Steady State Tokamak Equilibria Without External Current Drive

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Abstract

Steady-state tokamak equilibria without current drive are found. This is made possible by including the potato bootstrap current close to the magnetic axis. Tokamaks with this class of equilibria do not need seed current or current drive, and are intrinsically steady state.

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A tokamak is not known to be an intrinsically steady-state plasma confinement scheme in the sense that it needs external current drive, either inductive or non-inductive, or both, to maintain the equilibrium.[1] Although there are high-bootstrap-current-fraction ($\gtrsim 99\%$) tokamak equilibrium with a small amount of the seed current in the region close to the magnetic axis, they are still not intrinsically steady state.[2–4] To find an intrinsically steady-state tokamak equilibrium one needs to find at least a self-generated current in the region close to the magnetic axis. It turns out there does exist such a current.[5] The origin of this current is associated with the unique orbit topology in the region close to the magnetic axis.[6–8] Because the variation of the minor radius over the width of the orbit is significant there, the fraction of the trapped particles does not vanish. This is in contrast to the fraction of the trapped banana orbits which vanishes on the magnetic axis. Because trapped particles have the shape of a potato in the region close to the magnetic axis, we call these orbits the potato orbits to distinguish from the well-known banana orbits in the region away from the magnetic axis. It has been known that potato orbits associated with fusion-born alpha particles can drive a bootstrap current.[5] This result is extended to fuel ions and electrons.[6] The magnitude of the bootstrap current for potato electrons and potato ions is larger than that of alpha particles due to higher fuel density. For typical tokamaks, the potato bootstrap current is a significant fraction of the banana bootstrap current. This makes it feasible to have tokamak equilibria with only bootstrap current and diamagnetic current.[6] Here we demonstrate that such equilibria exist.

The potato orbits are quantitatively different from the banana orbits in that the position of a potato orbit ψ_p, measured in poloidal flux function ψ, is comparable to the width of a potato orbit $\Delta\psi_p$; while the position of a banana orbit ψ_b is much greater than the width of a banana orbit $\Delta\psi_b$ as shown in Fig. 1. Because of this difference, the radial variation of inverse aspect ratio ϵ can not be ignored in the calculation of the potato orbits. The trajectory of the particle orbit in tokamaks can be determined from three constants of motion: total particle energy $E = v^2/2 + e\Phi/M$, magnetic moment $\mu = v_\perp^2/2B$, and toroidal canonical momentum $P_\zeta = \psi - Iv_\parallel/\Omega$. Here, $B = |\mathbf{B}|$, $\mathbf{B}$ is the magnetic field, $v_\parallel(v_\perp)$ is parallel (perpendicular) particle speed, $I = R^2\boldsymbol{\nabla}\zeta\cdot\mathbf{B}$, ζ is the toroidal angle, R is the major radius, Ω is the gyrofrequency, e is the charge, M is the mass, $v^2 = v_\parallel^2 + v_\perp^2$, and Φ is the equilibrium electrostatic potential. Assuming $B = B_0/(1+\epsilon\cos\theta)$ where $\epsilon = r/R_0$, r is the minor radius, R_0 is R on the magnetic axis, and B_0 is B on the axis, we obtain a general orbit equation for a large aspect ratio ($\epsilon \ll 1$) tokamak from the constants of motion

$$(\psi-\psi_0)^2 + 2\,\frac{Iv_{\parallel 0}}{\Omega_0}(1+\epsilon_0\cos\theta_0)(\psi-\psi_0) + \frac{2I^2}{\Omega_0^2}(v_{\parallel 0}^2+\mu B_0)(\epsilon_0\cos\theta_0-\epsilon\cos\theta) = 0 + \mathcal{O}(\epsilon). \quad (1)$$

The subscript "0" in Eq. (1) indicates the quantity is evaluated at a reference point (ψ_0, θ_0). If ϵ is treated as a parameter, Eq. (1) describes standard banana orbits away from the magnetic axis. If the radial variation of $\epsilon = C_1\sqrt{\psi}$, where $C_1 = \sqrt{2q/I\delta R}$, δ is the elongation parameter and q is the safety factor, Eq. (1) is a quartic equation in $\sqrt{\psi}$ that describes a general tokamak orbit. This quartic equation can be simplified by choosing $\psi_0 \to 0$. In that limit, Eq. (1) reduces to a cubic equation in $x = \sqrt{\psi}$

$$x^3 + 2\,\frac{Iv_{\parallel 0}}{\Omega_0}\,x - \frac{2I^2C_1}{\Omega_0^2}(v_{\parallel 0}^2+\mu B_0)\cos\theta = 0, \quad (2)$$

which describes potato orbits close to the magnetic axis. As shown in Ref. 6, the solution to Eq. (2) is characterized by the parameter $\sigma\kappa$ where $\sigma = v_{\parallel 0}/|v_{\parallel 0}|$ and $\kappa = (8/27)(Iv_{\parallel 0}/\Omega_0)^3/[(I^2C_1/\Omega_0^2)^2(v_{\parallel 0}^2+\mu B_0)^2]$. Circulating particles are characterized by $-\infty < \sigma\kappa < -1$ [class (i) and (iv) in Fig. 1] and $0 < \sigma\kappa < \infty$ [class (ii)] in Fig. 1]. Trapped particles, i.e., the potato orbits, are characterized by $-1 < \sigma\kappa < 0$ [class (iii) in Fig. 1]. The bootstrap current is induced by the friction between trapped potato orbits and circulating particle orbits.

A convenient expression for the banana bootstrap current is given in Ref. 9. Here, we express the flux surface and radial averaged potato bootstrap current $\left\langle J_\parallel B\right\rangle_b$ derived in

Ref. 6 in a similar form

$$\left\langle J_{\parallel}B\right\rangle_b = J_0 \frac{\ell_{22}^b \widehat{\mu}_{11}^e}{\ell_{11}^{eb}\ell_{22}^{eb} - (\ell_{12}^{eb})^2} \left\{ \left(1 - \frac{\ell_{12}^{eb}}{\ell_{22}^{eb}} \frac{\widehat{\mu}_{12}^e}{\ell_{11}^e}\right) \left[A_1^e + \frac{1}{Z_i}\frac{T_i}{T_e}\left(A_1^i + \alpha_i A_2^i\right)\right] + \frac{\widehat{\mu}_{12}^e}{\widehat{\mu}_{11}^e}\left(1 - \frac{\ell_{12}^{eb}}{\ell_{22}^{eb}}\frac{\widehat{\mu}_{22}^e}{\widehat{\mu}_{12}^e}\right) A_2^e \right\}, \tag{3}$$

where $J_0 = -IcP_e$, c is the speed of light, B_t is the toroidal magnetic field strength, $P_e = N_eT_e$ is the electron pressure, N_e is the electron density, T_e is the electron temperature, $A_i^e = P_e'/P_e$, $A_1^i = P_i'/P_i$, $A_2^i = T_i'/T_i$, $A_2^e = T_e'/T_e$, P_i is the ion pressure, T_i is the ion temperature, Z_i is the ion charge, and prime denotes $d/d\psi$. The electron potato viscous coefficients $\widehat{\mu}_{ij}^e$ are $\widehat{\mu}_{11}^e = (N_eM_e/\tau_{ee})\, x_e(0.531 + 0.928Z_i)$, $\widehat{\mu}_{12}^e = -(N_eM_e/\tau_{ee})\, x_e(0.542 + 1.237Z_i)$, and $\widehat{\mu}_{22}^e = (N_eM_e/\tau_{ee})\, x_e(1.282 + 2.732Z_i)$, where M_e is the electron mass, τ_{ee} is the electron-electron collision time, $x_e = 2.2(I_0v_{te0}/\Omega_{e0})^{1/3}\cdot \left[q_0/(\delta_0 I_0 R_0)\right]^{1/3}$, v_{te0} is the electron speed, Ω_{e0} is the electron gyrofrequency, and the subscript "0" here indicates the quantities are evaluated at the magnetic axis.

The parameter α_i is defined as $\alpha_i = \frac{\ell_{22}^i(\widehat{\mu}_{22}^i/\widehat{\mu}_{11}^i}{\widehat{\mu}_{22}^i + \ell_{22}^i - (\widehat{\mu}_{12}^i)^2/\widehat{\mu}_{11}^i}$. The ion potato viscous coefficients are $\widehat{\mu}_{11}^i = (\sqrt{2}\, N_iM_i/\tau_{ii})\, 0.376\, x_i$, $\widehat{\mu}_{12}^i = -(\sqrt{2}\, N_iM_i/\tau_{ii})\, 0.383\, x_i$, and $\widehat{\mu}_{22}^i = (\sqrt{2}\, N_iM_i/\tau_{ii})\, 0.907\, x_i$, where M_i is the ion mass, N_i is the ion density, τ_{ii} is the ion-ion collision time, $x_i = 2.2(I_0v_{ti0}/\Omega_{i0})^{1/3}\cdot(q_0/\delta_0I_0R_0)^{1/3}$, v_{ti} is the ion thermal speed, and Ω_i is the ion gyrofrequency. The quantities ℓ_{ij}^{eb} are defined as $\ell_{ij}^{eb} = \ell_{ij}^e + \widehat{\mu}_{ij}^e$ where $\ell_{11}^e = N_eM_e/\tau_{ei}$, $\ell_{12}^e = -1.5\ell_{11}^e$, and $\ell_{22}^e = \left(13/4 + \sqrt{2}/Z_i\right)\ell_{11}^e$.

For the computational purpose, we have to connect potato asymptotic limit to the banana asymptotic limit. There is no unique way to accomplish this goal. Here, we simply join the viscosity coefficients $\widehat{\mu}_{ij}^e(\widehat{\mu}_{ij}^i)$ in the banana and potato limits by the following simple formula

$$\widehat{\mu}_{ij}^e = \left[\left(\widehat{\mu}_{ij}^e\right)_p^3 + \left(\widehat{\mu}_{ij}^e\right)_b^3\right]^{1/3}. \tag{4}$$

where $\left(\widehat{\mu}_{ij}^e\right)_p$ are electron potato viscosity coefficients and $\left(\widehat{\mu}_{ij}^e\right)_b$ are banana viscosity coefficients given in Ref. 9. The connection formula in Eq. (4) is motivated by the observation that the potato modification on the standard banana orbit is of the order of $[(I_0 v_{tj0} C_1^2/\Omega_{j0})^{1/3}/\sqrt{\epsilon}]^3$. Or in terms of the fraction of trapped particles f_t, $\left(f_t^p/f_t^b\right)^3$ where f_t^p is the fraction of the trapped potato which is proportional to x_j and f_t^b is the fraction of the trapped bananas which is proportional to $\sqrt{\epsilon}$. The correction of the banana orbit effects on the potato orbits is $\left(f_t^b/f_t^p\right)^4$ as shown in Ref. 8. Thus, our connection formula overestimates slightly in the transition region. One could construct a more sophisticated connection formula to account for this asymmetric asymptotic behaviors. But the resultant formula is more complicated and not necessarily more accurate.

For simplicity, we assume $T_e = T_i$ and assume that $\gamma = L_p/L_t = 0.5$. Here L_p and L_t are temperature and pressure gradient scale length. With these assumptions, parallel plasma current is completely determined by the pressure gradient. The quantity II' in Grad-Shafranov equation is determined by

$$\left\langle J_\parallel B \right\rangle = IcP' - \frac{c\langle B^2 \rangle}{4\pi} I', \tag{5}$$

with the vacuum value $I = RB_t$ as the boundary condition. The $\left\langle J_\parallel B \right\rangle$ on the left-hand side of Eq. (5) is given in Eq. (3) for our equilibrium calculations. Note that because there is no other current source besides the pressure gradient driven current, our equilibrium current density profile is exactly the same as the bootstrap current density profile. This is not an assumption, but is a natural consequence of a complete pressure-gradient-driven-current tokamak.

The fact that equilibrium exist follows from a theorem in Ref. 10. It is shown that as long as pressure P and current I are analytic functions of ψ and that plasma current does not vanish on the magnetic axis, tokamak equilibrium exist. The potato bootstrap current is approximately a constant in ψ if pressure is a parabolic function of minor radius r or a linear function in ψ in the region close to the magnetic axis. Thus, the property of the potato bootstrap current is consistent with the existence theorem of the equilibria.

To demonstrate the existence of the equilibria explicitly, we solve Grad-Shafranov equation numerically with a fixed boundary code (TOQ).[4] We have found equilibria in the parameter space we have searched. Here, we only show a typical one with an aspect ratio $A = 1.4$. The vacuum magnetic field on the axis is $2T$. In Fig. 2, we show the flux surface of this particular equilibrium which has an elongation parameter $\delta = 3.0$ and a triangularity parameter $\kappa = 0.522$ at the edge. The plasma beta β on the magnetic axis is $\beta_0 = 52\%$ and the average β is 32.2%. The pressure gradient P' profile employed is shown in Fig. 3. The increasing in the magnitude of P' in the region close to the magnetic axis is to reduce the q value on the magnetic axis. Note that the safety factor q profile is reversed as shown in Fig. 3. This is because the current density profile is hallow as shown in Fig. 4 due to the fact that the fraction of the trapped population decreases towards magnetic axis. The total current in this case is 9.6 MA. The reversed q profile is natural to this class of equilibria. In fact, all the equilibria we have found so far have reversed q profile. This does not imply, however, that there are no equilibria with monotonically increasing q profiles. It is just that we have not searched for them. The pressure profile is also shown in Fig. 4. This particular equilibrium is stable against high-n ballooning mode checked by the BALOO code.[4] We have not studied the kink stability property, which is beyond the scope of the present paper. However, we do plan to study the kink stability for this class of equilibria in the future. We would like to note that if kink modes are unstable, they could be stabilized by a close fitting wall.

In conclusion, we have found a class of steady-state tokamak equilibria without current drive. This is made possible by the existence of the significant amount of the potato bootstrap current on the magnetic axis. The ratio of the potato bootstrap current density to the banana bootstrap current density at normalized $\sqrt{\psi} = 0.5$ is about $f_t^p/f_t^b \sim (2q_0\rho_e/\delta_0 R_0)^{1/3}/\sqrt{\epsilon/2}$. For the equilibrium shown in Figs. 2–4, $f_t^p/f_t^b \sim 13\%$ with $T_e = 10\,\mathrm{keV}$, and $R_0 = 140\,\mathrm{cm}$. Note that because both q_0/δ_0 and $(\rho_e/R_0)^{1/3}$ are not sensitive to the size of the machine,

f_t^p/f_t^b is still substantial in reactors. This class of equilibria has naturally reversed q profile, and is stable against high-n ballooning mode. Tokamak can, therefore, in principle, be operated in this intrinsically steady-state mode in certain parameter space.

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FIGURE CAPTIONS

FIG. 1. Standard banana orbit and particle orbits close to the magnetic axis. Class (i), (ii) and (iv) orbits are circulating particles with $-\infty < \sigma\kappa < -1$ and $0 < \sigma\kappa < \infty$. Class (iii) trapped particles, i.e., potato orbits, are characterized by $-1 < \sigma\kappa < 0$. The standard banana orbit is (v).

FIG. 2. Flux surface of a steady-state tokamak equilibrium with $A = 1.4, \delta = 3.0$, and $\kappa = 0.522$.

FIG. 3. Safety factor q profile and pressure gradient P' as a function of normalized radius $\sqrt{\psi}$.

FIG. 4. Pressure profile and toroidal current density as a function of major radius R.