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LOCAL DISPERSION RELATION FOR THE VLASOV-MAXWELL EQUATIONS

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LOCAL WKB DISPERSION RELATION

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Abstract

A formalism for analyzing systems of integral equations, based on the Wentzel-Kramers-Brillouin (WKB) approximation, is applied to the Vlasov-Maxwell integral equations in an arbitrary β , spatially inhomogenous plasma model. It is shown that when treating frequencies comparable with and larger than the cyclotron frequency, relevant new terms must be accounted for to treat waves that depend upon local spatial gradients. For a specific model, the response for very short wavelength and high frequency is shown to reduce to the straight-line orbit approximation when the WKB rules are correctly followed.

I. INTRODUCTION

The Wentzel-Kramers-Brillion (WKB) method is a basic technique which can be used to study waves governed by linear operators (which in general are integral equations) in a spatially inhomogeneous plasma. It is extensively used to obtain a local dispersion relation that can describe the modes of a plasma. There is a subtle point with this method for modes whose frequencies depend upon the local scale length. One then must determine which part of an expression, that depends upon local spatial gradients, contributes to the local dispersion relation and which part is higher-order in the WKB expansion and should be ignored to lowest order. Recently[1-3], there has been extensive analysis using the WKB method in low frequency problems (i.e., frequencies ω less than the ion-cyclotron frequency, ω_{ci}) and, in this case, the subtlety in the method does not manifest itself. However, for higher frequencies (comparable and greater than the ion-cyclotron frequency) one has to be quite careful in the calculation to obtain a correct description of modes whose frequency depends upon the local scale length.

To obtain the correct local dispersion relation, we will start with the formalism of Berk and Book[4], which was recently extended to vector systems by Berk and Pfirsch.[5] These formalisms indicate a very specific method of evaluation for the local dispersion with the only approximation being that $kL_p \gg 1$, where k is the local wavenumber in the direction of the spatial inhomogeneity and L_p the macroscopic scale length. Unfortunately, in general for modes in a magnetic field, one is then forced to consider forms that are not usually analytically integrable. However, if two additional weak assumptions are imposed,

$$\frac{a}{L_p} \ll 1 \ , \qquad \frac{1}{k_\perp L_p} \, \mathrm{Min} \left(k_\perp a, \frac{\omega}{\omega_{ci}} \right) < 1 \quad , \qquad (1)$$

where a is the Larmor radius and $k_\perp$ the wavenumber perpendicular to the magnetic field, then structural forms can be obtained in terms of integrals commonly used.

In this work we will specifically treat a slab equilibrium in a sheared magnetic field. We will indicate the proper local dispersion relation, which includes the rather subtle spatially-dependent term. A simple example will be given of an electrostatic dispersion relation where we specifically show how additional terms enter in a crucial way. For completeness, we will also include a special case where the Berk-Book rules can be directly applied to obtain a local dispersion relation and the result will be compared to that obtained when the approximations in Eq. (1) are used. For a somewhat more extensive article on this subject, the reader is referred to Ref. 6.

II. QUADRATIC FORM

Following the Berk-Pfirsch method, we note that a general system of integral equations with one-dimensional spatial inhomogeneity may be expressed as

$$\int dx' \, \underset{\approx}{G} \left(x - x', \frac{x + x'}{2}, \omega \right) \cdot \underset{\sim}{\zeta}(x') = 0 \ . \qquad (2)$$

This form emphasizes the structure that is natural for developing the WKB formalism when the difference variable varies more rapidly than the sum variable. Previous investigations[5,7] have shown that in order to exhibit the intrinsic symmetries of the kernel $\underset{\sim}{G}$, it is convenient to work with quadratic functionals obtained from Eq. (2) by multiplying by the adjoint vector $\underset{\sim}{\zeta}^{+}(x)$ and integrating over x [$\underset{\sim}{\zeta}^{+}(x)$ is the solution of Eq. (2) when x and x' are interchanged]. The resulting quadratic form is

$$\int dx dx' \underset{\sim}{\zeta}^{+}(x) \cdot \underset{\approx}{G}\left(x - x', \frac{x + x'}{2}, \omega\right) \cdot \underset{\sim}{\zeta}(x') = 0 . \qquad (3)$$

Changing variables and using the Fourier representation of $\underset{\sim}{\zeta}(x)$, the quadratic form becomes

$$\int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} \int dx \exp[-i(k - k')x] \underset{\sim}{\zeta}^{+}(-k) \cdot \underset{\approx}{\Lambda}\left(\frac{k + k'}{2}, x, \omega\right) \cdot \underset{\sim}{\zeta}(k') = 0 , \qquad (4)$$

$$\underset{\sim}{\zeta}(k) = \int_{-\infty}^{\infty} dx \underset{\sim}{\zeta}(x) \exp(-ikx) , \qquad (5)$$

and

$$\underset{\approx}{\Lambda}(k,x,\omega) = \int_{-\infty}^{\infty} dz \underset{\approx}{G}(z,x,\omega) \exp(-ikz) . \qquad (6)$$

In the WKB formalism the determinant of $\underset{\approx}{\Lambda}(k,x,\omega)$ reduces to the local dispersion relation, which determines $k(x)$. In the next section it is shown that direct construction of the quadratic form results in a structure

$$\int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} \int dx \exp[-i(k - k')x] \underset{\sim}{\zeta}^{+}(-k) \cdot \underset{\approx}{\sigma}(k,k',x,\omega) \cdot \underset{\sim}{\zeta}(k') = 0 . \qquad (7)$$

Equation (7) may be transformed into Eq. (4) as follows: We define $k_+ = (k + k')/2$ and $k_- = k - k'$. Then, by expanding $\underset{\approx}{\sigma}$ about $k, k' = k_+$, Eq. (7) becomes

$$\int \frac{dk_+}{2\pi} \int \frac{dk_-}{2\pi} \int dx \, \exp(-ik_-x) \underset{\sim}{\zeta}^+(-\underset{\sim}{k}) \cdot \Bigg[\underset{\approx}{\sigma}(k_+, k_+, x, \omega) + \frac{k_-}{2} \left(\frac{\partial}{\partial k} - \frac{\partial}{\partial k'} \right)$$

$$\times \underset{\approx}{\sigma}(k, k', x, \omega) + \frac{k_-^2}{8} \left(\frac{\partial^2}{\partial k^2} - 2 \frac{\partial}{\partial k} \frac{\partial}{\partial k'} + \frac{\partial^2}{\partial k'^2} \right)$$

$$\times \underset{\approx}{\sigma}(k, k', x, \omega) + \ldots \Bigg] \Bigg|_{k=k'=k_+} \cdot \underset{\sim}{\zeta}(k') = 0 \; . \tag{8}$$

Using the identity $k_- \exp(-ik_-x) = i(\partial/\partial x)\exp(-ik_-x)$, and integrating by parts in x , Eq. (8) becomes

$$\int \frac{dk_+}{2\pi} \int \frac{dk_-}{2\pi} \int dx \exp(-ik_-x) \underset{\sim}{\zeta}^+(-\underset{\sim}{k}) \cdot \Bigg[\underset{\approx}{\sigma}(k_+, k_+, x, \omega) + \frac{i}{2} \left(\frac{\partial}{\partial k'} - \frac{\partial}{\partial k} \right) \frac{\partial}{\partial x}$$

$$\times \underset{\approx}{\sigma}(k, k', x\omega) + \ldots \Bigg] \Bigg|_{k=k'=k_+} \cdot \underset{\sim}{\zeta}(k') = 0 \; . \tag{9}$$

Comparing Eqs. (4) and (9) allows us to deduce that

$$\underset{\approx}{\Lambda}(k, x, \omega) = \left[\underset{\approx}{\sigma}(k, k, x, \omega) + \frac{i}{2} \left(\frac{\partial}{\partial k'} - \frac{\partial}{\partial k} \right) \frac{\partial}{\partial x} \underset{\approx}{\sigma}(k, k', x, \omega) + \ldots \right] \Bigg|_{k'=k} \; . \tag{10}$$

Equation (10), truncated after the second term, will be used to obtain the principal result of this paper. Applying the operation indicated in Eq. (10) leads to a relatively simple result when it is observed that the conductivity kernel $\underset{\approx}{\sigma}(k, k, x, \omega)$ has a structure of the form

$$\underset{\sim}{\underset{\sim}{\sigma}}(k,k,x,\omega) = \sum_{n=-\infty}^{\infty} \underset{\sim}{\underset{\sim}{\sigma}}_n(k,k,x,\omega) \quad , \tag{11}$$

where n is a summation over harmonics of the cyclotron frequency. As a result of applying Eq. (10), we find

$$\underset{\sim}{\underset{\sim}{\Lambda}}(\underset{\sim}{k},\underset{\sim}{x},\omega) = \sum_n \left\{ 1 + n \frac{\partial}{\partial x} \left[\frac{(\underset{\sim}{k} \times \underset{\sim}{b}) \cdot \hat{\underset{\sim}{x}}}{k_\perp^2(x)} \right] \right\} \underset{\sim}{\underset{\sim}{\sigma}}_n(\underset{\sim}{k},\underset{\sim}{k},x,\omega) , \tag{12}$$

where $\underset{\sim}{b}$ is the unit vector along the equilibrium magnetic field, $\underset{\sim}{k} = k_z \hat{\underset{\sim}{z}} + k_y \hat{\underset{\sim}{y}} + k \hat{\underset{\sim}{x}}$, and $k_\perp^2 = k^2 + (\underset{\sim}{k} \times \underset{\sim}{b} \cdot \hat{\underset{\sim}{x}})^2$. This result is demonstrated explicitly in the relatively complicated expression to be given in Eq. (16) for the electromagnetic plasma response function in a sheared magnetic field at arbitrary frequency. Recently, this correction has been incorporated in electromagnetic ballooning mode calculations.[9]

We note that Eq. (12) for $\underset{\sim}{\underset{\sim}{\Lambda}}$ is not exact. The exact form requires summing over all terms in the expression given in Eq. (8). However, we will show in a special example that the keeping of only the first correction term is adequate if $a/L_p \ll 1$ and $\omega k_y \, \mathrm{Min}(1,ka)/(\omega_c k_\perp^2 L_p) \ll 1$. We also point out that our method for constructing $\underset{\sim}{\underset{\sim}{\Lambda}}$ does not appear to introduce any spurious dissipative terms that are sometimes encountered due to inconsistent ordering in the WKB small parameter, $\varepsilon \equiv k^{-2}[dk(x)/dx] \approx 1/kL_p$.

III. ELECTROMAGNETIC FORM IN SHEARED MAGNETIC FIELD

The plasma is modeled by using a slab geometry in which all inhomogeneities are in the x-direction. The unperturbed state is described by the macroscopic quantities $n_0^{(s)}(x)$, $T^{(s)}(x)$ ($s \equiv$ species), and a self-consistent sheared magnetic field $\underset{\sim}{B}(x)$ which is assumed to vary slowly over a Larmor radius. The equilibrium magnetic field is of the form

$$\underset{\sim}{B} = B_y(x)\hat{\underset{\sim}{y}} + B_z(x)\hat{\underset{\sim}{z}} ,$$

with the magnetic shear parameter L_s defined by $L_s^{-1} = d\theta(x)/dx$ where $\theta(x) = \underset{\sim}{b}\times\hat{\underset{\sim}{x}}\cdot d\underset{\sim}{b}/dx$ with $\underset{\sim}{b} = \underset{\sim}{B}/|B|$. We also denote

$$\hat{\underset{\sim}{\eta}} = \underset{\sim}{b}\times\hat{\underset{\sim}{x}}$$

$$k_\eta = \underset{\sim}{k}\times\underset{\sim}{b}\cdot\hat{\underset{\sim}{x}} = \frac{-k_zB_y(x) + k_yB_z(x)}{|B(x)|}$$

$$k_\parallel = \underset{\sim}{k}\cdot\underset{\sim}{b} = \frac{k_zB_z(x) + k_yB_y(x)}{|B(x)|} .$$

To describe the perturbed field, we use the basis

$$\underset{\sim}{E}(\underset{\sim}{r}) = \{-\underset{\sim}{\nabla}_\perp\phi(x) - \underset{\sim}{\nabla}\times[\mathcal{A}(x)\underset{\sim}{b}] + E_\parallel(x)\underset{\sim}{b}\} \exp(ik_yy + ik_zz) \qquad (13)$$

where $\underset{\sim}{\nabla}$ acts on the exponentials as well as the arguments of the amplitudes. This basis is useful for describing the coupling of magnetic compressional perturbations. We will denote $\underset{\sim}{\eta}(r)$ as the triad $[\phi(x),\mathcal{A}(x),E_\parallel(x)]$, and one can show[8] that the adjoint triad is $\eta^+ = [-\phi(x),\mathcal{A}(x),E_\parallel(x)]$.

The kernel for the Vlasov-Maxwell equations in this basis is derived in Ref. 8 (Eqs. A.8-A.10). Written in the form of Eq. (7), we find that the kernel is given by

$$\underset{\approx}{\sigma}(k,k',x,\omega) = \underset{\approx}{\sigma}_V(k,k',x,\omega) + \underset{\approx}{\sigma}_p(k,k',x,\omega) \qquad (14)$$

where $\underset{\approx}{\sigma}_V(k,k',x,\omega)$ is the local response function (mostly vacuum terms) whose form is artificially bulky in this k representation and its detailed form need not concern us yet. The non-local plasma contribution $\underset{\approx}{\sigma}_p(k,k',x,\omega)$ is given by

$$\underset{\approx}{\sigma}_p(\underset{\sim}{k},\underset{\sim}{k}',x,\omega) = -\frac{4\pi}{c^2}\sum_s \int d^3\underset{\sim}{v} \sum_n \frac{D_n F_s}{(\omega-\omega_n)} \langle \underset{\sim}{q}_n(\underset{\sim}{k})\rangle^* \langle \underset{\sim}{q}_n(k')\rangle \qquad (15)$$

where

$$D_n F_s = \omega_n \frac{\partial F_s}{\partial E} + k_y \frac{\partial F_s}{\partial P_y} + k_z \frac{\partial F_s}{\partial P_z}$$

$$\omega_n = n\bar{\omega}_c + k_z\bar{v}_z + k_y\bar{v}_y$$

$$\bar{q} = \frac{1}{\tau}\int_{x^-}^{x^+} \frac{dx\, q(E,P_y,P_z,x)}{|v_x(E,P_y,P_z,x)|}$$

$$\tau = \int_{x^-}^{x^+} \frac{dx}{|v_x|}$$

$$v_x(E,P_y,P_z,x^{\pm}) = 0$$

$$\langle q_n(\underset{\sim}{k})\rangle = \frac{1}{\tau}\oint d\tau \begin{bmatrix} -i\underset{\sim}{k}_\perp\cdot\underset{\sim}{v}(\tau) \\ -i\underset{\sim}{k}\times\underset{\sim}{b}\cdot\underset{\sim}{v}(\tau) \\ \underset{\sim}{v}(\tau)\cdot\underset{\sim}{b} \end{bmatrix} \exp[i\underset{\sim}{k}\cdot\underset{\sim}{\delta r}(\tau) - in\bar{\omega}_c\tau]$$

$$\underset{\sim}{k} = k\hat{\underset{\sim}{x}} + k_y\hat{\underset{\sim}{y}} + \underset{\sim}{k}_z\hat{\underset{\sim}{z}}$$

$$\delta\underset{\sim}{r}(\tau) = \underset{\sim}{r}(\tau) - \bar{\underset{\sim}{v}}\tau$$

$$x,\underset{\sim}{v} \equiv \text{particle x-position and velocity at } \tau = 0 .$$

Now using Eq. (10), we find that the contribution of this term to the local dispersion function is,

$$\underset{\approx}{\Lambda}_p(k,x,\omega) = -\frac{4\pi}{c^2}\sum_s\int d^3\underset{\sim}{v}\sum_n\left(1 + \frac{\partial}{\partial x}\frac{nk_\eta}{k_\perp^2}\right)\frac{D_n f_s}{(\omega - \omega_n)}\left\langle \underset{\sim}{q}_n(k)\right\rangle^*\left\langle \underset{\sim}{q}_n(k)\right\rangle \tag{16}$$

where we have neglected first-order terms in $\underset{\approx}{\Lambda}_p$ which would vanish when the determinant of $\underset{\approx}{\Lambda}$ is constructed, we neglected second-order terms in the macroscopic scale length, and, finally, we note that instead of all phase-space quantities being a function of the constants of motion E , P_y , P_z , they should be taken as a function of E, $P_\parallel = (P_yB_y + P_zB_z)/B$, $A_\eta = (A_yB_z - A_zB_y)/B$ (note that A_η is a vector potential component at the guiding center position).

If we approximate the particle orbits as circular spirals, we note that $\langle q_n(k)\rangle$ can be written as

$$\left\langle \underset{\sim}{g}_n(\underset{\sim}{k})\right\rangle = \begin{bmatrix} -i\left(n\omega_{cs} + k_\eta\langle v_\eta\rangle\right) J_n(z) \\ -z\omega_{cs}\dfrac{\partial J_n(z)}{\partial z} + ik\langle v_\eta\rangle J_n(z) \\ v_\parallel J_n(z) \end{bmatrix} \exp\left[iz\ \sin(\phi-\theta) - in(\phi-\theta)\right] \tag{17}$$

where

$$\underset{\sim}{v} = (\mu B)^{1/2}\cos\phi\,\hat{\underset{\sim}{x}} + \left[(\mu B)^{1/2}\sin\phi + \langle v_\eta\rangle\right]\hat{\underset{\sim}{\eta}}$$

$$\underset{\sim}{k}_\perp = k_\perp \cos\theta\,\hat{\underset{\sim}{x}} + k_\perp \sin\theta\,\hat{\underset{\sim}{\eta}}$$

$$\langle v_\eta\rangle = \frac{\mu}{\omega_{cs}}\frac{dB}{dx}$$

$$z = \frac{k_\perp(\mu B)^{1/2}}{\omega_{cs}} .$$

Finally, we note that local term $\underset{\sim}{\Lambda}_V(k,x,\omega)$ is found to have the components

$$\Lambda_{V11} = k_\perp^2\left(k_\parallel^2 - \frac{\omega^2}{c^2} + \sum_x \frac{\omega_{ps}^2}{c^2}\right)$$

$$\Lambda_{V12} = -\Lambda_{V21} = 0$$

$$\Lambda_{V13} = -\Lambda_{V31} = -ik_\parallel k_\perp^2$$

$$\Lambda_{V22} = k_\perp^2\left(k_\perp^2 + k_\parallel^2 - \frac{\omega^2}{c^2} + \sum_s \frac{\omega_{ps}^2}{c^2}\right)$$

$$\Lambda_{V23} = \Lambda_{V32} = k_{\parallel} \frac{\partial k_{\eta}}{\partial x}$$

$$\Lambda_{V33} = k_{\perp}^2 - \frac{\omega^2}{c^2} + \sum_s \frac{\omega_{ps}^2}{c^2} \,. \tag{18}$$

The local dispersion relation is then determined by

$$\mathrm{Det}(\Lambda_{ij}) = \mathrm{Det}\left|\Lambda_{ijV} + \Lambda_{ijp}\right| = 0 \,. \tag{19}$$

IV. COLD PLASMA LIMIT

We now discuss the cold limit of a plasma in a magnetic field. Starting from fluid equations, the WKB method is straight-forward. However, the Vlasov equation only reproduces the fluid result when the corrections we have derived are taken into account.

We consider a neutral plasma slab, homogeneous in the y-z directions, a spatial variation in the x-direction, with a homogeneous magnetic field in the $\hat{z}$-direction. We consider electrostatic perturbations with $k_z = 0$. A straight-forward application of the cold fluid equations, assuming $n_e = n_i$, with $\omega/\omega_{ce} \ll 1$, but ω/ω_{ci} finite, yields the differential equation

$$\frac{\partial}{\partial x}\left[\left(\frac{\omega_{pi}^2}{\omega_{ci}^2 - \omega^2}\right)\frac{\partial \phi}{\partial x}\right] - \frac{k_y^2 \omega_{pi}^2 \phi}{\omega_{ci}^2 - \omega^2} - \phi \frac{\partial}{\partial x}\left[\frac{k_y \omega_{pi}^2 \omega}{(\omega_{ci}^2 - \omega^2)\omega_{ci}}\right] = 0 \tag{20}$$

where the subscript i refers to ions.

The local dispersion relation then yields

$$\frac{k_\perp^2\omega_{pi}^2}{\omega_{ci}^2 - \omega^2} + k_y \frac{\partial}{\partial x}\left[\frac{\omega_{pi}^2\omega}{(\omega_{ci}^2 - \omega^2)\omega_{ci}}\right] = 0 . \tag{21}$$

We note that if $\omega/\omega_{ci} \gg 1$, the local frequency is given by

$$\omega = -\left[\frac{k_y}{k_\perp^2}\frac{1}{\omega_{pi}^2}\frac{\partial}{\partial x}\left(\frac{\omega_{pi}^2}{\omega_{ci}}\right)\right]^{-1} . \tag{22}$$

This mode arises from the balance of the ion inertia term and the electron electric field drift, which is no longer cancelled by ions as $\omega \gg \omega_{ci}$. This mode is of importance in the stability analysis of EBT.[9]

Now we can examine the result of this problem from the Vlasov expression if we set $k_\parallel$ and ω/kc to zero and equate Λ_{11} component of Eq. (19) to zero (the appropriate operation if $A_\parallel = \mathscr{A} = 0$). If one then attempts to calculate the resulting expression in the cold plasma limit, but neglects the correction term, $n\left(k_y/k_\perp^2\right)(\partial/\partial x)$ for $n = \pm 1$, one finds a local dispersion relation of the form,

$$\frac{k_\perp^2\omega_{pi}^2}{\omega_{ci}^2 - \omega^2} = 0 . \tag{23}$$

Without the correction term, the equations do not have the physical fact that only electrons, and not ions, have a $c\underset{\sim}{E}\times\underset{\sim}{b}/B$ drift across the field lines at frequencies larger than the ion-cyclotron frequency. Hence, in

Eq. (20), the ion and electron drifts still cancel. However, by introducing the correction terms, one can show that Eq. (21) is reproduced directly from the Vlasov expression in the cold plasma limit.

V. EXACT WKB FORM FOR A SPECIAL CASE

The derivation of Eq. (16) relies on an expansion in powers of $nk_\eta/k_\perp^2 L_p = n\varepsilon k_\eta/k_\perp$ where L_p is the density variation scale length. To obtain a tractable form, we have assumed that $n\varepsilon$ is small, and have retained only first-order corrections. For certain types of high frequency $(\omega \gg \omega_{cs})$, short wavelength $(k_\perp a_i \gg 1)$ modes, $n\varepsilon$ may be of order unity so that the neglect of higher order corrections is not justified. The general treatment for this case is difficult, but for certain special cases the appropriate integrals can be performed.

In this section we study the Berk-Book form for the electrostatic problem for the special cases of a rigid-flow equilibrium without employing the expansion technique. The more complicated electromagnetic case is discussed in Ref. 6. For the derivation, we use the unperturbed distribution function

$$f_0^{(s)} = \frac{n_0}{\pi^{3/2} v_{T_s}^3} \exp - \left(\frac{v^2}{v_{T_s}^2} + \frac{x_G}{L_p} \right); \quad v_{T_s}^2 = \frac{2T_s}{m_s} , \qquad (24)$$

where we assume $x_G = x + v_y/\omega_{cs}$ and neglect temperature, magnetic field inhomogeneity, and external drifts.

To derive the electrostatic response, the perturbed charge density must be calculated. Hence, we first obtain the perturbed distribution function by integrating the linearized Vlasov equation along the unperturbed orbits. Using the identities $\phi(x) = \int dx' \delta(x - x')\phi(x')$ and $\delta(x - x') = \int (dk/2\pi)\exp[ik(x - x')]$, the result for $\hat{f}_1^{(s)} = f_1^{(s)}\exp(i\omega t - ik_y y - ik_z z)$ may be written as

$$\hat{f}_1^{(s)} = -\frac{n_0 q_s}{T_s} F^{(s)} \int \frac{dk}{2\pi} \int dx' \phi(x') \exp\left[ik(x - x') - \frac{(x + x')}{2L_p}\right]$$
$$\left\{ \exp -\left(\frac{v_y}{\omega_{cs} L_p}\right) + i(\omega - \omega_s) \int_{-\infty}^{t} dt \exp\left[-\frac{v_y + v_y(t')}{2\omega_{cs} L_p}\right] I(t,t') \right\} , \qquad (25)$$

where

$$F^{(s)} = \pi^{-3/2} v_{T_s}^{-3} \exp\left(-\frac{v^2}{v_{T_s}^2}\right) ,$$

$$\omega_s = -\frac{k_y v_{T_s}^2}{2m_s \omega_{cs} L_p} ,$$

$$I(t,t') = \exp i\{\underset{\sim}{k} \cdot [\underset{\sim}{r}(t') - \underset{\sim}{r}] - \omega(t' - t)\} ,$$

and

$$\underset{\sim}{k} = k\hat{\underset{\sim}{x}} + k_y \hat{\underset{\sim}{y}} + k_z \hat{\underset{\sim}{z}} .$$

The t' integration may be readily performed to obtain

$$\hat{f}_1^{(s)} =$$

$$-\frac{q_s}{T_s} n_0 F^{(s)} \int dx' \phi(x') \int \frac{dk}{2\pi} \exp\left[ik(x - x') - \frac{(x + x')}{2L_p}\right] \left(\exp\left(-\frac{v_y}{\omega_{cs} L_p}\right)\right.$$

$$- \sum_{m,n,p,q} \left[\frac{\omega - \omega_s}{\omega - (m - q)\omega_{cs} - k_z v_z}\right] J_n(\rho) J_m(\rho) J_p(i\zeta) J_q(i\zeta)$$

$$\left.\times \exp\left\{i[(n - m)(\phi - \theta) + (p + q)\phi]\right\}\right) , \qquad (26)$$

where $\rho = k_\perp v_\perp / \omega_{cs}$, $\zeta = v_\perp / 2\omega_{cs} L_p$, and $\theta = \tan^{-1}(k_y/k)$.

The perturbed charge density $\rho = \sum_s q_s \int d\underset{\sim}{v} f_1^{(s)}$ may be calculated from Eq. (26). Performing the azimuthal ϕ integration the perturbed charge density (suppressing the caret) is

$$\rho = -\sum_s \frac{n_0 q_s^2}{T_s} \int dx' \phi(x') \int \frac{dk}{2\pi} \exp\left[ik(x - x') - \frac{(x + x')}{2L_p}\right]$$

$$\times \left\{ \exp\left(\frac{b_s \varepsilon^2}{2}\right) - (\omega - \omega_s) \int d\underset{\sim}{u} F^{(s)} \sum_{n,p,q} J_n(\rho) J_{n+p+q}(\rho) J_p(i\zeta) J_q(i\zeta)\right.$$

$$\left.\times [\omega - (m + p)\omega_{cs} - k_z v_z]^{-1} \exp[i(p + q)\theta] \right\} , \qquad (27)$$

where $\varepsilon = (k_\perp L_p)^{-1}$, $b_s = k_\perp^2 T_s / m_s \omega_{cs}^2$, and $d\underset{\sim}{u} = 2\pi v_\perp dv_\perp dv_z$. Changing indices in the triple sum by letting $n + p \to n, p \to -p$, using the identity $J_{-p}(i\zeta) = J_p(-i\zeta)$, the sum rule

$$\sum_p J_p(x') J_{n+p}(x) \exp(ip\theta) = \exp\left[-in\left(\lambda + \frac{\theta}{2}\right)\right] J_{-n}(A) ,$$

with

$$A = \left(x^2 + x'^2 - 2xx' \cos\theta\right)^{1/2} ,$$

$$\lambda = \tan^{-1}\left\{\tan\left(\frac{\theta}{2}\right)\left[\frac{(x' + x)}{(x - x')}\right]\right\} ,$$

and performing the $v_\perp$ integration in Eq. (27) yields

$$\rho = -\sum_s \frac{n_0 q_s^2}{T_s} \int dx' \phi(x') \int \frac{dk}{2\pi} \exp\left[ik(x - x') - \frac{(x + x')}{2L_p}\right]$$

$$\times \left\{\exp\left(\frac{b_s \varepsilon^2}{2}\right) - (\omega - \omega_s) \int \frac{dv_z}{\pi^{1/2} v_{T_s}} \exp\left(-\frac{v_z^2}{v_{T_s}^2}\right)\right.$$

$$\left. \times \sum_m \mathscr{L}_n I_n \exp\left[-b_s\left(1 - \frac{\varepsilon^2}{4}\right)\right] \exp(2n\text{Im}\lambda)\right\} , \qquad (28)$$

where $I_n = I_n\left(b_s\{[1 - (\varepsilon^2/4)]^2 + \varepsilon^2\cos^2\theta\}^{1/2}\right)$, $\mathscr{L}_n = (\omega - n\omega_{cs} - k_z v_z)^{-1}$ and $\text{Im}\lambda$ denotes the imaginary part of λ. This result may be further simplified by assuming that $\varepsilon \ll 1$ and neglecting corrections of order ε^2 in Eq. (28) (note that $b_s\varepsilon^2$ may be of order unity). For small ε, it may be shown that $\text{Im}\lambda = -(\varepsilon/2)\sin\theta$, and we obtain the final result

$$\rho = -\sum_s \frac{n_0 q_s^2}{T_s} \int dx' \phi(x') \int \frac{dk}{2\pi} \exp\left[ik(x - x') - \frac{(x + x')}{2L_p} \right]$$

$$\times \left[\exp\left(\frac{b_s \varepsilon^2}{2}\right) - (\omega - \omega_s) \int \frac{dv_z}{\pi^{1/2} v_{T_s}} \exp\left(-\frac{v_z^2}{v_{T_s}^2}\right) \right.$$

$$\left. \times \sum_n \mathscr{L}_n \exp(-n\varepsilon \sin\theta) I_n(b_s) \exp(-b_s) \right] . \qquad (29)$$

To obtain the more approximate result of the previous sections, we set $\exp(-n\varepsilon \sin\theta) \simeq 1 - n\varepsilon \sin\theta$. Clearly, it is only valid if $n\varepsilon \ll 1$, or if the terms beyond n are no longer important in the sum. This consideration leads to the establishing of the second inequality of Eq. (1).

In the high frequency $(\omega \gg \omega_{cs})$, short wavelength $(k_\perp a_s \gg 1)$ limit, a large number of terms in the summation over n must be retained in the ion response. Hence, $n\varepsilon$ may be of order unity and the correction term $\exp(-n\varepsilon \sin\theta)$ is not expandable. In this limit ("unmagnetized species"), the perturbed charge density may be obtained by first using the asymptotic representation $I_n(b_s) \exp(-b_s) = (2\pi b_s)^{-1/2} \exp[-(n^2/2b_s)]$ and then employing the asymptotic identity[4]

$$\lim_{\delta \to 0} \sum_n \frac{f(\delta N)}{x - n} = P \int \frac{f(y)dy}{\delta x - y} + \pi \cot(\pi x) f(\delta x) , \qquad (30)$$

to evaluate the summation. Equation (30) follows from Cauchy's Residue

Theorem applied to the function $[\cot(\pi z)f(\delta x)/(z - x)]$. Performing these manipulations, the charge density becomes

$$\rho_s = -\frac{n_0 e^2}{T_s}\int dx'\phi(x')\int\frac{dk}{2\pi}\exp\left[ik(x - x') - \frac{(x + x')}{2L_p}\right]$$

$$\left[\exp\left(\frac{b_s\varepsilon^2}{2}\right) - (\omega - \omega_s)\left(\left(\int\frac{dv_z}{\pi^{1/2}v_{T_s}}\exp\left(-\frac{v_z^2}{v_{T_s}^2}\right)\right.\right.\right.$$

$$\left\{P\int\frac{dy}{\pi^{1/2}}\frac{\exp[-y^2 - (2b_s)^{1/2}\varepsilon\sin\theta\ y]}{\omega - k_z v_z - k_\perp v_{T_s} y} + \frac{\pi^{1/2}}{k_\perp v_{T_s}}\cot\left(\pi\frac{\omega - k_z v_z}{\omega_{cs}}\right)\right.$$

$$\left.\left.\left.\exp\left[\frac{-(\omega - k_z v_z)^2}{k_\perp^2 v_{T_s}^2} - \frac{(\omega - k_z v_z)}{k_\perp v_{T_s}}(2b_s)^{1/2}\varepsilon\sin\theta\right]\right\}\right)\right] . \quad (31)$$

The term $\cot[\pi(\omega - k_z v_z)/\omega_{cs}] \to -i$, if either $\mathrm{Im}\omega \gg \omega_{cs}$ or $\Delta(\omega - k_z v_z) \gg \omega_{cs}$, where $\Delta(\omega - k_z v_z)$ is the resonance width due to the spread in v_z . Employing this limit, the v_z integration is readily performed. Finally, using the identity $\pi^{-1/2}\int dz \exp(-z^2) = 1$ and setting $y = (\underset{\sim}{v}_\perp \cdot \underset{\sim}{k}_\perp)/k_\perp v_{T_s}$, $z = \hat{\underset{\sim}{z}}\cdot\underset{\sim}{v}\times\underset{\sim}{k}/k_\perp v_{T_s}$, the desired result for the perturbed density response is

$$\rho_s = -\frac{n_0 e^2}{T_s} \int dx' \phi(x') \int \frac{dk}{2\pi} \exp\left[ik(x - x') - \frac{(x + x')}{2L_p} \right]$$

$$\left\{ \exp\left(\frac{b_s \varepsilon^2}{2}\right) - (\omega - \omega_s) \int \frac{d\underset{\sim}{v}}{\pi^{3/2} v_{T_s}^3} \frac{\exp\left[-(v_\perp^2/v_{T_s}^2) - v_y/\omega_{cs} L_p \right]}{\omega - \underset{\sim}{k} \cdot \underset{\sim}{v}} \right\} . \qquad (32)$$

This result is the response of a completely unmagnetized species for a distribution of the form,

$$f_0^{(s)} = \frac{n_0(x)}{\pi^{3/2} v_{T_s}^3} \exp\left[-\left(\frac{v^2}{v_{T_s}^2} + \frac{v_y}{\omega_{cs} L_p} \right) \right] .$$

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