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# SOLITONS IN TURBULENT FLOW*

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## COHERENT STRUCTURES IN TURBULENCE

In this talk I will discuss the connection between two seemingly dissimilar concepts: turbulence and coherence. Historically, turbulent flow has been characterized by extreme incoherence or randomness, the most successful theoretical treatments assuming quasi-Gaussianity. At the opposite end of the spectrum of fluid motion are laminar flows such as isolated vortices, which we refer to as coherent structures. Perhaps the most coherent structure is the soliton which maintains its integrity as a manifestation of the integrability of the field theory that describes it.

Over the last ten years, there have been a large number of observations of organized flow in fluids commonly termed turbulent. An example is the experiments of Brown and Roshko (1974) (and many others) on mixing layers (the interface between two fluids moving at different velocities). Visualizing the fluid flow with injected dyes or reflecting particles shows that even for very large Reynolds' numbers, the turbulent shear layer is composed of vortex-like structures which roll up the interface between two fluids. The vorticies propagate downstream, interacting and combining. A similar case is the formation of "bursts" and turbulent spots in boundary layers (Cantwell _et al._, 1978; Cantwell, 1981). These local patches begin to appear intermittently near the critical Reynolds' number for transition to turbulence. There is some evidence that they persist even in fully-turbulent situations.

Finally, we mention the elegant experiments of Swinney, as reported at this Workshop, on Taylor-Couette flow. Here, remnants of Taylor vorticities are observed even at extremely high Reynolds' numbers.

Several examples of intermittent behavior in plasma turbulence may also be found. Satellite measurements of the electric fields in the auroral zone show localized pulses propagating parallel to the magnetic field which may be interpreted as double layers and ion-acoustic solitary waves (Temerin _et al_., 1982). The most striking observation is the high degree of intermittency in the electric field. If we define the intermittency coefficient as

$$I = \frac{\text{Number of Pulses} \times \text{Pulse Width}}{\text{Time}} \tag{1}$$

the measurements give $I \simeq 0.1 - 0.2$, as noted by Lotko (1982). Measurements made closer to home of turbulence in tokamaks have also begun to reveal a coherent component. Photographs of the edge region show localized filaments extended in the toroidal direction which are presumably regions of higher density (Zweben _et al_., 1982). Laser interferometry detects these density pulses and shows that they propagate poloidally and remain coherent for some distance, as reported by Surko and Slusher (1982). We have speculated that such coherent stuctures could be described as solitary drift waves (Meiss and Horton, 1983).

A major problem in determining the extent to which turbulent flow consists of coherent features is quantifying this idea. While most observations of coherence use flow visualization in distinguishing incoherent from coherent, it is not clear how mathematically to separate the two. One important, but vague, idea is the intermittency of the flow, as in Eq. 1. A better technique would be to construct the probability distribution of a field variable, say $P(\phi)$. A conventional turbulence theory would assume quasi-Gaussianity, while the coherent structure probability distribution would have non-Gaussian features. As an example, solitary waves tend to form with a distinct sign, and thus would lead to a large third moment. Berman <u>et al</u>. (1982) have used this technique as a diagnostic for the existence of electron-holes in a plasma situation.

INTEGRABLE <u>VERSUS</u> NONINTEGRABLE FIELD THEORIES

In this talk I will concentrate on the simplest models containing coherent structures: one-dimensional nonlinear fields. It is fitting that the premier coherent structure, the soliton, was discovered in the unsuccessful attempt to show numerically that nonlinear systems would be ergodic, and thus that statistical methods would apply.

Since the numerical discovery of the soliton by Zabusky and Kruskal (1965), analytical progress has shown that a class of nonlinear differential equations can be integrated using effectively linear techniques. These equations, the integrable theories, are distinguished by three properties (for reviews see Scott <u>et al</u>., 1973; or Ablowitz and Segur, 1981): the existence of an inverse-scattering transform--which allows formal solution of the initial value problem; a Backlund transformation--the nonlinear superposition principle; and an infinite set of independent conserved quantities. The soliton solutions of an integrable field theory are localized travelling waves which have the remarkable property that they are preserved upon collision. It can be shown that an arbitrary localized initial state evolves into a number of solitary waves and some small amplitude (roughly) linear waves.

As illustrative examples of the contrast between integrable and nonintegrable systems, consider the Korteweg de Vries (KdV) and regularized-long-wave (RLW) equations (Benjamin et al., 1972),

$$\phi_t + \phi_x + \phi_{xxx} - \phi\phi_x = 0 \qquad \text{(KdV)} \quad ; \tag{2}$$

$$\phi_t + \phi_x - \phi_{xxt} - \phi\phi_x = 0 \qquad \text{(RLW)} \quad . \tag{3}$$

Each of these equations has a solitary wave solution of the form

$$\phi_s(x,t) = A \operatorname{sech}^2[k(x - ct)] \tag{4}$$

where the amplitude $A$, width $k^{-1}$ and speed $c$, are related by

$$A = -3(c - 1)$$

$$k = \begin{cases} \frac{1}{2}(c - 1)^{1/2} & \text{(KdV)} \\ \frac{1}{2}\left(1 - \frac{1}{c}\right)^{1/2} & \text{(RLW)} \quad . \end{cases} \tag{5}$$

Note that for KdV, the speed must be greater than one, while for RLW $c$ may also be negative.

The two branches of the RLW solitary wave will be referred to as the KdV ($c > 1$) and plasma ($c < 0$) branches. The KdV branch solitary waves are quite similar to actual solutions of the KdV equation, since if we take the small wavenumber, $k \ll 1$, limit of the RLW equation it is equivalent to KdV. The plasma branch is so named because the RLW equation is applicable to drift waves in plasmas and the $c < 0$ solitary wave should be observable in this case (Petviashvili, 1977; Meiss and Horton, 1982).

Each of these equations has three conserved quantities obtained by symmetry considerations: mass, momentum, and energy

$$M = \int dx\phi$$

$$P = \begin{cases} \frac{1}{2}\int dx\phi^2 & \text{(KdV)} \\ \frac{1}{2}\int dx\left(\phi^2 + \phi_x^2\right) & \text{(RLW)} \end{cases}$$

$$H = \begin{cases} \int dx\left(-\frac{1}{2}\phi^2 + \frac{1}{2}\phi_x^2 + \frac{1}{6}\phi^3\right) & \text{(KdV)} \\ \int dx\left(-\frac{1}{2}\phi^2 + \frac{1}{6}\phi^3\right) & \text{(RLW).} \end{cases} \quad (6)$$

However, the KdV equation has an infinite sequence of such polynomial conserved quantities (Miura et al., 1968), each containing a term $1/n!\ \phi^n$. RLW, by contrast, has only three independent polynomial conserved quantities (Olver, 1980; Tsujishita, 1979).

Both equations are Hamiltonian systems, and Eqs. 2 and 3 can be written

$$\phi_t = [\phi, H]$$

where the Poisson bracket is defined (for review see Morrison, 1982) as

$$[A,B] = \int dx \frac{\delta A}{\delta\phi}\mathcal{O}\frac{\delta B}{\delta\phi} \quad .$$

$$\mathcal{O} = \begin{cases} \partial_x & \text{(KdV)} \\ \partial_x\left[1 - \partial_x^2\right]^{-1} & \text{(RLW) .} \end{cases}$$

Use of the inverse-scattering transform allows one to show that the KdV equation is a completely integrable Hamiltonian system: it can be transformed to action-angle form as shown by Zakharov and Faddeev (1971). No such transformation has been found for the RLW equation, and we strongly suspect that none exists.

One of the most striking manifestations of the nonintegrability of the RLW equation appears when we allow two solitary waves to collide. These collisions are inelastic: besides the two incoming solitary waves, small amplitude radiation, and sometimes new solitary waves are emitted (Morrison et al., 1983). Collisions between solitary waves with like-signed velocities are nearly elastic (Abdullov et al., 1976). In fact, when the speeds are only moderately larger than one, the emitted radiation is difficult to detect. This is probably because, for these parameters, the RLW equation is near to the integrable KdV equation.

Collisions between one positive and one negative velocity solitary wave can result in four out-going solitary waves as well as radiation. Figure 1 shows a collision with $c_1 = 2.5$ and $c_2 = -1.5$. Notice that during the collisions, sharp peaks are produced, and so care must be taken to ensure numerical convergence. After the collision, four peaks which obey the amplitude, width, and speed relations of Eq. 5 are seen, as well as a localized bundle of radiation. The resulting speeds are $c_1 = 2.31$, $c_2 = -1.20$, $c_3 = 1.32$, and $c_4 = -0.37$. The radiation packet spreads out roughly in accord with the linear group velocity.

We have found an entire region of the parameter space ($c_1$, $c_2$) above a certain threshold, for which two new solitary waves are produced (Morrison et al., 1983). Below that threshold only radiation is emitted.

These results can be stated in terms of the S-matrix, which relates the state at $t=-\infty$ to that at $t=+\infty$. For the KdV equation, the preservation of solitons implies the S-matrix is diagonal (at least in the soliton sector, treatment of the radiation is perhaps more difficult). For RLW there are non-diagonal terms in the S-matrix for solitary wave production

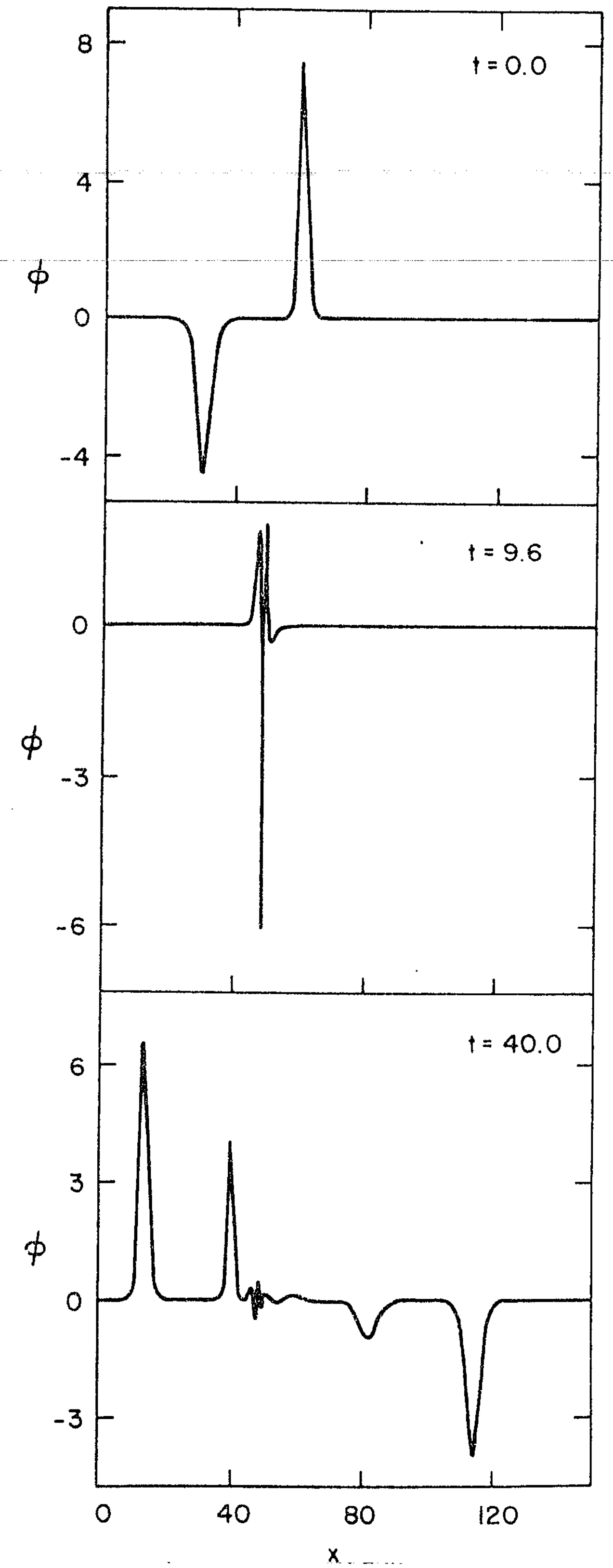


Fig. 1 Collision of two RLW solitary waves, $c_1 = 2.5$ , $c_2 = -1.5$ , resulting in four outgoing solitary waves plus radiation.

as well as radiation production. In addition, time-reversal symmetry implies that it is possible to construct new solitary waves from solitary waves and radiation, and possibly from radiation alone.

Similar considerations apply to other nonintegrable systems, such as the Zakharov equations, where ion-acoustic radiation is formed during solitary wave collisions (Thornhill and ter Haar, 1978), and the double sine-Gordon equation for which bound states can be formed (Kumar and Holland, 1982). It is probable that inelasticity of this sort is a general property of field theories.

The primary question, for a statistical mechanic, is how can the long-time (equilibrium?) state be characterized. Consider the RLW equation on a periodic space $0 < x < L$ where $\phi(x,t) = \phi(x + L,t)$, and an arbitrary initial state with fixed values of M, P, and H: is the asymptotic state independent of the initial? We suspect that when the momentum is small (we use P because it is positive definite) the evolution is nearly integrable (KdV-like) and regular regions in phase space do not allow ergodicity. As the momentum increases, the regular regions will shrink, plasma branch solitary waves can be excited for $P \gtrsim 14.01$ (Meiss and Horton, 1982), and the flow looks more ergodic. For small times, the number of solitary waves will fluctuate as collisions occur. The suspicion is that even for long times, the number of solitary waves will be nonzero.

Statistical mechanics of field theories has been discussed for the "$\phi^4$" theory in the context of solid state physics, and techniques for evaluating the partition function have been developed (Currie et al., 1980). It can be seen that the free energy can be decomposed into a (dressed) solitary wave part and a (dressed) phonon part. We have begun a similar calculation for RLW (Meiss and Horton, 1982).

## TURBULENCE AND THE FORCED RLW EQUATION

So far we have confined our discussion to conservation systems, and were led to consider the validity of equilibrium statistical mechanics. A more common situation in plasma physics is the presence of an external energy source which leads to linear instability. In this section, I will argue that the saturation of such an instability, in the context of the RLW equation, leads to the formation of coherent structures. The discussion is based on a collaboration with T. Kamimura and W. Horton (Meiss et al., 1983).

The model is Eq. 3 with an added source term

$$\phi_t + \phi_x - \phi_{xxt} - \phi\phi_x = \hat{S}\phi \quad , \tag{7}$$

where the linear operator $\hat{S}$ is given by its Fourier transform

$$\hat{S}\phi(x,t) = \frac{1}{2\pi}\int S\phi_{k_x} \exp(-ik_x x)\,dk_x$$

$$S = \delta \frac{k_x^4}{\left(1 + k_x^2\right)^2} - \mu k_x^4 \quad . \tag{8}$$

The linearized version of Eq. 7 has the dispersion relation

$$\omega = \omega_r + i\gamma \quad ,$$

$$\omega_r = \frac{k_x}{1 + k_x^2} \quad ,$$

$$\gamma = \frac{k_x^4}{1+k_x^2}\left[\frac{\delta}{\left(1+k_x^2\right)^2} - \mu\right] . \tag{9}$$

For $\delta < \mu$ the modes $k_x < k_0 = [(\delta/\mu)^{\frac{1}{2}} - 1]^{1/2}$ are unstable; large wavenumber modes are damped at a rate proportional to $\mu k_x^2$ . This growth rate, shown in Fig. 2, is a model for drift waves: the instability $\propto \delta$ arising from inverse electron dissipation and the damping $\propto \mu$ provided by ion viscosity. More generally, $\gamma$ has features in common with most instabilities: forcing at low and damping at high wavenumbers. We expect the dynamics to be governed by the width of the unstable spectrum $k_0$, and the maximum growth rate $\gamma(k_p) = \gamma_p$ .

Prior to the treatment of the full evolution implied by Eq. 7, consider the effect of $\hat{S}$ on a single solitary wave solution. For $\mu < \delta << 1$, we can assume that the solitary wave maintains its form while its amplitude, width, and speed slowly change with time. The standard technique, (Ott and Sudan, 1970) for obtaining equations describing the adiabatic change of parameters derives from the modified conservation laws, Eq. 6, which now vary according to

$$\begin{aligned}
\frac{d}{dt} M &= 0 \\
\frac{d}{dt} P &= \int \phi \hat{S} \phi dx \\
\frac{d}{dt} H &= \int \left[\int^x \phi_t(x')dx'\right] \hat{S}\phi dx \quad .
\end{aligned} \tag{10}$$

We assume that the field has the form

$$\phi = A(t)\ \mathrm{sech}^2\left\{k(t)[x - \theta(t)]\right\} + \delta\phi$$

where $A$, $k$, and $\dot{\theta} = c(t)$ are related by the unperturbed relations, Eq. 5, and $\delta\phi$ represents the non-adiabatic change.

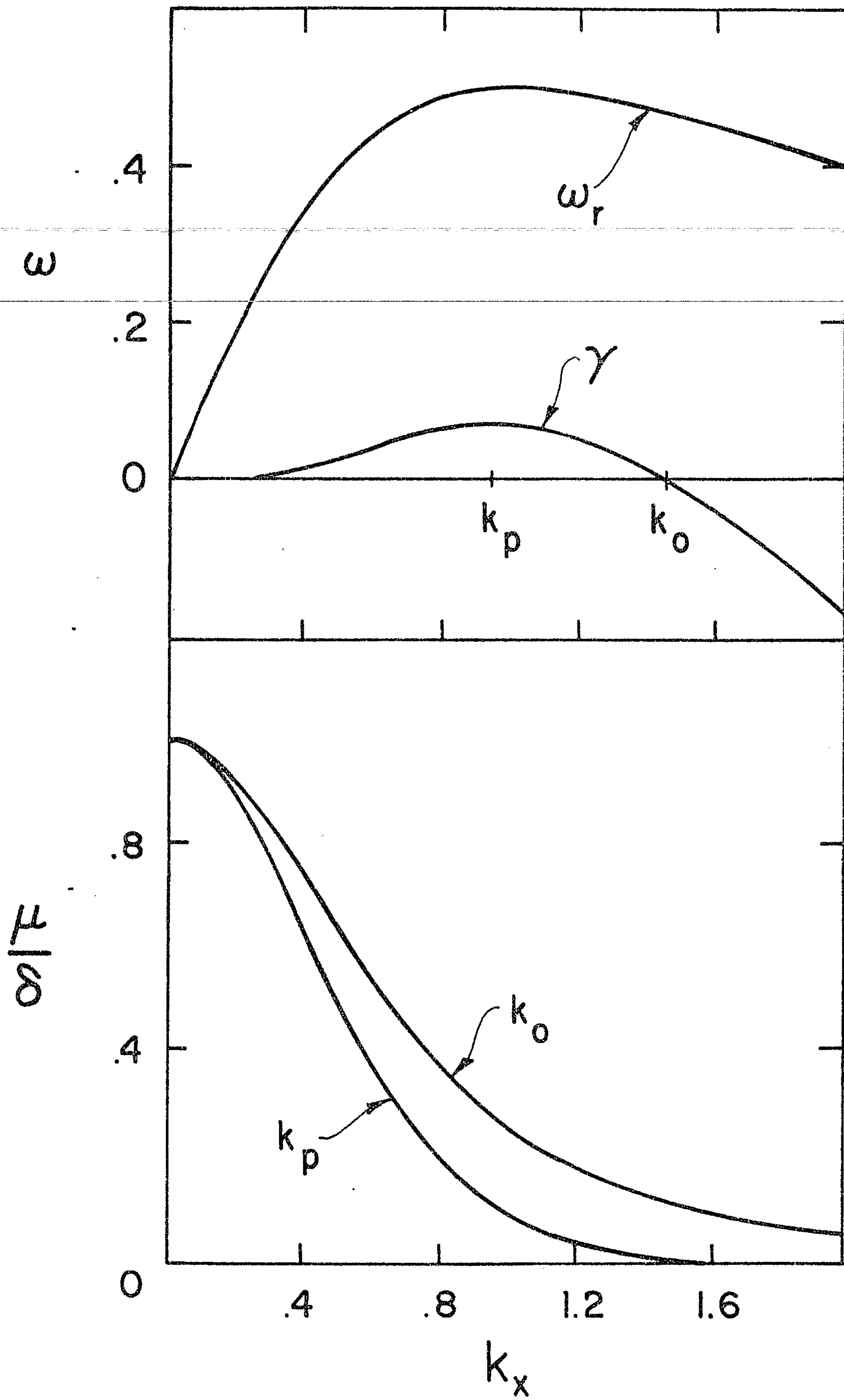


Fig. 2 Model dispersion relation from Eq. 9 for $\delta = 1.0$, $\mu = 0.1$. Lower figure shows wavenumbers of peak growth rate, $k_p$, and zero growth rate as a function of $\mu/\delta$.

Substitution of this into Eq. 10 yields three o.d.e.'s for the parameters of $\phi$ . Mass conservation implies that any area change in the solitary wave is compensated for by $\delta\phi$

$$\int \delta\phi dx = \frac{A(0)}{k(0)} - \frac{A(t)}{k(t)} .$$

Typically, $\delta\phi$ takes the form of a "tail", or shelf emitted behind the solitary wave. It takes a more complete perturbation theory (see Keener and McLaughlin, 1977; Karpman and Maslov, 1978) to determine the structure of the tail. Unfortunately, these theories are based on the inverse-scattering transform, which is unavailable for the RLW equation.

Momentum conservation yields

$$\frac{dP}{dt} = \delta \frac{64}{21} A^2 k^3 \left[R(k) - \frac{\mu}{\delta}\right] ; \qquad (11)$$

$$P = \frac{2}{3} \frac{A^2}{k} \left(1 + \frac{4}{5} k^2\right) ; \qquad (12)$$

$$R(k) = 42\pi \int_0^\infty dz \frac{z^6 \operatorname{csch}(\pi z)}{\left[1 + (2kz)^2\right]^2} . \qquad (13)$$

The function $R(k)$ asymptotically obeys

$$R(k) = \begin{cases} 1 - \frac{56}{5} k^2 + \mathcal{O}(k^4) \\ \frac{7}{4} k^{-4} + \text{logarithmic corrections.} \end{cases}$$

Remarkably, the energy equation is identical to the momentum equation by virtue of the fact that, for a solitary wave, $dH/dP = -c$ (Morrison et al., 1983).

Equation 11 implies that for a given $\mu/\delta$ one particular solitary wave is in equilibrium with the forcing. This solitary wave has $k = k_s$ where $R(k_s) = \mu/\delta$ . The saturation condition, shown in Fig. 3, is roughly given by $k_s \simeq k_0/2$ . It is

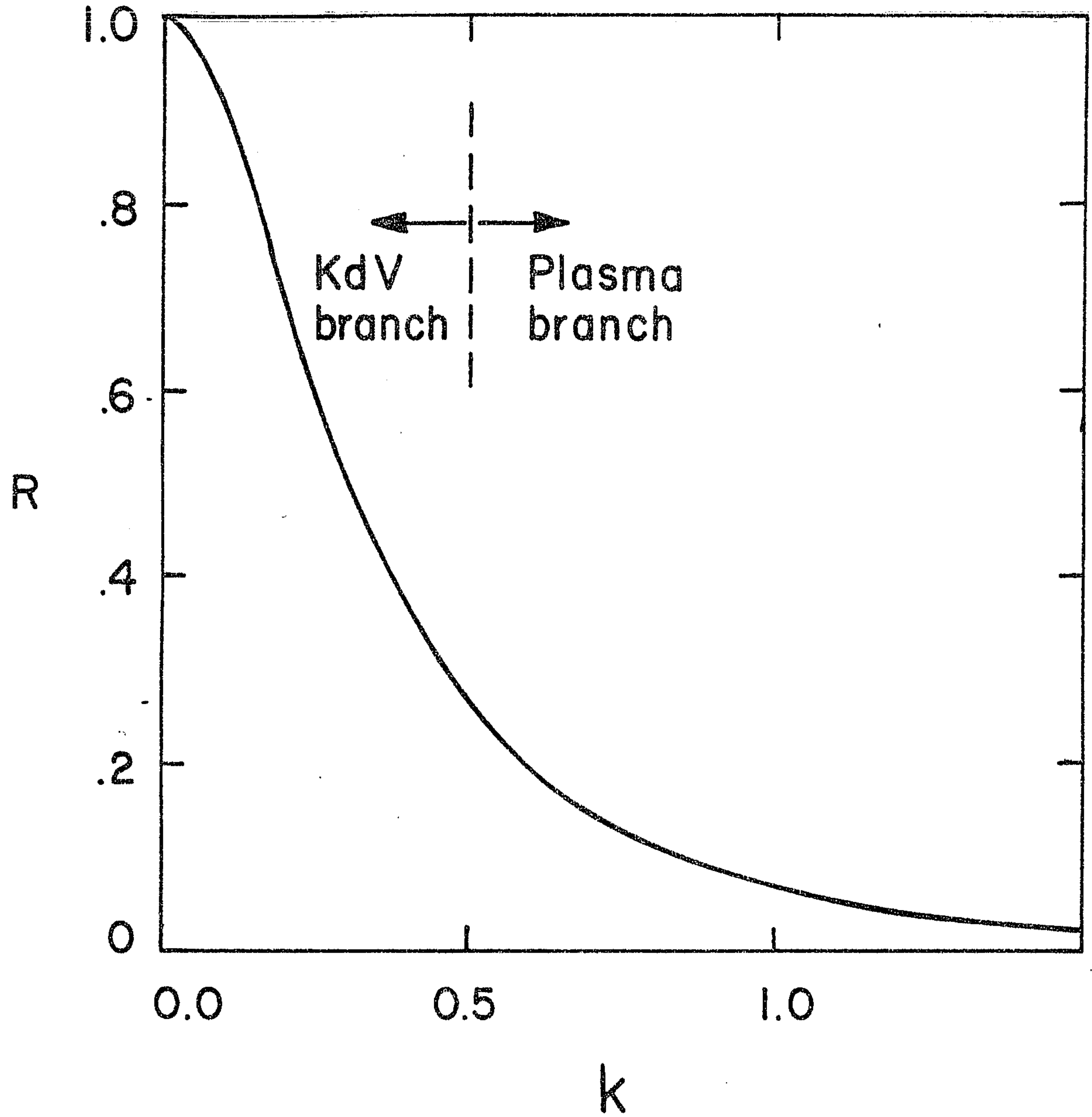


Fig. 3 Source function, R(k), from Eq. 13. The saturation width is determined by $R(k_s) = \mu/\delta$ .

significant that saturation depends only on the ratio $\mu/\delta$ (hence only on the width of the unstable spectrum $k_0$), and not on the magnitude of the growth rate. To attain saturation, the solitary wave picks a width, $k_s^{-1}$, so that its spectrum has the property that the energy lost from high wavenumbers is exactly balanced by the input to low wavenumbers. The coupling between stable and unstable regions of the spectrum is provided by the coherent nonlinearity of the solitary wave. This picture is appropriate if the nonlinear time scale is shorter than the linear growth rate, so that the solitary wave can maintain its shape:

$$A \gg \frac{\gamma_p}{\omega_p} .$$

The opposite condition, that the linear time scales are more rapid than nonlinear ones, would give incoherent turbulence.

Using the relationships between amplitude, width, and speed in Eq. 5, the momentum equation (Eq. 11) can be expressed as an equation for $c$

$$\dot{c} = \delta \frac{20}{7} \frac{(c-1)^3}{24c^2 - 8c - 1} \left\{ R[k(c)] - \frac{\mu}{\delta} \right\} . \tag{14}$$

It is easy to see that when $k_s < 1/2$ and $c > 1$, the point $k = k_s$ is a stable attractor of Eq. 14. Plasma branch $(c < 0)$ solitary waves are attracted to the point

$$c = \frac{2 - \sqrt{10}}{12}$$

(where $\partial P/\partial c = 0$) but since $\dot{c} \to \infty$, Eq. 14 ceases to be valid. Probably such solitary waves are destroyed by the source.

When $k_s > 1/2$, or equivalently $\delta > 3.727\mu$, then even the KdV branch solitary waves are unstable according to Eq. 14. In fact, for this case, the velocity increases exponentially in time

$$c(t) \propto \exp(\gamma t) \quad , \quad \gamma = 0.119(\delta - 3.727\mu) \quad ;$$

for $c >> 1$. Therefore, the value $\delta/\mu = 3.727$ is a critical value dividing the region of stable saturation from explosive instability.

From the chaotic perspective it is more relevant to study evolution from initial small amplitude noise. To correspond with the simulations we consider a periodic system of length L and assume $\phi(x,0) = A_0 \sum_j \cos[(2\pi jx/L) + \theta_j]$ where $A_0 << 1$ and the $\theta_j$ are picked randomly. During the first few e-folding times $(\gamma_p^{-1})$ the stable modes will quickly damp away and the spectrum will become peaked about $k_p$ with a width $\Delta k \sim (\gamma_p'' t)^{-\frac{1}{2}}$. As the peak mode reaches an amplitude of order one, mode coupling begins to spread energy into the stable region of wavenumber space.

At this point, providing $\delta/\mu < 3.727$ and $\gamma_p/\omega_p << 1$, it is plausible (and we observe numerically) that solitary waves will begin to form. Actually, since the system is periodic, it is more appropriate to assume that the cnoidal wave forms:

$$\phi = \phi_0 + A\mathrm{cn}^2[k(y - ct)|m] \quad ;$$

$$A = \frac{12k^2}{4\lambda k^2 - 1} \quad ;$$

$$c = \frac{1 - \phi_0}{1 + (1 - 2m)4k^2} \quad ; \qquad (15)$$

where $\lambda = m - 2 + 3[E(m)/K(m)]$, cn is a Jacobi elliptic function, and E and K are complete elliptic integrals. As $m \to 0$, this solution becomes a linear wave, obeying the correct dispersion relation, while as $m \to 1$ it becomes the solitary wave solution. There are three independent parameters: m, $\phi_0$ , and k .

The conservation laws provide evolution equations for the parameters of the cnoidal wave in the same manner as before. Now $\phi_0$ is determined by mass conservation and represents the "tail" formation of the solitary wave case. The equations predict a saturated state with an amplitude dependent only on $\mu/\delta$ and L. The critical value of $\mu/\delta$ for explosive instability is somewhat larger than before, due to periodicity, but if $L >> 1$, this effect is relatively small.

Preliminary numerical experiments tend to confirm the cnoidal wave model for saturation. As nonlinear saturation occurs, the single mode spectrum $\phi \sim A_0 \exp(\gamma_p t)\cos(k_p x)$ begins to develop sharper valleys and flatter peaks, which is similar to the shape of a periodic array of KdV branch ($A < 0$) solitary waves. The wavenumber spectrum broadens approaching that of the cnoidal wave

$$\left|\phi_j\right|^2(t) = \left\{\frac{2\pi^2 j}{Lmk^2} \operatorname{csch}\left[\frac{2\pi j}{Lk} K'(m)\right]\right\}^2 .$$

Because the spectrum is not initially a single mode at $k_p$ (or perhaps due to a modulational instability) the peaks of the cnoidal wave, once formed, develop different amplitudes. Peaks with large amplitudes propagate more rapidly than those with small amplitudes and so collisions occur. Collisions are inelastic just as those between solitary waves, though now the inelasticity is enhanced by the source term. Smaller peaks lose energy at each collision and seem to have a lifetime of about eight collisions. The number of solitary peaks decreases linearly during this stage, and the field begins to look more like a rarified gas of solitary waves. Further numerical work is in progress to determine the final steady state. In our preliminary runs the saturation amplitude seems to agree with the model, but more analysis is needed. (Meiss et al., 1983)

Overall, the picture we propose can be formulated in terms of the two important parameters of the source; $\gamma_p/\omega_p$ and $k_0$. For a narrow unstable spectrum and large growth rates, linear terms in the RLW equation predominate and incoherent turbulence theory should apply. However, as the width of the unstable spectrum increases (increasing the saturation amplitude) or if $\gamma_p/\omega_p$ decreases, nonlinearity acts to create coherent structures outside the scope of ordinary turbulence theory. At this stage, the saturated state may be more appropriately described as a gas of localized coherent objects.

## ACKNOWLEDGMENTS

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