Transport theory in the collisionless limit

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April 14, 1998

Abstract

Traditional transport theory provides a closure of fluid equations that is valid in the collisional, short mean–free–path limit. The possibility of extending an analogous closure to long mean–free path is examined here. An appropriate kinetic equation, using a model collision operator, is solved rigorously for arbitrary collisionality but weak, Maxwellian source terms. The corresponding particle and heat flows are then expressed in terms of the density and temperature profiles. The transport matrix is found to be symmetric even at vanishing collision frequency; in the collisionless limit it takes the form of nonlocal operators. The operator corresponding to thermal conductivity agrees with one found previously by Hammett and Perkins [1]. However particle diffusion, which turns out to satisfy a local Fick's law for any finite collision frequency, becomes singular at vanishing collisionality, where the pressure gradient vanishes. We conclude that the fluxes can generally be expressed in terms of particle and energy sources, but not always in terms of pressure and temperature profiles.

I Introduction

This work considers particle diffusion and thermal conduction parallel to the magnetic field of a magnetized plasma. Our lowest order description is thus one–dimensional and equivalent to the one–dimensional transport problem in a neutral gas. However we emphasize the limiting regime of vanishing collisions, as pertains to a hot plasma. Thus we consider a plasma in which

$$\lambda \gg L$$

where

$$\lambda \equiv \frac{v_t}{\nu} \tag{1}$$

is the mean–free path, with v_t a thermal speed and ν the collision frequency, and L is the scale length for density and temperature.

The conventional closure for a collisionless plasma is the kinetic description associated with Vlasov and Landau. Our objective is to derive from the Vlasov equation a *fluid* description, in terms of the plasma pressure and temperature, as well as the particle and heat fluxes. The fluid description has the advantage of involving only the coordinate dimensions, rather than the larger dimensionality of phase space. Thus we derive a collisionless version, in one dimension, of the parallel transport equations of Braginskii [2] or Spitzer [3].

The present work is related to other recent investigations of the same physics, such as that of Hammett and Perkins [1]. In order to emphasize key features of transport as a function of λ/L, we simplify the situation, considering the quasistatic state and omitting perturbation of the electromagnetic field. On the other hand we treat particle and heat transport in a systematic way, deriving the (2×2) transport matrix for arbitrary collisionality. Our results agree with classical collisional transport theory in the short mean–free path limit, and modestly generalize those of Hammett and Perkins at long mean–free path. The general transport matrix displays some features that are not apparent in previous work. In particular we find that particle diffusion, in distinction to heat conduction, obeys a classical, short mean–free path law for all values of the collision frequency.

Conventional fluid closure is based, first of all, on the fact that the lowest–order distribution in a collision–dominated system is approximately Maxwellian. Since a Maxwellian distribution is specified by its density (n), temperature (T) and flow velocity, a fluid closure in terms of those variables makes sense. When collisional effects are very weak, on the other hand, the distributions need not be nearly Maxwellian, and a conveniently small set of parameters specifying even the lowest–order distributions might not exist. In the fully nonMaxwellian case it is not clear whether transport theory can be constructed, or whether it would be useful.

Yet there are various circumstances in which a Maxwellian distribution is pertinent even when the mean–free path is long. Magnetically confined fusion plasmas, for example, are approximately Maxwellian because, however much the mean–free path exceeds scale lengths, the confinement time of the plasma is long compared to collision times. Similarly collisions are not needed to Maxwellianize a plasma when its sources—such as ionization of neutrals, or influx from a wall—are Maxwellian. The Maxwellian–source case is considered here.

Assuming, then, that the distributions are nearly Maxwellian, an oft–noted aspect of transport at long mean–free path is that it is nonlocal. Since λ measures the distance that a typical particle explores between collisions, transport is affected by a sizable range of the pressure and temperature profiles when λ is long. Only when $\lambda \ll L$—only, that is, in the so–called Chapman–Enskog regime [4]—can the particles and heat fluxes at x be expressed in terms of the gradients at x. Our conclusions will qualify this statement, but the inadequacy of a local description for the collisionless case is nonetheless clear.

A more fundamental distinction of the collisionless case involves the question of what drives transport. We are used to the short mean–free–path viewpoint, in which transport responds to local gradients, or more generally to profile information. But it is clear that such gradients occur only because of *sources*: without localized inputs of particle or heat neither gradients nor diffusion can occur. In a collisional system the sources, while conceptually necessary, play little role in the analysis. But an analysis for arbitrary collisionality must treat the source terms explicitly as driving terms in the linearized kinetic equation. In particular the assumed small amplitude of the source terms provides the essential small parameter of the generalized transport analysis, replacing the short mean–free path.

The sources appear explicitly in the kinetic equation

$$v\,\frac{\partial f}{\partial x} - C(f) = S \tag{2}$$

where C represents the collision operator, v is the velocity coordinate and $f(x,v)$ is the distribution function. The quantity $S(x,v)$ is a *local* source of particles and heat.

Chapman–Enskog theory systematically uses the small parameter λ/L to expand the distribution

$$f = f_M + f_1$$

where f_M is a local Maxwellian and $f_1 \cong (\lambda/L)f_M$ is a perturbation, and thus to derive the well–known equation ("Spitzer" problem)

$$C(f_1) = v\,\frac{\partial f_M}{\partial x} \tag{3}$$

in which f_1 is driven by the local gradients. But even for $\lambda/L \ll 1$ it is clear (and made explicit in Sec. II) that the latter are in turn driven by S, whose effects they therefore mediate.

Before attacking the solution of (2), we comment on a final distinctive feature of long mean–free–path theory. Why compute such fluid moments as the heat flux, q? At short mean–free path, the answer is that knowing

$$q = -\kappa\,\frac{dT}{dx}$$

gives closure to moment equations: a 3^{rd}–order moment of f has been expressed in terms of a 2^{nd}–order moment. It is not obvious that a similar closure advantage pertains at long mean–free path: the obvious collisionless version of Chapman–Enskog theory yields circularity rather than closure.

The issue is especially clear in the $\nu \to 0$ limit. Then (2) implies

$$f_1 = \int ds \, \frac{S}{v}$$

and we can easily compute q. But the result is hardly a closure relation, being merely the one–dimensional version of the elementary moment equation

$$\nabla \cdot \mathbf{q} = S.$$

Hence one needs to specify which moment equations one is trying to close, and make sure that any kinetic theory actually addresses closure—one must know, in other words, why one is computing f_1.

One answer to this question is implicit in previous work, such as Ref. [1]: by expressing q in terms of the temperature profile rather than S, one obtains fluid equations that are valid for arbitrary spatial variation of the sources, and that map conveniently onto the short mean–free–path–fluid description. The same point of view is adopted here. We find however, that the goal of eliminating the source terms from fluid theory is not always achievable.

Our purpose is to develop the simplest collisionless closure that is self–consistent. It is both instructive and convenient to include collisions, treating the $\nu \to 0$ limit as a special case. Thus we obtain a description of transport that is valid for any collisionality. On the other hand we simplify by using a Krook–model collision operator with constant collision frequency [5]. The Krook model misses effects, such as off–diagonal transport coefficients, that depend upon the energy variation of Coulomb scattering. But it is not misleading in the collisionless limit, and it describes diagonal transport processes at short mean–free path with good qualitative accuracy.

II Kinetic equation

The Krook–model collision operator [5] that conserves particles and energy is well known. Acting on a nonMaxwellian perturbation f_1 it takes the form

$$C(f_1) = -\nu \left[f_1 - \left(\frac{\Delta n(x)}{n_0} + \left(s^2 - \frac{1}{2} \right) \frac{\Delta T(x)}{T_0} \right) f_M \right]$$

where

$$f_M(n_0, T_0) \equiv \frac{n_0}{\pi^{3/2} v_t^3} e^{-v^2/v_t^2}$$

is a Maxwellian distribution with constant density n_0 and constant temperature T_0, and ν is the collision frequency. The perturbations Δn and ΔT depend upon position as moments of f_1 (see below). The thermal speed is defined by

$$v_t \equiv \sqrt{\frac{2T_0}{m}}$$

where m is the particle mass, and

$$s \equiv \frac{v}{v_t}$$

is the normalized particle speed. It is obvious that the operator C drives f_1 toward the form of a perturbed Maxwellian,

$$f_1 \to f_M(n_0 + \Delta n, T_0 + \Delta T) - f_M(n_0, T_0)$$

and that it conserves particles and energy,

$$\int dv C = 0, \tag{4}$$

$$\int dv \left(s^2 - \frac{1}{2} \right) C = 0, \tag{5}$$

provided the perturbed quantities satisfy

$$\Delta n = v_t \int ds f_1, \tag{6}$$

$$n_0 \Delta T = 2 v_t T_0 \int ds \left(s^2 - \frac{1}{2} \right) f_1. \tag{7}$$

We also assume the source to have Maxwellian velocity dependence:

$$S(x,v) = \widehat{f}_M(T_0)\left[S_0(x) + \left(s^2 - \frac{1}{2}\right)S_2(x)\right].$$

Here we distinguish the particle source, S_0, and the energy or heat source, S_2. We have also introduced the convenient notation

$$\widehat{f}_M \equiv \frac{f_M}{n_0}$$

for a Maxwellian normalized to unit density.

The source acts as a perturbation on a homogeneous Maxwellian equilibrium, described by the lowest–order form of (2):

$$v\frac{\partial f_M}{\partial x} - C(f_M) = 0.$$

Here "lowest–order" refers to the small parameter measuring the amplitude of S. The resulting linear perturbation of the distribution function, f_1, satisfies

$$v\frac{\partial f_1}{\partial x} - C(f_1) = S. \tag{8}$$

This linear equation is easily solved by Fourier transformation. We use the convention

$$f_k = \int dx e^{ikx} f_1(x)$$

and find that

$$f_k = -\frac{\widehat{f}_M}{\nu - ikv}F_k(s). \tag{9}$$

Here we have introduced the function

$$F_k(s) \equiv \sigma_0 + \nu w_0 + \left(s^2 - \frac{1}{2}\right)(\sigma_2 + \nu w_2) \tag{10}$$

where

$$w_0 \equiv v_t \int ds f_k \tag{11}$$

$$w_2 = 2v_t \int ds \left(s^2 - \frac{1}{2}\right) f_k \tag{12}$$

measure respectively the Fourier–transformed density and temperature perturbations. The quantities σ_{0k} and σ_{2k} similarly denote the transforms of the particle and energy sources. It is worth pointing out that these transforms exist only when the sources are spatially *local*, decaying for large $|x|$. Indeed a spatially uniform source would drive neither gradients nor transport.

After substituting (9) into (11) and (12) we obtain two coupled equations for w_0 and w_2. The solution to this system is expressed in terms of the function

$$\overline{Z}(\zeta) \equiv \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} ds \frac{e^{-s^2}}{s - \zeta} \tag{13}$$

with variable

$$\zeta \equiv \frac{-i}{k\lambda}. \tag{14}$$

It is clear that $\overline{Z}$ is closely related to the ordinary Z–function; it differs only in that ζ is in the *lower*–half plane (for positive k), whence

$$\overline{Z}(\zeta) = Z(\zeta) - \Theta(k) 2i\sqrt{\pi} e^{-\zeta^2}. \tag{15}$$

Here $\Theta(k)$ is the Heaviside step–function.

Thus we find

$$w_0(\zeta) = \frac{2\sigma_0\zeta[-2\zeta^3 + (1 + \zeta^2 - 2\zeta^4)\overline{Z}] + \sigma_2[2\zeta + (2\zeta^2 - 1)\overline{Z}]}{\nu[2(\zeta^2 - 1) + (2\zeta^3 - 3\zeta)\overline{Z}]} \tag{16}$$

$$w_2(\zeta) = \frac{(2\sigma_0 - \sigma_2)\zeta[2\zeta + (2\zeta^2 - 1)\overline{Z}]}{\nu[2(\zeta^2 - 1) + (2\zeta^3 - 3\zeta)\overline{Z}]}. \tag{17}$$

These results can be substituted into (9) and (10) to provide an explicit expression for the Fourier–transform of the distribution; our kinetic analysis is complete.

Equations (16) and (17) explicate a comment made in the introduction: at any collisionality, it is the sources that support density and temperature gradients. Transport is driven ultimately by such local inputs of particles and heat; the gradients that often accompany this process can be viewed—and at low collisionality should be viewed—as artifacts.

III Transport matrix

We denote the particle and heat fluxes by Γ and q respectively:

$$\Gamma \equiv \int dvvf \tag{18}$$

$$q \equiv \int dvv \left(s^2 - \frac{3}{2}\right) f. \tag{19}$$

Particle and energy conservation, together with the kinetic equation (2), immediately provide simple formulae for the Fourier transforms of these fluxes:

$$\Gamma_k = \frac{i\sigma_0}{k} \tag{20}$$

$$q_k = \frac{i(\frac{1}{2}\sigma_2 - \sigma_0)}{k}. \tag{21}$$

Conventional Chapman–Enskog theory uses (the collisional version of) (9) to compute the particle and heat fluxes as integrals of the distribution function. The same calculation is easily performed here, for arbitrary collisionality, but it is no longer helpful. Indeed, after combining (10), (16) and (17) one obtains expression for Γ and q that merely reproduce the conservation laws (20) and (21). Thus the Chapman–Enskog recipe provides in this case a helpful check on the algebra but no progress at all regarding closure. As we have emphasized, kinetic theory plays a rather different role here than it does at short mean–free path.

The natural generalization of conventional transport theory relates the fluxes to the density and temperature profiles, whose transforms are given by w_0 and w_2. Because the transforms of the fluxes are trivially provided by the conservation laws, this generalized closure scheme uses kinetic theory to compute the transformed *profiles* rather than the fluxes.

The conventional "thermodynamic forces" are given by the pressure and temperature gradients:

$$A_1 \equiv \frac{d\,\log p}{dx} = \frac{d\,\log n}{dx} + \frac{d\,\log T}{dx}$$

$$A_2 \equiv \frac{d\,\log T}{dx},$$

The corresponding Fourier transforms are

$$A_{1k} = -ik(w_0 + w_2)/n_0 \tag{22}$$

$$A_{2k} = -ikw_2/n_0. \tag{23}$$

The general transport matrix is a set of operators (functionals) L_{ij} expressing the fluxes Γ_k and q_k in terms of the the forces:

$$\Gamma = -L_{1j}[A_j] \tag{24}$$

$$q = -L_{2j}[A_j] \tag{25}$$

At short mean–free path these operators degenerate into multiplicative coefficients, satisfying Onsager symmetry,

$$L_{ij} = L_{ji}. \tag{26}$$

In that case transport has a local description. At long mean–free path we expect the flows to respond nonlocally to the pressure and temperature profiles, so that the transport operators L_{ij} are expected to involve integrals over some spatial range. Nonetheless we will find that Onsager symmetry, as a relation between operators, holds for any value of λ/L.

We note that other choices for the driving forces are possible, but Onsager symmetry is preserved only with a "canonical" choice. The *density* and temperature gradients are not canonical in this sense.

IV Heat flux

We begin with the flow of heat, which can be seen from (17) and (21) to involve only A_{2k}. We find that

$$q_k = -\frac{\nu\lambda^2}{2} Q(\zeta) n_0 A_{2k} \tag{27}$$

where

$$Q(\zeta) \equiv \zeta \, \frac{2(\zeta^2 - 1) + \zeta(2\zeta^2 - 3)\overline{Z}}{2\zeta + (2\zeta^2 - 1)\overline{Z}}. \tag{28}$$

Equation (27) is exact (within the model collision operator) for any collisionality, and might in principle be used to derive interpolation formulae for heat flow over a range of collisionalities. Here we are content to examine its form in the collisional and collisionless limits.

Suppose first that the mean–free path is short. Then we let $\zeta \to \infty$ in the function $Q(\zeta)$ to find

$$\lim_{\zeta\to\infty} Q = \frac{3}{2} \tag{29}$$

(We note that it is necessary to keep terms in $\overline{Z}$ of order ζ^{-7} to compute this limit.) Thus (27) provides the collisional heat flux $q_k = -(3/2)\nu\lambda^2 n_0 A_{2k}$, which is the Fourier transform of the familiar relation

$$q = -\frac{3}{2} \, \nu\lambda^2 n_0 A_2. \tag{30}$$

In other words our kinetic theory has reproduced, in the appropriate limit, the classical law of parallel heat conduction.

At long mean–free path ($\zeta \ll 1$), we note from (15) that

$$\overline{Z} = -i\sqrt{\pi}\,\frac{k}{|k|} - 2\zeta + O(\zeta^2)$$

and thus find that

$$q_k = \frac{i v_t}{\sqrt{\pi}} \, \frac{k}{|k|} \, n_0 A_2. \tag{31}$$

It is helpful to express this result as $q_k = r_k n_0 A_2$ with

$$r_k = \frac{iv_t}{\sqrt{\pi}} \frac{k}{|k|}.$$

Now the inverse transform is

$$q(x) = n_0 \int_{-\infty}^{\infty} dx' r(x') \Delta T(x - x')$$

so we only need the inverse transform of $k/|k|$. This function is evidently not square–integrable, but it has a well–behaved inverse transform in the generalized function sense (see, for example, Gel'fand [6]):

$$r(x) = \frac{v_t}{\pi^{3/2}} P\left(\frac{1}{x}\right)$$

where the P denotes a principal value. The resulting heat flow

$$q(x) = \frac{n_0 v_t}{\pi^{3/2}} \int_0^{\infty} \frac{T(x - x') - T(x + x')}{x'} \tag{32}$$

was previously derived by Hammett and Perkins. Note that the direction of the flow is such as to dissipate the temperature gradient.

Thus (27), having been derived without a collisionality ordering, indeed contains the physics of both short and long mean–free path.

V Particle flux

The particle flux is more interesting. The key observation is that the Fourier–transformed thermodynamic force A_{ik}, corresponding to the pressure gradient, involves only the particle source, σ_0. The heat source does not enter. Indeed one finds from (16) and (17) that

$$w_0 + w_2 = \frac{-2\zeta^2 \sigma_0}{\nu}. \tag{33}$$

It follows, in view of (20) and (22), that the particle flux is remarkably simple:

$$\Gamma_k = -\frac{1}{2} \nu \lambda^2 n_0 A_{ik}. \tag{34}$$

Because the coefficient here is independent of k, the inverse Fourier transform is obvious and simply reproduces Fick's law:

$$\Gamma = -\frac{1}{2}\,\nu\lambda^2 n_0 \nabla\,\log p. \tag{35}$$

Its diffusion coefficient,

$$D_c = \frac{1}{2}\,\nu\lambda^2 = \frac{1}{2}\,\frac{v_t^2}{\nu}$$

is classical and, at short mean–free path, easily understood from random walk arguments. Of course its occurrence in an expression valid for arbitrary mean–free path is surprising. Two comments are in order:

1. A Krook–model plasma with Maxwellian sources is described by Fick's law, with the same diffusion coefficient $D_c(\nu)$, for any finite collision frequency. In this respect the particle flux is very different from the heat flux: the former never involves a nonlocal operator, as appears in (32).

2. The diffusion coefficient $D_c(\nu)$ becomes infinite in the collisionless limit. That the collisionless particle flux is nonetheless finite reflects the fact that the pressure perturbation is proportional to ν. Thus a collisionless plasma has a finite particle flux, due to the inhomogeneity of the source σ_0, but vanishing pressure gradient.

Thus our goal—to express the fluxes in terms of the profiles rather than the sources—is only partly successful. It works nicely for the heat flux, but produces a singular result for the collisionless particle flux. The relation between the particle flux and its canonical driving force is singular at $\nu = 0$, where free–streaming removes the pressure gradient while Γ remains finite.

Because the fluxes are actually driven by local sources, not gradients, we should not be surprised at the singularity. But its occurrence carries a cautionary message: transport closure at long mean–free path cannot be formulated, generally, in terms of profile information alone.

Acknowledgments

I thank Bill Dorland, Swadesh Mahajan, Per Helander, and A. Thyagaraja for helpful conversations. This work was supported by the US Department of Energy Contact No. DE–FG03–96ER–54346.

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