Asymptotic Persistence of collective modes in shear flows

Swadesh M. Mahajan
Institute for Fusion Studies, The University of Texas at Austin, Austin, Texas, USA and Abdus Salam International Centre for Theoretical Physics, Trieste, Italy

Andria D. Rogava
Centre for Plasma Astrophysics, KULeuven, Heverlee, Belgium, and Abdus Salam International Centre for Theoretical Physics, Trieste, Italy

Abstract

A new nonasymptotic method is presented that reveals an unexpected richness in the spectrum of fluctuations sustained by a shear flow with nontrivial arbitrary mean kinematics. The principal characteristic of the revealed exotic collective modes is their asymptotic persistence. "Echoing" as well as unstable (including parametrically driven) solutions are displayed. Further areas of application, for both the method and the new physics, are outlined.

It is well known that the predictions of the classical asymptotic stability theory often fail to match the relevant experimental results for a variety of shear flows. This failure, generally attributed to the non self-adjoint character of the governing equations [1,2], suggests that alternative nonasymptotic routes to solve the initial value problem must be explored. A method originally proposed by Lord Kelvin [3] to study the nonexponential temporal evolution of fluctuations has recently been exploited to unearth fascinating aspects of fluctuation dynamics both in neutral fluids [4-6] and in plasmas [7-9].

In this Letter, we follow the spirit of the Kelvin approach, but extend the domain of investigation to shear flows with arbitrary mean kinematics. We propose a simple method which reduces the initial value problem to a set of manageable ordinary differential equations in time. The nontrivial velocity inhomogeneity imparts an immense richness to the temporal behaviour of the perturbations. The transient growths, seen in simple
parallel shear flows, can either become asymptotic growths or can appear in a periodic pattern in a sheared flow.

After giving a brief synopsis of the general approach, we will discuss in some detail the relatively simple, two-dimensional hydrodynamic system. An analysis of this example illustrates the salient features of new physics without much algebraic complication.

In the close neighborhood of a point \( A(x_0, y_0, z_0) \) \(|x - x_0|/|x_0|\ll 1\), etc., the spatial variation of a general mean velocity field \( U(x, y, z) \) could be approximated by the linear terms in its Taylor expansion. Then a set of nine constants \( U_{i,k}(x_0, y_0, z_0) \) \((i, k = x, y, z)\) forming the Shear Matrix \( S \):

\[
S = \begin{pmatrix}
U_{x,x} & U_{x,y} & U_{x,z} \\
U_{y,x} & U_{y,y} & U_{y,z} \\
U_{z,x} & U_{z,y} & U_{z,z}
\end{pmatrix} = \begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{pmatrix},
\]

will fully characterize the flow in the region. For flows with \textit{homogeneous} equilibrium density, the velocity field is divergence-free \( \nabla \cdot U = 0 \) which translates into the constraint that the Shear Matrix is traceless. The components of the mean velocity may, now, be expressed as \( U_i(x, y, z) \simeq U_{0i} + a_{ik}x_k \) where \( U_{0i} = U_i(x_0, y_0, z_0) \) and \( x_i \equiv x_i - x_{0i} \). This will convert the linearized convective derivative operating on the perturbations to \( D u_i + a_{ik}u_k \), where \( D \) stands for the spatially inhomogeneous operator \( D \equiv \partial_t + U_i(x, y, z)\partial_i \).

Our first task is to find a transformation that will annihilate the spatial dependence in the operator \( D \). For any fluctuation \( F(x, y, z; t) \), the ansatz

\[
F(x, y, z; t) \equiv \hat{F}(k(t), t) \exp[i\varphi], \quad \varphi \equiv k_i(t)x_i - U_{0i} \int k_i(t)dt,
\]

seems to do the job provided the wave vector \( k \) acquires the time dependence given by

\[
\partial_t k + S^T \cdot k = 0,
\]

where \( S^T \) is the transposed shear matrix. These equations, even in their the most general form, have simple analytic solutions that embrace all possible kinds of background flows. For example, the plane Couette flow, corresponding to \( a_{12} = A \) and all other \( a_{ik} = 0 \), leads to the obvious solutions: 
\( k_x(t) = k_x(0) \), \( k_z(t) = k_z(0) \), and \( k_y(t) = k_y(0) - Atk_x(0) \). For
shear flows with more sophisticated kinematics, the temporal dependence of $k(t)$ would be much more complex encompassing both exponentially evolving and periodic behaviour.

The convective derivative now becomes an ordinary derivative in time: $D F = \exp[i \varphi] \partial_t F$. The resulting nonautonomous ordinary differential equations (ODE’s) can be analyzed to ‘smoke out’ some overlooked modes of collective behaviour, excited by nontrivial velocity fields.

The scope of this method is best demonstrated by examining the fluctuation dynamics of a two-dimensional hydrodynamic system with a homogeneous mean density. With the ambient flow velocity given by $U(x,y) \equiv U_x(x,y)e_x + U_y(x,y)e_y$ with $a_{11} = -a_{22} \equiv \sigma$, $a_{12} \equiv a$ and $a_{21} \equiv b$, we can readily derive the set of linearized equations governing the evolution of small-scale perturbations in this flow. On applying ansatz (2), we convert the system to the set of first order ODE’s:

\begin{align}
\partial_\tau \rho' + \rho_0 (\partial_x u_x + \partial_y u_y) &= 0, \quad (4a) \\
\partial_\tau u_x + \sigma u_x + a u_y &= -c_s^2 \partial_x (\rho'/\rho_0), \quad (4b) \\
\partial_\tau u_y + b u_x - \sigma u_y &= -c_s^2 \partial_y (\rho'/\rho_0), \quad (4c)
\end{align}

with the components of the variable wave vector obeying

\begin{align}
\partial_\tau K_x + \varepsilon K_x + R_1 v_y &= -K_x \varrho, \quad (5a) \\
\partial_\tau v_x + \varepsilon v_x + R_1 v_y &= -K_x \varrho, \quad (5b) \\
\partial_\tau v_y + R_2 v_y - \varepsilon v_y &= -K_y \varrho, \quad (5c)
\end{align}

where we have used the dimensioless notation: $\tau \equiv c_s k_x(0)t$, $\varepsilon \equiv \sigma/c_s k_x(0)$, $R_1 \equiv a/c_s k_x(0)$, $R_2 \equiv b/c_s k_x(0)$, $K_{x,y} \equiv k_{x,y}/k_x(0)$, $\varrho \equiv i(\rho'/\rho_0)$, and $v_{x,y} \equiv u_{x,y}/c_s$. 

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Equations (5) and (6) contain two conserved quantities: the first one \( \Delta \equiv K_y \partial_x K_x - K_x \partial_x K_y \) is associated with (6), while the second \( C \equiv K_y v_x - K_x v_y + (R_1 - R_2) \Phi \) follows from (5) and (6) and links the physical variables of the system with one another.

Defining \( K^2 \equiv K_x^2 + K_y^2 \), and \( \varrho \equiv K \Psi \) we can derive from (5-6), the following explicit second-order ODE [\( \Lambda^2 \equiv R_1 R_2 + \sigma^2 \)]:

\[
\partial_r^2 \Psi + \left( K^2 - \Lambda^2 + \frac{3 \Delta^2}{\Lambda^2} \right) \Psi = \frac{2 \mathcal{C} \Delta}{\Lambda^3},
\]

which determines the time behaviour of the perturbations.

The constant of integration \( \mathcal{C} \), appearing in the inhomogeneous term, can be interpreted as the source of vortical fluctuations. If one considers (7) as the oscillator equation then the inhomogeneity term represents time-dependent external force, while the function in square brackets stands for the square of the oscillator’s time-dependent effective frequency. The entire time dependence of both coefficients in Eq. (7) comes through \( K^2 \). The actual form of \( K^2 \) crucially depends on the value and sign of the parameter \( \Lambda^2 \). In particular, here are the following three cases:

(a) \( \Lambda^2 = 0 \). In this simple case \( K^2 = K^2(0) + \partial_r K^2(0) \tau + (R_1 - R_2) \Delta \tau^2 \). This form of \( K \) pertains to the extensively studied case of plane Couette flow.

(b) \( \Lambda^2 > 0 \). Here \( K^2 = \delta + A \operatorname{Cosh}(2 \Lambda \tau + \psi_0) \) with \( \delta \equiv -(R_1 - R_2) \Delta / 2 \Lambda^2 \) and \( A, \psi_0 \), determined by initial values of the wave numbers. This case allows a simple asymptotic analysis. For \( \Lambda \tau \to \infty \), \( K^2 \approx a e^{2 \Lambda \tau} \), and (7) may be approximated by a Bessel equation

\[
\partial_r^2 \Psi + \left[ a e^{2 \Lambda \tau} - \Lambda^2 \right] \Psi = 0,
\]

with the solution

\[
\Psi = J_1 \left( a^{1/2} \frac{e^{\Lambda \tau}}{\Lambda} \right) \to \left( \frac{2 \Lambda}{e^{\Lambda \tau} \pi a^{1/2}} \right)^{1/2} e^{-\Lambda \tau/2} \cos \left[ \frac{a^{1/2}}{\Lambda} e^{\Lambda \tau} - \frac{3 \pi}{4} \right],
\]

leading to an exponential growth for the physical density perturbation:

\[
\varrho = K \Psi \approx e^{\Lambda \tau/2} \cos \left[ \frac{a^{1/2}}{\Lambda} e^{\Lambda \tau} - \frac{3 \pi}{4} \right].
\]
The numerical solution displayed in Fig. 1, where we have plotted $\varrho$ as a function of time, clearly confirms the predictions of Eq. (9). It must be stressed, however, that for this class of flows ($K^2 \rightarrow e^{2\lambda t}$), the viscous damping will tend to kick in due time and will eventually damp the mode. This is, indeed, found to be the case, when viscosity is incorporated into the original setup.

(c) $-\omega^2 \equiv \Lambda^2 < 0$. The solution for $K^2$ is, now, periodic: $K^2 = \delta + B\cos(2\omega \tau + \phi_0)$, with $B$, $\phi_0$, depending, again, on the initial values of the wave numbers. Equation (7) acquires the form of an inhomogeneous Hill equation and its numerical analysis reveals the following three subclasses of solutions:

1. **Vortical solutions** interacting with the Acoustic wave. These solutions are expected whenever $C$ is large and the frequency $\omega \equiv [-\left(R_1 R_2 + \epsilon^2\right)]^{1/2}$ is small. These are new, very peculiar versions of the Kelvin modes [3,5,9] with the difference that the new vortex (Fig. 2) is not a transient, it does have a transient growth, but it repeats with the periodicity $2\omega$. The vortex forms at a given time, gives up its energy back to the flow and the acoustic oscillations, and echoes back after a time $T = \pi/\omega$. We call this picturesque phenomenon *Asymptotic Persistence*, and we believe it is a hallmark of the multidimensional shear flows.

2. **Unstable Acoustic Waves**: It is well-known that equations with periodic coefficients allow unstable solutions in certain ranges of parameters. Notice that for $\Delta = 0$, (7) becomes a *Mathieu* equation whose solutions are very well known. It is reasonable to expect that even for $\Delta \neq 0$, the solutions will retain the peculiarities of the Mathieu solution, e.g., the regions of instability. Numerical solutions show that (7) has a number of unstable regions in the parameter space defined by $[\omega, R_2$ and $K_y(0)]$. One such parametrically unstable solution is displayed in Fig. 3, where the perturbation is exponentially growing in time.

3. **Stable Acoustic Waves**: the basic solutions of Eq. (7) are some combinations of the vortical and acoustic type of perturbations. For a variety of initial conditions,
acoustic type will dominate the vortical type. In this case the frequency as well as
the amplitude of the mode changes periodically. A pair of typical plots is shown
on Figures 4a and 4b. These figures are drawn for the same set of parameters
as Fig. 3, but with the difference that $K_y(0) = 10$ for Fig. 4a, and $K_y(0) = 6$
for Fig. 4b. Comparison of these solutions with the unstable one shows that the
acoustic waves become unstable for relatively narrow and restricted ranges of the
system parameters. This feature clearly reflects on the parametric nature of the
solution displayed in Fig. 3.

For all the cases discussed, the fluctuations do not go away; they are not transient,
they amplify and persist. Similar behaviour is seen in plasmas.

It should be remembered that not all $U(x,y,z)$’s are possible; the relevant flow must
satisfy the stationary state equations for a given physical system. In this Letter, we
consciously did not circumscribe the range of possible mean velocity fields. We simply
concentrated on developing an approach for studying the initial value problem valid for
arbitrary ambient shear flows.

It seems reasonable and tempting to argue that the phenomenon of asymptotic persist-
tence of fluctuations is quite general; it will manifest itself in all systems (both in neutral
fluids and in plasmas) with sheared background mean flows. Since the sheared flows are
being recognised as major determinants of the fate of fusion plasmas, we believe that
this shear generated and maintained fluctuation spectrum could be a crucial new element
in understanding the anomalous transport in magnetic-confinement experiments. Astro-
physical systems stand out as yet another class of examples of shear flows with complex
kinematics, where asymptotically persisting fluctuations must lead to the delineation of
processes with direct observational consequences.

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REFERENCES

3. Lord Kelvin (W. Thomson), Phil. Mag. 24, Ser. 5, 188 (1887).

Figure Captions

Fig. 1 Density perturbation \( \varrho(\tau) = \mathcal{K}(\tau)\Psi(\tau) \), for \( R_1 = 0.1, R_2 = 0.05, \varepsilon = 0, \mathcal{K}_y(0) = 0.1, C = 0, \Psi(0) = 10^{-2} \), and \( \partial_\tau \Psi(0) = 0 \).

Fig. 2 Asymptotic persistence of “echoing” Kelvin vortices for density perturbation \( \varrho(\tau) \) when \( R_1 = 0.1, R_2 = -0.01, \varepsilon = 0, \mathcal{K}_y(0) = 1, C = 1, \Psi(0) = 3.87 \times 10^{-2} \), and \( \partial_\tau \Psi(0) = 8.66 \times 10^{-3} \).
Fig. 3 Parametrically unstable acoustic wave solution plotted for $\varrho(\tau)$ when $R_1 = 2.0$, $R_2 = -2 \times 10^{-2}$, $\varepsilon = 0$, $\mathcal{K}_y(0) = 8$, $\mathcal{C} = 1$, $\Psi(0) = 10^{-4}$, and $\partial_{\tau} \Psi(0) = 0$.

Fig. 4 Stable acoustic wave solutions presented by $\varrho(\tau)$ graphs, for the same sample of system parameters as on Fig. 3, except that here $\mathcal{K}_y(0) = 10$ (on Fig. 4a) and $\mathcal{K}_y(0) = 6$ (on Fig. 4b).