Double Curl Beltrami Flow—Diamagnetic Structures

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Abstract

It is shown that in an ideal coupled magnetofluid, the equilibrium magnetic (velocity) field is described by a two-parameter, double curl ($\nabla\times\nabla\times\quad$) system of equations. The new system allows, amongst others, a novel, fully diamagnetic, pressure confining, minimum $|B|$ configuration with velocity fields comparable (in appropriate units) to the magnetic fields.

The coupling between the magnetic field and the flow velocity (through the nonlinear induction effect and its reciprocal Lorentz force) tends to impart considerable complexity to magnetohydrodynamics (MHD). In spite of this complexity, MHD does allow regular and ordered behavior. An example is the appearance of the equilibrium magnetic field ($\mathbf{B}$) satisfying the "Beltrami condition,"

$$\nabla\times\mathbf{B}=\alpha\mathbf{B}, \tag{1}$$

when the flow energy can be neglected. Here, α is a scalar field that must satisfy $\mathbf{B}\cdot\nabla\alpha=0$ to insure $\nabla\cdot\mathbf{B}=0$. The Beltrami magnetic field represents a "stationary (no-flow)" force-free

(the Lorentz force $\mathbf{J} \times \mathbf{B} = (c/4\pi)(\nabla \times \mathbf{B}) \times \mathbf{B} \equiv 0$) macroscopic plasma state. Woltjer [1] derived (1) by minimizing the magnetic energy with the constraint that the local magnetic "helicity" is conserved. Taylor [2] introduced the far-reaching concept of relaxation: he conjectured that a small amount of resistivity present in a realistic plasma would tend to relax all the local helicity constraints leaving only the conservation of global helicity intact. The minimization of the field energy with the global constraint, then, leads to the "relaxed state" characterized by a spatially homogeneous α in (1), i.e., a "constant-α Beltrami field."

The aim of this paper is to show that a more adequate formulation of the plasma dynamics allows a much wider class of special equilibrium solutions. The set of new solutions contains field configurations which can be qualitatively different from the constant-α-Beltrami magnetic fields (which are naturally included in the set). The larger new set may help us understand a variety of structures generated in plasmas. It also opens up the possibility of experimenting with altogether different configurations some of which may lead to a novel regime of high-pressure plasma confinement.

When electron inertia is neglected, the normalized two-fluid equations with arbitrary flows can be written as

$$\mathbf{E} + \frac{\mathbf{v}_e \times \mathbf{B}}{c} = -\frac{1}{n}\nabla(p_e + g_e) \tag{2}$$

$$\frac{\partial \mathbf{v}_i}{\partial t} - \frac{e}{M}\mathbf{E} - \mathbf{v}_i \times \left[\frac{e}{Mc}\mathbf{B} + \nabla \times \mathbf{v}_i\right] = -\frac{1}{Mn}\nabla\left(p_i + g_i + \frac{v_i^2}{2}\right) \tag{3}$$

where $\mathbf{v}_i(\mathbf{v}_e)$, and $p_i(p_e)$ are respectively the ion (electron) velocity and pressure, M is the ion mass, n is the constant particle density, and $g_i(g_e)$ represent all other gradient forces including an externally applied electrostatic field. To derive (2) and (3), we have used the vector identity $\mathbf{v}_i \cdot \nabla \mathbf{v}_i = \nabla(v_i^2/2) - \mathbf{v}_i \times (\nabla \times \mathbf{v}_i)$. In order to write (2) and (3) simply, and fully in terms of the fluid velocity $\mathbf{V} = (m\mathbf{v}_e + M\mathbf{v}_i)/(m + M) \simeq \mathbf{v}_i$, and the magnetic field $\mathbf{B}$, we affect the following changes: Writing $\mathbf{E} = -(1/c)\partial\mathbf{A}/\partial t$ [where $\mathbf{A}$ is the vector potential ($\boldsymbol{\nabla} \times \mathbf{A} = \mathbf{B}$)], $\mathbf{v}_i = \mathbf{V}$, $\mathbf{v}_e = \mathbf{V} - \mathbf{J}/ne = \mathbf{V} - (c/4\pi ne)\boldsymbol{\nabla} \times \mathbf{B}$ (using Ampére's law),

and then normalizing $\mathbf{B}$ to some arbitrary B_0, $\mathbf{V}$ to the Alfvén speed $v_A = B_0/\sqrt{4\pi M n}$, the space and time scales respectively to the ion skin depth $\lambda_i = c/\omega_{pi} = c(4\pi n_0 e^2/M)^{1/2}$, and the cyclotron time $\tau_c = (Mc/eB_0)$, we obtain

$$\frac{\partial \mathbf{A}}{\partial t} - (\mathbf{V} - \boldsymbol{\nabla} \times \mathbf{B}) \times \mathbf{B} = \nabla (\beta_e + \widehat{g}_e) \tag{4}$$

$$\frac{\partial}{\partial t} (\mathbf{A} + \mathbf{V}) - \mathbf{V} \times (\mathbf{B} + \nabla \times \mathbf{V}) = \nabla \left(\beta_i + \widehat{g}_i + \frac{1}{2} V^2 \right) \tag{5}$$

where β_i and β_e are the ion and electron pressures normalized to $B_0^2/4\pi$, and $\widehat{g}_i(\widehat{g}_e)$ are the normalized gradient forces. To derive (4) and (5), we have also assumed that the particle density is spatially uniform. Our intention, in this letter, is to concentrate on an algebraically simple system to delineate new and interesting physics. The effects of electron inertia, non-uniform density etc., will be dealt with in a forthcoming detailed publication.

Taking the curl of these equations, we can cast them in a revealing symmetric form

$$\frac{\partial}{\partial t} \boldsymbol{\Omega}_j - \nabla \times (\mathbf{U}_j \times \boldsymbol{\Omega}_j) = 0 \tag{6}$$

in terms of a pair of generalized vorticities

$$\boldsymbol{\Omega}_1 = \mathbf{B}, \qquad \boldsymbol{\Omega}_2 = \mathbf{B} + \boldsymbol{\nabla} \times \mathbf{V}, \tag{7}$$

and effective velocities,

$$\mathbf{U}_1 = \mathbf{V} - \boldsymbol{\nabla} \times \mathbf{B}, \qquad \mathbf{U}_2 = \mathbf{V}. \tag{8}$$

The first equation is the induction equation with the difference that we do not neglect $\boldsymbol{\nabla} \times \mathbf{B}$ ($\sim \mathbf{J}$) with respect to $\mathbf{V}$ as is often done in MHD. This departure from the standard one–fluid treatment is crucial; it is the source of the $\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{B}$ term in Eq. (11), and hence of the possible diamagnetic structures. The second equation is the Lorentz force equation which also includes the standard fluid force $\mathbf{V} \times (\nabla \times \mathbf{V})$. Let us introduce $\widehat{\mathbf{U}}_j = \mathbf{U}_j - \mu_j \boldsymbol{\Omega}_j (j = 1, 2)$, and rewrite (6) as

$$\frac{\partial}{\partial t} \boldsymbol{\Omega}_j - \nabla \times (\widehat{\mathbf{U}}_j \times \boldsymbol{\Omega}_j) = 0 \qquad (j = 1, 2), \tag{9}$$

where μ_1 and μ_2 are scale parameters that can be considered as intensives in a possible thermodynamic interpretation (to be discussed later). The simplest equilibrium solution to (9) is $\widehat{\mathbf{U}}_j = 0 (j = 1, 2)$, or equivalently the system of linear equations in $\mathbf{B}$ and $\mathbf{V}$ ($a = 1/\mu_1$ and $b = 1/\mu_2$)

$$\begin{aligned} \mathbf{B} &= a(\mathbf{V} - \nabla \times \mathbf{B}), \\ \mathbf{B} + \nabla \times \mathbf{V} &= b\mathbf{V}, \end{aligned} \tag{10}$$

which describes, explicitly, the strong coupling between the magnetic and the fluid aspects of the plasma. It is from this coupling that new physics is expected to arise. Equations (10) can be combined to yield, in either $\mathbf{V}$ or $\mathbf{B}$, the second order partial differential ($\alpha = b - (1/a)$ and $\beta = 1 - b/a$)

$$\nabla \times (\nabla \times \mathbf{B}) - \alpha \nabla \times \mathbf{B} + \beta \mathbf{B} = 0, \tag{11}$$

which, will, naturally lead to magnetic field (and flow velocity) structures far richer than the ones contained in the "constant-α Beltrami-Taylor" (B–T) system.

The equilibrium solution (10), when substituted into (2) and (3) leads to the Bernoulli conditions $\nabla(\beta_e + \widehat{g}_e) = 0 = \nabla(\beta_i + g_i + V^2/2)$ suggesting a mechanism for creating pressure (β) gradients in this extended relaxed state. In the simplest case ($\widehat{g}_1 = 0 = \widehat{g}_e$),

$$\beta_i + \tfrac{1}{2} V^2 = \text{ constant} \tag{12}$$

revealing that an appropriate sheared velocity field can sustain a desired ion pressure gradient. Equations (10)–(11) will serve as a basis for designing a highly effective plasma confinement machine.

Before investigating the explicit solutions of (11), we would like to make the following comments:

(1) The set of equations (10) can be derived by following the Taylor prescription of relaxed equilibria applied to (9). It is straightforward to show that with appropriate boundary conditions, there are two bilinear constants of motion, the usual total magnetic helicity

$h_1 = -\frac{1}{2}\int \mathbf{A}\cdot\mathbf{B}\, d^3x$, and the generalized helicity $h_2 = \frac{1}{2}\int(\mathbf{A}+\mathbf{V})\cdot(\mathbf{B}+\nabla\times\mathbf{V})d^3x$ (or any appropriate combination of h_1 and h_2) [4]. Minimization of the total energy $E = \frac{1}{2}\int\left(B^2+V^2\right)d^3x$ with the constraints of constant h_1 and h_2 will directly lead us to (10). The constants a and b are related to the Lagrange multipliers needed in the constrained minimization. We recently found that Steinhauer and Ishida [5] have developed a similar procedure for dealing with the magnetofluids.

(2) The general steady-state solution allowed by (9) consists of a set of nonlinear equations $\mathbf{U}_j = A_j(\mathbf{x})\mathbf{\Omega}_j$, and $\mathbf{\Omega}_j\cdot\nabla A_j(\mathbf{x}) = 0$ $(j = 1,2)$. The linear equilibrium solution (10) is the special case where $A_j(\mathbf{x}) \equiv \mu_j =$ constant $(j = 1,2)$. In a thermodynamic sense, the spatially inhomogeneous (homogeneous) A_j imply a nonequilibrium (equilibrium) state. The latter corresponds to the Euler-Lagrange equations associated with the global free energy $F = E - \sum_j \mu_j h_j$ with μ_j acting as Lagrange multipliers. The system can be viewed as a 'grand-canonical ensemble' in which the injection of a 'helicity' h_j creates an equivalent energy $\mu_j h_j$. The equations then follow as the global "relaxed state."

Before writing down some highly revealing solutions, we analyze the mathematical structure of the double Beltrami flow (11). We rewrite it in the form

$$(\nabla\times-\lambda_+)(\nabla\times-\lambda_-)\mathbf{B} = 0. \tag{13}$$

where $\lambda_\pm = [\alpha\pm\sqrt{\alpha^2-4\beta}/2]$. At the boundary $\partial\Omega$ of the three-dimensional bounded domain Ω, we assume $\mathbf{n}\cdot\mathbf{B} = 0$, $\mathbf{n}\cdot(\nabla\times\mathbf{B}) = 0$, $\mathbf{n}\cdot(\nabla\times\nabla\times\mathbf{B}) = 0$, where $\mathbf{n}$ is the unit normal vector onto $\partial\Omega$. The first and second conditions, respectively, imply that the magnetic field and the electric current are confined in Ω. The third condition follows, for smooth solutions, from the first and second conditions and (13). If Ω is simply connected (like a ball), the boundary value problem (13)–(14) has nontrivial solution ($\mathbf{B}\not\equiv 0$ in Ω), only if at least one of $\lambda_\pm$ belongs to the point spectrum (set of eigenvalues) associated with the self-adjoint part of the curl operator. The point spectrum is a discrete set of real numbers [3], which we

denote by $\sigma_p(\text{curl})$. When Ω is multiply connected (like a toroid), however, λ_+ and λ_- can take arbitrary real values (and, more over, complex values) for (13)–(14) to have nontrivial solutions. In what follows, we consider a multiply connected domain Ω, and assume that $\lambda_\pm$ does not belong to σ_p (curl). If ν be the topological genus (first Betti number) of Ω (a simple toroidal domain has $\nu = 1$), the boundary value problem (13)–(14) will have 2ν degrees of freedom characterized as follows. Let S_ℓ ($\ell = 1, \cdots, \nu$) be the cuts of Ω such that $\Omega \setminus \cap_{\ell=1}^{\nu} S_\ell$ becomes a simply connected domain. On each cut, we define fluxes (currents)

$$\Phi_\ell^B = \int_{S_\ell} \mathbf{n} \cdot \mathbf{B}\, ds, \quad \Phi_\ell^J = \int_{S_\ell} \mathbf{n} \cdot (\nabla \times \mathbf{B})\, ds \quad (\ell = 1, \cdots, \nu), \tag{14}$$

where $\mathbf{n}$ is the unit normal vector onto S_ℓ and ds is the surface element on S_ℓ. By the divergence-free property of $\mathbf{B}$ and $\nabla \times \mathbf{B}$, together with (14), these fluxes are homotopy invariants with respect to S_ℓ, i.e., they are unchanged when S_ℓ is modified as far as its topological place is unchanged. For a given set of values $\Phi_1^B, \cdots, \Phi_\nu^B, \Phi_1^J, \cdots, \Phi_\nu^J$, we can solve (13)–(14) uniquely, and if at least one of these values is non-zero, the solution $\mathbf{B}$ is nontrivial ($\not\equiv 0$) [6]. This assertion can be derived by generalizing Theorem 2 of Yoshida-Giga [3]. Detailed proof will be given elsewhere.

The magnetic (velocity) fields described by (13) have two characteristic length scales ($\lambda_\pm^{-1}$) determined by the amounts of helicity and of generalized helicity [manifested through the values of (a, b) and hence of (α, β)] present in the system. The consequences of two inherent scales are best illustrated by studying the explicit solutions of (13) in simple coordinate systems. An obvious choice would have been the generalization of the well-known cartesian three-dimensional ABC flow. In spite of its tremendous interest, however, we concentrate, in this paper, on the one-dimensional cylindrical system. The cylinder should be seen as the limiting case of a large aspect ratio torus. We choose this example because the main message of this paper is to show that new and exciting, high-pressure confining, highly compact magnetic configurations can be created in the laboratory.

With boundary conditions, $B_z(r=0) = B_0 \equiv 1$, and $J_z = (\nabla \times B)_z(r=0) = s$, Eq. (13) yields

$$\begin{aligned} B_t &= (\lambda_+ - \lambda_-)^{-1} \left\{ (S - \lambda_-) J_0(\lambda_+ r) - (S - \Lambda_+) J_0(\lambda_- r) \right\}, \\ B_p &= (\lambda_+ - \lambda_-)^{-1} \left\{ (S - \lambda_-) J_1(\lambda_+ r) - (S - \Lambda_+) J_1(\lambda_- r) \right\}, \end{aligned} \tag{15}$$

where J_0 and J_1 are the ordinary Bessel functions, and we have chosen to write $B_{z(\theta)}$ as $B_{t(p)}$ where $t(p)$ stands for toroidal (poloidal). The rest of this paper is an exploration of (15).

Let us begin by deriving from (15), the well-known solutions for the reversed field pinch [2]; the solutions which are seen as a display of the remarkable property of the plasma to organize itself. We remind the reader that the B-T type of solutions ($\mathbf{V} \to 0$) are characterized by a single real valued scale parameter. From (10) and (11)), we see that the limit $b \gg |\nabla| \gg b^{-1}$ with $b \to \infty$ (or equivalently $\alpha \to b$, $b \to -b/a$) produces the desired result $\nabla \times \mathbf{B} = (\mathbf{B}/a)$. Notice, however, that this is a 'singular' limit of Eq. (11); the highest derivative term $\nabla \times (\nabla \times \mathbf{B})$ has been neglected!

The inescapable conclusion, therefore, is that even a small velocity field could cause 'singular' perturbations to the flowless magnetic field structures. For a large finite b (and moderate $|a| \ll |b|$), $\lambda_+ \simeq b$ and $\lambda_- \simeq a^{-1}$ implying that the magnetic field [Eq. (15)] varies not only on the moderate scale $|a|$ but also on the finer scale $|b|^{-1}$. This latter part, unless dissipated by finite resistivity, should emerge as a wavy part superimposed upon a moderately smooth part. This feature should be common to most magnetofluid equilibria and will be dealt with elsewhere.

There does exist a unique situation, however, when the fast component may disappear. If the boundary value $J_z(r=0) = s$ were exactly equal to λ_-, then (15) tells us that the highly-varying part of B_t and B_p will vanish, and we would be left with only the smooth part characterized by the scale a^{-1}. Here we see a glimpse of how a smooth magnetic field could emerge (as it seems to do, for example, in a reversed field pinch) in a system with nonzero flow. Aided, perhaps, by turbulence, the system may organize itself in such a way that the

central current acquires the needed correct value. In Fig. 1, we can see an example of such a smooth equilibrium. The size of the system is $10\lambda_i$ ($\sim 25\,\mathrm{cm}$ for $n \sim 10^{14}$). The velocity fields are small but not negligible. This configuration produces very insignificant pressure confinement by the Bernoulli mechanism.

The most interesting and novel aspects of (15) emerge, when the term $\nabla \times (\nabla \times \mathbf{B})$ plays a fundamental role; that is when both terms of (15) are in full play. (This happens when $|B|$ and $|V|$ are comparable.) For this regime, we shall present two representative cases. The system size is taken to be $2\lambda_i$, and the edge $\beta_i(r = 2) = 0$. The parameters were so chosen that the toroidal current profile follows the beta profile (so that the current can be ohmically driven), and $V_t(r = 0) = 0$. The latter was done to optimize central beta. In the first case [Fig. 2], $\lambda_\pm$ are real, while in the second case [Fig. 3], they are a complex conjugate pair (even for complex $\lambda_+, \lambda_- = \lambda_+^*$, the physical quantities $B_t,\ B_p$... remain real). For both of these examples, we notice:

1. the magnetic field increase away from the center [$B_t(r = 0) = 1$];

2. The magnitude of the velocity field (refer to the right vertical axis], normalized to the central Alfvén speed, is sizeable, and is monotonically increasing;

3. the configurations through the Bernoulli mechanism [Eq. (11))]; produce excellent and almost identical pressure confinements [Fig. 4] with a central $\beta_i \simeq 1.25$.

Notice that for the Bernoulli condition Eq. (11), the standard one–fluid model, is perfectly adequate if the flow energy is not neglected. However, for the Bernoulli mechanism to provide plasma confinement, the velocity fields must increase away from the plasma center. This becomes generally possible when the term proportional to $\mathbf{J} \sim \boldsymbol{\nabla} \times \mathbf{B}$ (absent in one–fluid models) is retained in $\mathbf{U}_1$ [Eq. (6)].

It follows then, that by the strong coupling of the fluid-kinetic and magnetic aspects of the plasma, a highly confining, fully diamagnetic (fields in plasma are everywhere smaller

than the edge plasmas), and everywhere minimum $|B|$ configuration [this will persist even in a large aspect ratio torus as long as the $1/R$ variation is less than the difference $B_{\text{edge}} - B(0)$] can naturally emerge for a very compact plasma.

These highly compact diamagnetic structures with scale length of a few λ_i, which require strong plasma flows, deserve a thorough investigation. Being the states of lowest free energy, they are expected to be MHD stable. We believe that these configurations point to a possible new path in our quest for controlled thermonuclear fusion. A tentative theoretician's machine-design is being work on.

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[4] For a vorticity $\mathbf{\Omega}$, the "helicity" is defined as $h_j = \pm\frac{1}{2}\int(\mathrm{curl}^{-1}\mathbf{\Omega})\cdot\mathbf{\Omega}d^3x$, where curl^{-1} is the inverse operator of the curl that is represented by the Biot-Savart integral. We choose the sign in an appropriate way.

[5] L. C. Steinhauer and A. Ishida, Phys. Rev. Lett. **79**, 3423 (1997).

[6] If one of λ_+ and λ_- belongs to the point spectrum $\sigma_p(\mathrm{curl})$, we cannot assign independent values to the fluxes Φ^B_ℓ, Φ^J_ℓ $(\ell = 1, c\ldots\nu)$. For example, if $\lambda_+ \in \sigma_p(\mathrm{curl})$, we must assume $\Phi^J_\ell - \lambda_-\Phi^B_\ell = 0$ $(\ell = 1, c\ldots\nu)$. Therefore, the degree of freedom becomes ν. If both λ_+ and λ_- belong to $\sigma_p(\mathrm{curl})$, we must assume $\Phi^J_\ell = \Phi^B_\ell = 0$ $(\ell = 1, c\ldots\nu)$, and hence, the degree of freedom is zero. Indeed, the nontrivial solution is given by the two eigenfunctions of the self-adjoint curl operator, which carry no flux through every cross-section of the domain.

FIGURE CAPTIONS

FIG. 1. A radial plot of B_t, B_p (left vertical axis) and V_t, V_p (right vertical axis). The system size is $10\lambda_i$ and $B_t(r=0)=1$. The parameters are $a=-4.4,\ b=60$, and $s=.2440$, implying $\lambda_+=59.98$ and $\lambda_-=.2440$. s was chosen to be equal to λ_- to eliminate the fast-varying component.

FIG. 2. A radial plot of B_t, B_p (left vertical axis) and V_t, V_p (right vertical axis). The system size is $2\lambda_i$, $B_t(r=0)$, the velocity is measured in units of the central Alfvén speed. The parameters are $a=-1.9,\ b=1.5$, and $s=0.52$, implying $\lambda_+=-0.64$ and $\lambda_-=0.32$.

FIG. 3. A radial plot of B_t, B_p (left vertical axis) and V_t, V_p (right vertical axis). The system size is $2\lambda_i$, $B_t(r=0)=1$, the velocity is measured in units of central Alfvén speed. The parameters are $a=-2,\ b=1.4$, and $s=0.5$, implying $\lambda_\pm=-0.45\pm 0.31i$.

FIG. 4. A radial plot of β_i for cases displayed in Figs. 2 and 3. The toroidal current (not shown) follows the pressure profile.