_{m \geq 1} \frac{1}{m!} Q_{\tau\lambda_1 \dots \lambda_m}^{(m)}(u) \xi_e^{\lambda_1} \dots \xi_e^{\lambda_m}, \quad (6.68)$$

where

$$Q_{\tau\lambda}^{(1)}(u) := \mathbb{D} \left(A_{\tau\lambda}^\rho k_\rho(u) - \bar{g}_{\tau\lambda} \Phi(u) \right), \quad (6.69)$$

and

$$\begin{aligned} Q_{\tau\lambda_1 \dots \lambda_m}^{(m)}(u) &:= A_{\tau\lambda_1}^{\tau_1} A_{\tau_1\lambda_2}^{\tau_2} \dots A_{\tau_{m-3}\lambda_{m-2}}^{\tau_{m-2}} A_{\tau_{m-2}\lambda_{m-1}}^{\tau_{m-1}} \\ &\quad \times \mathbb{D}^m \left(A_{\tau_{m-1}\lambda_m}^\rho k_\rho(u) - \bar{g}_{\tau_{m-1}\lambda_m} \Phi(u) \right), \quad (6.70) \end{aligned}$$

for $m \geq 2$. If we define $k_0(u) := -\Phi(u)$, we can also write the $Q^{(m)}$ in terms of the D tensors, defined by (5.28), as

$$Q_{\lambda_1 \dots \lambda_m \lambda_{m+1}}^{(m)}(u) := \sum_{\rho=0}^{n-1} D_{\lambda_1 \dots \lambda_m \lambda_{m+1}}^{(m)\rho} \mathbb{D}^m k_\rho(u).$$

The sum in m in (6.68) terminates since the $A_{(\mu)}$ are nilpotent. The $Q^{(m)}(u)$ are symmetric in all their lower indices.

Note that (6.68) is not a closed-form solution for the equilibria: depending on the specific form of the Hamiltonian, the equation may be straightforward or difficult to solve, or possibly not have any solutions at all. The situation is the same as for Eqs. (6.6) and (6.7) (the energy-Casimir limit for CRMHD), which were solved for p_e and v_e in (6.9).

The fact that the coextension, which we used to find Casimir invariants in Chapter 5, appears in this calculation is not surprising, since the energy-Casimir method result is recovered by letting $\Psi(u) = u$, which simply says that \mathbb{D} is replaced by d/du .

We still have to satisfy (6.61) ($\dot{\varpi}_e = 0$) to get an equilibrium. Substituting in the results of (6.64) and (6.68), we get the condition

$$[\Psi'(u) H_{,n} + \Phi'(u) \varpi_e, u] + \left[k_\mu(u) + \sum_{m \geq 1} \frac{1}{m!} Q_{\mu\lambda_1 \dots \lambda_m}^{(m)}(u) \xi_e^{\lambda_1} \dots \xi_e^{\lambda_m}, \xi_e^\mu \right] = 0.$$

This can be solved, using the same techniques as for $H_{,0}, \dots, H_{,n-1}$ above, to give

$$H_{,n} = k_n - \mathbb{D}\Phi \varpi_e + \mathbb{D}k_\mu \xi_e^\mu + \sum_{m \geq 1} \mathbb{D}Q_{\lambda_1 \dots \lambda_m \lambda_{m+1}}^{(m)} \frac{\xi_e^{\lambda_1} \dots \xi_e^{\lambda_{m+1}}}{(m+1)!}. \quad (6.71)$$

We now have expressions for the equilibria of arbitrary nonsingular extensions, given by (6.64), (6.68), and (6.71). We can proceed to determine their stability.

Formal Stability

The dynamically accessible variations are obtained from (5.2), and are just equations (6.43) and (6.44) modified appropriately,

$$\delta\varpi_{\text{da}} = [\chi_0, \varpi]^\dagger + [\chi_\mu, \xi^\mu]^\dagger + [\chi_n, \psi]^\dagger, \quad (6.72)$$

$$\delta\xi_{\text{da}}^\mu = [\chi_0, \xi^\mu]^\dagger + \widetilde{W}_\lambda{}^{\mu\nu} [\chi_\nu, \xi^\lambda]^\dagger + g^{\mu\nu} [\chi_\nu, \psi]^\dagger, \quad (6.73)$$

$$\delta\psi_{\text{da}} = [\chi_0, \psi]^\dagger. \quad (6.74)$$

Notice that unlike the pure semidirect sum case given by (6.43) and (6.44), the dynamically accessible variations for ξ^1, \dots, ξ^n are now potentially *independent*.

We can use expression (6.45) for $\delta^2 H_{\text{da}}$ of the pure semidirect sum, modified to admit a cocycle,

$$\begin{aligned} \delta^2 H_{\text{da}} = & \frac{1}{2} \left\langle \delta\varpi_{\text{da}}, H_{,00} \delta\varpi_{\text{da}} + 2H_{,0\mu} \delta\xi_{\text{da}}^\mu + 2H_{,0n} \delta\psi_{\text{da}} + [\chi_0, H_{,0}] \right\rangle \\ & + \frac{1}{2} \left\langle \delta\xi_{\text{da}}^\mu, H_{, \mu\nu} \delta\xi_{\text{da}}^\nu + 2H_{, \mu n} \delta\psi_{\text{da}} + [\chi_0, H_{, \mu}] + [\chi_\mu, H_{,0}] + W_\mu{}^{\sigma\tau} [\chi_\sigma, H_{, \tau}] \right\rangle \\ & + \frac{1}{2} \left\langle \delta\psi_{\text{da}}, H_{, nn} \delta\psi_{\text{da}} + [\chi_0, H_{, n}] + [\chi_n, H_{,0}] + g^{\sigma\tau} [\chi_\sigma, H_{, \tau}] \right\rangle. \end{aligned} \quad (6.75)$$

As we did for the semidirect sum case, we want to express all the brackets in terms of dynamically accessible variations. We know we must be able to do this by the theorem proved at the end of Section 6.3.1.

The starting point is the $[\chi_n, H_{,0}]$ term, since it contains χ_n and thus can only be expressed in terms of $\delta\varpi_{\text{da}}$, given by Eq. (6.72). We do not present the calculation in detail here because it involves a great deal of algebra, none of which is very illuminating. We have to make liberal use of the identity

$$A_{\mu\tau}^\sigma \mathbb{D}Q_{\sigma\lambda_1 \dots \lambda_m}^{(m)} = Q_{\mu\tau\lambda_1 \dots \lambda_m}^{(m+1)}, \quad \text{for } m \geq 1,$$

easily verified from the definition of $Q^{(m)}$, Eq. (6.70).

The final form of the second variation of the Hamiltonian is

$$\begin{aligned}
\delta^2 H_{\text{da}} = & \frac{1}{2} \left\langle \delta \varpi_{\text{da}} H_{,00} \delta \varpi_{\text{da}} + 2\delta \varpi_{\text{da}} H_{,0\mu} \delta \xi_{\text{da}}^\mu + 2\delta \varpi_{\text{da}} \left(H_{,0n} + \mathbb{D}\Phi \right) \delta \psi_{\text{da}} \right\rangle \\
& + \frac{1}{2} \left\langle \delta \xi_{\text{da}}^\mu \left(H_{,\mu\nu} - Q_{\mu\nu}^{(1)} - \sum_{m \geq 1} \frac{1}{m!} Q_{\mu\nu\lambda_1 \dots \lambda_m}^{(m+1)} \xi_e^{\lambda_1} \dots \xi_e^{\lambda_m} \right) \delta \xi_{\text{da}}^\nu \right\rangle \\
& + \left\langle \delta \xi_{\text{da}}^\mu \left(H_{,\mu n} - \mathbb{D}k_\mu - \sum_{m \geq 1} \frac{1}{m!} \mathbb{D}Q_{\mu\lambda_1 \dots \lambda_m}^{(m)} \xi_e^{\lambda_1} \dots \xi_e^{\lambda_m} \right) \delta \psi_{\text{da}} \right\rangle \\
& + \frac{1}{2} \left\langle \delta \psi_{\text{da}} \left(H_{,nn} - \mathbb{D}k_n + \mathbb{D}^2\Phi \varpi_e - \mathbb{D}^2 k_\mu \xi_e^\mu - \sum_{m \geq 2} \frac{1}{m!} \mathbb{D}^2 Q_{\lambda_1 \dots \lambda_m}^{(m-1)} \xi_e^{\lambda_1} \dots \xi_e^{\lambda_m} \right) \delta \psi_{\text{da}} \right\rangle.
\end{aligned} \tag{6.76}$$

This very general expression allows us to see exactly where the cocycles modify the energy expression. Obtaining a useful result out of it is difficult, so we will do what we usually do: we simplify the problem! The case we will treat in more detail is the vanishing coextension case.

6.3.6 Vanishing Coextension

We consider the case where the coextension $A \equiv 0$ but g is nonsingular, as is the case for CRMHD (see Section 5.5.1). A schematic representation of this type of extension is shown in Figure 6.3. Then from (6.69) we have

$$Q_{\tau\lambda}^{(1)}(u) = -\mathbb{D}\Phi(u) \bar{g}_{\tau\lambda},$$

and from (6.70) we have $Q^{(m)}(u) \equiv 0$ for $m \geq 2$. We still have $\psi_e = \Psi(u)$, and the equilibrium relations (6.68) and (6.71) simplify to

$$H_{,\tau} = k_\tau - \mathbb{D}\Phi \bar{g}_{\tau\lambda} \xi_e^\lambda, \tag{6.77}$$

$$H_{,n} = k_n - \mathbb{D}\Phi \varpi_e + \mathbb{D}k_\mu \xi_e^\mu - \frac{1}{2} \mathbb{D}^2\Phi \bar{g}_{\mu\lambda} \xi_e^\mu \xi_e^\lambda, \tag{6.78}$$

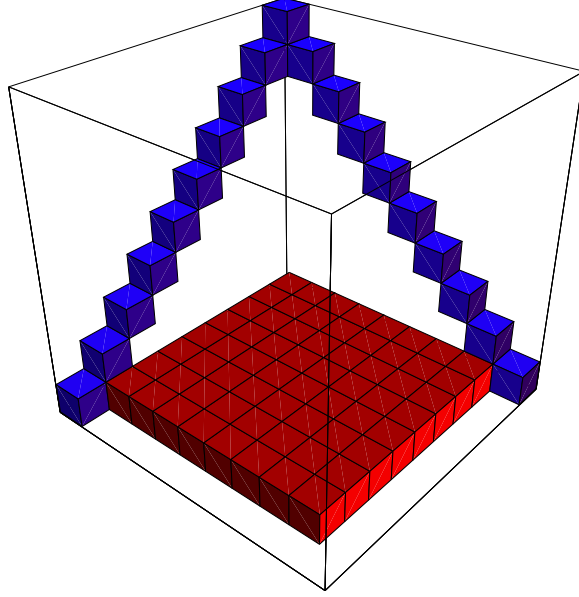


Figure 6.3: Schematic representation of the 3-tensor W for a semidirect extension with *vanishing coextension* ($A \equiv 0$). The axes are as in Figure 4.2. The red cubes represent the $n - 1 \times n - 1$ matrix $g^{\mu\nu}$, assumed here nonsingular. Note that compressible reduced MHD, in Figure 2.1, has this structure.

where as in Section 6.3.5 the greek indices run from 1 to $n - 1$. The second order variation of the Hamiltonian, Eq. (6.76), “reduces” to

$$\begin{aligned} \delta^2 H_{\text{da}} = & \frac{1}{2} \left\langle \delta \varpi_{\text{da}} H_{,00} \delta \varpi_{\text{da}} + 2 \delta \varpi_{\text{da}} H_{,0\mu} \delta \xi_{\text{da}}^\mu + 2 \delta \varpi_{\text{da}} (H_{,0n} + \mathbb{D}\Phi) \delta \psi_{\text{da}} \right\rangle \\ & + \frac{1}{2} \left\langle \delta \xi_{\text{da}}^\mu (H_{, \mu\nu} + \mathbb{D}\Phi \bar{g}_{\mu\nu}) \delta \xi_{\text{da}}^\nu \right\rangle + \left\langle \delta \xi_{\text{da}}^\mu (H_{, \mu n} - \mathbb{D}k_\mu + \mathbb{D}^2\Phi \bar{g}_{\mu\lambda} \xi_e^\lambda) \delta \psi_{\text{da}} \right\rangle \\ & + \frac{1}{2} \left\langle \delta \psi_{\text{da}} (H_{,nn} - \mathbb{D}k_n + \mathbb{D}^2\Phi \varpi_e - \mathbb{D}^2 k_\mu \xi_e^\mu + \frac{1}{2} \mathbb{D}^3\Phi \bar{g}_{\mu\nu} \xi_e^\mu \xi_e^\nu) \delta \psi_{\text{da}} \right\rangle. \quad (6.79) \end{aligned}$$

Again, to make progress we must further specialize the form of the Hamiltonian.

RMHD-like System

Let us take the RMHD-like Hamiltonian (6.57). We first need to find the equilibria, which we accomplish by substituting (6.57) into the equilibrium

conditions (6.64), (6.77) and (6.78),

$$-\phi_e + \mathcal{V}_{,0} = -\Phi(u), \quad (6.80)$$

$$\mathcal{V}_{,\tau} = k_\tau(u) - \mathbb{D}\Phi(u) \bar{g}_{\tau\lambda} \xi_e^\lambda, \quad (6.81)$$

$$-J_e + \mathcal{V}_{,n} = k_n(u) - \mathbb{D}\Phi(u) q_e + \mathbb{D}k_\mu(u) \xi_e^\mu - \frac{1}{2} \mathbb{D}^2\Phi(u) \bar{g}_{\mu\lambda} \xi_e^\mu \xi_e^\lambda, \quad (6.82)$$

Since we have not specified the exact dependence of \mathcal{V} on the ξ^μ , we cannot solve these for the ξ_e^μ . For the pure semidirect sum case, we had $\xi_e^\mu = \Xi(u)$, regardless of the form of the Hamiltonian. The presence of the nondegenerate cocycle leads to potentially much richer equilibria.

For the perturbation energy, we can use the result (6.60) in (6.79) to obtain

$$\begin{aligned} \delta^2 H_{\text{da}} = & \frac{1}{2} \left\langle |\nabla \delta\phi_{\text{da}} - \nabla(\mathcal{K} \delta\psi_{\text{da}})|^2 + (1 - \mathcal{K}^2) |\nabla \delta\psi_{\text{da}}|^2 + \mathcal{V}_{,00} |\delta q_{\text{da}}|^2 \right. \\ & + \mathcal{F} |\delta\phi_{\text{da}} - \mathcal{K} \delta\psi_{\text{da}}|^2 + 2\mathcal{V}_{,0\mu} \delta q_{\text{da}} \delta\xi_{\text{da}}^\mu \\ & + (\mathcal{V}_{,\mu\nu} + \mathbb{D}\Phi \bar{g}_{\mu\nu}) \delta\xi_{\text{da}}^\mu \delta\xi_{\text{da}}^\nu + 2(\mathcal{V}_{,\mu n} - \mathbb{D}k_\mu + \mathbb{D}^2\Phi \bar{g}_{\mu\lambda} \xi_e^\lambda) \delta\xi_{\text{da}}^\mu \delta\psi_{\text{da}} \\ & \left. + \left(\mathcal{V}_{,nn} - \mathbb{D}k_n + \mathbb{D}^2\Phi q_e - \mathbb{D}^2k_\mu \xi_e^\mu + \frac{1}{2} \mathbb{D}^3\Phi \bar{g}_{\mu\nu} \xi_e^\mu \xi_e^\nu + \mathcal{K} (\nabla^2\mathcal{K} - \mathcal{F}\mathcal{K}) \right) \right. \\ & \left. \times |\delta\psi_{\text{da}}|^2 \right\rangle, \quad (6.83) \end{aligned}$$

where

$$\mathcal{K}(u) := \mathcal{V}_{,0n} + \mathbb{D}\Phi(u).$$

Immediately we see that the stability conditions $|\mathcal{K}| \leq 1$, $\mathcal{F} \geq 0$, and $\mathcal{V}_{,00} \geq 0$ still hold. However, until we have a closed form for the equilibria we cannot make definite stability predictions. We now proceed to use a more restricted class of Hamiltonians for which the equilibria can be found explicitly.

Quadratic Potential

An important case we can do explicitly is when \mathcal{V} is quadratic,

$$\mathcal{V} = \frac{1}{2} \xi^\mu \mathfrak{V}_{\mu\nu}(\mathbf{x}) \xi^\nu + \mathbf{v}_\sigma(\mathbf{x}) \xi^\sigma,$$

where \mathfrak{V} is a symmetric matrix, in which case we have

$$\mathcal{V}_{,\tau} = \mathfrak{V}_{\tau\nu} \xi^\nu + \mathbf{v}_\tau,$$

and $\mathcal{V}_{,0} = \mathcal{V}_{,n} = 0$. Inserting this into (6.81), we obtain

$$(\mathfrak{V}_{\tau\lambda} + \mathbb{D}\Phi \bar{g}_{\tau\lambda}) \xi_e^\lambda = k_\tau - \mathbf{v}_\tau.$$

Assuming \mathfrak{V} is nondegenerate, the matrix

$$\mathcal{W}_{\tau\lambda} := \mathfrak{V}_{\tau\lambda} + \mathbb{D}\Phi \bar{g}_{\tau\lambda} \tag{6.84}$$

will be invertible except possibly at some points. We denote its inverse by $\mathcal{W}^{\tau\lambda}$, and (6.3.6) has solution

$$\xi_e^\lambda(\mathbf{x}) = \mathcal{W}^{\lambda\tau} (k_\tau(u) - \mathbf{v}_\tau(\mathbf{x})). \tag{6.85}$$

We emphasize how different this expression is to the pure semidirect sum result, $\xi_e^\lambda(\mathbf{x}) = \Xi(u)$. In (6.85) the equilibrium solution ξ^λ can explicitly depend on \mathbf{x} through the Hamiltonian. This can never occur for equilibria of the pure semidirect sum, regardless of the form of the Hamiltonian.

The most interesting feature of the new equilibria (6.85) is the fact that there are new *resonances* in the system—solutions for which $\mathcal{W}^{\lambda\tau}$ will blow up. This is what occurred for CRMHD in Section 6.2, where we had a singularity in the solution (6.9) of v_e and p_e , associated with the acoustic resonance. As

the equilibrium solution approaches this resonance, we can expect the system to become less stable.

We can use the solution (6.85) in (6.82) to obtain a closed-form result for J_e ,

$$J_e = -k_n + \mathbb{D}\Phi (\nabla^2\Phi - \mathcal{F}\Phi + f) - \mathbb{D}k_\mu \mathcal{W}^{\mu\tau} (k_\tau - \mathbf{v}_\tau) + \frac{1}{2} \mathbb{D}^2\Phi (k_\tau - \mathbf{v}_\tau) \mathcal{W}^{\tau\mu} \bar{g}_{\mu\lambda} \mathcal{W}^{\lambda\sigma} (k_\sigma - \mathbf{v}_\sigma), \quad (6.86)$$

where $J_e = \nabla^2\Psi(u)$. Using Eq. (6.34) for ω_e and the analogous relation for J_e , we have that (6.86) can be rewritten

$$\frac{((\Psi')^2 - (\Phi')^2)}{\Psi'} \nabla^2 u + \frac{(\Psi' \Psi'' - \Phi' \Phi'')}{\Psi'} |\nabla u|^2 = -k_n + \mathbb{D}\Phi (-\mathcal{F}\Phi + f) - \mathbb{D}k_\mu \mathcal{W}^{\mu\tau} (k_\tau - \mathbf{v}_\tau) + \frac{1}{2} \mathbb{D}^2\Phi (k_\tau - \mathbf{v}_\tau) \mathcal{W}^{\tau\mu} \bar{g}_{\mu\lambda} \mathcal{W}^{\lambda\sigma} (k_\sigma - \mathbf{v}_\sigma). \quad (6.87)$$

This is a nonlinear PDE to be solved for $u(\mathbf{x})$ with arbitrary functions $\Phi(u)$, $\Psi(u)$, and $k_\mu(u)$, and given functions $\mathcal{W}^{\tau\mu}(\mathbf{x})$, $\mathbf{v}_\sigma(\mathbf{x})$, $\mathcal{F}(\mathbf{x})$, and $f(\mathbf{x})$. Needless to say, solving (6.87) in general is extremely difficult. There are, however, classes of solution that can be obtained analytically. We now examine one of these special cases.

A particularly simple case are the aforementioned Alfvén solutions, for which

$$\Psi'(u) = c\Phi'(u), \quad (6.88)$$

where c is a constant. We also obtain

$$\mathbb{D}\Phi(u) = \frac{1}{\Psi'(u)} \frac{d\Phi}{du}(u) = \frac{1}{c\Phi'(u)} \Phi'(u) = c^{-1},$$

so that $\mathbb{D}^m \Phi = 0$ for $m > 1$. Thus, assuming that Ψ' and Φ' are proportional for the dynamical accessibility method is analogous to assuming that $\Phi(\psi_e)$ is linear for the energy-Casimir method. From (6.88), we might be tempted to simply write $\Phi = \Phi(\Psi)$, and indeed this is true. However, this is not useful because in general we still cannot rewrite u in terms of Ψ , since $\Psi = \Psi(u)$ may not be invertible. If Ψ is invertible, then we recover the energy-Casimir result completely.

Using (6.88) in the equilibrium condition (6.87) gives

$$(1 - c^{-2}) (\Psi' \nabla^2 u + \Psi'' |\nabla u|^2) = -k_n - c^{-2} (\mathcal{F}u - cf) - k'_\mu \mathcal{W}^{\mu\tau} (k_\tau - \mathbf{v}_\tau), \quad (6.89)$$

so that the quadratic term (proportional to $\mathbb{D}^2 \Phi$) disappears.

Several systems have $\mathcal{W}^{\mu\tau}$ independent of \mathbf{x} . It may then also happen that we can choose the $k_\mu(u)$ such that

$$k'_\mu \mathcal{W}^{\mu\tau} \mathbf{v}_\tau = c^{-2} (\mathcal{F}u - cf), \quad (6.90)$$

After this is effected, Eq. (6.89) no longer depends explicitly on \mathbf{x} , and has solutions such as the Kelvin–Stuart cat’s eye. This procedure can be carried out for CRMHD, for which $\mathcal{F} = f = 0$. In that case, (6.90) becomes (6.11).

Stability for Quadratic Potential

Assuming that we still have the quadratic potential of the previous section, we now show that the (acoustic) resonance which occurred for CRMHD is a generic feature of Lie–Poisson systems with cocycles.

We take the energy expression (6.83), use the fact the $\mathcal{V}_{,0} = \mathcal{V}_{,n} = 0$,

and obtain

$$\begin{aligned} \delta^2 H_{\text{da}} = & \frac{1}{2} \left\langle |\nabla \delta \phi_{\text{da}} - \nabla (\mathcal{K} \delta \psi_{\text{da}})|^2 + (1 - \mathcal{K}^2) |\nabla \delta \psi_{\text{da}}|^2 \right. \\ & + \mathcal{F} |\delta \phi_{\text{da}} - \mathcal{K} \delta \psi_{\text{da}}|^2 + \mathcal{W}_{\mu\nu} \delta \xi_{\text{da}}^\mu \delta \xi_{\text{da}}^\nu + 2(\mathbb{D}^2 \Phi \bar{g}_{\mu\lambda} \xi_e^\lambda - \mathbb{D} k_\mu) \delta \xi_{\text{da}}^\mu \delta \psi_{\text{da}} \\ & \left. + \left(\mathbb{D}^2 \Phi q_e - \mathbb{D} k_n - \mathbb{D}^2 k_\mu \xi_e^\mu + \frac{1}{2} \mathbb{D}^3 \Phi \bar{g}_{\mu\nu} \xi_e^\mu \xi_e^\nu + \mathcal{K} (\nabla^2 \mathcal{K} - \mathcal{F} \mathcal{K}) \right) |\delta \psi_{\text{da}}|^2 \right\rangle, \end{aligned}$$

where we used the definition of \mathcal{W} , Eq. (6.84), and we have not made any assumptions about the form of Φ and Ψ . The equilibrium solutions ξ_e^λ satisfy (6.85).

If we assume $\mathcal{K} \leq 1$ and $\mathcal{F} \geq 0$, then to obtain part of the sufficient conditions for stability we require that \mathcal{W} be positive-definite. But when \mathcal{W} becomes singular we cannot guarantee this. This was the case with CRMHD.

Note that $\det \mathcal{W} = 0$ does *not* imply that the system will be unstable beyond the resonance. It is, however, a strong indication that it might be.

Chapter 7

Conclusions

Using the tools of Lie algebra cohomology, we have classified low-order extensions. We found that there were only a few normal forms for the extensions, and that they involved no free parameters. This is not expected to carry over to higher orders ($n > 4$). The classification includes the Leibniz extension, which we have shown is the maximal extension. One of the normal forms is the bracket appropriate to compressible reduced MHD.

We then developed techniques for finding the Casimir invariants of Lie–Poisson brackets formed from Lie algebra extensions. We introduced the concept of coextension, which allowed us to explicitly write down the solution of the Casimirs. The coextension for the Leibniz extension can be found for arbitrary order, so we were able obtain the corresponding Casimirs in general.

By using the method of dynamical accessibility, we derived general conditions for the formal stability of Lie–Poisson systems. In particular, for compressible reduced MHD, we found the presence of a cocycle could only make a certain class of solutions more unstable. In general, cocycles were shown to lead to resonances, such as the acoustic resonance for CRMHD.

The dynamical accessibility approach also allowed us to get a clearer picture of the role of cocycles: in a pure semidirect extension, the absence of a

cocycle means that the system necessarily describes an advective system, and the dynamically accessible variations are not independent. In contrast, for the nonsingular cocycle case all of the perturbations are independent. The form of the stability condition is thus much more complex.

It would be interesting to generalize the classification scheme presented here to a completely general form of extension bracket [72, 77]. Certainly the type of coordinate transformations allowed would be more limited, and perhaps one cannot go any further than cohomology theory allows.

Though we have gone a long way in this respect, the interpretation of the Casimir invariants has yet to be fully explored, both in a mathematical and a physical sense. Mathematically, we could give a precise geometrical relation between cocycles and the form of the Casimirs. The cocycle and Casimirs should yield information about the holonomy of the system. For this one must study extensions in the framework of their principal bundle description [21]. Physically, we would like to attach a more precise physical meaning to these conserved quantities. The invariants associated with simultaneous eigenvectors can be regarded as constraining the associated field variable to move with the fluid elements [68]. The field variable can also be interpreted as partially labeling a fluid element. Some attempt has been made in formulating the Casimir invariants of brackets in such a manner [52, 90], but for the more complicated invariants a general treatment is still not yet available.

Appendices

Appendix A

Proof of the Jabobi Identity

We want to show that the Lie–Poisson bracket

$$\{F, G\}_{\pm}(\xi) = \pm \left\langle \xi, \left[\frac{\delta F}{\delta \xi}, \frac{\delta G}{\delta \xi} \right] \right\rangle, \quad (2.1)$$

where $\xi \in \mathfrak{g}^*$, and $F : \mathfrak{g}^* \rightarrow \mathbb{R}$ and $G : \mathfrak{g}^* \rightarrow \mathbb{R}$ are functionals, satisfies the Jacobi identity

$$\{\{F, G\}_{\pm}, H\}_{\pm} + \{\{G, H\}_{\pm}, F\}_{\pm} + \{\{H, F\}_{\pm}, G\}_{\pm} = 0.$$

The inner bracket $[\cdot, \cdot]$ is the bracket of the Lie algebra \mathfrak{g} , so it satisfies the Jacobi identity. The overall sign of the bracket is inconsequential, so we choose the $+$ bracket. We first compute the variation of $\{F, G\}$,

$$\begin{aligned} \delta\{F, G\} &= \left\langle \delta\xi, \left[\frac{\delta F}{\delta \xi}, \frac{\delta G}{\delta \xi} \right] \right\rangle + \left\langle \xi, \left[\frac{\delta^2 F}{\delta \xi \delta \xi} \delta\xi, \frac{\delta G}{\delta \xi} \right] \right\rangle + \left\langle \xi, \left[\frac{\delta F}{\delta \xi}, \frac{\delta^2 G}{\delta \xi \delta \xi} \delta\xi \right] \right\rangle \\ &= \left\langle \delta\xi, \left[\frac{\delta F}{\delta \xi}, \frac{\delta G}{\delta \xi} \right] \right\rangle - \left\langle \left[\frac{\delta G}{\delta \xi}, \xi \right]^{\dagger}, \frac{\delta^2 F}{\delta \xi \delta \xi} \delta\xi \right\rangle + \left\langle \left[\frac{\delta F}{\delta \xi}, \xi \right]^{\dagger}, \frac{\delta^2 G}{\delta \xi \delta \xi} \delta\xi \right\rangle \\ &= \left\langle \delta\xi, \left[\frac{\delta F}{\delta \xi}, \frac{\delta G}{\delta \xi} \right] - \frac{\delta^2 F}{\delta \xi \delta \xi} \left[\frac{\delta G}{\delta \xi}, \xi \right]^{\dagger} + \frac{\delta^2 G}{\delta \xi \delta \xi} \left[\frac{\delta F}{\delta \xi}, \xi \right]^{\dagger} \right\rangle, \end{aligned}$$

where we have used the definition of the coadjoint bracket (2.3) and the self-adjoint property of the second derivative operator. Thus, we have

$$\frac{\delta\{F, G\}}{\delta \xi} = \left[\frac{\delta F}{\delta \xi}, \frac{\delta G}{\delta \xi} \right] - \frac{\delta^2 F}{\delta \xi \delta \xi} \left[\frac{\delta G}{\delta \xi}, \xi \right]^{\dagger} + \frac{\delta^2 G}{\delta \xi \delta \xi} \left[\frac{\delta F}{\delta \xi}, \xi \right]^{\dagger}.$$

We can now evaluate the first term of the Jacobi identity,

$$\begin{aligned}
\{\{F, G\}, H\} &= \left\langle \xi, \left[\frac{\delta\{F, G\}}{\delta\xi}, \frac{\delta H}{\delta\xi} \right] \right\rangle \\
&= \left\langle \xi, \left[\left[\frac{\delta F}{\delta\xi}, \frac{\delta G}{\delta\xi} \right] - \frac{\delta^2 F}{\delta\xi\delta\xi} \left[\frac{\delta G}{\delta\xi}, \xi \right]^\dagger + \frac{\delta^2 G}{\delta\xi\delta\xi} \left[\frac{\delta F}{\delta\xi}, \xi \right]^\dagger, \frac{\delta H}{\delta\xi} \right] \right\rangle \\
&= \left\langle \xi, \left[\left[\frac{\delta F}{\delta\xi}, \frac{\delta G}{\delta\xi} \right], \frac{\delta H}{\delta\xi} \right] \right\rangle + \left\langle \left[\frac{\delta H}{\delta\xi}, \xi \right]^\dagger, \frac{\delta^2 F}{\delta\xi\delta\xi} \left[\frac{\delta G}{\delta\xi}, \xi \right]^\dagger \right\rangle \\
&\quad - \left\langle \left[\frac{\delta H}{\delta\xi}, \xi \right]^\dagger, \frac{\delta^2 G}{\delta\xi\delta\xi} \left[\frac{\delta F}{\delta\xi}, \xi \right]^\dagger \right\rangle.
\end{aligned}$$

Upon adding permutations of F , G , and H , the second-derivative terms cancel and we are left with

$$\left\langle \xi, \left[\left[\frac{\delta F}{\delta\xi}, \frac{\delta G}{\delta\xi} \right], \frac{\delta H}{\delta\xi} \right] + \left[\left[\frac{\delta G}{\delta\xi}, \frac{\delta H}{\delta\xi} \right], \frac{\delta F}{\delta\xi} \right] + \left[\left[\frac{\delta H}{\delta\xi}, \frac{\delta F}{\delta\xi} \right], \frac{\delta G}{\delta\xi} \right] \right\rangle,$$

which vanishes by the Jacobi identity in \mathfrak{g} .

Appendix B

Proof of $W^{(1)} = I$

Our goal is to demonstrate that through a series of lower-triangular coordinate transformations we can make $W^{(1)}$ equal to the identity matrix, while preserving the lower-triangular nilpotent form of $W^{(2)}, \dots, W^{(n)}$.

We first show that we can always make a series of coordinate transformations to make $W_\lambda^{11} = \delta_\lambda^1$. First note that if the coordinate transformation M is of the form $M = I + L$, where I is the identity and L is lower-triangular nilpotent, then $\widetilde{W}^{(1)} = M^{-1} W^{(1)} M$ still has eigenvalue 1, and the matrices

$$\widetilde{W}^{(\mu)} = M^{-1} W^{(\mu)} M, \quad \mu > 1$$

are still nilpotent.

For $\lambda > 1$ we have

$$\overline{W}_\lambda^{11} = \widetilde{W}_\lambda^{11} + \widetilde{W}_\lambda^{1\nu} L_\nu^1 = \widetilde{W}_\lambda^{11} + \sum_{\nu=2}^{\lambda-1} \widetilde{W}_\lambda^{1\nu} L_\nu^1 + L_\lambda^1, \quad (\text{B.1})$$

where we used $\widetilde{W}_\lambda^{1\lambda} = 1$. Owing to the triangular structure of the set of equations (B.1) we can always solve for the L_λ^1 to make $\overline{W}_\lambda^{11}$ vanish. This proves the first part.

We now show by induction that if $W_\lambda^{11} = \delta_\lambda^1$, as proved above, then the matrix $W^{(1)}$ is the identity. For $\lambda = 1$ the result is trivial. Assume

that $W_\mu^{1\nu} = \delta_\mu^\nu$, for $\mu < \lambda$. Setting two of the free indices to one, Eq. (2.21) can be written

$$\begin{aligned} W_\lambda^{\mu 1} W_\mu^{1\sigma} &= W_\lambda^{\mu\sigma} W_\mu^{11} \\ &= W_\lambda^{\mu\sigma} \delta_\mu^1 = W_\lambda^{1\sigma}. \end{aligned}$$

Since $W^{(1)}$ is lower-triangular the index μ runs from 2 to λ (since we are assuming $\lambda > 1$):

$$\sum_{\mu=2}^{\lambda} W_\lambda^{\mu 1} W_\mu^{1\sigma} = W_\lambda^{1\sigma},$$

and this can be rewritten, for $\sigma < \lambda$,

$$\sum_{\mu=2}^{\lambda-1} W_\lambda^{\mu 1} W_\mu^{1\sigma} = 0.$$

Finally, we use the inductive hypothesis

$$\sum_{\mu=2}^{\lambda-1} W_\lambda^{\mu 1} \delta_\mu^\sigma = W_\lambda^{\sigma 1} = 0,$$

which is valid for $\sigma < \lambda$. Hence, $W_\lambda^{\sigma 1} = \delta_\lambda^\sigma$ and we have proved the result. ($W_\lambda^{\lambda 1}$ must be equal to one since it lies on the diagonal and we have already assumed degeneracy of eigenvalues.)

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Vita

Jean-Luc Thiffeault was born in Montréal, Québec, Canada on February 24, 1971, the son of Irène Thiffeault and Claude Thiffeault. After completing his work at Collège Bois-de-Boulogne, Montréal, Québec, in 1990, he entered McGill University in Montréal. He received a degree of Honors Bachelor of Science from that school in May, 1993. In August 1993, he entered The Graduate School at the University of Texas. He received his Master of Arts in August of 1995 under the supervision of Wendell Horton.

Permanent address: 15, 2^e rue
Batiscan, Québec
Canada
G0X 1A0

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