# Strong echo effect and nonlinear transient growth in shear flows

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The nonlinear interaction of two disturbances excited successively in a two-dimensional Couette flow is shown to lead to a transient energy growth. This phenomenon, which is called echo effect and exists in several other physical systems, is interesting because the energy growth appears long after the energy associated with the original disturbances has decayed. Here, the echo effect is studied analytically and numerically in a situation where the nonlinear response has the same order of magnitude as the two excitations. A system of amplitude equations describing the nonlinear interactions between three sheared modes is derived and employed to examine the physical mechanism of the echo. The qualitative validity of this system is confirmed by numerical simulations. The influence of viscous dissipation on the echo effect is also considered.

## 1 Introduction

It is now widely recognized that the kinetic energy of an infinitesimal disturbance in a shear flow can be significantly amplified even if the flow is spectrally stable. Such an amplification, which is then followed by a decay in the long-time limit and is generally referred to as transient growth, has originally been identified by Orr [14] in a two-dimensional Couette flow. In recent years, it has been studied in connection with various problems in hydrodynamics stability (e.g. [6, 4, 15, 18, 2]) and in meteorology (see the review [7]).

The physical mechanism behind transient growth is particularly transparent for two-dimensional incompressible flows, which are governed by a vorticity equation. In such flows, the disturbance vorticity is advected and sheared by the basic-flow velocity, and the disturbance energy crucially depends on the phase mixing that appears when the streamfunction is derived from the vorticity. In the long-time limit, the scale of the vorticity field systematically decreases with time, leading to an enhanced phase mixing and thus to a decrease of the disturbance streamfunction and energy [3] — this decrease is sometimes referred to as Landau damping, by analogy with the similar phenomenon of decrease of electrostatic energy in plasmas [12]. However, if the initial disturbance is dominated by vorticity lines tilted against

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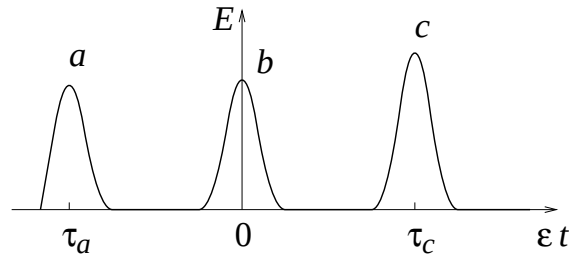


Figure 1: Schematic representation of the echo effect showing the time evolution of the disturbance energy. At $t = \epsilon^{-1}\tau_a$ and $t = 0$, the shear flow is disturbed by spatialy-periodic excitations. The energy of the response to those excitations decays through Landau damping, but because of nonlinear effects, a third peak — the echo — appears in the energy after a time $t = \epsilon^{-1}\tau_c$.

the shear, the phase mixing does not evolve monotonically: it temporarily cancels when, after some time, the vorticity lines are perpendicular to the shear, leading to a peak in the disturbance streamfunction and energy which is the mark of transient growth.

A crucial feature of (linear) transient growth is the fact that it requires very specific initial conditions: the initial disturbances must be coherent, have a very small scale and a specific orientation for the amplification to be significant [16]. In this paper, we discuss a different mechanism that also leads to a transient amplification of the disturbance, but does not require such a small-scale excitation of the flow. This mechanism, called echo effect, is essentially nonlinear; it is characterized by a transient growth of the disturbance energy following two successive excitations, as illustrated schematically in figure 1. The two successive (spatially-periodic, impulsive) excitations, denoted by $a$ and $b$, are applied at time $t = \epsilon^{-1}\tau_a < 0$ and $t = 0$, respectively. Through nonlinear interaction, they produce a delayed response, denoted by $c$, the echo. The energy peak associated with the echo is isolated, because it appears when the direct responses to the excitations $a$ and $b$ have already decayed away through Landau damping.

The echo effect is in fact common to a variety of physical systems [9, 1]; in particular, it has been studied in plasmas modelled by the Vlasov–Poisson equations [8, 13]. In view of the strong analogy between these equations and the equations describing the evolution of disturbances in two-dimensional shear flows, the existence of an echo effect in shear flows is not entirely surprising. It has in fact been previously studied by Lifschitz [11] who considered a two-dimensional Couette flow in a channel. Following the plasma derivation, he assumed weak amplitudes for the two excitations and used a regular perturbation expansion to calculate the nonlinear response leading to the echo. When this treatment is valid, the amplitude of the energy peak corresponding to the echo is much smaller than the maximum amplitude of the forced responses $a$ and $b$ (although when the echo appears the amplitude of these responses may have decayed sufficiently and be negligible).

In this paper, we shall be concerned with a somewhat different — and more spectacular — situation, which we refer to as *strong* echo and arises when the three energy peaks in the energy have comparable amplitudes. This is possible provided that the amplitude of the two excitations be $O(\epsilon)$, with $\epsilon \ll 1$, while the time-lag between them is $O(\epsilon^{-1})$ ($\tau_a$ and $\tau_c$ introduced in figure 1 are then $O(1)$ quantities). However, in that case, the perturbative treatment of Ref. [11] in not strictly valid; we therefore employ numerical simulations to demonstrate the existence of a strong echo. We also considerably simplify the problem by dealing with a Couette flow in an unbounded domain as opposed to a channel. With this geometry, the disturbance vorticity and streamfunction can be expanded in terms of sheared modes (the exact solutions

originally discovered by Kelvin [17]), whose form is very simple; the echo effect may then be interpreted as the nonlinear interaction between three sheared modes. Exploiting this, we derive a system of three amplitude equations (analogous to the three-wave equations for wave triads) which capture the essence of the echo effect. Although these amplitude equations cannot be obtained completely rigorously, they provide results which compare fairly well with numerical simulations. As will be seen, the echo effect involves disturbances with small spatial scales, and hence may be expected to be significantly affected by viscous dissipation. We investigate this influence by deriving an estimate for the viscous damping of the echo and confirm our findings numerically.

## 2 Sheared modes in Couette flow

The nonlinear evolution of disturbances in a two-dimensional Couette flow $U = \Lambda y$ is governed by the vorticity equation

$$\left(\frac{\partial}{\partial t} + y\frac{\partial}{\partial x}\right)\omega + \frac{\partial(\psi,\omega)}{\partial(x,y)} = \mu\nabla^2\omega, \quad \text{with} \quad \nabla^2\psi = \omega, \tag{1}$$

where $\psi$ is the streamfunction and $\mu$ the inverse of a Reynolds number. This equation has been rendered dimensionless using $\Lambda^{-1}$ as a timescale and a reference length $L$ as a lengthscale. The dimensional viscosity $\nu$ is thus related to $\mu$ through $\mu = \nu/(\Lambda L^2)$. A standard procedure to study Couette flows is to introduce the convected coordinates (the importance of which was noted by e.g. [5] in this problem)

$$X := x - yt, \quad Y := y, \quad T := t,$$

which transform (1) into

$$\frac{\partial\omega}{\partial T} + \frac{\partial(\psi,\omega)}{\partial(X,Y)} = \mu\nabla^2\omega \tag{2}$$

and give the form

$$\nabla^2 = (1+T^2)\frac{\partial^2}{\partial X^2} - 2T\frac{\partial^2}{\partial X\partial Y} + \frac{\partial^2}{\partial Y^2}$$

to the Laplacian. Considering a periodic domain in $X$ and $Y$, we can expand the vorticity and the streamfunction in Fourier series according to

$$\begin{aligned} \omega &= \epsilon \sum_{k_a=-\infty}^{+\infty}\sum_{l_a=-\infty}^{+\infty} A_a(T)\exp[\mathrm{i}(k_aX + l_aY)], \\ \psi &= \epsilon \sum_{k_a=-\infty}^{+\infty}\sum_{l_a=-\infty}^{+\infty} \frac{-A_a(T)}{k_a^2 + (l_a - k_aT)^2}\exp[\mathrm{i}(k_aX + l_aY)], \end{aligned} \tag{3}$$

where the subscript $a$ of $A_a$ designates the pair $(k_a, l_a)$ and $A_{-a} = (A_a)^*$. Here, we have introduced a formal parameter $\epsilon \ll 1$ as we are concerned with weak amplitude disturbances. Note that the extension of (3) to an unbounded domain is immediately obtained by replacing the summations by integrals. Introducing (3) into (2) leads to the amplitude equations

$$\begin{aligned} \frac{\mathrm{d}A_a}{\mathrm{d}T} &= \frac{\epsilon}{2}\sum_{k_b=-\infty}^{+\infty}\sum_{l_b=-\infty}^{+\infty}\left[\frac{1}{k_b^2 + (l_b - k_bT)^2} - \frac{1}{k_c^2 + (l_c - k_cT)^2}\right](k_cl_b - k_bl_c)\,A_b^*A_c^* \\ &- \mu\left[k_a^2 + (l_a - k_aT)^2\right]A_a, \end{aligned} \tag{4}$$

where $c$ is defined by the interaction conditions

$$k_a + k_b + k_c = 0, \quad l_a + l_b + l_c = 0. \tag{5}$$

(We consider sum interactions only, allowing for both positive and negative values of the wavenumbers.)

We now concentrate on a strictly inviscid fluid with $\mu = 0$. As is well-known, a single mode with constant amplitude $A_a(T) = A_a(0)$ is then an exact solution of (4). In the original variables $(x, y, t)$, this solution consists of a sheared mode whose vorticity

$$\omega = \epsilon A_a(0) \exp\{\mathrm{i}[k_a x + (l_a - k_a t)y]\} \tag{6}$$

behaves like a passive tracer: it is constant along straight lines which are simply tilted by the shear — the slope of these lines evolves as $1/(t - l_a/k_a)$. The time evolution of the disturbance kinetic energy $\frac{1}{2}\iint |\nabla\psi|^2 \mathrm{d}x\mathrm{d}y$ of a sheared mode is given by

$$E_a(t) = \frac{\epsilon^2 |A_a(0)|^2}{k_a^2 + (l_a - k_a t)^2} = \frac{k_a^2 + l_a^2}{k_a^2 + (l_a - k_a t)^2} E_a(0).$$

Although in the long-time limit the energy decays like $t^{-2}$, it can be temporarily amplified if $k_a/l_a > 0$, i.e. if the vorticity lines are initially tilted against the shear. It is this transient growth, identified originally by Orr [14], that has motivated much of the literature on sheared modes [6, 4, 15, 18, 2].

# 3 Echo effect

A superposition of sheared modes is generally not an exact solution of the nonlinear equation (1). Analyzing spectral equations analogous to (4), Tung [19] nevertheless concluded that it represents an $O(\epsilon)$ approximation to an exact solution and is uniformly valid in time, because of the explicit decrease of the nonlinear terms. However, as pointed out by Haynes [10] in his study of the stability of sheared modes, this conclusion assumes that all the wavenumbers $l_a$ are $O(1)$. If at least one wavenumber $l_a$ is $O(\epsilon^{-1})$, i.e. if initially the disturbance partly consists of strongly tilted vorticity lines, the nonlinearity can have non-trivial consequences at leading order which cannot be captured by a regular perturbation expansion. The echo effect which we now describe relies on this fact and requires the simultaneous presence of two modes: a mode $a$ with $l_a = O(\epsilon^{-1})$ and a mode $b$ with $l_b = O(1)$. A natural way to achieve this configuration is to force the two modes at two successive instants separated by a time of $O(\epsilon^{-1})$. Successive excitation have been traditionally considered to study other echo effects both theoretically and experimentally [9, 1, 8, 13, 11]; however, in these studies, the time lag is assumed to be much smaller than $\epsilon^{-1}$, leading to an echo amplitude much smaller than the maximum amplitudes of the forced responses.

Consider mode $a$ forced at $t = \epsilon^{-1}\tau_a < 0$, $\tau_a = O(1)$, with vorticity lines perpendicular to the shear. The subsequent evolution of the vorticity is given by (6), with

$$l_a = \epsilon^{-1} k_a \tau_a, \quad A_a(0) = A_a(\epsilon^{-1}\tau_a), \tag{7}$$

while the energy decays according to

$$E_a(t) = \frac{\epsilon^2 |A_a(\epsilon^{-1}\tau_a)|^2}{k_a^2[1 + (t - \epsilon^{-1}\tau_a)^2]} = \frac{1}{1 + (t - \epsilon^{-1}\tau_a)^2} E_a(\epsilon^{-1}\tau_a).$$

At $t = 0$, when mode $a$ has vorticity lines strongly tilted in the direction of the shear (their slope is $-\epsilon\tau_a^{-1} > 0$) and a very small energy ($E_a(0) \sim \epsilon^2 E_a(\epsilon^{-1}\tau_a)$), we excite the second mode $b$, with $k_b \neq k_a$

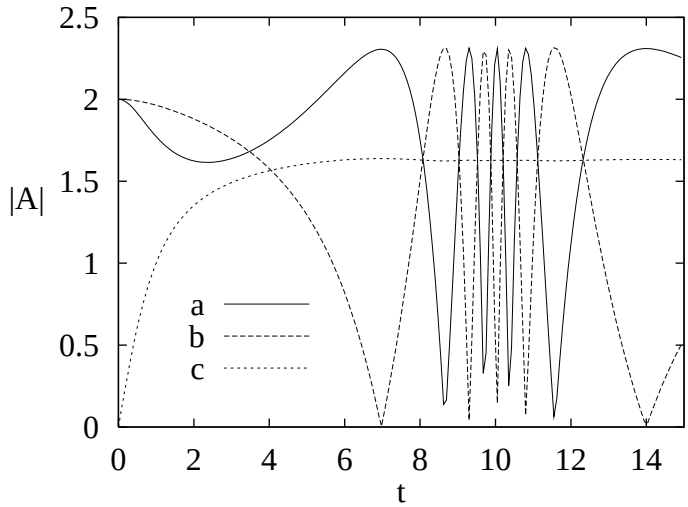


Figure 2: Time evolution of the amplitudes of the three modes $a$, $b$ and $c$ as predicted by the truncated system (10).

and $l_b = 0$. The linear evolution of this mode is again a simple tilt of the vorticity lines, with a decay of the energy. However, the nonlinear interaction of $a$ and $b$ generate a third mode $c$, whose wavenumbers satisfy (5). Defining,

$$\epsilon^{-1}\tau_c := \frac{l_c}{k_c} = \epsilon^{-1}\frac{k_a\tau_a}{k_a + k_b}, \tag{8}$$

where $\tau_c$ is $O(1)$, it can be seen that the vorticity of $c$ takes the form

$$\omega_c = \epsilon A_c(t)\exp\left\{\mathrm{i}[k_c x + (l_c - k_c t)y]\right\} = \epsilon A_c(t)\exp\left\{\mathrm{i}k_c[x + (\epsilon^{-1}\tau_c - t)y]\right\},$$

where the amplitude $A_c(t)$ is determined by the nonlinear interactions. Correspondingly, the energy of $c$ is given by

$$E_c(t) = \frac{\epsilon^2|A_c(t)|^2}{k_c^2[1 + (t - \epsilon^{-1}\tau_c)^2]}.$$

If $\tau_c > 0$, which is achieved if

$$\frac{k_a\tau_a}{k_a + k_b} > 0 \quad \text{i.e. if } \; k_a k_b < 0 \; \text{ and } \; |k_a| < |k_b|, \tag{9}$$

one can expect the appearance of a peak in $E_c$ for $t \approx \epsilon^{-1}\tau_c$. At that time, both $E_a$ and $E_b$ have decreased to $O(\epsilon^4)$ while $E_c$ is $O(\epsilon^2)$ since, as shown below, $A_c(\epsilon^{-1}\tau_c) = O(1)$. The peak in $E_c$ can thus be viewed as the echo of two modes which have already damped away (see figure 1). Note that, because sheared modes are exact nonlinear solutions of (1) (with $\mu = 0$), we can treat the generation of the echo as an initial-value problem, assuming that both disturbances $a$ and $b$ are initialized at $t = 0$ with $l_a$ given by (7) and $l_b = 0$.

# 4 Truncated model

To investigate the echo effect in more details, we first consider the truncation of (4) to the triad of modes $a, b, c$. Neglecting $O(\epsilon)$ terms and using (7) and (8), the evolution equations for $A_a, A_b, A_c$ can be written

$$\left\{\begin{array}{l} \dfrac{\mathrm{d}A_a}{\mathrm{d}t} = -J\left\{\dfrac{1}{k_b^2(1+t^2)} - \dfrac{1}{k_c^2\left[1+(t-\epsilon^{-1}\tau_c)^2\right]}\right\} A_b^* A_c^* \\ \dfrac{\mathrm{d}A_b}{\mathrm{d}t} = -\dfrac{J}{k_c^2\left[1+(t-\epsilon^{-1}\tau_c)^2\right]} A_c^* A_a^* \\ \dfrac{\mathrm{d}A_c}{\mathrm{d}t} = \dfrac{J}{k_b^2(1+t^2)} A_a^* A_b^* \end{array}\right. \tag{10}$$

where $J := k_b k_c \tau_c = O(1)$. Clearly, the initial forcing of $A_c$ is $O(1)$ and, in general, one can expect the mode $c$ to quickly reach an $O(1)$ amplitude. This can be confirmed by direct numerical solutions of (10). As an example, we consider the modes with

$$k_a = -2,\ l_a = 10,\ k_b = 3,\ l_b = 0,\ k_c = -1,\ l_c = -10,$$

which satisfy (5) and (9), such that $\epsilon^{-1}\tau_a = -5$ and $\epsilon^{-1}\tau_c = 10$. We take $\epsilon = 0.1$ and choose the initial amplitudes $A_a = 2$ and $A_b = 2\mathrm{i}$. Figure 2 displays the time evolution of the three amplitudes. As expected, mode $c$ attains an amplitude comparable to that of $a$ and $b$ due to their nonlinear forcing. For $t = O(\epsilon^{-1})$, $A_c$ is approximately constant (because of the explicit time decrease of the nonlinear interaction term), but $A_a$ and $A_b$ are strongly modulated for $t \approx \epsilon^{-1}\tau_c = 10$. The echo effect appears in the total energy, given by

$$E = \frac{\epsilon^2|A_a|^2}{k_a^2[1+(t-\epsilon^{-1}\tau_a)^2]} + \frac{\epsilon^2|A_b|^2}{k_b^2(1+t^2)} + \frac{\epsilon^2|A_c|^2}{k_c^2[1+(t-\epsilon^{-1}\tau_c)^2]},$$

whose evolution is shown in figure 3 (dashed curve). Initially, the energy is dominated by the contribution of mode $b$ and decays as predicted by the linear theory (dotted curve). For longer time, however, the contribution of mode $c$ is dominant and the energy exhibits a clear peak for $t \approx \epsilon^{-1}\tau_c$ corresponding to the echo. (The energy had also peaked after the excitation of mode $a$, at $t = \epsilon^{-1}\tau_a < 0$.)

It is possible to derive an approximate analytical solution of (10) and thus to obtain an estimate for the echo amplitude by noting that with $\epsilon \ll 1$ the nonlinear terms are significant only for $t = O(1)$ and $t - \epsilon^{-1}\tau_c = O(1)$. For $t = O(1)$, $A_b$ is almost constant and simply plays a catalytic role in the interaction between modes $a$ and $c$. Similarly, for $t - \epsilon^{-1}\tau_c = O(1)$, $A_c$ is almost constant; the evolution of $A_a$ and $A_b$ during this second time period is irrelevant for the echo effect. Focusing on $t = O(1)$, we approximate (10) by the linear system

$$\left\{\begin{array}{l} \dfrac{\mathrm{d}A_a}{\mathrm{d}t} = \dfrac{-J}{k_b^2(1+t^2)} A_b^* A_c^* \\ \dfrac{\mathrm{d}A_c}{\mathrm{d}t} = \dfrac{J}{k_b^2(1+t^2)} A_a^* A_b^* \end{array}\right.$$

where $A_b = A_b(0)$ is kept constant. The solution corresponding to the initial conditions $A_a = A_a(0)$ and $A_c = 0$ is

$$A_a(t) = A_a(0)\cos(\alpha \arctan t), \qquad A_c(t) = \frac{A_a^*(0)A_b^*(0)}{|A_b(0)|}\sin(\alpha \arctan t),$$

where $\alpha := |A_b|J/k_b^2$. Since $A_c$ does not change significantly for $t \gg 1$, we obtain the following estimates for the echo amplitude:

$$|A_c(\epsilon^{-1}\tau_c)| \approx |A_a(0)\sin(\alpha\pi/2)| \quad \rightarrow \quad E_{\mathrm{echo}} \approx \frac{\epsilon^2|A_a(0)|^2}{k_c^2}\sin^2(\alpha\pi/2). \tag{11}$$

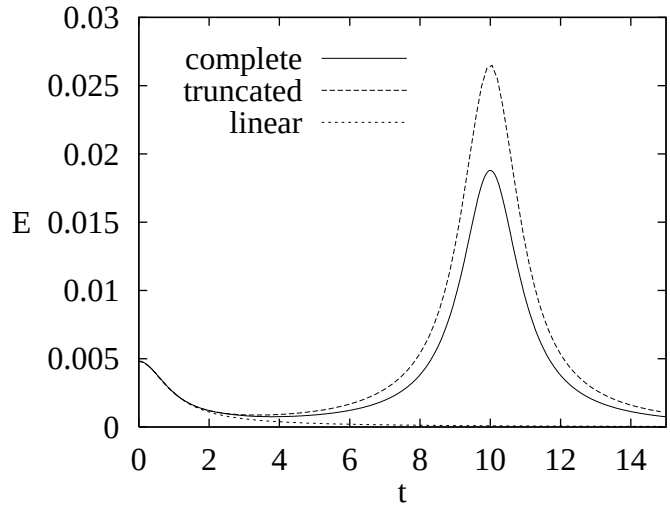


Figure 3: Time evolution of the total disturbance energy in the system governed by the complete equations (4), in the truncated system (10) and according to the linear theory.

In the above example, $\alpha = 2/3$, so that the estimates are $|A_c((\epsilon^{-1}\tau_c)| \approx \sqrt{3} = 1.73$ and $E_{\text{echo}} \approx 0.03$. Both values compare well with what has been found by solving (10) numerically, as seen from figures 2 and 3, although $\epsilon$ is only marginally small.

A few remarks can be made about the approximate result (11). First, it indicates that the echo amplitude depends on the initial amplitudes of modes $a$ and $b$ in two distinct manners: the echo amplitude is directly proportional to $A_a(0)$, whereas $A_b(0)$ merely determines a time scale for the evolution of $c$. Note also the particular dependence of $|A_c|$ on $\alpha$, i.e. on $|A_b|$ for fixed wavenumbers (and thus fixed $J$); the echo is maximized for $\alpha = (2n+1)$, where $n$ is an integer, and it disappears for $\alpha = 2n$. Finally, we mention that standard results about (weak) echo, which neglect the feedback of $c$ on $a$ and $b$, correspond to the limit $\alpha \ll 1$ in (11). This leads to the estimates

$$|A_c(\epsilon^{-1}\tau_c)| \approx \frac{\pi k_c \tau_c |A_a(0) A_b(0)|}{2k_b} = 2.09 \quad \rightarrow \quad E_{\text{echo}} \approx E_c(\epsilon^{-1}\tau_c) \approx \frac{\epsilon^2 \pi^2 \tau_c^2 |A_a(0) A_b(0)|^2}{4k_b^2} = 0.044.$$

which provide the correct order of magnitude although they are significantly overestimated as one could expect.

# 5 Numerical results

The system (10), which has been derived as an ad hoc truncation of (4), misses important parts of the dynamics of (4), in particular the generation of a mode $d$, with $k_d = k_b - k_a$ and $l_d = l_b - l_a$, corresponding to the interaction of $a$ with $-b$. This and other neglected interactions are likely to weaken the echo. It is therefore important to study the echo effect directly with the complete system (4), or in pratical terms, with a truncation of (4) keeping many modes. It turns out that convergent results are obtained with a limited number of modes; those presented below have been obtained for the same parameters as before, with a $25 \times 25$ truncation, and reflect the behaviour of the complete system (the disturbance energy is

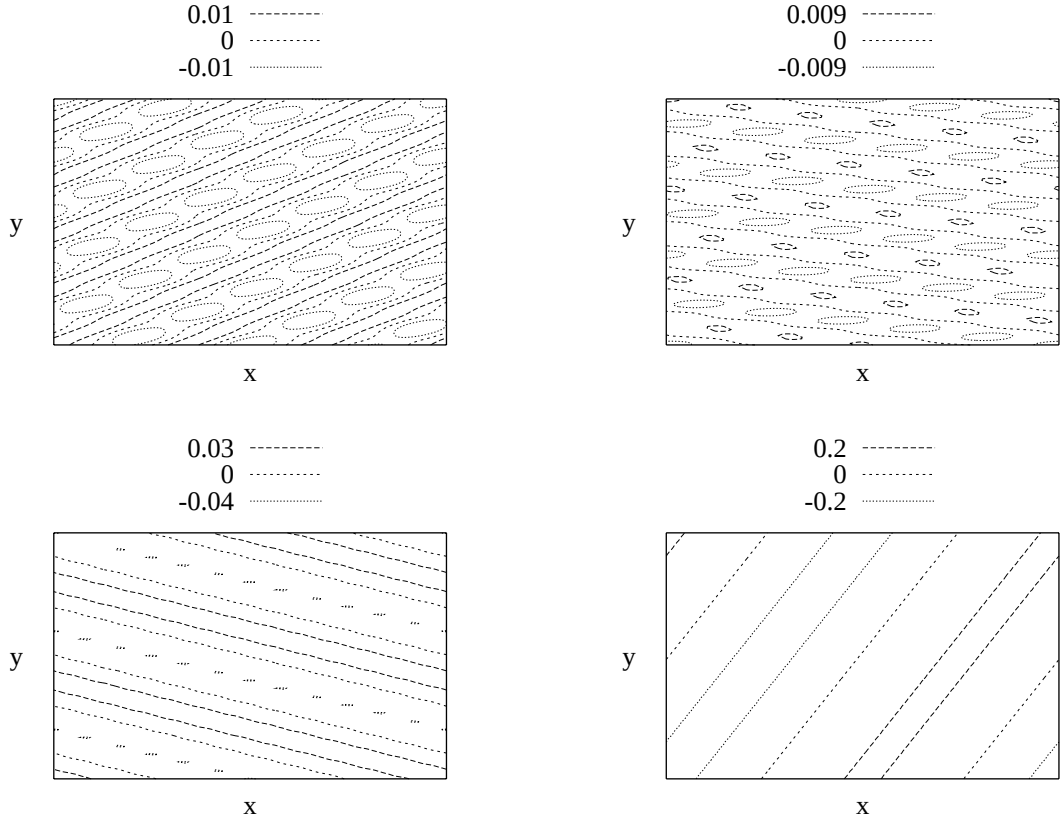


Figure 4: Disturbance streamfunction for $t = 1.5, 4.5, 7.5$ and $10.5$. Note the change in the contour levels.

particularly stable when resolution is varied.) The evolution of the energy so obtained is displayed in figure 3 (solid curve). As anticipated, the truncated model significantly overestimates the energy peak at $t \approx \epsilon^{-1}\tau_c$. Nevertheless, the new results confirm the qualitative validity of the truncated model and the existence of a strong echo effect. Note that figure 3 also shows the energy evolution predicted by the linear theory; this theory only capture the Landau damping of mode $b$ and thus completely misses the appearance of the echo.

It is interesting to examine the evolution of the structure of the streamfunction and vorticity during the simulation. Figure 4 shows the streamfunction at $t = 1.5, 4.5, 7.5$ and $10.5$. At $t = 1.5$, the streamfuction is dominated by mode $b$, since Landau damping has already acted strongly on mode $a$, while mode $c$ only begins to emerge. At $t = 4.5$, the streamfunction amplitudes of modes $b$ and $c$ are similar, but weak because both have relatively small spatial scales. At $t = 7.5$ and $t = 10.5$, little remains of mode $b$ in terms of the streamfunction which is dominated by mode $c$. The amplitude of the streamfunction increases as the spatial scale of this mode increases; it reaches its maximum for $t = 10$ before decreasing. Note the factor 20 between the amplitude of the streamfunction for $t = 1.5$ and for $t = 10.5$ which indicates the importance of the echo effect.

It is difficult to plot the evolution of the vorticity field similarly, since it contains very small scales associated with mode $a$, and, after some time, with mode $b$. Figure 5, which shows the vorticity field with

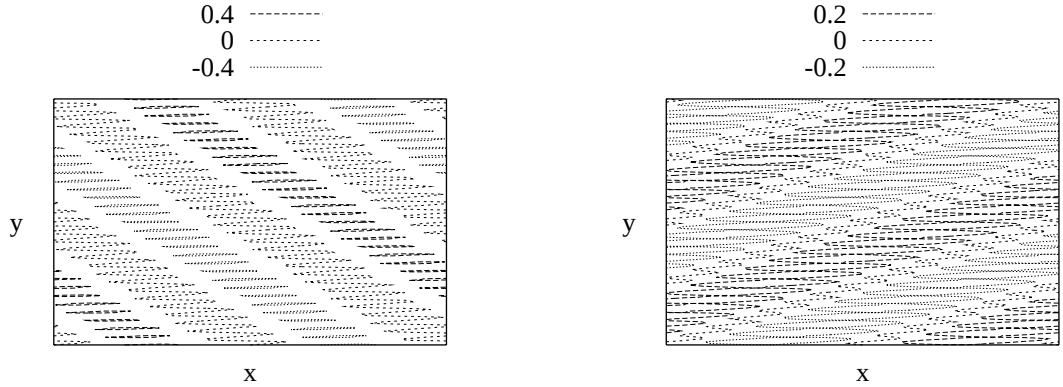


Figure 5: Vorticity field with wavenumber $k_c$ for $t = 7.5$ and 12.

$x$-wavenumber $k_c$ only, focuses on mode $c$. The two panels correspond to $t = 7.5$ (before the echo) and $t = 12.5$ (after the echo). Although a dominant orientation of the vorticity lines can be distinguished and corresponds to the behaviour expected from mode $c$, this figure emphasizes that several sheared modes, with wavenumber $k = k_c$ but different wavenumbers $l$, are in fact superposed.

As figures 4 and 5 clearly illustrate, the echo effect relies on the presence in the flow of disturbances with very small scales. Since such disturbances are strongly affected by dissipation, it is important to consider the echo effect with a non-zero viscosity. A rough estimate of the influence of viscosity can be derived from (4) by noting that, in the linear approximation, any mode amplitude is damped by a factor $\exp[-\mu(k_a^2 + l_a^2 - k_a l_a t + k_a^2 t^2/3)t]$. The modes $a$ and $c$ involved in the echo have wavenumbers $l_a$ and $l_c$ which are $O(\epsilon^{-1})$, and mode $c$ must remain excited until $t \approx \epsilon^{-1}\tau_c$. Therefore, at the moment of the echo, dissipation is responsible for an overall damping factor of $\exp(-\mu\epsilon^{-3})$. A condition for the echo to occur in a viscous fluid is thus $\mu\epsilon^{-3} \leq 1$. The validity of this estimate can be confirmed by direct numerical resolution of (4) (again with a $25 \times 25$ truncation) for different values of the viscosity parameter $\mu$. The corresponding evolution of the total energy is shown in figure 6. With the non-zero values chosen for $\mu$ which are $O(\epsilon^3)$, the energy peak significantly decreases, although it remains well defined. It is clear that for $\mu \ll \epsilon^3 = 0.001$, the echo would be virtually unaffected by dissipation, whereas for $\mu \gg \epsilon^3$ it would entirely disappear.

# 6 Conclusion

In this paper, we have studied the echo effect in a two-dimensional Couette flow both analytically and numerically. This phenomenon can be regarded as a nonlinear mechanism of transient growth: a disturbance strongly tilted against the shear, and thus susceptible to experience a significant energy amplification, is generated by the nonlinear interaction of two other disturbances which are excited successively. The time lag between these two excitations, when large as is assumed here, provides a natural way of introducing in the system disturbances with very different scales. This leads to a significant nonlinear effect although the initial excitation amplitude is weak. The echo effect, as the instability of sheared modes [10], thus illustrates the difficulties that may arise when linearizing evolution equations for disturbances in stable shear flows: stability guarantees that some norm of the disturbance — the enstrophy for Couette flows

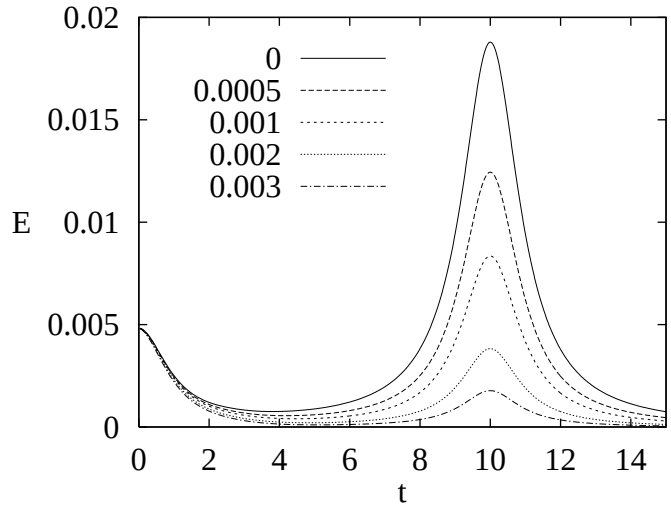


Figure 6: Time evolution in the system governed by the complete system (4) for different values of the viscosity parameter $\mu$.

— is bounded, but this does not preclude the nonlinear terms to be large since they involve vorticity gradients which increase with time.

Our investigation of the echo effect is particularly simple, mainly thanks to the very simple form taken by sheared modes in two-dimensional Couette flows when the domain is unbounded. The physical mechanisms involved, however, are generic to all monotonic shear flows, so that echoes can be expected to occur in a variety of situations. Geophysical flows seem especially interesting in that respect, since they are very little affected by dissipation; this can motivate an extension of our work to include the effects of rotation, curvature ($\beta$-effect) and stratification. The possibility of spatial echoes [13] also deserves investigations, notably because they would be well suited for an experimental demonstration of the echo effect in shear flows.

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