HOT PLASMA DECOUPLING CONDITION FOR LONG WAVELENGTH MODES

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Abstract

The stability of layer modes is analyzed for z-pinch and bumpy cylinder models. These modes are long wavelength across the layer and flute-like along the field line. The stability condition can be expressed in terms of the ratio of hot to core plasma density. It is shown that to achieve conditions close to the Nelson, Lee-Van Dam core beta limit, one needs a considerably smaller hot to core plasma density than is required to achieve stability at zero core beta.

I. Introduction

Stability of a hot plasma annulus immersed in a core plasma density depends upon various criteria that have been discussed by several workers.[1-8] An important class of modes are long wavelength modes across the annulus, known as layer modes. A discussion of these modes have been given for a z-pinch model in Ref. (6). In this work we, (1), extend the analysis of layer modes for the z-pinch model and find a significant modification of the previous analysis, and (2), we apply the analysis to a more realistic bumpy cylinder model using equations derived by Antonsen and Lee.[9] The results indicate how small the ratio of hot to core plasma density needs to be to achieve stability. We also show that in order to achieve stability near the Nelson, Lee-Van Dam core beta limit, the hot to core density ratio needs to be considerably below that ratio needed at zero core beta.

In Section II, we derive our criteria, and in Section III, we discuss the results.

II. Derivation of Layer Mode Dispersion Relation

We take a model where in equilibrium a hot plasma layer surrounds a core plasma component. The plasma is taken as constant density from the axis to the surface containing the peak of the hot plasma pressure. The core plasma density and pressure then decreases to zero in the outer part of the hot plasma annulus within a scale length comparable to the scale length of hot plasma pressure gradient.

The basic equation from the z-pinch model, with the assumptions,

$$\frac{\omega}{\omega_{ci}}, \quad \frac{\omega^2}{k^2 V_A^{\,2}} \to 0; \quad \frac{\beta_c}{\beta_h}, \quad \kappa\Delta_b \ll 1, \quad \text{is}$$

$$\omega^2 \frac{d}{dr} r\rho \frac{d\xi}{dr} + \left[k^2 \frac{d}{dr} (P_{\perp h} + P_{\parallel h} + P_c) \left[\frac{-\frac{\omega}{\omega_{cv}} - \tilde{\beta}_c}{1 - \frac{\omega}{\omega_{cv}} - \tilde{\beta}_c} \right] + k^2 \frac{dP_c}{dr} - k^2 r\rho \right] \xi = 0 \qquad (1)$$

where ρ is the mass density, r the radial coordinate (the curvature, κ, in the z-pinch model is $-1/r$), P the particle pressure, the subscripts h and c refer to hot and core species,

$$\tilde{\beta}_c = \frac{- r \frac{d}{dr} (P_c/B^2)}{\left[1 + (dP_{\parallel h}/dr)/(dP_{\perp}/dr)\right]}$$

and

$$\omega_{cv} = \frac{k(P_{\perp h} + P_{\parallel h})}{\rho \, r \, \omega_{ci} p}$$

with ω_{ci} the ion cyclotron frequency and p the density ratio of hot component to core component.

We can solve this equation analytically for a hot electron layer within a thickness Δ if we assume $|k|\Delta < 1$. The analysis divides into two parts, $\tilde{\beta}_c \ll 1$ and $\tilde{\beta}_c \sim 1$. For simplicity we take ω_{cv} as a constant, and $\tilde{\beta}_c$ as a non-zero constant in the outer half of the layer and zero in the inner half of the layer.

a. $\tilde{\beta}_c \ll 1$

In this case we have to go to second order in a layer expansion. For a thin layer we order,

$$\xi = \xi_0 + \varepsilon\xi_1 + \varepsilon^2\xi_2$$

$$\tilde{\beta}_c \sim \varepsilon, \quad \beta_c/\beta_h \sim \varepsilon, \quad k\Delta \sim \varepsilon^2$$

Hence the following equations are satisfied:

$$\omega^2 \frac{d}{dr} r\rho \frac{d\xi_0}{dr} = 0$$

$$\omega^2 \frac{d}{dr} r\rho \frac{d\xi_1}{dr} = k^2 \frac{d}{dr}(P_{\perp h}+P_{\parallel h}) \frac{\omega\ \xi_0}{(\omega_{cv} - \omega)}$$

$$\omega^2 \frac{d}{dr} r\rho \frac{d\xi_2}{dr} = k^2 \frac{d}{dr} (P_{\perp h} + P_{\parallel h}) \frac{\omega\ \xi_1}{\omega_{cv} - \omega}$$

$$+ \xi_0 \left[k^2 \frac{d}{dr} (P_{\perp h} + P_{\parallel h}) \frac{\tilde{\beta}_c \omega_{cv}^2}{(\omega_{cv} - \omega)^2} + k^2 \frac{dP_c}{dr} \left(\frac{\omega}{\omega_{cv} - \omega} - 1 \right) \right] \tag{2}$$

Then we take $\xi_0 = 1$, integrate r in Eq. (2) between R-Δ and R+Δ, with $\rho(R+\Delta) = 0$, $P(R+\Delta) = 0$, $\rho(R-\Delta) = \rho(R) = \rho_0$, $P_h(R-\Delta) = 0$, $P_h(R) = P_{ho}$, and thereby obtain the following result,

$$r\rho \frac{d\xi_1(r)}{dr} = \frac{-k^2 (P_{\perp h} + P_{\parallel h})}{\omega(\omega - \omega_{cv})} \tag{3}$$

$$(R-\Delta)\rho_0 \frac{d\xi_2(R-\Delta)}{dr} = -\int_{R-\Delta}^{R+\Delta} dr\, k^2 \frac{d\xi_1}{dr} \frac{(P_{\perp h} + P_{\parallel h})}{\rho\omega(\omega - \omega_{cv})} \tag{4}$$

$$+ k^2 \frac{(P_{\perp h0} + P_{\parallel h0}) \tilde{\beta}_c \omega_{cv}^2}{(\omega_{cv} - \omega)^2} + k^2 P_c \frac{2\omega - \omega_{cv}}{\omega_{cv} - \omega}$$

We define the impedence function as $K = \frac{1}{\xi(R-\Delta)} \frac{d\xi(R-\Delta)}{dr}$ and note that $K \simeq |k|$. In general we assume it is a positive number independent of ω. Then by substituting equation (3) into equation (4), we obtain

$$RK\rho_0 = \int_{R-\Delta}^{R+\Delta} \frac{drk^4}{r\rho} \left(\frac{P_{\perp h} + P_{\parallel h}}{\omega(\omega - \omega_{cv})} \right)^2$$

$$+ k^2 \frac{(P_{\perp h0}+P_{\parallel h0})\tilde{\beta}_c\, \omega_{cv}^2}{\omega^2(\omega_{cv}-\omega)^2} + k^2 P_c \frac{(2\omega-\omega_{cv})}{(\omega_{cv}-\omega)\omega^2} \tag{5}$$

We define:

$$\Delta' = \frac{R\rho_0}{(P_{\perp h0}+P_{\parallel h0})^2} \int_{R-\Delta}^{R+\Delta} \frac{dr}{r\rho} (P_{\perp h}+P_{\parallel h})^2 \approx \Delta \ , \quad \delta = |k|\Delta'$$

$$\gamma_{MHD}^2 = |k|(P_{\perp h0}+P_{\parallel h0})/\rho_0 \ , \quad \Omega = \omega/\gamma_{MHD} \ , \quad \Omega_{cv} = \omega_{cv}/\gamma_{MHD} \ , \quad \alpha = \frac{\Delta B^2}{R(P_{\perp h0}+P_{\parallel h0})} \ .$$

Note that γ_{MHD}^2 is roughly the growth of a hot electron disk in standard MHD theory.

With the new variables we then find the dispersion relation

$$\frac{K}{|k|} - \frac{\delta+\tilde{\beta}_c\Omega_{cv}^2}{\Omega^2(\Omega-\Omega_{cv})^2} - \frac{\alpha\, \tilde{\beta}_c(2\Omega-\Omega_{cv})}{\Omega^2(\Omega_{cv}-\Omega)} = 0 \tag{6}$$

In analyzing this equation, we can first neglect the last term ($\alpha \ll 1$) and we find that the dispersion relation for the unstable mode is

$$\Omega(\Omega-\Omega_{cv}) + \left[\frac{k}{K}\left(\delta+\tilde{\beta}_c\Omega_{cv}^2\right)\right]^{1/2} = 0, \tag{7}$$

with the stability condition

$$\Omega_{cv}^2 > 4\left[\frac{k}{K}\left(\delta+\tilde{\beta}_c\Omega_{cv}^2\right)\right]^{1/2} \tag{8a}$$

or

$$\Omega_{cv} > \mathrm{Max}\left[2\left(\frac{k}{K}\delta\right)^{1/4},\quad 4\left(\frac{k}{K}\tilde{\beta}_c\right)^{1/2}\right] \tag{8b}$$

This result is new for $\tilde{\beta}_c \lesssim \frac{(K/k)^{1/2}\delta^{1/2}}{4}$, while the opposite limit has been discussed in Ref. 1.

From Eq. (8b) we see that for extremely low $\tilde{\beta}_c$, the decoupling condition can be satisfied for $\Omega_{cv} > 2(\frac{k}{K}\delta)^{1/4}$. However, instability can arise for relatively low $\tilde{\beta}_c$, viz.

$$\tilde{\beta}_c > \tilde{\beta}_{cr} \equiv \frac{\Omega_{cv}^2}{16}\frac{K}{|k|} - \frac{\delta}{\Omega_{cv}^2} \tag{9}$$

a result valid as long as $\tilde{\beta}_{cr} < 1$.

In the unstable region the frequency, Ω, is given by

$$\Omega = \frac{\Omega_{cv}}{2} + i\left\{\left[\frac{k}{K}\left(\delta+\tilde{\beta}_c\Omega_{cv}^2\right)\right]^{1/2} - \frac{\Omega_{cv}^2}{4}\right\}^{1/2} \tag{10a}$$

$$\text{if} \quad \mathrm{Max}\left[2\left(\frac{k}{K}\delta\right)^{1/4},\ 4\left(\frac{k}{K}\tilde{\beta}_c\right)^{1/2}\right] \gtrsim \Omega_{cv} \gtrsim \left(\tilde{\beta}_c^2\,\frac{\alpha^2 k}{K} - \delta\right)^{1/2} / \tilde{\beta}_c^{1/2}$$

$$\Omega \simeq i\left(2\frac{\tilde{\beta}_c\alpha k}{K}\right)^{1/2}, \quad \text{if} \quad \Omega_{cv} \lesssim \left(\tilde{\beta}_c^2\,\frac{\alpha^2 k}{K} - \delta\right)^{1/2} / \tilde{\beta}_c^{1/2} \tag{10b}$$

b. $\tilde{\beta}_c \gg \left(\frac{k}{K}\right)^{1/4}\delta^{1/2}$

In this limit we let $\xi = 1 + \varepsilon\xi_1$, $k\Delta \sim \varepsilon$, and integrate Eq. (1) across the layer. This yields (recall $\tilde{\beta}_c$ = constant if $R < r < R+\Delta$; $\tilde{\beta}_c \approx 0$ otherwise),

$$\omega^2\rho_0\left[\frac{r d\xi_1}{dr}\right]_{r=R-\Delta} \equiv (R-\Delta)\omega^2 K\rho_0$$

$$= k^2(P_{\perp h0}+P_{\parallel h0})\left[\frac{\frac{\omega}{\omega_{cv}} + \tilde{\beta}_c}{1 - \frac{\omega}{\omega_{cv}} - \tilde{\beta}_c} - \frac{\frac{\omega}{\omega_{cv}}}{1 - \frac{\omega}{\omega_{cv}}}\right] \tag{11}$$

$$- k^2 P_c\left[\frac{-\frac{\omega}{\omega_{cv}} - \tilde{\beta}_c}{1 - \frac{\omega}{\omega_{cv}} - \tilde{\beta}_c}\right] - k^2 P_c$$

Combining terms and normalizing variables yields

$$\frac{K}{|k|}\Omega^2 - \frac{\tilde{\beta}_c\Omega_{cv}^2}{[\Omega_{cv}(1-\tilde{\beta}_c)-\Omega](\Omega_{cv}-\Omega)} + \frac{\alpha\tilde{\beta}_c[\Omega_{cv}(1-2\tilde{\beta}_c)-2\Omega]}{\Omega_{cv}(1-\tilde{\beta}_c)-\Omega} \qquad (12)$$

To analyze this equation we assume $\frac{K}{|k|}\Omega_{cv} \gg 1$, and we will ascertain that the instability threshold arises for $\tilde{\beta}_c$ near unity. With the assumption that the quantities α, Ω/Ω_{cv}, $1-\tilde{\beta}_c$ are all small, we have,

$$\Omega^3 - \Omega^2\Omega_{cv}(1-\tilde{\beta}_c) + \frac{|k|}{K}\Omega_{cv} = 0 \qquad (13)$$

The stability condition is then,

$$\tilde{\beta}_c < 1 - \frac{3}{2}\cdot 2^{1/3}\left(\frac{|k|}{K}\frac{1}{\Omega_{cv}^2}\right)^{1/3} \equiv \tilde{\beta}_{cr} \qquad (14)$$

For $\tilde{\beta}_c = 1$ the growth rate is maximum at the value,

$$\mathrm{Im}\Omega = i\,\frac{\sqrt{3}}{2}\left(\Omega_{cv}\frac{|k|}{K}\right)^{1/3} .$$

When $\tilde{\beta}_c \gg 1$ the growth rate asymptotes to the value

$$\mathrm{Im}\Omega = i\left(\frac{k}{|K|}\right)^{1/2} .$$

We note that the standard MHD theory of our system would yield a growth rate,

$$\mathrm{Im}\Omega = i\left(\frac{k}{|K|}\delta\right)^{1/2} .$$

Previously, it was observed that if $\tilde{\beta}_c \gg 1$, short wavelength modes had the same growth rate as that predicted by standard MHD theory. Here we see that for the layer modes of a hot component plasma larger growth rates are obtained.

c. Equations with Axial Variation

Antonsen and Lee[9] have derived the generalization of equation (1) to an equilibrium with axial variation. With the same assumptions as went into Eq. (1), plus the assumption $\beta_h \ll 1$, the governing equation was found to be,

$$\omega^2(\langle\rho r^2\rangle U')' - m^2\omega^2 \langle\rho/B^2r^2\rangle U + \frac{\left[\frac{\omega}{\bar{\omega}_{cv}} + \bar{\tilde{\beta}}_c\right]m^2}{\left[1-\frac{\omega}{\bar{\omega}_{cv}} - \bar{\tilde{\beta}}_c\right]} \left\langle\frac{\kappa}{rB}(P_{\perp h}+P_{\parallel h})\right\rangle' U = 0 \quad (15)$$

with

$$\alpha' \equiv \frac{\partial\alpha}{\partial\psi}, \quad d\psi = rBdr, \quad \langle\alpha\rangle = \int\frac{d\ell}{B}\alpha$$

$$\bar{\tilde{\beta}}_c = \frac{\langle \frac{P_{\perp h}}{B^2} \rangle' \, P_c'}{\langle \frac{\kappa}{rB} (P_{\perp h}' + P_{\parallel h}') \rangle} \, , \quad \frac{1}{\bar{\omega}_{cv}} = \frac{e \langle n_h \rangle'}{m \langle \frac{\kappa}{rB} (P'_{\perp h} + P'_{\parallel h}) \rangle}$$

with n_h the hot density component and e its electric charge.

Equation (15) is exactly the same form as Eq. (1) (with $P_c/P_h \rightarrow 0$). We can use the previous analysis if we redefine the parameters so that,

$$\bar{\gamma}^2_{MHD} = -|m| \, \frac{\langle \frac{\kappa}{rB} (P_{\perp h} + P_{\parallel h}) \rangle_0 \psi_o}{\langle r^2 \rho \rangle_0} \, , \quad \Delta\tilde{\psi} = \frac{\langle r^2 \rho \rangle_0}{\langle \frac{\kappa}{rB} (P_{\perp h} + P_{\parallel h}) \rangle_0^2} \int d\psi \, \frac{\langle \frac{\kappa}{rB} (P_{\perp} + P_{\parallel h}) \rangle^2}{\langle r^2 \rho \rangle}$$

$$\Omega_{cv} = \frac{\bar{\omega}_{cv}}{\bar{\gamma}_{MHD}} , \quad \Omega = \frac{\omega}{\bar{\gamma}_{MHD}} \, , \quad \delta = |m| \, \frac{\Delta\tilde{\psi}}{\psi_0}$$

where the subscript zero refers to the flux position ψ_0 where $P_{\perp h}$ is a maximum. The boundary condition is,

$$\frac{U'(\psi_0 - \Delta\psi)}{U'(\psi_0 - \Delta\psi)} \equiv M\psi_0 \approx |m|\psi_0$$

Now, if in the previous analysis, we replace $\frac{K}{|k|}$ and $\tilde{\beta}_c$ by $\frac{M}{|m|}$ and $\bar{\tilde{\beta}}_c$, all parameters are identical and we can use the results of the previous section.

III. Summary

From the results of this work we conclude that if the curvature drift frequency can be greater than twice the growth rate predicted from conventional MHD theory, the system will be stable if the core pressure gradient is small enough. Achieving this criteria is defined as the decoupling condition. If the decoupling condition is strongly satisfied, then the system can support a core pressure gradient satisfying the Nelson, Lee-Van Dam limit ($\beta_c < 1$). However, if the decoupling condition is only moderately satisfied the tolerable core pressure gradient can be considerably less than the Nelson, Lee-Van Dam limit.

If $\beta \to 0$, the basic decoupling for $|k|\Delta < 1$, in the z-pinch model is,

$$p/q < (p/q)_{cr} \equiv \frac{(K\Delta)^{1/2}}{4} \qquad (16)$$

where $p = n_c/n_h$, $q_0 = \dfrac{(P_{\perp h}+P_{\parallel h})}{e_h n_h R\omega_{ci}\Delta B}$ and $K \sim k$.

For the bumpy cylinder model, for $m\,\dfrac{\Delta\psi}{\psi_0} < 1$, Eq. (16) still applies if we define the weighted average quantities,

$$q_0 = \frac{\psi_0}{\Delta\psi\omega_{ci0}}\,\frac{\langle\frac{\kappa}{rB}(P'_{\perp h}+P'_{\parallel h})\rangle}{e_h\langle n_h'\rangle}$$

$$q_0 p = \frac{\psi_0^2}{\Delta\psi} \frac{\langle \frac{\kappa}{rB} (P_{\perp h} + P_{\parallel h}) \rangle}{\langle r^2 \rho \rangle \, \omega_{ci0}^2}$$

$$K\Delta \to \frac{\Delta\psi_0}{\psi_0} M, \quad M \sim m$$

and ω_{ci0} refers to a typical cyclotron frequency. Stability can be maintained if the core beta parameter, β_c ($\bar{\beta}_c$ in the line averaged case), satisfies

$$\beta_c < \mathrm{Min}\left[1 - 1.9\left(\frac{p}{q_0 K\Delta}\right)^{1/3}, \left(q_0 \frac{K\Delta}{16p} - \frac{p}{q_0}\right)\right] \tag{17}$$

We note that the constraint on p/q needed to achieve decoupling up to $\beta_c \approx 1$ is,

$$p/q \lesssim \frac{K\Delta}{16} = \frac{(K\Delta)^{1/2}}{4} (p/q)_{cr} \tag{18}$$

Hence, the achievement of stable containment up to the Nelson, Lee-Van Dam limit leads to a considerably more stringent decoupling condition than that for $\beta_c = 0$.

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