Turbulent transport in mixed states of convective cells and sheared flows

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Abstract

Low-order mode coupling equations are used to describe recent computer simulations of resistive-g turbulent convection that show bifurcations for the onset of steady and pulsating sheared mass flows. The three convective transport states are identified with the tokamak confinement regimes called low mode (L-mode), high mode (H-mode), and edge-localized modes (ELMs). The first bifurcation (L $\rightarrow$ H) and the second bifurcation (H $\rightarrow$ ELMs) conditions are derived analytically and compared with direct solutions of the 6-ode mode coupling equations. First an exact expression is given for the energy transfer rate from the fluctuations to the sheared mass flow through the triplet velocity correlation function. Then the time scale expansion required to derive the Markovian closure formula is given. Markovian closure formulas form the basis for the thermodynamic-like L-H models used in several recently proposed models.

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I. INTRODUCTION

Recent computer simulations of resistive-g turbulence[1−5] show that above a critical pressure gradient the plasma convection is unstable to the onset of sheared mass flows. This spontaneous generation of sheared mass flows is a robust phenomena occurring in both two-dimensional (2D) and three-dimensional (3D) simulations of resistive-g turbulence in a sheared magnetic field, drift-resistive ballooning[6] (DRBM), and in the simulations of the toroidal ion-temperature-gradient (ITG) driven turbulence.[7,8] The shear flow generation mechanism is the same for these transport models, and since the resistive-g system with two partial differential equations (PDE's) is considerably simpler than the three[7] and four[8] pde's needed to describe DRBM and ITG turbulence we restrict the present study to the resistive-g transport model. Thus, we view the resistive-g, or equivalently the resistive interchange, equations for the coupling of the vorticity and the pressure convection as a paradigm for the turbulent transport in confined plasmas.

The resistive-g system in a sheared magnetic field is a two-parameter family of pde's closely analogous to the Rayleigh-Bénard system parameters by the Rayleigh number ($\mathrm{Ra} = g\alpha T' L^4/\mu\kappa$) and the Prandtl number ($\mathrm{Pr} = \mu/\kappa$). Thus it is useful to describe the plasma system in terms of the effective $\mathrm{Ra}^{\mathrm{eff}}$ and $\mathrm{Pr}^{\mathrm{eff}}$ even though the analogy is not complete. A principal difference arises from the inhomogeneity of the plasma equilibrium introduced by the magnetic shear length L_s which then replaces the scale height L in the Rayleigh-Bénard due to the localization of the eigenmodes. In the ITG turbulence[9,10] more complex considerations determine the effective radial wavenumber k_x. In this work we avoid these complications by reverting to sinusoidal radial modes adding the caution that the interpretation of the value of k_x occurring in the formulas presented here depends on the details of the magnetic shear and the toroidicity and is a difficult problem in itself.

Likewise the choice of the parallel wavenumber is a closely related difficulty in the plasma analog which we simplify here by taking $k_{||} = 0$ for the pure radial modes and $k_{||} = \pi/L_{||}$. For the interior plasma $L_{||}$ is determined by the mode structure discussed above, and in the scrape-off-layer (SOL) plasma $L_{||}$ is the connection length to the divertor plates.[3]

Returning to the comparison of the plasma convection with the Rayleigh-Bénard system the second principal difference is that the plasma vorticity ($\Omega = \widehat{\mathbf{e}}_z \cdot \nabla \times \mathbf{v} = c\nabla_\perp^2 \phi/B$) is damped both by collisional viscosity $\mu_\perp \nabla_\perp^2 \Omega = -k_\perp^2 \mu_\perp \Omega_k$ where ion-ion collisions give $\mu_\perp = 0.3\nu_{ii}\rho_i^2$ and by the collisional decay from parallel plasma currents at the rate determined by the dielectric constant $\epsilon_\perp$ shielded electrical conductivity $\sigma_s = n_e e^2/m_e\nu_e$. Here σ_s is the Spitzer conductivity and $\epsilon_\perp = 1 + 4\pi\rho_m c^2/B^2$ which greatly exceeds unity in tokamaks. This vorticity damping mechanism follows from $\nabla_\perp \cdot \mathbf{j} + \nabla_{||} j_{||} = 0$ with $\mathbf{j}_\perp = \epsilon_\perp \partial \mathbf{E}_\perp/\partial t$ and $\mathbf{j}_{||} = \sigma_s E_{||}\mathbf{b}$ to obtain $\nu_\Omega = -k_{||}^2\sigma_s/k_\perp^2\epsilon_\perp$. Thus, the effective Prandtl number for the damping of the fluctuations is $\mathrm{Pr}^{\mathrm{eff}} = \mu_\perp^{\mathrm{eff}}/\kappa$ with $\mu_\perp^{\mathrm{eff}} = \mu_\perp + k_{||}^2\sigma_s/k_\perp^4\epsilon_\perp$ where $\widehat{\sigma}_{||} = \sigma_s/\epsilon_\perp = T_e/m_e\nu_e\rho_s^2$. The role of the electrical conductivity damping is examined in Pogutse *et al.*[3] for both the local Spitzer conductivity (σ_s) and the nonlocal sheath conductivity (σ_e). The sheath conductivity replaces the Spitzer conductivity as the dominant damping mechanism at high electron temperatures ($T_e \gtrsim 100\,\mathrm{ev}$) in the edge or SOL plasma. Inside the separatrix the Spitzer conductivity is dominant. In the core the effective conductivity becomes reduced again by the drift wave term $\nabla_{||} p_e$ in Ohm's law which is taken into account in the DRBM and ITG transport models.

Notwithstanding these differences in the structures of the eigenmodes and the vorticity damping in the plasma compared with the neutral fluid, it is clear that the mechanism for the onset of tilted vortices from the nonlinear convection is the same in these transport systems. Measurements from the onset of tilted vortices and plumes in the Rayleigh-Bénard convection experiment are reported by Krishnamurti and Howard.[11] Thus in addition to computer simulation of plasma convection, we have analogous hydrodynamic experiments confirming the basic mechanism. The hydrodynamics are modeled by Howard and Krishnamurti[12] by a low-order system of mode coupling equations similar to those used in the present work. For the classical Rayleigh-Taylor problem with free slip boundary conditions, Finn[13] reports the generation of shear flow at $\mathrm{Pr} = 1$ and proposes a simple two-mode model.

The new results reported here are a detailed study of the mechanism for the rate of the energy transfer between the three energy components of the system: the potential energy $U(t)$, the energy in the turbulent fluctuations $W(t)$, and the kinetic energy $F(t)$ in the mean flows. The rate of energy transfer is determined by the three field correlation functions arising from the convection of the vorticity and the pressure. We test three closure approximations for the triplet (third order) velocity correlation function. It is the closure of these triplet correlation functions that allows the introduction of a reduced thermodynamic model for the L-H-ELMs dynamics that have been written down without analysis of the closure problem in earlier works.

To be more explicit the three field velocity correlator that takes energy out of the turbulence W into the shear flow F has appeared in earlier works as $\tau_c\, FW$, FW/U and $FW/U^{1/2}$. These are the closure formulas for the triplet correlator which is defined in Sec. IV. Here τ_c is the correlation time taken as a constant in the first case and as related to the mean driving pressure gradient-curvature driving energy U through $\tau_c = \ell_c/U^{1/2}$ in the third case. Here we compare these closure formulas with the mode coupling dynamics.

Finally, we discuss the properties of the thermodynamic models. In a broad sense these reduced equations are the analogs of the wave-kinetic equations where interaction with the pure radial modes giving shear flow replacing one of the waves. In both cases irreversibility is introduced by the Markovian closure approximation. Thus, we look for the corresponding entropy production functional for the system. As in the three-wave kinetic equation the parameter range for the validity of the thermodynamic description is not well defined. Here the bifurcation diagram derived in Sec. III provides a useful guide.

The work is organized as follows: In Sec. II we discuss the justification for introducing a low-order mode coupling description giving the 6-ode system for resistive-g transport. In Sec. III we derive the bifurcation criteria for the first and second instabilities of the mode coupling equations and compare the direct numerical solutions (DNS) with the analytic (A) criteria. We show the parametric collisionless dependence of the limit cycle frequency and the fractional change in the thermal flux with the input power. In Sec. IV we introduce three closure formulas and compare them with the DNS. Based on the closures we give thermodynamic-like equations and discuss their properties. The irreversibility introduced by the Markovian closure gives an entropy production functional. In Sec. V we give the conclusions commenting on the relationship with earlier models and with the phenomenology of the L-H-ELMs dynamics.

II. LOW-ORDER MODE COUPLING EQUATIONS

The numerical simulations of the resistive-g dynamics show that the first few low m modes ($k_y = 2m\pi/L_y$) contain most of fluctuation energy in the saturated state. For example, the $3D$ sheared magnetic field simulations[2] found it sufficient to use representations with $|m| \leq 11$ and $|n| \leq 2m$. In the saturated state the $m = 1$ modes at the separated rational surfaces given by $n = 0, \pm 1, \pm 2$ were the dominant fluctuations. The simulations of Pogutse *et al.*[3] with no magnetic shear found it sufficient to use a 32×32 grid in $\mathbf{x}$-space to represent the turbulent convection. Using a minimum of six grid points to represent a sinusoidal period this is equivalent to a 5×5 grid in $\mathbf{k}$-space. In their simulations, $k_{||} = 2\pi/L_{||}$ is a constant.

Even in the limit where the resistive-g mode transforms into the ITG mode, the gyrofluid simulation of Waltz *et al.*[8] studies grid convergence and concludes that a $40 \times 10(k_x, k_y)$ grid gives good convergence. The saturated states show again most of the fluctuation energy in a low-k_y $\quad(k_y\rho_s = 0.2)$ mode. Thus we conclude that it is useful and reasonable to study the low-order mode coupling dynamics in some details for these systems.

The resistive-g paradigm for turbulent plasma convection is given by

$$\partial_t \nabla^2\phi + [\phi, \nabla^2\phi] + g\,\frac{\partial p}{\partial y} = (\sigma_{||}k_{||}^2 + \mu_\perp \nabla_\perp^4)\phi \tag{1}$$

$$\frac{\partial p}{\partial t} + [\phi, p] + R\,\frac{\partial \phi}{\partial y} = \kappa_\perp \nabla^2 p \tag{2}$$

where the vorticity equation (1) follows from $\nabla \cdot \mathbf{j} = 0$ and the pressure convection equation (2) is for the energetically dominant component which is typically the ions in the edge and SOL of a tokamak. The precise form of the equation varies as given in Refs. 1–8. We note that the dimensionless variables for the time scale $t_0 = (L_p R_c)^{1/2}/c_s \sim$ few microseconds in Refs. 1–6 whereas the space scale normalization differs between the references. These differences are not important, however, for the following calculations.

For the resistive-g turbulent convection we represent the state of system with two complete fluctuations dynamics (ϕ_1, p_1) and (ϕ_2, p_2) chosen with $\mathbf{k}_1 = (k_x, k_y)$ and $\mathbf{k}_2 = (2k_x, k_y)$. The consideration of the mode coupling terms then shows that the convective nonlinearity of the vorticity $\mathbf{v}_E \cdot \boldsymbol{\nabla}_\perp \Omega$ and the pressure $\mathbf{v}_E \cdot \boldsymbol{\nabla}_\perp p$ create the convective flows $\phi_0 \sin(k_x x)$ and the flattening of the background pressure gradient $p_0 \sin(2k_x x)$. All higher order components $(2k_x, 2k_y)$, $(3k_x, 0)$, … are dropped (Galerkin approximation) being outside the truncated six-dimensional state space of $(\phi_0, \phi_1, \phi_2, p_0, p_1, p_2)$. The procedure is standard and the resulting mode coupling coefficients are the same as given in Ref. 12. The potential and pressure are represented by

$$\phi = \phi_0 \sin(k_x x) + \phi_1 \sin(k_x x)\sin(k_y y) + \phi_2 \sin(2k_x x)\cos(k_y y), \tag{3}$$

$$p = p_0 \sin(2k_x x) + p_1 \sin(k_x x)\cos(k_y y) + p_2 \sin(2k_x x)\sin(k_y y). \tag{4}$$

And the dynamics of (ϕ, p) is given by

$$k_x^2 \frac{d\phi_0}{dt} = -\frac{3}{4}\, k_x^3 k_y \phi_1 \phi_2 - k_x^2(\nu_L + \mu k_x^2)\phi_0, \tag{5}$$

$$k_1^2 \frac{d\phi_1}{dt} = k_y g p_1 + \frac{k_x k_y}{2}\,(3k_x^2 + k_y^2)\phi_0\phi_2 - k_1^4 \mu \phi_1, \tag{6}$$

$$k_2^2 \frac{d\phi_2}{dt} = -k_y g p_2 - \frac{k_x k_y}{2}\, k_y^2 \phi_0 \phi_1 - k_2^4 \mu \phi_2, \tag{7}$$

$$\frac{dp_0}{dt} = \frac{k_x k_y}{2}\, \phi_1 p_1 - 4k_x^2 \kappa p_0, \tag{8}$$

$$\frac{dp_1}{dt} = k_y R \phi_1 - k_x k_y \phi_1 p_0 - \frac{k_x k_y}{2}\, \phi_0 p_2 - k_1^2 \kappa p_1, \tag{9}$$

$$\frac{dp_2}{dt} = -k_y R \phi_2 + \frac{k_x k_y}{2}\, \phi_0 p_1 - k_2^2 \kappa p_2, \tag{10}$$

where $k_1^2 = k_x^2 + k_y^2$ and $k_2^2 = 4k_x^2 + k_y^2$. Here $R = -d \ln P_0/dx$ is the background pressure gradient and $g = v_T^2[\hat{\mathbf{e}}_x \cdot (\mathbf{b} \cdot \boldsymbol{\nabla} \mathbf{b})] = v_T^2/R_c$ is the average curvature of magnetic field line.

The 6 ordinary differential equations (ODE) model defined by Eqs. (3)–(8) has many nice properties in terms of the bifurcations from L-H-ELM states which appear as attractors in the 6-dimensional state space. There are two extensions of the representation in (3)–(4) that have been considered based on the conservation laws of the system (1)–(2). By adding a seventh equation for the pressure flattening component $p_4 \sin(4k_x x)$, which would make the isolated harmonic set $\{\phi_2, p_2, p_4\}$ a complete Lorenz system, the resulting 7-ODE model has better energy conservation properties during the L-H transitions. The analysis of this $d = 7$ system and higher order generalizations is given in Thiffeault and Horton.[14] Hermiz *et al.*[15] show that the conservation of the mean vorticity at finite viscosity requires the addition of $\phi_3(t) \sin(3k_x x)$ to Eq. (4). The steady state solution for ϕ_3 gives the modification of the shear flow profile with $\phi_3 = -\phi_0/(27)$ and the instability onset condition discussed after Eq. (20).

III. BIFURCATION CRITERIA

A. First instability

For sufficiently low driving force $I \equiv gR$ from the pressure gradient in the unfavorable curvature, only the first mode ϕ_1, p_1 is linearly unstable. The dispersion relation for the perturbations $\phi_1 e^{\lambda t}, p_1 e^{\lambda t}$ is

$$k_1^2 \lambda^2 + k_1^4 (\mu + \kappa) \lambda + k_1^6 \mu \kappa - k_y^2 R g = 0, \tag{11}$$

giving the condition for the first bifurcation to steady convection

$$I \geq I_c = \frac{k_1^6 \mu \kappa}{k_y^2}. \tag{12}$$

For some purposes, it is useful, such as in Fig. 1, to define the normalized Rayleigh number

$$\mathrm{Ra} = \frac{I}{I_c} = \frac{Rg}{(k_1^6 \mu \kappa / k_y^2)}. \tag{13}$$

In the case of finite $k_{\|}^2$, we note that the definition of I_c becomes

$$I_c = \frac{k_1^2 \kappa (k_1^4 \mu + k_{\|}^2 \sigma_{\|})}{k_y^2}. \tag{14}$$

For $1 < \mathrm{Ra} < \mathrm{Ra}_2$ the system (3)–(8) evolves to one of the two Lorenz attractor branches

$$\phi_1 = \pm \sqrt{\frac{8}{k_1^4 \mathrm{Pr}} (I - I_c)} = \pm \sqrt{\frac{8 I_c}{k_1^4 \mathrm{Pr}} (\mathrm{Ra} - 1)}, \tag{15}$$

$$p_1 = \frac{k_1^4 \mu}{k_y g} \phi_1 = \pm \frac{k_1^4 \mu}{k_y g} \sqrt{\frac{8 I_c}{k_1^4 \mathrm{Pr}} (\mathrm{Ra} - 1)}, \tag{16}$$

$$p_0 = \frac{I_c}{k_x g} (\mathrm{Ra} - 1), \tag{17}$$

and any initial ϕ_0, ϕ_2, p_2 decay to zero. The choice of the branch depends on the initial data or the residual diamagnetic drift term and is reported in Bazdenkov *et al.*[16]

The two convective transport fluxes for the system are the thermal flux

$$q_x = \angle p v_x = \tfrac{1}{2} k_y \left[p_1 \phi_1 \sin^2(k_x x) - p_2 \phi_2 \sin^2(2 k_x x) \right], \tag{18}$$

and the momentum flux

$$\pi_{xy} = \angle v_x v_y = \tfrac{1}{2} k_x k_y \phi_1 \phi_2 \sin(k_x x). \tag{19}$$

On the sub-space of the Lorenz attractor the momentum flux vanishes and the thermal flux in this steady state of convection is

$$q_x^L = \frac{1}{4} k_y (p_1 \phi_1 - p_2 \phi_2) = \frac{2\kappa}{g} (I - I_c).$$

From the solutions of Eqs. (13)–(15) we see that for weak viscous-conductivity damping the flow velocity term ϕ_1 becomes large such that $\mathbf{E} \times \mathbf{B}$ rotation rate ω_E in the cell can exceed the linear growth rate $k_x k_y \phi_1 > \gamma^l$ where $\gamma^l \equiv \max(\lambda)$. Also the gradient in the y-direction of the pressure becomes very strong with $k_y p_1 > R$. Thus, we may expect the high flow velocities and steep pressure gradients in the fundamental mode to become unstable to a secondary instability.

B. Second instability

Now we take the fixed-point given by $\phi_0 = \phi_2 = p_2 = 0$ and (ϕ_1, p_1, p_0) in Eqs. (13)–(15) as known and consider the stability of this fixed point to small amplitude perturbation $(\phi_0, \phi_2, p_2)e^{\lambda t}$. The stability criterion is a function of the flows in the primary or first unstable mode. The dispersion relation for the secondary mode is the cubic equation given by

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0, \tag{20}$$

where

$$\begin{aligned} A &= k_2^2(\mu + \kappa) + k_x^2\mu, \\ B &= -\frac{k_y^2}{k_2^2}\, I - \frac{3k_x^2 k_y^2}{k_1^4 k_2^2 \mathrm{Pr}}\,(I - I_c) + k_x^2 k_2^2 \mu(\mu + \kappa) + k_2^4 \mu\kappa, \\ C &= k_x^2 k_2^4 \mu^2 \kappa - \frac{k_x^2 k_y^2 \mu}{k_2^2}\, I - \left(\frac{3k_x^2 k_y^4 \kappa^2}{k_1^4 \mu} + \frac{3k_x^2 k_y^2 \kappa}{k_2^2}\right)(I - I_c). \end{aligned} \tag{21}$$

where the excess $(I - I_c)$ determines the strength of the primary flow ϕ_1 and gradients (p_1, p_0), and thus the stability of steady state in Eqs. (13)–(15).

The cubic equation (20) loses its stability in two ways. For $C < 0$ one eigenvalue λ becomes positive with $\lambda_{\mathrm{unstable}} \approx -C/B$ and the solutions evolve to a new fixed point with finite steady shear flow. For $C > AB$, there is the Hopf bifurcation with a pair of complex conjugate eigenvalues (λ, λ^*) going unstable, $\mathrm{Re}(\lambda) > 0$. After Ra is increased to the point that $C > AB$, there is a Hopf bifurcation. The solutions go to a limit cycle with angular frequency $\mathrm{Im}(\lambda) = \sqrt{B}$ giving frequency of the limit cycle for small amplitude oscillation near the critical condition for the Hopf bifurcation.

In general, the bifurcation curve is determined by $C = 0$ and $C = AB$ which are best determined numerically due to the complexity of the coefficients in Eq. (21). Here we give the analytic results for a set of reference parameters. For $r = k_x/k_y$, the second instability is determined by $C = 0$ and gives the critical resistive-g driving strength

$$I \geq \frac{\left[r^2(4r^2+1)^3\mathrm{Pr}^2 + 3r^2(r^2+1)^3\mathrm{Pr} + 3r^2(r^2+1)(4r^2+1)\right]}{r^2(r^2+1)^3\mathrm{Pr}^2 + 3r^2(r^2+1)^3\mathrm{Pr} + 3r^2(r^2+1)(4r^2+1)} I_c \qquad (22)$$

or the equivalent ratio for $\mathrm{Ra}_2 = I/I_c$. For the specific case $k_x = k_y, r = 1$ we have $I \geq \frac{5}{2}\left(\frac{25\mathrm{Pr}^2+24/5\mathrm{Pr}+6}{4\mathrm{Pr}^2+12\mathrm{Pr}+15}\right) I_c$.

For low viscous damping of the vorticity, the threshold given by Eq. (22) approaches the threshold for the first instability as $\mathrm{Pr} \ll 1$. For large viscous damping $\mathrm{Pr} \gg 1$ the threshold becomes for $r = 1$

$$\mathrm{Ra} \geq \left(\frac{125}{8}\right) \approx 15.6.$$

Hermiz *et al.* show that the addition of the ϕ_3-vorticity conserving mode modifies the threshold (20) for the onset of shear flow as follows. The coefficients of the Pr^2-terms are unchanged, the coefficients of the Pr terms change from $3 \to 10/3$ and the coefficient of the terms independent of Pr change from $3 \to 2(5-4r^2)$. Thus, the effect of ϕ_3 is significant for $\mathrm{Pr} \leq 1$ and stabilizes the cells with $r = k_x/k_y \gtrsim 1$ with respect to the onset of shear flow.

In Fig. 1, we show the bifurcation diagram with the critical curves for $\mathrm{Ra} = I/I_c$ versus $\mathrm{Pr} = \mu/\kappa$ determined by Eqs. (10) and (22) for the first and second instability and the third instability calculated from $C = AB$ giving the transition to oscillatory states identified with ELMs in the edge region of tokamaks. The condition for this third instability is straightforward to calculate from the formulas from A, B, C in Eq. (21), but too complicated to warrant writing out. Finally, in the upper part of Fig. 1, we show the region of chaotic behavior determined from the numerical solution. The boundary for the chaotic shifts strongly with an increasing number of modes so that we do not emphasize this regime. For the reference parameter used in computing Fig. 1, the gap between second threshold bifurcation and the third instability where the H-mode convection occurs is relatively narrow. The gap for the H-mode increases when the ratio k_x/k_y is increased.

In relation to the partial differential equation the transition to the shear flow state between the second and third instability is robust being clear in both the 2D and 3D simulation of Sugama and Horton[1,2] with magnetic shear and in the shearless simulations of Pogutse *et al.*[3] The ELMs regime past the third instability is seen in the simulations of Pogutse *et al.*[3] The corresponding intermittent behavior in ITG turbulence is shown in the long-time runs of Su *et al.*[7] Takayama *et al.*[5] show intermittent vortex structures appearing in the H-mode state which they identify as ELMing activity in two-dimensional interchange driven convection. The weak intermittency seen in the ELM-like solutions of Takayama *et al.*[5] is presumably a reflection of the chaos in the low-order truncated systems.

An important part of the identification and classification of the ELMs involves the variation of the ELMing frequency $\nu(P_{in})$ with increasing power delivered to the transport layer. For this purpose we have added a constant term P_{in} to the right-hand side of Eq. (6) and obtained the frequency of the limit cycle as a function of P_{in}. The prediction for $\nu(P_{in})$ from the 6-ODE model are shown in Fig. 2. The frequency first decreases with increasing power, which is the behavior reported in the turbulence model of Sugama and Horton,[17] and is the signature of type III ELMs.[18] For $P_{in}/P_{th} \gtrsim 1.6$ the ELM frequency increases with increasing P_{in}, for these particular parameters, which is the signature of type I ELMs. For large P_{in}, faster relaxation oscillations are required to transport the higher input power $P_{in}\sin(2k_x x)$. At low P_{in} the behavior is determined by $\mathrm{Im}(\lambda) = \sqrt{\overline{B}}$ near the critical condition for Hopf bifurcation, where $\overline{B}$ is the B term in Eq. (21) with the effect of P_{in}. In the ELM literature many different (grassy, mossy, etc) time series are observed in the recombination radiation and $\delta B_\theta(t)$ signals used to identify ELMs.

The state of convection flows is described by the (ϕ_0, ϕ_1, ϕ_2) state space diagram in Fig. 3. Along the stable, attracting curves in Fig. 3, the system changes from closed, rectangular cells with no momentum flux $\angle v_x v_y = 0$ to tilted cells with a net momentum flux $\pi_\perp = \angle v_x v_y \neq 0$ driving the shear flow $\overline{v}_y = d\phi_0/dx$ against the viscous damping. Note that for the background flow the conductivity damping vanishes and only the collisional Pfirsch-Schlüter, or neoclassical damping, which is a frictional damping between trapped and passsing ions, remains strong in axisymmetric tokamak. In Fig. 4 we show the transition of the convective flow pattern to this H-mode regime. The importance of the tilting of the cells or plumes for the creation of the shear flow is what is clearly shown in the analogous Rayleigh-Bénard experiments of Krishnamurti and Howard.[11] The onset of cell tilting and momentum transport is investigated in plasmas in Horton *et al.*,[19] Finn *et al.*[20] and Drake *et al.*[21] have also done some modeling.

In Fig. 5 we show that the change of the thermal flux in a L-H transition is a strong function of Pr. After Pr passes a critical value $(\mathrm{Pr})_c$ as we increase Pr, the heat flux *increases* despite that the fluctuation intensity decreases in order to conserve the energy. The reason for this change is not totally clear but is also found in the energy conserving 7-ODE mode.[14] We argue that the flux is not only related to the fluctuation intensity, but also to the relative phase of p and ϕ which also suffers a modification during the L-H transition. This serves as a caution that in the conventional thermodynamical modelings of L-H transition that focus only on the fluctuation intensity, the whole picture of L-H transition is rather incomplete because the phase information of the fluctuating variables is missing.

In Fig. 6 the evolution of the three energy components shear flow F, fluctuation energy W, and potential energy U is shown. The thermodynamical variables F, W, U [defined precisely in Eqs. (21)–(23)] change rapidly during the L-H transition, which is expected. The total energy $(F + W - U)$ suffers a much smaller change during the transition in this 6-D model. The dominant fractional change is the drop in the fluctuation energy W and the increase of the shear flow energy F.

The transfer of convective kinetic energy between the shear flow and the $k_y \neq 0$ fluctuations is controlled by the three triplet correlation functions. The quasilinear model is investigated in Pogutse *et al.*[3] and K-epsilon turbulent closure model is used in Sugama and Horton.[17] Now we investigate the closure problem in more detail.

IV. CLOSURES FOR REDUCED DYNAMICAL MODELS

In preparation for deriving a reduced dynamical description we investigate the flow of energy through the physically distinct energy components consisting of the effective potential energy $U(t)$, the turbulent kinetic energy $W(t)$ and the kinetic energy $F(t)$ in the mean flows. The three components are defined by

$$U = \frac{g}{R}\, p_0^2, \tag{23}$$

$$W = \tfrac{1}{2}\,(k_1^2\phi_1^2 + k_2^2\phi_2^2 - \frac{g}{R}\, p_1^2 - \frac{g}{R}\, p_2^2), \tag{24}$$

$$F = k_x^2\phi_0^2. \tag{25}$$

where U arises from the change in the local average gradient (quasilinear flattening), W from the sum of kinetic and potential energy in the fluctuations (with W positive-definite for stable profiles $gR < 0$) and F from the kinetic energy in the mean (sheared) flow $\overline{\mathbf{V}} = v_y(x,t)\widehat{\mathbf{e}}_y$.

A straightforward calculation of $\dot{U}$, $\dot{W}$ and $\dot{F}$ shows that the rate of energy transfer is determined by the two triplet correlation functions

$$\angle p_0 p_1 \phi_1 \quad \text{and} \quad \angle \phi_0 \phi_1 \phi_2. \tag{26}$$

Here the ensemble average $\angle \cdots$ may be defined over a set of initial conditions (ϕ^i, p^i).

Calculating the rate of change of U, W, F from Eqs. (3)–(8) and adding in external sources P_U and P_F of energy and momentum, we obtain

$$\frac{dU}{dt} = \frac{g}{R}\, k_x k_y (p_0 p_1 \phi_1) - \epsilon_U + P_U, \tag{27}$$

$$\frac{dW}{dt} = \frac{3}{2}\, k_x^3 k_y (\phi_0\phi_1\phi_2) + \frac{g}{R}\, k_x k_y (p_0 p_1 \phi_1) - \epsilon_W, \tag{28}$$

$$\frac{dF}{dt} = -\frac{3}{2}\, k_x^3 k_y (\phi_0\phi_1\phi_2) - \epsilon_F + P_F, \tag{29}$$

where the dissipation rates are given by

$$\epsilon_U = \frac{8g}{R}\, k_x^2 \kappa p_0^2,$$

$$\epsilon_W = \mu(\kappa_1^4\phi_1^2 + \kappa_2^4\phi_2^2) - \frac{g}{R}\,\kappa(p_1^2 + p_2^2), \tag{30}$$

$$\epsilon_F = 2k_x^2(\nu_L + k_x^2\mu)\phi_0^2. \tag{31}$$

The ensemble averaged equations follow from Eqs. (27)–(29) with the triplet correlation functions defined in Eq. (26) give the average power transfers between U and W and F. By defining W as in Eq. (22) the nonlinear transfers between ϕ_1^2 and ϕ_2^2 and p_1^2 and p_2^2 by the shear flow, ϕ_0 are eliminated leaving the two turbulent fluxes in Eqs. (16) and (17) as the nonlinear energy transfer mechanism.

The negative contribution to ϵ_W from dissipation of the pressure fluctuations can be understood from the condition for a steady state solution given by the Lorenz attractor. We can rewrite $\dot{W} = 0$ to be

$$-\frac{g}{R} \angle q_x \frac{dp}{dx} = \frac{g}{R} k_x k_y (p_0 p_1 \phi_1) = \epsilon_W$$

and calculate using Eq. (14) for p_1/ϕ_1 that

$$\epsilon_W = \mu \kappa_1^4 \phi_1^2 \left(1 - \frac{I_c}{I}\right).$$

Thus, the dissipation ϵ_W increases from zero as I exceeds I_c with that value of ϵ_W just such as to balance the convective turbulent dissipation from $\angle q_x p'$ with q_x given in Eq. (16).

Now it is possible to repeat this calculation for the steady state (fixed point fp) given in Sec. III B for the second instability. The procedure is to solve the full set of $\dot{\phi}_i = 0, \dot{p}_i = 0$ equations for the (ϕ_i^{fp}, p_i^{fp}) and write out the values of q_x and π_{xy} from Eqs. (18) and (19). The resulting equation is a sixth-order equation in ϕ_1 which is rather complicated for root-finding. Even if we had the analytical results, they are not very useful and miss the point that in the actual system conditions vary through either the parameters, such as R and μ, or the injected power-momentum sources P_U, P_F. Therefore, the useful result is to truncate the system by expressing the triplet correlation functions in Eq. (26) in terms of the energy variables U, W, F and the system parameters.

In earlier works such approximations have been made without explicitly investigating the nature of the closure approximation. Thus, one finds for example in Pogutse *et al.*[3] that

$$\angle \phi_0 \phi_1 \phi_2 \to \tau_c W F$$

with constant τ_c, and in Suguma and Horton[17] that

$$\angle \phi_0 \phi_1 \phi_2 \to \frac{WF}{U^{\frac{1}{2}}}$$

where the correlation time is taken from $\tau_c = l_c/U^{\frac{1}{2}}$. Here we investigate the closure approximation explicitly. We start by showing in Fig. 7 the details of the evolution of the $\phi_0(t)$ and $\phi_1(t)$ components in frame (a) and the associated drop $q_x(t)$ in frame (b) for the L $\rightarrow$ H transition described by U, W, F in Fig. 6. The comparison of the triplet transfer functions computed from the full $\{\phi_i(t), p_i(t)\}$ orbits with closure models is shown in frames (a)–(d) of Fig. 8. The closure approximation is defined in each frame. The quasilinear formula in frame (a) underestimates the transfer rate by 56%, the K-epsilon model in frame (b) underestimates the transfer rate by 24% and the empirical formula given by $FW/U^{1/2}$ overestimates the rate by 8%. For other parameters the results vary indicating the need for a theory for the closure formula.

In the following we look at the closure problems in some detail. We first ask when such local-time closure formulas are expected to apply theoretically and then derive a set of nonlinear, time-local closure relations.

A. The Markovian expansion parameter

Here we assume, as appears consistent with simulations, that the harmonic mode ϕ_2, p_2 is linearly stable (damped as $e^{-\nu_2 t}$) and has a low energy content compared to the fundamental mode ϕ_1, p_1. We calculate ϕ_2, p_2 as driven by ϕ_0, ϕ_1, p_0 and introduce a Markovian (local in time) expansion in the time history integrals. The result gives formula for the kinetic energy transfer rate

$$T_F = \angle \frac{\partial v_y}{\partial x} \pi_{xy} = (\tau_c/\Delta^2) W(t) F(t)$$

where τ_c is the nonlinear effective triplet interaction time and Δ is the effective width of the shear flow layer. The structure of this formula for T_F is essentially the same as studied extensively for three-wave interactions[22,23] where the problem is to determine the nonlinear τ_c.

The driven response of the damped harmonic mode is given by

$$
\begin{pmatrix} \dfrac{\partial}{\partial t} + k_2^2\,\mu & \dfrac{k_y g}{k_2^2} \\ \\ k_y\,R & \dfrac{\partial}{\partial t} + k_2^2\kappa \end{pmatrix} \begin{pmatrix} \phi_2 \\ \\ \\ p_2 \end{pmatrix} = M\phi_0 \begin{pmatrix} \dfrac{-k_y^2\phi_1}{k_2^2} \\ \\ \\ p_1 \end{pmatrix}, \tag{32}
$$

where $M = k_x k_y/2$ and $k_2^2 = 4k_x^2 + k_y^2$.

An exact closure in the sense of eliminating the harmonic mode

$$
X_2(t) = \begin{pmatrix} \phi_2(t) \\ \\ p_2(t) \end{pmatrix} \tag{33}
$$

in terms of $\phi_0(t)$ and the fundamental mode

$$
Y_1(t) = \begin{pmatrix} \dfrac{-k_y^2}{4k_x^2 + k_y^2}\ \phi_1(t) \\ \\ p_1(t) \end{pmatrix} \tag{34}
$$

can be carried out as follows. Introducing the L_2 matrix defined by

$$
L_2 = \begin{pmatrix} k_2^2\,\mu & k_y g/k_2^2 \\ \\ k_y\,R & k_2^2\,\kappa \end{pmatrix} \tag{35}
$$

and the initial value $X_2(t_0 \to -\infty) = 0$, the solution of Eq. (30) is

$$
X_2(t) = M\int_0^\infty d\tau\, e^{-\tau L_2}\phi_0(t-\tau)Y_1(t-\tau) \tag{36}
$$

where the matrix propagator is $\exp(-\tau L_2) = \sum_n (-\tau L_2)^n/n!$.

To extract the ϕ_2 and p_2 components of Eq. (34) we introduce the quantum mechanics matrix notation $\angle 1|e^{-\tau L_2}|Y_1$ and $\angle 2|e^{-\tau L_2}|Y_1$ for contractions of the matrix equation with the basis vectors $\langle 1| = (1,0)$ and $\langle 2| = (0,1)$. With this notation we have for the non-Markovian energy transfer rate $T_F = -3Mk_x^2\phi_0\phi_1\phi_2$ between Eqs. (26) and (27) the formula

$$T_F(t) = -3k_x^2 M^2\, \phi_0\phi_1 \int_0^\infty d\tau \phi_0(t-\tau)\angle 1|e^{-\tau L_2}|Y_1(t-\tau). \tag{37}$$

Equation (35) is an exact, non-Markovian expression for the energy transfer between W and F valid for system parameters $\{\mu\} = \{R, g, \mu, \kappa, k_x, k_y\}$ where mode-2 is stable, i.e. the roots of $\det(pI + L_2) = 0$ have $\mathrm{Re}(p) < 0$. When the least damped root

$$p_+ = \left[k_2^4(\mu-\kappa)^2/4 + (k_y^2 gR/k_2^2)\right]^{1/2} - k_2^2(\mu+\kappa)/2 < 0$$

decays faster than the time rate of change of $\phi_0 Y_1$ then the system enters the Markovian domain where $T_F(t)$ depends on the local values $\phi_0^2(t)$ and $\phi_1^2(t)$. This is the reduced, strongly dissipative behavior assumed in the earlier low order state space studies.

The integral representation in Eq. (35) makes clear how to define the expansion parameter $\epsilon_M \ll 1$ used to obtain the Markovian closure. For sufficiently fast damping $p_+ < 0$ compared with $\gamma_E = |\partial_t \ell n(\phi_0 Y_1)|$ we can expect the expansion $Y_1(t-\tau) = \sum_n (-\tau)^n Y_1^{(n)}(t)/n!$ to converge. The successive terms contribute $\int_0^\infty d\tau\tau^n\, e^{-\tau L_2} = n! L_2^{-n-1}$ to the Markovian expansion of Eq. (34). The first N-terms give

$$X_2 = M L_2^{-1}\phi_0(t)Y_1(t) - M L_2^{-2}\partial_t(\phi_0 Y_1) + \cdots + (-1)^N L_2^{-N-1}\partial_t^N(\phi_0 Y_1). \tag{38}$$

The expansion parameter ϵ_M is then

$$\epsilon_M = ||L_2^{-1}||\max_t |\partial_t \ell n(\phi_0 Y_1)|\,. \tag{39}$$

For $\epsilon_M \ll 1$ we have the Markovian limit from the first term in Eq. (36) and the local transfer function

$$T_F^{(M)}(t) = -3\, k_x^2 M^2\, \frac{\angle 1|L_2^{\mathrm{cof}} Y_1}{\det(L_2)}\, \phi_0^2(t)\phi_1(t)$$

where $D_2 = \det(L_2) = k_2^4\mu\kappa - k_y^2 gR/k_2^2 > 0$. Working out the $\angle 1|L_2^{\mathrm{cof}}|Y_1$ we obtain the formula

$$T_F^{(M)} \cong \frac{3}{2}\, \frac{k_x^2 k_y^2 F(t) K_1(t)}{\det(L_2)} \left(\kappa + \frac{(gR)^{1/2}}{k_2^2}\right) \tag{40}$$

where $F = k_x^2\phi_0^2$ and $K_1 = \frac{1}{2}k_1^2\phi_1^2$. Thus, we see from Eq. (38) that the thermal flux is a driving force for the energy transfer from the turbulence W to the shear flow F. In obtaining Eq. (38) we have taken that $\delta p_1/\delta\phi_1$ is close to the linear value.

In the next section we develop this reduction with the generalization of allowing for a renormalization of the mode-1 attractor due to the evolution of ϕ_0 and p_0. Here we use $\det(L_2) \approx k_2^4\kappa\mu$ to reduce the transfer function T_F in Eq. (38) further

$$T_F^{(M)} = \frac{3}{4}\,\frac{r^2}{\mu(4r^2+1)^2}\,W(t)F(t) \tag{41}$$

where we use $k_x = rk_y$ and take $K_1 \cong W$ and $k_2^2\kappa > (gR)^{1/2}$ to obtain this simple estimate. From Eq. (39) it follows that the flow energy transfer rate $\dot{F} = T_F$ is peaked around $\theta = \tan^{-1}(k_y/k_x) \approx \pi/4$ and varies in $|\mathbf{k}|$ as $\nu_F := \dot{F}/F \cong (3/16)\tau_c\angle(\nabla\times\mathbf{v}_E)^2$ that is with the mean square vorticity in the fluctuating flows. For the saturation level given in Eqs. (13)–(15) we can estimate $\angle v_E^2 = l_c^2(I - I_c) \approx l_c^2 v_T^2/L_pR_c$ and the transfer rate $\nu_F \sim \tau_c(gR) \sim (gR)^{1/2}(1 - I_c/gR)$. Here $(gR)^{1/2} = v_T/\sqrt{L_pR_c}$ is the resistive-g fast interchange time scale determined by the thermal velocity $v_T = \sqrt{(T_i+T_e)/m_i}$ and the geometric mean $\sqrt{L_pR_c}$ of the pressure gradient scale length L_p and the radius of curvature of the magnetic field R_c. In the simulations of Refs. 1 and 2 the unit of time is $(gR)^{-1/2}$.

B. Derivation of the dynamical system in the $\epsilon_M \to 0$ limit

In this section we assume that mode ϕ_2, p_2 is strongly damped as defined by $\epsilon_M \ll 1$ in Sec. IVA. Consequently $d\phi_2/dt$ and dp_2/dt in Eqs. (5) and (8) rapidly evolve to zero giving local equations for ϕ_2, p_2. This approximation of a Markovian limit for the small scale modes reduces the dynamical space to a four-dimension space $(\phi_1, p_1, \phi_0, p_0)$.

Our purpose is to reduce the dynamical space to a three-dimensional space so that the state of the system could be determined by the three energy components U, W, and F.

Before discussing this reduction, we rewrite Eqs. (3)–(8) in a simplified dimensionless form by setting:

$$\phi_0 = \lambda_0\varphi_0; \quad \phi_1 = \lambda_1\varphi_1; \quad \phi_2 = \lambda_2\varphi_2;$$
$$p_0 = \eta_0\rho_0; \quad p_1 = \eta_1\rho_1; \quad p_2 = \eta_2\rho_2 \tag{42}$$

with

$$\lambda_0 = k_1k_2 \Big/ \left(Mk_y\sqrt{3k_x^2+k_y^2}\right)$$
$$\lambda_1 = \sqrt{\frac{2}{3}}\,k_2 \Big/ (Mk_y)$$
$$\lambda_2 = \sqrt{\frac{2}{3}}\,k_1 \Big/ \left(M\sqrt{3k_x^2+k_y^2}\right) = \eta_2 \tag{43}$$
$$\eta_0 = \frac{2}{3}\,k_1^2k_2^2 \Big/ \left[Mk_y^2(3k_x^2+k_y^2)\right]$$
$$\eta_1 = \sqrt{\frac{2}{3}}\,k_1^2k_2 \Big/ \left[Mk_y(3k_x^2+k_y^2)\right]$$

where $M = k_xk_y/2$.

We obtain the following set of dynamical equations for $\varphi_0, \varphi_1, \varphi_2$:

$$\frac{d\varphi_0}{dt} = -\varphi_1\varphi_2 - \gamma_0\varphi_0$$
$$\frac{d\varphi_1}{dt} = \varphi_0\varphi_2 + \beta_1\rho_1 - \gamma_1\varphi_1 \tag{44}$$
$$\frac{d\varphi_2}{dt} = -\varphi_0\varphi_1 - \beta_2\rho_2 - \gamma_2\varphi_2$$

with

$$\gamma_0 = \gamma_L + \mu k_x^2; \qquad \gamma_1 = k_1^2\mu; \qquad \gamma_2 = k_2^2\mu;$$
$$\beta_1 = k_yg/(3k_x^2+k_y^2); \qquad \beta_2 = k_yg/k_2^2$$

and similar dynamical equations for $\rho_0,\ \rho_1, \rho_2$:

$$\frac{d\rho_0}{dt} = \varphi_1\rho_1 - \gamma_0'\rho_0$$
$$\frac{d\rho_1}{dt} = \alpha_1\varphi_1 - \xi_1\rho_0\varphi_1 - \rho_2\varphi_0 - \gamma_1'\rho_1 \tag{45}$$
$$\frac{d\rho_2}{dt} = -\alpha_2\varphi_2 + \xi_2\varphi_0\rho_1 - \gamma_2'\rho_2$$

with

$$\gamma'_0 = 4k_x^2\kappa \qquad \gamma'_1 = k_1^2\kappa; \qquad \gamma'_2 = k_2^2\kappa$$

$$\alpha_1 = k_y R\left(\frac{3k_x^2 + k_y^2}{k_1^2}\right); \qquad \alpha_2 = k_y R.$$

$$\xi_1 = \frac{4}{3}\frac{k_2^2}{k_y^2}; \qquad \xi_2 = \frac{k_1^2 k_2^2}{k_y^2(3k_x^2 + k_y^2)}. \tag{46}$$

The equations (42) and (43) are defined in 12-dimensional parameter space $\{\mu\} = \{\gamma_i, \gamma'_i, \alpha'_i, \beta_i, \xi_i\}$. We assume $\gamma_2,\ \gamma'_2 \gg \sqrt{\alpha_1\beta_1}$ so that the decay times $1/\gamma_2$ and $1/\gamma'_2$ are shorter than the characteristic time of evolution of the whole system of equation. Consequently, we can write at the lowest order:

$$\begin{aligned} \gamma_2\varphi_2 + \beta_2\rho_2 &= -\varphi_0\varphi_1 \\ \gamma'_2\rho_2 + \alpha_2\varphi_2 &= \xi_2\varphi_0\rho_1 \end{aligned} \tag{47}$$

from which we obtain:

$$\begin{aligned} \varphi_2 &= -\left(\gamma'_2\varphi_1 + \beta_2\xi_2\rho_1\right)\left(\frac{\varphi_0}{\gamma_2\gamma'_2 - \alpha_2\beta_2}\right) \\ \rho_2 &= \left(\varphi_1\alpha_2 + \gamma_2\xi_2\rho_1\right)\left(\frac{\varphi_0}{\gamma_2\gamma'_2 - \alpha_2\beta_2}\right). \end{aligned} \tag{48}$$

We cast ρ_2 and φ_2 into the dynamical equations for φ_1, ρ_1 which give:

$$\begin{aligned} \frac{d\varphi_1}{dt} &= -\Gamma_1\varphi_1 + B_1\rho_1 \\ \frac{d\rho_1}{dt} &= -\Gamma'_1\rho_1 + A_1\varphi_1 \end{aligned} \tag{49}$$

where:

$$\begin{aligned} \Gamma_1 &= \gamma_1 + \frac{\varphi_0^2\gamma'_2}{\gamma_2\gamma'_2 - \alpha_2\beta_2} \\ B_1 &= \beta_1 - \frac{\varphi_0^2\beta_2\xi_2}{\gamma_2\gamma'_2 - \alpha_2\beta_2} \\ \Gamma'_1 &= \gamma'_1 + \frac{\gamma_2\xi_2\varphi_0^2}{\gamma_2\gamma'_2 - \alpha_2\beta_2} \\ A_1 &= \alpha_1 - \xi_1\rho_0 - \frac{\alpha_2\varphi_0^2}{\gamma_2\gamma'_2 - \alpha_2\beta_2}. \end{aligned} \tag{50}$$

The assumption $\gamma_2, \gamma_2' \gg (\alpha_1\beta_1)^{1/2}$ implies $(\gamma_2')^2, \gamma_2^2 \gg \alpha_2\beta_1$ and consequently $\gamma_2\gamma_2' \gg \alpha_2\beta_2$ where we used $\alpha_1 > \alpha_2$ and $\beta_1 > \beta_2$. It is seen from Eq. (48) that the growth of φ_0 will increase the damping coefficients and simultaneously reduce the destabilizing terms A_1 and B_1, leading to a saturation of the interchange instability and increasing the characteristic time of evolution. For reducing the system dimensions one step further, we use this qualitative analysis as follows. We look at the behavior of φ_1, ρ_1 in the tangent space, i.e. we solve the equation

$$\begin{aligned} \frac{d\delta\varphi_1}{d\delta t} &= -\Gamma_1\delta\varphi_1 + B_1\delta\rho_1 \\ \frac{d\delta\rho_1}{d\delta t} &= -\Gamma_1'\delta\rho_1 + A_1\delta\varphi_1 \end{aligned} \tag{51}$$

where $\delta\varphi_1, \delta\rho_1$ are functions of δt and $\Gamma_1, B_1, \Gamma_1', A_1$ are taken at a given time t. The general solutions for $\delta\rho_1, \delta\varphi_1$ can be written

$$\begin{aligned} \delta\varphi_1 &= u_1(t) + u_2(t) \\ \delta\rho_1 &= u_1(t)\,\frac{s_1+\Gamma_1}{B_1} + u_2(t)\,\frac{s_2+\Gamma_1}{B_1} \end{aligned} \tag{52}$$

where s_1 and s_2 are the roots of the secular equations

$$(s+\Gamma_1)(s+\Gamma_1') = A_1B_1 \tag{53}$$

and $u_i(t) = u_i(0)\exp(\int^t s_i(t')dt')$. As we have $\gamma_2\gamma_2' > \alpha_2\beta_2$, then Γ_1 and Γ_1' are positive and consequently, at least, one of the two roots s_1, s_2 has a negative real part. Thus, for large δt, we have

$$\delta\rho_1 \approx \delta\varphi_1\,\frac{s_1+\Gamma_1}{B_1} \approx \delta\varphi_1\,\frac{A_1}{s_1+\Gamma_1'} \tag{54}$$

where s_1 is the root such that $\mathrm{Re}\, s_1 > \mathrm{Re}\, s_2$.

We now consider the system of equations for φ_1 and ρ_1 and we ask when the behavior of φ_1 and ρ_1 can be considered as similar to the behavior of $\delta\rho_1$ and $\delta\varphi_1$. We know that this will be the case at any fixed point and in the vicinity of the fixed points as the solution leaves it or approaches it. It will also be the case when φ_0^2 will be large enough to reduce

B_1, leading to a slow growth or damping of ϕ_1. So we can consider that it is the case at the beginning of the evolution as well as close to the instability saturation level and probably also when the final state is not stationary provided averaging is performed on the time scale of the oscillations.

Consequently, we try the following closure condition for system (42)–(43):

$$\rho_1 = \varphi_1 \frac{A_1}{s_1 + \Gamma_1'} = C_1(U,F)\varphi_1. \tag{55}$$

Again we note that the increase of $F = k_x^2\phi_0^2$ and $U = (g/R)p_0^2$ will decrease A_1 and slow down the evolution of the system. We have now reduced the initial system to the three-dynamical variables U, F and W. We use these closure relationships to compute the time behavior of W, F, and U. We first notice that from Eqs. (21)–(22) and Eq. (40) we have:

$$\begin{aligned} U &= \frac{g}{R}\eta_0^2\rho_0^2 \\ F &= k_x^2\lambda_0^2\varphi_0^2 \\ W &= \tfrac{1}{2}\Lambda_1^2\varphi_1^2 \end{aligned} \tag{56}$$

with

$$\Lambda_1^2 = k_1^2\lambda_1^2 - \frac{g}{R}\eta_1^2 C_1^2 + \varphi_0^2 \Bigg\{ k_2^2\lambda_2'^2\gamma_2^2 - \frac{g}{R}\eta_2'^2\alpha_2^2 + 2C_1\xi_2\gamma_2 \left[\beta_2 k_2^2\lambda_2'^2 - \frac{g}{R}\eta_2'^2\alpha_2\right]$$
$$+ \ C_1^2\left[k_2^2\lambda_2'^2\xi_2^2\beta_2^2 - \eta_2'^2\gamma_2^2\xi_2^2\right]\Bigg\} \tag{57}$$

where $\eta_2'^2 = \lambda_2'^2 = \lambda_2^2/D_2^2$ where $D_2 = \gamma_2\gamma_2' - \alpha_2\beta_2$. We see that Λ^2 depends only on U and F though φ_0^2 and $C_1(\varphi_0^2, \rho_0)$ defined in Eq. (53).

We can now write the closed set of equations for U, F, W, namely

$$\begin{aligned} \frac{dU}{dt} &= F_U - \varepsilon_U + P_U \\ \frac{dW}{dt} &= -F_W + F_U - \varepsilon_W \\ \frac{dF}{dt} &= F_W - \varepsilon_F + P_F \end{aligned} \tag{58}$$

with

$$
\begin{aligned}
F_U &= 2\frac{k_x k_y \eta_0 \eta_1 \lambda_1 C_1}{\Lambda_1^2}\left(\frac{g}{R}\right)^{1/2} W U^{1/2} \\
F_W &= 3k_x k_y \,\frac{\lambda_1 \lambda_2}{\Lambda_1^2}\,\frac{\gamma_2' + \beta_2 \xi_2 C_1}{\gamma_2' \gamma_2 - \alpha_2 \beta_2}\, F W.
\end{aligned}
\tag{59}
$$

The nonlinear fluxes are in the form of a renormalized closure for the triplet correlation functions in Eqs. (25)–(27). The critical approximation to the model is given in Eq. (46), which has been tested by comparing the exact φ_2, ρ_2 with those given by Eq. (46). Qualitatively, the difference is that the true φ_2, ρ_2 are slower responding to the change in the primary modes, but the variation remains faithful over long times. Thus, the onset and turn-off of the shear flow is too fast in the model. The amount of the error seems controlled by the inequalities given in the derivation of the model. Thus, formally these renormalized fluxes connect to the nonlinear regimes of the K-epsilon model. The dissipation rates $\varepsilon_U = 8k_x^2 \kappa U$ and $\varepsilon_F = 2(\nu_L + k_x^2 \mu) F$ remain linear; however, the dissipation rate $\epsilon_W = \epsilon_W(F, W)$ becomes a nonlinear function of the flow F and turbulence level W.

From these developments of F_W and ε_W on F and W it is clear that the closure model contains the exchange of stability property between the two fixed points $(U_L, W_L, 0)$ and (U_H, W_H, F_H) defined as the L and H confinement modes. The H-mode state becomes the stable attractor when the viscous damping $\varepsilon_F = 2(\nu_L + k_x^2 \mu) F$ is sufficiently weak that the turbulence level obtained from the balance $F_{W_H} = \varepsilon_{F_H}$ is lower than the turbulence level obtained by saturation on the nonlinear (Lorenz attractor) obtained from the convective flattening of the local pressure gradient. This H-mode state is the flow-state attractor in the extended state space of the dynamics.

It remains to investigate for which parameter $\{\mu\} = \{\gamma_i, \gamma_i', \alpha_i, \beta_i, \xi_i\}$ the reduced model is faithful to the full 6-ODE model. While based on the 6-ODE dynamics, it is also interesting to consider the reduced thermodynamic model in its own right asking where in $\{\mu\}$-space its dynamics is comparable to the K-epsilon model of Sugama and Horton.

V. DISCUSSION AND CONCLUSIONS

In this work we derive a thermodynamic state space description for mixed states of temperature gradient driven turbulent convection and sheared mass flows by introducing a Markovianization of the *small scale* modes in a well-known low-order mode coupling model[12] of the Rayleigh-Bénard system. The mode coupling equations contain the Lorenz attractor on a sub-manifold and the shear-flow attractor in the full $d = 6$ phase space. The three-dimensional thermodynamic state space follows the three physically distinct energy components: potential energy U, the turbulent energy W and the kinetic energy F contained in the sheared mass flows. In contrast to weak turbulence theory for the wave-kinetic equation the closure presented here contains nonlinear functions of F and U in the triplet closure that are similar in structure to the Padé-approximants of renormalized turbulence theory. The mathematical procedure is to determine a Markovian expansion parameter $\epsilon_M \ll 1$ in the smallest scale modes and to systematically eliminate these damped modes in the limit that $\epsilon_M \to 0$. The resulting thermodynamic model given in Eqs. (56)–(57) remains highly nonlinear in its energy transfer rates.

While the thermodynamic model derived here as a projection on the attracting manifold describes the lowest order modes of the system rather than a full set of turbulent fluctuations, components, it has the advantage over the previous turbulence closure models of avoiding the unknown dimensionless coefficients in the dynamical model. In the K-epsilon model[17] there are a set $\{C_K, C_F\}$ of dimensionless parameters that must be specified. Here the reduced model has the explicit coefficients that are defined in the original underlying dynamics.

The new reduced model defined by Eqs. (56) and (57) has the L to H mode bifurcation of the original 6-ODE system. In the new model it remains to investigate the stability of the H-mode state to the Hopf bifurcation to an ELMy state. The structure of the new model is sufficiently close to that of the Sugama-Horton model that the properties are expected to be similar. The SH-model shows the bifurcation of the H-mode attractor to type III ELMs and the well-known[18] hystersis of the lower critical power $P_U^{\mathrm{H}\to\mathrm{L}}$ for the inverse H to L transition than the $P_U^{\mathrm{L}\to\mathrm{H}}$ critical power. In a future work we will make a numerical comparison of the new model with the original 6-ODE model and the K-epsilon model.

It is interesting to note that the possibility of hystersis in the thermodynamic model is present despite the Markovian (memoryless) approximation on the small scale modes. This hystersis issue and the question of the domain D_μ in the high dimensional parameter $\{\mu\}$-space for which the 6-ODE model and the thermodynamic model qualitatively agree will be investigated in a future study. Here we restrict the work to the development of the new model and the formal comparison of its structure with the K-epsilon model and the quasilinear model as approximations to the temperature gradient driven convection given by the original coupled partial differential equations.

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FIGURE CAPTIONS

FIG. 1. Bifurcation diagram showing the onset of convection at the normalized Rayleigh number Ra = 1; the analytic (A) and numerical (N) curves for the onset of steady-state tilted cell convection; and the boundary of the oscillatory states with time-dependent high-low thermal flux and momentum transport states. The small domain inside the dashed lines for Ra $>$ 20 and $0.5 <$ Pr < 2 in this diagram contains chaotic solutions.

FIG. 2. Angular frequencies from the time series of $\phi_0(t)$ and $\phi_1(t)$ as a function of increasing gR past the point of the Hopf bifurcation by the third instability. Adding an external input power parameter P_{in} in the $\dot{p}_0$ equation results in an effective $R_{\rm eff}$, so this plot can also be interpreted as an equivalent variation of the angular frequency of the ELM-type oscillation versus P_{in}.

FIG. 3. Schematic phase-transition diagram for the fundamental mode ϕ_1 and shear flow mode ϕ_0 states as a function of the driving force $I = gR$ at fixed Pr. The Lorenz attractor is an invariant subspace until the second instability threshold is passed which corresponds to the L $\rightarrow$ H transition. Subsequently, the stable shear flow attractor has a Hopf bifurcation into a limit cycle designated as the H $\rightarrow$ ELMs transition.

FIG. 4. Comparison of the convective flows in the L-mode state ($\phi_0 = 0$) and the H-mode state ($\phi_0 \neq 0$) for $gR = 3.75$ and Pr = 1.0. The tilting of the cellular flow is clear in part (4b). In part (4a), the thermal flux is $q_x = 0.44$ and in part (4b) $q_x = 0.34$.

FIG. 5. The maximum fractional change in the convective thermal flux as a function of Pr. The flux drop strongly decreases with increasing Pr and becomes an increase for $\mathrm{Pr} \geq (\mathrm{Pr})_c \approx 3.35$.

FIG. 6. The evolution of F, W, U for the case Pr = 1.0, Ra = 3.75. The total energy

$(F+W-U)$ shows a small change during the (L $\rightarrow$ H) transition comparing with the rapid change of F, W, U.

FIG. 7. The dynamics of the L to H transition. (a) the $\phi_0(t)$ and $\phi_1(t)$ solutions exponential departure from the neighborhood of the unstable Lorenz manifold and approach to the stable H-mode manifold. (b) the drop in the convective thermal flux.

FIG. 8. Comparison by ratio tests of the accuracy of the Markovian closure formulae for the triplet correlation functions: (a) the quasilinear formula FW, (b) the K-epsilon $FW/U^{1/2}$ closure formula, (c) the empirical closure formula FW/U, and (d) the quasilinear thermal flux $U^{1/2}W$ formula compared with the thermal flux triplet correlation function.