DOE/ET/53088 IFSR #83

NON-INTRINSIC AMBIPOLAR DIFFUSION IN TURBULENCE THEORY

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March 1983

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Abstract

Ambipolar flow in a turbulent plasma is investigated by combining a WKB treatment of the waves with a turbulent collision operator resulting from either quasi-linear theory or certain renormalized turbulence theories. If the wave momentum has a flow from outgoing waves, then particle diffusion is not intrinsically ambipolar, and the time variation of the electric potential profile is determined by the turbulent spectrum. However, in most cases of practical interest, as in the drift wave problem, this effect is small, and in steady state equal rates of stochastic diffusion is predicted for electrons and ions.

It is well known that collisional transport theory predicts intrinsic ambipolar diffusion to lowest order in Larmor radius in systems where the equilibrium has at least one degree of symmetry. This property has also been attributed to systems where the transport is due to short wavelength turbulence.[1,2] However, we show here that a self-consistent development of the turbulent transport equations can lead to a non-ambipolar flow. As the global system must maintain quasi-neutrality, a time varying potential is induced to guarantee such a state. Previous results are recovered in most cases of practical interest as the radial variation of parameters over the extent of the eigenfunctions is small. By neglecting this variation, one then has intrinsically ambipolar diffusion as observed by previous authors.[1,2] These results can be demonstrated both for quasi-linear theory and for a specific renormalized turbulence theory[3], that treats stochastic electron orbits. Other renormalized theories[4] are not sufficiently complete to allow an explicit demonstration here, but we expect that if they are developed properly, similar results will arise.

We first consider the combination of WKB theory, where wavenumbers along the inhomogeneous direction are taken into account, with quasi-linear theory. It has been shown[5] that this leads to a quasi-linear equation for the distribution function f in the following model. A slab plasma is considered which is homogeneous in the y,z directions and inhomogeneous in the x-direction, while the magnetic field is sheared in the y,z directions with mod-B constant (the inclusion of magnetic shear is a straightforward extension of Ref. 5) and the perturbed fields are electrostatic with potentials of the form,

$$\hat{\psi}(x,t)\ \exp\left(i \int^{x} k_x dx + ik_y y + ik_z z - i \int^{t} \omega dt'\right) \qquad (1)$$

The addition of electromagnetic waves would not alter the development below, but is omitted for simplicity.

The quasi-linear equation is

$$\frac{\partial F_s}{\partial t} = C_{QLT}(f_s) \equiv \pi \frac{q_s^2}{m_s^2} \sum_{n,k} \left[2n\,\omega_{cs} \frac{\partial}{\partial v_\perp^2} + 2\,k_\parallel(x) v_\parallel \frac{\partial}{\partial v_\parallel^2} + \frac{k_y}{\omega_{cs}} \frac{\partial}{\partial x} \right]$$

$$J_n^2\left(\frac{k_\perp(x)v_\perp}{\omega_{cs}}\right) |\hat{\psi}_k(x,t)|^2\, \delta\left(\omega - n\omega_{cs} - k_\parallel v_\parallel\right) \left[2n\,\omega_{cs} \frac{\partial}{\partial v_\perp^2} + 2k_\parallel(x) v_\parallel \frac{\partial}{\partial v_\parallel^2} + \frac{k_y}{\omega_{cs}} \frac{\partial}{\partial x} \right] f_s \tag{2}$$

where the summation on k is the symbolic sum over all modes that exist at position x, $k_\parallel(x) = \left[k_y B_y(x) + k_z B_z(x)\right]/B$ and F_s, is given in terms of the particle distribution, f_s by

$$F_s = f_s + \frac{1}{2} \frac{q_s^2}{m_s^2} \sum_{k,n} \left(2n\,\omega_{cs} \frac{\partial}{\partial v_\perp^2} + 2k_\parallel v_\parallel \frac{\partial}{\partial v_\parallel^2} + \frac{k_y}{\omega_{cs}} \frac{\partial}{\partial x} \right)$$

$$|\hat{\psi}_k(x,t)|^2\, J_n^2 \frac{\partial}{\partial \omega} \frac{P}{(\omega - n\omega_{cs} - k_\parallel v_\parallel)} \left(2n\,\omega_{cs} \frac{\partial}{\partial v_\perp^2} + 2\,k_\parallel v_\parallel \frac{\partial}{\partial v_\parallel^2} + \frac{k_y}{\omega_{cs}} \frac{\partial}{\partial x} \right) f_s \,. \tag{3}$$

We note that F_s, a modified distribution function that subtracts the non-resonant or "sloshing" term, is constant in time in the absence of dissipation.

The frequency is determined by the condition,

$$\mathrm{Re}\, D[\omega, k(x)] \equiv D_R[\omega, k(x)] = 0 \tag{4}$$

and $|\hat{\psi}(x,t)|^2$ satisfies the relation

$$\frac{\partial N_k}{\partial t} + \frac{d}{dx}\left(v_{gk}N_k\right) = 2k_I v_g N_k \tag{5}$$

with

$$N_k = \frac{|\hat{\psi}_k|^2}{8\pi}\frac{\partial D_R}{\partial \omega} \equiv \text{adiabatic density,} \qquad v_{gk} = \frac{-\partial D_R/\partial k_x}{\partial D_R/\partial \omega} \equiv \text{group velocity ,}$$

$$k_I = \frac{-D_I}{D_R/\partial k_x} \equiv \text{spatial amplification factor,} \qquad D_I = \text{ImD} \;; \tag{6}$$

$$D(\omega,k,x) = k^2 + \sum_s \frac{4\pi q^2}{m} \sum_n \int d^3v \frac{J_n^2}{(\omega - n\omega_{cs} - k_\parallel v_\parallel)}$$
$$\times \left(2n\omega_{cs}\frac{\partial}{\partial v_\perp^2} + 2k_\parallel v_\parallel \frac{\partial}{\partial v_\parallel^2} + \frac{k_y}{\omega_{cs}}\frac{\partial}{\partial x}\right) f_s(v_\perp^2, v_\parallel^2, x) \; . \tag{7}$$

We now construct the time rate of change of density and find $\partial n_s/\partial t = \int d^3v \, \partial f_s/\partial t = -\partial \Gamma_s/\partial x$ where the particle flux Γ_s is

$$\Gamma_s = \frac{-c}{B} \sum_k \frac{k_y}{q_s} \left(\frac{\partial N_k^s}{\partial t} + 2D_I^s \frac{|\psi_k|^2}{8\pi}\right) \tag{8}$$

where the superscript "s" refers to the contribution of species s to N_k and D_I. The term in Eq. (8) proportional to $\partial/\partial t$ is the reversible component of diffusion due to the growth or damping of a wave, while the term proportional to D_I is irreversible diffusion due to particle-wave resonance.

The time rate of change of charge density, $\partial\sigma/\partial t \equiv \partial/\partial t \sum_s n_s q_s$ is found from Eqs. (5) and (8) to be

$$\frac{\partial\sigma}{\partial t} = -\sum_{k} \frac{ck_y}{B} \frac{\partial^2}{\partial x^2}\left(N_k v_{gk}\right) \quad . \tag{9}$$

Hence, a charge imbalance arises if $\partial\left(\sum N_k v_{gk}\right)/\partial x$ varies spatially. For standing waves, with evanescent decay, waves have reflection points so that $N_k(v_g) = N_k(-v_g)$. Hence, the right-hand side of Eq. (8) is zero. However, for convective waves or modes with outgoing wave boundary conditions, Eq. (8) is not intrinsically zero and the quasi-linear theory then predicts non-ambipolar flow, unless a potential can develop to restore local charge balance. A time-varying field causes only ion motion to balance the non-ambipolar flow, and thus we have

$$\frac{\partial\sigma}{\partial t} = -\frac{n_i q_i c}{\omega_{ci} B} \frac{\partial}{\partial t} \frac{\partial^2\Phi}{\partial x^2} - \sum_{\underset{\sim}{k}} \frac{ck_y}{B} \frac{\partial^2}{\partial x^2}\left(N_k v_{gk}\right) \quad . \tag{10}$$

Demanding $\partial\sigma/\partial t = 0$, constrains $\Phi(x)$ to satisfy,

$$\frac{\partial}{\partial t} \frac{\partial}{\partial x} \phi = -\sum \frac{k_y B}{m_i n_i c} \frac{\partial}{\partial x}\left(N_k v_{gk}\right), \tag{11}$$

assuming $\partial \ln n(x)/\partial x \ll \partial \ln\left(\sum_k N_k v_{gk}\right)/\partial x$.

The non-ambipolar flow we have calculated can be interpreted as a flow arising from a viscous stress. In equilibrium, pressure balance gives $\underset{\sim}{j} \times \underset{\sim}{B}/c = \underset{\sim}{\nabla} \cdot \underset{\approx}{P}$ where $\underset{\approx}{P}$ is the particle pressure tensor (or momentum flux). We see that the waves $\underset{\sim}{\hat{y}}$ flux of $\underset{\sim}{\hat{x}}$ directed momentum is $\underset{\approx}{T} \equiv \sum_k \underset{\sim}{\hat{x}}\, \underset{\sim}{\hat{y}}\, N_k k_y\, v_{gx}$. However, $\underset{\approx}{T} = -\underset{\approx}{P}$, if overall momentum is to be conserved. Hence, using the force balance equation, we find $j_x = c/B\ \partial\left(\sum_k N_k k_y\, v_{gx}\right)/\partial x$ which agrees, after using the continuity equation, with Eq. (9).

We now consider the specific problem of low frequency drift waves in a sheared magnetic field.[6,7] In a tokamak, the waves are paramterized by poloidial (m) and toroidal (n) mode numbers. Spectral properties and wave dynamics depend on poloidal number, m, but the toroidal mode number enters only in determining the rational surface. Thus, in the slab model, where $1/q = (1+x/L_s)q_0$, the rational surface x_{mn}, such that $q(x_{mn}) = m/n$, is given by $x_{mn} = L_s \frac{n}{m} q_0 - L_s$. Now, the sum over modes, $\underset{\sim}{k}$, in Eq. (2) will effectively sum over rational surfaces, and this can be written as an integral $\sum_{\underset{\sim}{k}} \rightarrow \int J\, dk_y\, k_y\, dx_j$, where J is a constant Jacobian $\equiv R_p^2/L_s q_0$ in the tokamak, while in our slab model we take J=1.

Drift waves are driven unstable by negative dissipation from the electrons near the rational surface. Wave energy (and momentum) is convected away from the rational surface as outgoing waves and eventually absorbed by the ion wave particle resonance in the outer regions of each eigenfunction. This picture is expressed by Eq. (5), which can be rewritten

$$\frac{\partial N_k}{\partial t}(x-x_j, x_j) + \frac{d}{dx}(v_{gk}N_k) = -2D_I(x-x_j, x_j)\frac{|\psi_k(x-x_j, x_j)|^2}{8\pi}, \quad (12)$$

where x_j denotes the rational surface. It is typical for drift waves in a sheared field, that the principal functional dependence of Eq. (12) varies primarily with $x-x_j$, with only a slow dependence on the explicit x_j variable. Hence the eigenfunctions centered at different x_j's , but within an eigenfunction scale length of each other, will have close to the same functional dependence on x_j. We shall show that this property has the following consequences: (1) the non-ambipolar diffusion component is small compared with the particle diffusion of each species, (2) for stationary turbulence the intrinsic diffusion of electrons due to diffusion near $x = x_j$ is nearly equal to the ion diffusion due to the resonant interaction of ions in the outer regions of each eigenfunction.

To demonstrate statement (1) we note that since $|\psi|^2$ vanishes sufficiently far from the mode rational surface, Eq. (12) implies on integrating over x , that

$$\sum_s \frac{c}{B} \int dx\, k_y \left[\frac{\partial N^s}{\partial t} (x-x_j, x_j) + 2\, D_I^s(x-x_j, x_j)\, |\psi(x-x_j, x_j)|^2 \right] = 0 \quad . (13)$$

On the other hand, from Eq. (8) we have,

$$\sum_s q_s \Gamma_s = \frac{-c}{B} \sum_{k_y} k_y\, dx_j \left[\frac{\partial N^s}{\partial t} (x-x_j, x_j) + 2D_I^s(x-x_j, x_j)\, |\psi(x-x_j, x_j)|^2 \right] \tag{14}$$

Hence to the extent the dependence on the second x_j is small over the range of the eigenfunctions, $\sum_s q_s \Gamma_s \approx \lambda/R_p\, q_s \Gamma_s$, where λ is the scale of a typical eigenfunction and R_p the macroscopic radial scale length.

The second assertion readily follows from Eqs. (13) and (14) as in steady state the $\partial/\partial t$ terms vanish, and Eqs. (12) and (13) show that the resonant electron flux coming primarily from the center of the eigenfunction, nearly equals the resonant ion flux coming from the "wings" of the eigenfunction. This demonstrates that electrons and ions are both participating in an irreversible diffusion process and that ions can then have a significant heat conductivity. It should be noted that recently there has been experimental evidence that the ion thermal conduction channel may be a significant heat loss in a tokamak discharge.[8]

One now infers on physical grounds that these results are of a general nature that surpass quasi-linear theory. The underlying reason for intrinsic ambipolarity here, as in the collisional problem and the local theory of drift waves, is momentum conservation. Here, however, the momentum exchange between species is non-local and the conservation law is global as indicated in Eq. (13). Because of this ambipolarity can break down locally as indicated in Eq. (9). Although each eigenfunction generates equal ion and electron diffusion, it does so at different spatial points. Diffusion of the two species at the same spatial point is due to different eigenfunctions. When the eigenfunctions have unequal amplitudes due to slow radial variations of the spectrum small non-ambipolar flows will occur. We therefore expect that any nonlinear (renormalized) theory that conserves momentum and energy will have the same conservation properties as demonstrated above in quasi-linear theory.

In particular for the collision operator of Ref. (3) derived for intrinsically stochastic electron orbits using the Normal Stochastic Approximation (NSA)[9], the demonstration parallels the above almost exactly. The energy conservation properties[10] of this operator follow from a non-local WKB analysis similar to that employed here. The operator, for the electrons, which replaces the right side of Eq. (2), is given by

$$C_{NSA}(f_e) = \frac{q_e^2}{m_e^2} \sum_{\underset{\sim}{k},\omega} \left[2k_\parallel(x)v_\parallel \frac{\partial}{\partial v_\parallel^2} + \frac{k_y}{\omega_{ce}} \frac{\partial}{\partial x} \right]$$

$$\times \, \mathrm{Re}\, h_{\underset{\sim}{k}\omega}(x, v_\parallel)\, S_{\underset{\sim}{k}\omega}(x) \left[2k_\parallel(x)v_\parallel \frac{\partial}{\partial v_\parallel^2} + \frac{k_y}{\omega_{ce}} \frac{\partial}{\partial x} \right] f_e \quad , \tag{15}$$

where $S_{\underset{\sim}{k}\omega}(x)$, the spectrum of potential fluctuations, replaces $|\psi_{\underset{\sim}{k}}|^2$ above, and $h_{\underset{\sim}{k}\omega}$ is the resonance function, $h_{\underset{\sim}{k}\omega}(x,v_\parallel) = \int_0^\infty d\tau \exp[i(\omega - k_\parallel(x)v_\parallel)\tau - \frac{1}{3}(k'_\parallel v_\parallel)^2 D\tau^3]$. where spatial diffusion coefficient, D, in $h_{\underset{\sim}{k}_i\omega}$ is determined self-consistently[3] from the spectral sum in Eq. (15).

The spectrum, $S_{\underset{\sim}{k}\omega}$, satisfies the transport equation [Eq. (5)][10], with the electron contribution to the dielectric, Eq. (7), given by

$$D^e(\omega,k,x) = \frac{i4\pi q_e^2}{m_e} \int d^3\underset{\sim}{v}\; h_{\underset{\sim}{k}\omega} \left[2k_\parallel(x)\frac{\partial}{\partial v_\parallel^2} + \frac{k_y}{\omega_{ce}}\frac{\partial}{\partial x}\right] f_e \quad . \tag{16}$$

Finally, the sloshing term in Eq. (3) is replaced by

$$\frac{1}{2}\frac{q_e^2}{m_e^2}\sum_{\underset{\sim}{k},\omega}\left(2k_\parallel v_\parallel \frac{\partial}{\partial v_\parallel^2} + \frac{k_y}{\omega_{ce}}\frac{\partial}{\partial x}\right) S_{\underset{\sim}{k}\omega}(x)\frac{\partial}{\partial\omega}\,\mathrm{Im}\, h_{\underset{\sim}{k}\omega}(x,v_\parallel)\left(2k_\parallel v_\parallel\frac{\partial}{\partial v_\parallel^2} + \frac{k_y}{\omega_{ce}}\frac{\partial}{\partial x}\right) f_e \quad .$$

The formulas here for the turbulent electron response have been written in a way that directly parallels the quasi-linear equations and do not exactly coincide with the formulas of Ref. 10. The difference between these two forms is of order $k_x x_c \ll 1$, where $x_c \equiv (D/k_\parallel v_e)^{1/3}$ is the correlation distance and is negligible within the WKB approximation.

One now easily sees that all the above formulas and results are duplicated by identifying $|\psi_{\underset{\sim}{k}}|^2$ with the spectrum, $S_{\underset{\sim}{k}\omega}$, including a frequency sum, and modifying the electron dielectric. In particular, the structure wherein the particle flux, Eq. (8), is determined by the dissipative part of the dielectric is retained. This structure assures global energy and momentum conservation and, thereby, all the results of this paper.

To estimate the magnitude of the time varying potential we assume the following scaling characteristic of drift waves.

$$\frac{\partial D}{\partial k_x} \simeq k_x \rho_i^2 \frac{\omega_{pe}^2}{v_e^2} \sim \frac{k_x \rho_i}{k_y} \frac{\omega_{pe}^2}{v_e^2}, \qquad \frac{\partial}{\partial x} \sum_k N_k v_{gk} \simeq \sum_k \frac{N_k v_{gk}}{R_p}$$

$$\frac{\partial \Phi}{\partial x} \sim \Phi / R_p, \qquad R_p \equiv \text{radial scale length}, \qquad k_x \sim \frac{R_p}{L_s \rho_i}$$

We then find, $\partial(q\Phi/T_e)/\partial t \approx \omega_{ci} \sum_k |q\psi_k/T_e|^2 \; L_n/L_s$. This scaling appears to induce appreciable variations of the ambipolar potential , especially near the edge of the plasma, where the fluctuation level has been observed to be particulary high.[11]

In summary we have shown that with net wave momentum flow, the turbulent transport equations induce a net ambipolar flow that will give rise to time varying potential. However, if a steady state in drift wave turbulence is to exist, the transport is required to be ambipolar with both electrons and ions diffusing stochastically. This can be accomplished by having the ions absorb the outgoing wave momentum generated by electrons at the rational surfaces, or by perhaps more complicated nonlinear processes.

Acknowledgments

This work was supported by United States Department of Energy contracts DE-FG05-80ET-53088 (IFS) and DE-AC02-78ET-51013 (MIT).

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