Can a "superconductor" always expel the generalized magnetic field?

S.M. Mahajan

Institute for Fusion Studies, The University of Texas at Austin, Texas, USA

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Abstract

The conservation of generalized helicity in a perfectly conducting fluid may act as an electrodynamic barrier for the transition to the London (superconducting) state when the system is immersed in a topologically nontrivial magnetic field (with a nonzero generalized helicity). An experiment is proposed to test whether the mechanism responsible (quantum correlations) for superconductivity respects the electrodynamic constraint.

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Perfect diamagnetism (Meissner–Ochsenfeld effect) coupled with perfect conductivity defines a superconducting state. A superconductor is, thus, a perfect conductor that expels any magnetic field from its interior (we shall restrict ourselves to type-1 superconductors only). The latter attribute of the superconducting state stems from a remarkable magnetic constitutive relationship embodied in the London equation [1]. It is universally believed that the transition from the normal to the superconducting state is purely a quantum–mechanical phenomenon outside the purview of classical electrodynamics. We do know of at least one attempt to derive the London equation from classical electrodynamics [W.F. Edwards, Phys. Rev. Lett. **47**, 1963 (1981). The Edwards paper, however, was found to be erroneous [see, for example, J.B. Taylor, Nature **29**, 681 (1982)].

In this paper, we shall attempt to investigate several properties of the perfect/superconducting states from purely classical considerations. To avoid any confusion, we emphatically state that this work is not another attempt at a classical "derivation" of superconductivity, which we believe to have a quantum origin. The principal motivation of this paper is to find whether a detailed analysis of the electrodynamics of perfect conductors might shed some new light on the superconducting state. We shall, therefore, study the electrodynamics of the perfectly conducting state in order to:

1. understand the electrodynamic significance of the Meissner–Ochsenfeld effect, in fact, of its theoretical underpinning, the London equation, first proposed phenomenologically and then derived from the microscopic theories of superconductivity [2];

2. derive an electrodynamic constraint, based on the conservation of generalized magnetic helicity. This constraint might tend to hinder the transition from the normal perfectly conducting state to the London state, and hence to the putative superconducting state;

3. derive the generalization of the London equation consistent with the electrodynamic constraint;

4. speculate whether the "superconducting transition" will lead to the London state if,

at the transition threshold, the sample were embedded in a 'complicated' magnetic field. It is, of course, assumed that the transition does occur for a "simple" magnetic field. What constitutes a 'complicated' magnetic field will be clear a little later. The magnetic field could be generated externally as well as by inducing an internal current in the sample;

5. and finally propose an experiment to test whether the quantum mechanical processes (which causes the transition) do find a way to relax the classical constraint and expel the "complicated" magnetic field.

Consider a perfectly conducting charged fluid made up of carriers (electron–pairs of some variety) with a number density n, elementary charge $-e^*$, and elementary mass m^*. We shall assume the number density to be uniform. In dimensionless notation: the electromagnetic fields $\mathbf{B}$ and $\mathbf{E}$ normalized to an arbitrary B_0, the length scales (∇^{-1}) normalized to the collisionless skin depth $\lambda_e = c/\omega_p$, $\omega_p = e^*(4\pi n/m^*)^{1/2}$, and the carrier speed $\mathbf{u}_e = \mathbf{u}\, v_0$ with $v_0 = B_0/(4\pi n m^*)^{1/2}$: the equation of motion of this fluid can be written as

$$\frac{\partial \mathbf{P}}{\partial t} = \frac{\partial}{\partial t}[\mathbf{A} - \mathbf{u}] = \nabla\left(\frac{u^2}{2} + g\right) + \mathbf{u} \times \boldsymbol{\Omega}, \tag{1}$$

where $\mathbf{P}$ is the canonical momentum, $\mathbf{A}$ is the appropriately normalized vector potential ($\mathbf{B} = \nabla \times \mathbf{A}$), ∇g represents all the other gradient forces, and

$$\boldsymbol{\Omega} = \nabla \times \mathbf{P} \equiv \mathbf{B} - \boldsymbol{\nabla} \times \mathbf{u} \tag{2}$$

is the 'generalized magnetic field' (GM) experienced by the charged fluid. In deriving (1), the convective derivative $(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u}$ was replaced by its vector equivalent $\nabla(u^2/2) - \mathbf{u} \times (\nabla \times \mathbf{u})$. Taking the curl of Eq. (1), we obtain the evolution equation for GM,

$$\frac{\partial \boldsymbol{\Omega}}{\partial t} = \nabla \times [\mathbf{u} \times \boldsymbol{\Omega}], \tag{3}$$

which along with (2), and Ampére's law (quasistatic processes) written in our dimensionless units,

$$\boldsymbol{\nabla} \times \mathbf{B} = -\mathbf{u} \tag{4}$$

forms the core of the perfectly conducting single–fluid electrodynamics. Notice that, in standard units, the current density $\mathbf{J} = -ne^*\mathbf{u}_e$.

A simple examination of (3) reveals that $\boldsymbol{\Omega} = 0$ is a very special equilibrium solution of (3), i.e. if at any time $\boldsymbol{\Omega} = 0$, then it always remains zero. It is fitting, perhaps, that this special solution defines the superconducting state, for $\boldsymbol{\Omega} = 0$ is nothing but the London equation

$$\boldsymbol{\Omega} = \mathbf{B} - \boldsymbol{\nabla} \times \mathbf{u} = 0, \tag{5}$$

which, on combining with (4), leads us to the Meissner–Ochsenfeld fields

$$\nabla \times \nabla \times \mathbf{B} = -\mathbf{B}, \tag{6}$$

the very hallmark of superconductivity. Equation (6) contains the result that in any superconducting sample of dimensions $a \gg \lambda_e$ (∇^{-1} is normalized to λ_e), the fields inside will be exceedingly small rising to match the externally applied field in a skin length near the surface.

The preceding discussion suggests another possible definition of the superconducting state; it is the state in which the generalized magnetic field is strictly zero everywhere including the region near the surface. Notice that, apart from a constant factor, the generalized magnetic field is equivalent to the generalized vorticity (magnetic plus fluid). Thus the superconducting state does not permit any generalized vorticity in its entire interior.

Electrodynamically, the superconducting state appears to be a very special solution among the many possible for a perfectly conducting state. Since the constancy of $\boldsymbol{\Omega}$, when $\boldsymbol{\Omega} = 0$, is guaranteed, it stands to reason that by choosing the correct and obvious initial condition, one could ensure that the perfectly conducting state becomes superconducting. From this perspective, quantum correlations simply conspire to create this very special initial configuration in the conventional solid state superconductors [3].

In order to better understand the fundamental role played by the initial conditions in the dynamics of ideal conductors, we must look for the invariants of motion because the initial state may be defined by allocating some specific values to these invariants. From Eqs. (1) and (3), it is quite straightforward to derive

$$\frac{\partial}{\partial t}\mathbf{P}\cdot\mathbf{\Omega} = \nabla\cdot\left[\mathbf{\Omega}(\frac{u^2}{2}+g)+\mathbf{P}\times(\mathbf{u}\times\mathbf{\Omega})\right] \tag{7}$$

with the implication that (with the boundary conditions $\mathbf{n}\cdot\mathbf{\Omega} = 0 = \mathbf{n}\cdot\mathbf{u}$, where $\mathbf{n}$ is the unit normal to the surface) the generalized helicity (GH)

$$\int \mathbf{P}\cdot\mathbf{\Omega}\, d^3x = \sigma_g \tag{8}$$

is an invariant of the motion, and must be conserved in time. The generalized helicity is simply the generalization of the magnetic helicity, $\int \mathbf{A}\cdot\mathbf{B}\, d^3x$, whose conservation in ideal magnetohydrodynamics [4] leads to magnetic field structures very different from the magnetic fields which are known to exist in the superconducting state.

The concept of magnetic (generalized) helicity, which measures the knottedness of the field lines, provides us with a criterion for classifying the magnetic structures. Due to their topological simplicity, the fields with zero helicity will be called "simple." By the same token, a generalized magnetic field will be termed "complicated" when it is characterized by a nonzero GH (σ_g). It should be remarked that it is possible to have a gauge invariant definition of helicity, but for the present paper, that sophistication is not of essence.

The GM structures found in the superconducting state are clearly "simple"; the GH associated with the London state is trivially zero [Eqs. (8) and (5)]. It follows, then, that the London state is electrodynamically accessible only from "simple" initial configurations for which $\sigma_g = 0$; ideal electrodynamics would not permit "complicated" field configurations with $\sigma_g \neq 0$ to ever transform into the London state.

In order to appreciate the significance of this conclusion, let us recreate the scenario of an experiment in which a given sample of a superconducting material, immersed in a magnetic field, is being cooled below the transition temperature T. A little above T, it can be safely assumed that the resistivity has become so small that the sample can be treated as an almost perfect conductor. There is evidence from experiments on high temperature plasmas that the the total volume integral helicity (σ_g) is quite well conserved in the presence of small residual resistivity . As this perfectly conducting specimen is cooled towards and past T, we know that the transition to the London state takes place if the material is immersed in a "simple" field. In fact this is the reason why the material is supposed to be superconducting. We also know, from previous discussion, that this transition is compatible with ideal electrodynamics.

Let us now place this very material (which becomes London in a "simple" field) in a "complicated" generalized field (see examples given below), and then rapidly cool it through T in a time shorter than the resistive diffusion time which is comparatively long at such low resistivity. We may expect:

1. that the transition to London state, being forbidden by ideal electrodynamics, does not take place, and the magnetic field is not expelled from the interior of the sample. It implies that the mechanism (quantum correlations) which was responsible for conventional superconductivity respects the electrodynamic constraint and cannot affect the transition to the "superconducting" state. We already know that London transition does not take place in the presence of fields larger than the critical field for that material. Now we know that it is not only the magnitude of the field, but also its topological structure which may prevent the superconducting transition; the latter seems to do it considerably more efficiently,

2. that the transition to London state, although forbidden by ideal electrodynamics, does take place, and the field is expelled from the interior of the sample. This has a profound implication: the mechanism responsible for superconductivity has found a way to override the electrodynamic constraint!

Either of the results that the experiment may reveal is fascinating: the first case opens up a totally new area of the internal magnetics of "superconductors," while the second possibility will force us to discover how the quantum–transition may bring about what was electrodynamically forbidden.

We continue by deriving and solving (in simple representative geometry) a possible set of general equations for systems with arbitrary GH. The London ($\mathbf{\Omega} = 0$), and other "simple" systems will be special cases of the formalism. We follow the standard procedure; by minimizing the combined magnetic energy and the kinetic energy of the electron fluid with the constraint that σ_g is conserved,

$$\delta \int \left\{ \left(\frac{B^2}{2} + \frac{u^2}{2} \right) + \frac{1}{2\mu} (\mathbf{A} - \mathbf{u}) \cdot (\mathbf{B} - \nabla \times \mathbf{u}) \right\} d^3x = 0, \tag{9}$$

we obtain, for independent variations $\delta\mathbf{A}$ and $\delta\mathbf{u}$,

$$\mathbf{\Omega} = \nabla \times \mathbf{u} - \mathbf{B} = -\mu \mathbf{u}, \tag{10}$$

$$\nabla \times \mathbf{u} - \mathbf{B} = \mu \boldsymbol{\nabla} \times \mathbf{B}; \tag{11}$$

the latter (11) becomes identical with (10) because of Amperé's law (2). Thus the variation principle leads to only one independent Eq. (10), which when substituted into (3), shows that the variational solution is automatically an equilibrium solution. For this solution

$$\mathbf{P} \cdot \mathbf{\Omega} = (\mathbf{A} - \mathbf{u}) \cdot \mathbf{\Omega} = -\mu (\mathbf{A} - \mathbf{u}) \cdot \mathbf{u} = \mu \left[\mathbf{u}^2 - \mathbf{u} \cdot \mathbf{A} \right]. \tag{12}$$

Equations (10)–(12) show that μ, the inverse of the Lagrange multiplier, is a measure of both the GM and the GH (volume integral of $\mathbf{P} \cdot \mathbf{\Omega}$). In addition, they reveal that a finite current ($\mathbf{u}$ measures the current) is a necessary condition for the existence of what we have called a "complicated" field configuration. As long as the current and the generalized momentum are not strictly perpendicular to one another, a nonzero GH will be the expected result.

Combining (4) and (10), we derive the equation

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{B} + \mu \boldsymbol{\nabla} \times \mathbf{B} + \mathbf{B} = 0 \tag{13}$$

$$(\boldsymbol{\nabla} \times +\lambda_+)(\boldsymbol{\nabla} \times -\lambda_-)\mathbf{B} = 0 \tag{14}$$

with

$$\lambda_\pm = -\frac{\mu}{2} \pm \frac{1}{2}\sqrt{\mu^2 - 4}, \tag{15}$$

which determines the structure of the "complicated" magnetic field. It can be readily verified that $\mathbf{u}$ obeys exactly the same equation. Equation (13) is obviously a generalization of (6) for finite GH measured by μ. For $\mu = 0$, $\lambda_\pm = \pm i$, and the field reverts to the standard fully diamagnetic structure.

Fortunately exact solutions of (13) or (14) can be written down in several simple geometries. We shall present, here, the simplest one-dimensional solutions in cylindrical geometry. For technical reasons involving the solvability of (13) [5], the cylinder should be seen as the limit of a torus. For that reason we shall use the words toroidal (t) and axial, and poloidal (p) and azimuthal synonymously. The general solution is

$$\begin{aligned} B_t &= A\, J_0(\lambda_+ r) + P\, J_0(\lambda_- r), \\ B_p &= A\, J_1(\lambda_+ r) + P\, J_1(\lambda_- r), \end{aligned} \tag{16}$$

where r is the radial variable. Corresponding expressions for u_t and u_p can be readily written down. Since $B_r = 0 = u_r$, the solution automatically satisfies the surface conditions $\mathbf{n} \cdot \mathbf{u} = 0 = \mathbf{n} \cdot \mathbf{B}$. The constants A and P are determined by specifying, for example, the values of B_t and B_p at $r = a$, the radius of the specimen.

A few representative examples of (16) are plotted in Figs. 1–3. For all these cases, we assume $B_t(r = a) = 1$, and $B_p(r = a) = 0$. The size of the system is $a = 10$, and we have deliberately chosen a to be not so large that the field variations can be visualized. In Fig. 1, we take $\mu = 0$ ($\lambda_\pm = i$), and it reproduces the standard field structure (consistent with the London equation) for a cylindrical sample immersed in an axial (toroidal) magnetic field of unit strength. For this case, the toroidal current $u_t = 0$. It turns out as μ is increased, there is very little change in the field structure till we reach μ of order unity. In Fig. 2, we display the case where $\mu = 1.25$. We notice that B_t has begun to penetrate the interior (is not just limited to the surface region) along with the appearance of a toroidal current. The quenching of diamagnetism continues as μ is increased towards the value 2. For $\mu = 1.85$, we see that in Fig. 3 both B_t and B_p permeate the entire region. For $\mu \geq 2$ (not shown), $\lambda_\pm$ becomes real, and the field structures becomes paramagnetic — a complete departure from the London fields. The numerical plots clearly show that we need a toroidal current to create the "complicated" field structures.

We can now conceptualize the critical experiment. We take a cylindrical (thin torus) sample of a substance known to be a type-1 superconductor and begin cooling it. Just before the threshold, we induce an axial current in the sample (from the definition of $\mathbf{u}$ and its numerical value, the toroidal drift speed of $\cong 10\, B_0$ cm. per second is needed to insure the requisite current) along with immersing it in an externally created axial field. Then we suddenly cool it below the transition temperature and observe if the magnetic field has been fully expelled or not. Either result will be interesting.

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FIGURE CAPTIONS

FIG. 1. Plots of the fields (B_t and B_p) and currents (u_t and u_p) versus the radius r for $\mu = 0$ — the standard London case.

FIG. 2. Plots of the fields (B_t and B_p) and currents (u_t and u_p) versus the radius r for $\mu = 1.25$ — the fields are beginning to penetrate.

FIG. 3. Plots of the fields (B_t and B_p) and currents (u_t and u_p) versus the radius r for $\mu = 1.85$ — the fields permeate the whole region