Asymptotic Persistence of collective modes in shear flows

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Abstract

A new nonasymptotic method is presented that reveals an unexpected richness in the spectrum of fluctuations sustained by a shear flow with nontrivial *arbitrary* mean kinematics. The prinicipal characterstic of the revealed exotic collective modes is their *asymptotic persistence*. "Echoeing" as well as unstable (including parametrically driven) solutions are displayed. Further areas of application, for both the method and the new physics, are outlined.

It is well known that the predictions of the classical asymptotic stability theory often fail to match the relevant experimental results for a variety of shear flows. This failure, genearlly attributed to the non self-adjoint character of the governing equations [1,2], suggests that alternative nonasymptotic routes to solve the initial value problem must be explored. A method originally proposed by Lord Kelvin [3] to study the nonexponential temporal evolution of fluctuations has recently been exploited to unearth fascinating aspects of fluctuation dynamics both in neutral fluids [4-6] and in plasmas [7-9].

In this *Letter*, we follow the spirit of the Kelvin approach, but extend the domain of investigation to shear flows with arbitrary mean kinematics. We propose a simple method which reduces the initial value problem to a set of managable ordinary differential equations in time. The nontrivial velocity inhomogeneity imparts an immense richness to the temporal behaviour of the perturbations. The transient growths, seen in simple

parallel shear flows, can either become asymptotic growths or can appear in a periodic pattern in a sheared flow.

After giving a brief synopsis of the general approach, we will discuss in some detail the relatively simple, two-dimensional hydrodynamic system. An analysis of this example illustrates the salient features of new physics without much algebraic complication.

In the close neighborhood of a point $A(x_0, y_0, z_0)$ ($|x - x_0|/|x_0| \ll 1$, etc.), the spatial variation of a general mean velocity field $\mathbf{U}(x, y, z)$ could be approximated by the linear terms in its Taylor expansion. Then a set of nine constants $U_{i,k}(x_0, y_0, z_0)$ $(i, k = x, y, z)$ forming the *Shear Matrix* $\mathcal{S}$:

$$\mathcal{S} \equiv \begin{pmatrix} U_{x,x} & U_{x,y} & U_{x,z} \\ U_{y,x} & U_{y,y} & U_{y,z} \\ U_{z,x} & U_{z,y} & U_{z,z} \end{pmatrix} \equiv \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \tag{1}$$

will fully characterize the flow in the region. For flows with *homogeneous* equilibrium density, the velocity field is divergence-free $\nabla \cdot \mathbf{U} = 0$ which translates into the constraint that the Shear Matrix is traceless. The components of the mean velocity may, now, be expressed as $U_i(x, y, z) \simeq U_{0i} + a_{ik}x_k$ where $U_{0i} = U_i(x_0, y_0, z_0)$ and $x_i \equiv x_i - x_{0i}$. This will convert the linearized convective derivative operating on the perturbations to $\mathcal{D}u_i + a_{ik}u_k$, where $\mathcal{D}$ stands for the spatially inhomogeneous operator $\mathcal{D} \equiv \partial_t + U_i(x, y, z)\partial_i$.

Our first task is to find a transformation that will annihilate the spatial dependence in the operator $\mathcal{D}$. For any fluctuation $F(x, y, z; t)$, the ansatz

$$F(x, y, z; t) \equiv \hat{F}(\mathbf{k}(t), t) \exp[i\varphi], \qquad \varphi \equiv k_i(t)x_i - U_{0i} \int k_i(t)dt, \tag{2}$$

seems to do the job provided the wave vector $\mathbf{k}$ acquires the time dependence given by

$$\partial_t \mathbf{k} + \mathcal{S}^T \cdot \mathbf{k} = 0, \tag{3}$$

where $\mathcal{S}^T$ is the transposed shear matrix. These equations, even in their the most general form, have simple analytic solutions that embrace all possible kinds of background flows. For example, the plane Couette flow, corresponding to $a_{12} = A$ and all other $a_{ik} = 0$, leads to the obvious solutions: $k_x(t) = k_x(0)$, $k_z(t) = k_z(0)$, and $k_y(t) = k_y(0) - Atk_x(0)$. For

shear flows with more sophisticated kinematics, the temporal dependence of $\mathbf{k}(t)$ would be much more complex encompassing both exponentially evolving and periodic behaviour.

The convective derivative now becomes an ordinary derivative in time: $\mathcal{D}F = \exp[i\varphi]\partial_t \hat{F}$. The resulting nonautonomous ordinary differential equations (ODE's) can be analyzed to 'smoke out' some overlooked modes of collective behaviour, excited by nontrivial velocity fields.

The scope of this method is best demonstrated by examining the fluctuation dynamics of a two-dimensional hydrodynamic system with a homogeneous mean density. With the ambient flow velocity given by $\mathbf{U}(x,y) \equiv U_x(x,y)\mathbf{e}_x + U_y(x,y)\mathbf{e}_y$ with $a_{11} = -a_{22} \equiv \sigma$, $a_{12} \equiv a$ and $a_{21} \equiv b$, we can readily derive the set of linearized equations

$$\mathcal{D}\rho' + \rho_0(\partial_x u_x + \partial_y u_y) = 0, \tag{4a}$$

$$\mathcal{D}u_x + \sigma u_x + a u_y = -c_s^2 \partial_x(\rho'/\rho_0), \tag{4b}$$

$$\mathcal{D}u_y + b u_x - \sigma u_y = -c_s^2 \partial_y(\rho'/\rho_0), \tag{4c}$$

governing the evolution of small-scale perturbations in this flow. On applying ansatz (2), we convert the system to the set of first order ODE's:

$$\partial_\tau \varrho = \mathcal{K}_x v_x + \mathcal{K}_y v_y, \tag{5a}$$

$$\partial_\tau v_x + \varepsilon v_x + R_1 v_y = -\mathcal{K}_x \varrho, \tag{5b}$$

$$\partial_\tau v_y + R_2 v_y - \varepsilon v_y = -\mathcal{K}_y \varrho. \tag{5c}$$

with the components of the variable wave vector obeying

$$\partial_\tau \mathcal{K}_x + \varepsilon \mathcal{K}_x + R_2 \mathcal{K}_y = 0, \tag{6a}$$

$$\partial_\tau \mathcal{K}_y + R_1 \mathcal{K}_x - \varepsilon \mathcal{K}_y = 0, \tag{6b}$$

where we have used the dimensioless notation: $\tau \equiv c_s k_x(0)t$, $\varepsilon \equiv \sigma/c_s k_x(0)$, $R_1 \equiv a/c_s k_x(0)$, $R_2 \equiv b/c_s k_x(0)$, $\mathcal{K}_{x,y} \equiv k_{x,y}/k_x(0)$, $\varrho \equiv i(\rho'/\rho_0)$, and $v_{x,y} \equiv \hat{u}_{x,y}/c_s$.

Equations (5) and (6) contain *two* conserved quantities: the first one $\Delta \equiv \mathcal{K}_y \partial_\tau \mathcal{K}_x - \mathcal{K}_x \partial_\tau \mathcal{K}_y$ is associated with (6), while the second $\mathcal{C} \equiv \mathcal{K}_y v_x - \mathcal{K}_x v_y + (R_1 - R_2)\varrho$ follows from (5) and (6) and links the physical variables of the system with one another.

Defining $\mathcal{K}^2 \equiv \mathcal{K}_x^2 + \mathcal{K}_y^2$, and $\varrho \equiv \mathcal{K}\Psi$ we can derive from (5-6), the following explicit second-order ODE $[\Lambda^2 \equiv R_1 R_2 + \sigma^2]$:

$$\partial_\tau^2 \Psi + \left[\mathcal{K}^2 - \Lambda^2 + \frac{3\Delta^2}{\mathcal{K}^4}\right]\Psi = \frac{2\mathcal{C}\Delta}{\mathcal{K}^3}, \tag{7}$$

which determines the time behaviour of the perturbations.

The constant of integration $\mathcal{C}$, appearing in the inhomogeneous term, can be interpreted as the source of vortical fluctuations. If one considers (7) as the oscillator equation then the inhomogeneity term represents time-dependent external force, while the function in square brackets stands for the square of the oscillator's time-dependent effective frequency. The entire time dependence of both coefficients in Eq. (7) comes through $\mathcal{K}^2$. The actual form of $\mathcal{K}^2$ crucially depends on the value and sign of the parameter Λ^2. In particular, here are the following three cases:

(a) $\Lambda^2 = 0$. In this simple case $\mathcal{K}^2 = \mathcal{K}^2(0) + \partial_\tau \mathcal{K}^2(0)\tau + (R_1 - R_2)\Delta\tau^2$. This form of $\mathcal{K}$ pertains to the extensively studied case of plane Couette flow.

(b) $\Lambda^2 > 0$. Here $\mathcal{K}^2 = \delta + \mathcal{A}\mathrm{Cosh}(2\Lambda\tau + \psi_0)$ with $\delta \equiv -(R_1 - R_2)\Delta/2\Lambda^2$ and $\mathcal{A}$, ψ_0, determined by initial values of the wave numbers. This case allows a simple asymptotic analysis. For $\Lambda\tau \to \infty$, $\mathcal{K}^2 \approx a e^{2\Lambda\tau}$, and (7) may be approximated by a *Bessel* equation

$$\partial_\tau^2 \Psi + \left[a e^{2\Lambda\tau} - \Lambda^2\right]\Psi = 0, \tag{8}$$

with the solution

$$\Psi = J_1\left(\frac{a^{1/2}}{\Lambda} e^{\Lambda\tau}\right) \to \left(\frac{2\Lambda}{\pi a^{1/2}}\right)^{1/2} e^{-\Lambda\tau/2} \cos\left[\frac{a^{1/2}}{\Lambda} e^{\Lambda\tau} - \frac{3\pi}{4}\right],$$

leading to an exponential growth for the physical density perturbation:

$$\varrho = \mathcal{K}\Psi \approx e^{\Lambda\tau/2}\mathrm{Cos}\left[\frac{a^{1/2}}{\Lambda} e^{\Lambda\tau} - \frac{3\pi}{4}\right]. \tag{9}$$

The numerical solution displayed in Fig. 1, where we have plotted ϱ as a function of time, clearly confirms the predictions of Eq. (9). It must be stressed, however, that for this class of flows ($\mathcal{K}^2 \to e^{2\lambda\tau}$), the viscous damping will tend to kick in due time and will eventually damp the mode. This is, indeed, found to be the case, when viscosity is incorporated into the original setup.

(c) $-\omega^2 \equiv \Lambda^2 < 0$. The solution for $\mathcal{K}^2$ is, now, periodic: $\mathcal{K}^2 = \delta + \mathcal{B}\mathrm{Cos}(2\omega\tau + \phi_0)$, with $\mathcal{B}$, ϕ_0, depending, again, on the initial values of the wave numbers. Equation (7) acquires the from of an inhomogeneous Hill equation and its numerical analysis reveals the following three subclasses of solutions:

1. *Vortical solutions* interacting with the Acoustic wave. These solutions are expected whenever $\mathcal{C}$ is large and the frequency $\omega \equiv [-(R_1 R_2 + \varepsilon^2)]^{1/2}$ is small. These are new, very peculiar versions of the Kelvin modes [3,5,9] with the difference that the new vortex (Fig. 2) is not a transient, it does have a transient growth, but it repeats with the periodicity 2ω. The vortex forms at a given time, gives up its energy back to the flow and the acoustic oscillations, and echoes back after a time $T = \pi/\omega$. We call this picturesque phenomenon *Asymptotic Persistence*, and we believe it is a hallmark of the multidimensional shear flows.

2. *Unstable Acoustic Waves:* It is well-known that equations with periodic coeffcients allow unstable solutions in certain ranges of parameters. Notice that for $\Delta = 0$, (7) becomes a *Mathieu* equation whose solutions are very well known. It is reasonable to expect that even for $\Delta \neq 0$, the solutions will retain the peculiarities of the Mathieu solution, e.g., the regions of instability. Numerical solutions show that (7) has a number of unstable regions in the parameter space defined by [ω, R_2 and $\mathcal{K}_y(0)$]. One such parametrically unstable solution is displayed in Fig. 3, where the perturbation is exponentially growing in time.

3. *Stable Acoustic Waves:* the basic solutions of Eq. (7) are some combinations of the vortical and acoustic type of perturbations. For a variety of initial conditions,

acoustic type will dominate the vortical type. In this case the frequency as well as the amplitude of the mode changes periodically. A pair of typical plots is shown on Figures 4a and 4b. These figures are drawn for the same set of parameters as Fig. 3. but with the difference that $\mathcal{K}_y(0) = 10$ for Fig. 4a, and $\mathcal{K}_y(0) = 6$ for Fig. 4b. Comparison of these solutions with the unstable one shows that the acoustic waves become unstable for relatively narrow and restricted ranges of the system parameters. This feature clearly reflects on the parametric nature of the solution displayed in Fig. 3.

For all the cases discussed, the fluctuations do not go away; they are not *transient*, they amplify and persist. Similar behaviour is seen in plasmas.

It should be remembered that not *all* $\mathbf{U}(x, y, z)$'s are possible; the relevant flow must satisfy the stationary state equations for a given physical system. In this *Letter*, we consciously *did not* circumscribe the range of possible mean velocity fields. We simply concentrated on developing an approach for studying the initial value problem valid for arbitrary ambient shear flows.

It seems reasonable and tempting to argue that the phenomenon of asymptotic persistence of fluctuations is quite general; it will manifest itself in all systems (both in neutral fluids and in plasmas) with sheared background mean flows. Since the sheared flows are being recognised as major determinants of the fate of fusion plasmas, we believe that this shear generated and maintained fluctuation specturm could be a crucial new element in understanding the anamolous transport in magnetic-confinement experiments. Astrophysical systems stand out as yet another class of examples of shear flows with complex kinematics, where asymptotically persisiting fluctuations must lead to the delineation of processes with direct observational consequences.

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Figure Captions

Fig. 1 Density perturbation $\varrho(\tau) = \mathcal{K}(\tau)\Psi(\tau)$, for $R_1 = 0.1$, $R_2 = 0.05$, $\varepsilon = 0$, $\mathcal{K}_y(0) = 0.1$, $\mathcal{C} = 0$, $\Psi(0) = 10^{-2}$, and $\partial_\tau \Psi(0) = 0$.

Fig. 2 Asymptotic persistence of "echoeing" Kelvin vortices for density perturbation $\varrho(\tau)$ when $R_1 = 0.1$, $R_2 = -0.01$, $\varepsilon = 0$, $\mathcal{K}_y(0) = 1$, $\mathcal{C} = 1$, $\Psi(0) = 3.87\times10^{-2}$, and $\partial_\tau \Psi(0) = 8.66\times10^{-3}$.

Fig. 3 Parametrically unstable acoustic wave solution plotted for $\varrho(\tau)$ when $R_1 = 2.0$, $R_2 = -2\times10^{-2}$, $\varepsilon = 0$, $\mathcal{K}_y(0) = 8$, $\mathcal{C} = 1$, $\Psi(0) = 10^{-4}$, and $\partial_\tau\Psi(0) = 0$.

Fig. 4 Stable acoustic wave solutions presented by $\varrho(\tau)$ graphs, for the same sample of system parameters as on Fig. 3, except that here $\mathcal{K}_y(0) = 10$ (on Fig. 4a) and $\mathcal{K}_y(0) = 6$ (on Fig. 4b).