DOE/ET/53088/69 IFSR #69

FINITE LARMOR RADIUS STABILITY THEORY OF EBT PLASMAS

H. L. Berk
C. Z. Cheng*
M. N. Rosenbluth
J. W. Van Dam

Institute for Fusion Studies

November 1982

FINITE LARMOR RADIUS STABILITY THEORY OF EBT PLASMAS

H. L. Berk

C. Z. Cheng*

M. N. Rosenbluth

J. W. Van Dam

Institute for Fusion Studies
The University of Texas at Austin
Austin, Texas 78712

November 1982

* Permanent Address: Princeton Plasma Physics Laboratory, Princeton, New Jersey 08544

Abstract

An eikonal ballooning mode formalism is developed to describe curvature-driven modes of hot electron plasmas in bumpy tori. The formalism treats frequencies comparable to the ion cyclotron frequency, as well as arbitrary finite Larmor radius and field polarization, although the detailed analysis is restricted to $E_{\parallel} = 0$. Moderate hot electron finite Larmor radius effects are found to lower the background beta core limit, whereas strong finite Larmor radius effects produce stabilization. The critical finite Larmor radius parameter with weak curvature is

$$FR = \frac{k_{\perp}^2 \rho_h^2 R/\Delta_b}{1 + P_{\parallel}'/P_{\perp}'}$$

where $k_{\perp}$ is the perpendicular wavenumber, ρ_h the hot electron Larmor radius, R the magnetic field radius of curvature at the hot electron layer, Δ_b the magnetic field scale length in the diamagnetic well, and $P_{\parallel,\perp}'$ are the parallel and perpendicular pressure gradients. The interchange instability arises if

$$1 > FR > 1 - \frac{2\beta_c R}{[\Delta(1 + P_{\parallel}'/P_{\perp}')]} ,$$

whereas all modes are stable if $FR > 1$, where β_c is the core plasma beta and Δ is the core plasma pressure gradient length.

I. INTRODUCTION

One of the most important problems concerning hot electron plasmas, such as in the Elmo Bumpy Torus (EBT), is to determine the parameter regimes for stable plasma operation with respect to curvature-driven modes, such as interchange modes, ballooning modes, and the compressional Alfvén wave.[1-7] In EBT, the plasma contains a very hot electron population which digs a magnetic well in each of the mirror cells. This hot electron population is characterized by having a magnetic drift frequency, ω_{dh}, that is larger than the frequency, ω, of the typical MHD perturbations and, in present EBT experiments, is comparable to the ion-cyclotron frequency, ω_{ci}. Therefore, the conventional fluid and kinetic energy principles[8], which assume that ω is much larger than the magnetic drift frequency of the plasma species, are not appropriate for the stability analysis. There is now considerable effort to develop an alternate analysis. One method is a kinetic energy principle[9,10], which employs the kinetic guiding-center model for hot electrons, with the ordering $\omega < \omega_{dh}$. However, this kinetic energy principle gives predictions that are too pessimistic due to the neglect of finite Larmor radius (FLR) effects of ions and hot electrons, as well as the neglect of finite hot electron density. The effect of finite hot electron density has been studied in modal analysis studies of z-pinch[11] and slab models.[12] The effect of the FLR was neglected in previous work. This paper, as well as recent work by Tsang and Catto[7] and El Nadi[4], studies the important effect of FLR. In addition, we will incorporate more realistic geometrical effects along the magnetic field lines.

In past stability studies it has been a frequent practice to neglect FLR effects and to take the ideal MHD assumption that the parallel perturbed electric field, $E_{\parallel}$, vanishes. This assumption is valid for MHD-like modes. As has been shown in kinetic studies of ballooning modes in tokamaks[13] and recently in tandem mirrors[14], the $E_{\parallel} = 0$ assumption is not valid for a large class of trapped-particle modes. Since the curvature-driven modes involve perturbations with frequencies ranging from the ion diamagnetic drift frequency to above the ion-cyclotron frequency, it is important to develop equations that are valid over a wide range of frequencies. In this paper we present the relevant eikonal-ballooning mode equations that include finite Larmor radius, finite $E_{\parallel}$, and high-frequency effects $(\omega \sim \omega_{ci})$, appropriate to the relevant modes of EBT. This naturally leads to a complicated set of equations that will require extensive analysis and numerical work. To simplify analysis we will study in detail only the case of $E_{\parallel} = 0$ and finite but small Larmor radius. The study of other relevant cases will be left to future investigations.

A simplifying assumption used in this work is to assume that a given plasma species satisfies either the inequality $\omega - \omega_d > \omega_b$ or $\omega - \omega_d < \omega_b$, where ω_b is the transit (bounce) frequency of a passing (trapped) particle in a single mirror cell. Usually the hot electrons will satisfy the second inequality, while the background plasma can be in either of the two regions.

The structure of the paper is as follows. In Sec. II, the coordinate system and MHD equilibria for a bumpy cylinder model will be described. In Sec. III, a gyro-kinetic equation[15,16] valid for arbitrary frequency, with anisotropic equilibrium pressure and ambipolar electrostatic potential, is presented. In Sec. IV, the solutions of the gyro-kinetic equation are obtained and are used to construct the current and charge densities needed for Maxwell's equations. The eigenmode equations in various frequency regimes are thereby obtained. In Sec. V, a local stability analysis including the finite Larmor radius effects will be presented for the interchange modes and the compressional Alfvén waves in the low $(\omega \ll \omega_{ci})$ and high $(\omega \gtrsim \omega_{ci})$ frequency limits. In Sec. VI, a line-averaged analysis in the high hot electron bounce frequency limit for interchange and compressional waves on each field line is given.

The principal physical result of this paper is that hot electron FLR effects stablize all curvature-driven modes when, roughly, $k_\perp^2\rho_h^2 \gtrsim \kappa\Delta_b$, where $k_\perp$ is the perpendicular eikonal wavenumber, ρ_h is the hot electron Larmor radius, κ is the field line curvature, and Δ_b is the magnetic scale length. The line-averaged analysis confirms the local analysis and shows how to include the appropriate weightings along a field line.

II. COORDINATES AND EQUILIBRIUM

We will analyse EBT in its large-aspect ratio limit for which a bumpy cylinder model applies (see Fig. 1). The magnetic field can be expressed as

$$\underset{\sim}{B} = \underset{\sim}{\nabla}\psi \times \underset{\sim}{\nabla}\theta \tag{1}$$

where ψ labels the magnetic flux surfaces and θ is the ignorable poloidal angle. The quantity s is the distance along a field line. In this coordinate system, $\underset{\sim}{B}\cdot\underset{\sim}{\nabla} = B\partial/\partial s|_{\psi,\theta}$.

The equilibrium of the system is specified by the particle guiding-center distribution functions for each species, $F(E,\mu,\psi_g)$. The variables are the constants of motions: energy $E = v^2/2 + e\Phi(\psi,s)/m$, magnetic moment $\mu = v_\perp^2/2B$, and the magnetic flux position of the guiding center, ψ_g . The equilibrium satisfies the following two conditions:

(1) quasi-neutrality:

$$\rho = \sum_j e_j N_j = 0 \tag{2}$$

where j is summed over all species and ρ is the charge density,

$$N_j = \sum_\sigma 2\pi \int_0^\infty dE \int_0^{E/B} \frac{Bd\mu}{|v_{\parallel j}|} F_j(E,\mu,\psi) ,$$

with

$$v_{\parallel j} = \sigma\sqrt{2}\left(E - \mu B - \frac{e_j\Phi}{m_j}\right)^{1/2}$$

and

$$\sigma = \frac{\underset{\sim}{v}\cdot\underset{\sim}{B}}{|\underset{\sim}{v}\cdot\underset{\sim}{B}|} .$$

(2) force balance:

$$\underset{\sim}{\nabla}_\perp\left(\frac{B^2}{2} + 4\pi P_\perp\right) = \underset{\sim}{\kappa}\left[B^2 + 4\pi\left(P_\perp - P_\parallel\right)\right] \tag{3}$$

where $\kappa = (b\cdot\nabla)b$ is the curvature, $b = B/|B|$, and $P_\parallel$ and $P_\perp$ are, respectively, the parallel and perpendicular pressures given by

$$\begin{Bmatrix} P_\parallel \\ P_\perp \end{Bmatrix} = \sum_{j,\sigma} 2\pi m_j \int \frac{dE d\mu B}{|v_{\parallel j}|} F_j \begin{Bmatrix} 2\left(E - \mu B - \dfrac{e_j\Phi}{m_j}\right) \\ \mu B \end{Bmatrix} .$$

In addition, it follows from force balance along the field line that, for each species,

$$\frac{\partial P_\parallel}{\partial s} = (P_\parallel - P_\perp)\frac{1}{B}\frac{\partial B}{\partial s} . \tag{4}$$

For a general equilibrium solution, the potential Φ and the distribution functions F_j are constrained to satisfy Eqs. (2) and (3). It has been shown that Eqs. (2) and (3) lead to a partial differential equation for ψ which, when there is zero parallel current, is of the form[17]

$$\nabla^2\psi + \nabla\psi\cdot\nabla\ln\left(\frac{\sigma}{r^2}\right) = -4\pi\,\frac{r^2}{\sigma}\frac{\partial P_\parallel}{\partial\psi} \tag{5}$$

where $r = |\nabla\theta|^{-1}$, $\sigma = 1 + 4\pi(P_\perp - P_\parallel)/B^2$. Usually, Eq. (5) requires a numerical solution for a complete specification of ψ , although analytic techniques are viable in a long-thin approximation. The exact form of the equilibrium will not be essential to our stability analysis, but the equilibrium constraints given in Eqs. (2), (3), and (4) will be extensively used.

III. BASIC EQUATIONS

We consider waves which can be described by the eikonal representation, i.e., a perturbed quantity $\xi(\psi,\theta,s)$ is expressed as $\xi(s)\exp[iS(\psi,\theta)]$ with $\underset{\sim}{k}_\perp = \underset{\sim}{\nabla}\psi(\partial S/\partial\psi) + \underset{\sim}{\nabla}\theta(\partial S/\partial\theta) \equiv k_\psi \underset{\sim}{\nabla}\psi + k_\theta \underset{\sim}{\nabla}\theta$. We then consider waves with $k_\perp \rho^2/L \ll 1$ (L is the macroscopic scale length and ρ is the particle Larmor radius). The perturbed vector potential is chosen as

$$\underset{\sim}{A} = A_\parallel \underset{\sim}{b} + \underset{\sim}{\nabla} \times \mathscr{A} \underset{\sim}{b} \ ,$$

and the perturbed electric field is given by

$$\underset{\sim}{E} = -\underset{\sim}{\nabla}\phi + \frac{i\omega}{c} \underset{\sim}{A} \ .$$

Then the perturbed distribution function f_j for a given species will satisfy the equation (we delete the subscript denoting species):

$$f = \frac{e}{m}\frac{\partial F}{\partial E}\phi + \frac{e}{mB}\frac{\partial F}{\partial \mu}\left(\phi - \frac{v_\parallel A_\parallel}{c}\right) - r\,\frac{\partial \mathscr{A}}{\partial \psi}\frac{\partial F}{\partial \psi} + \sum_{\ell=-\infty}^{\infty}\left(\left\{g_\ell - \frac{e}{mB}\frac{\partial F}{\partial \mu}\left[\left(\phi - \frac{v_\parallel A_\parallel}{c}\right)J_\ell - \frac{k_\perp v_\perp \mathscr{A}}{c}J'_\ell\right]\right\}\exp(iL_\ell)\right) \qquad (6)$$

where $J_\ell = J_\ell(z)$ is the Bessel function of order ℓ , $z = k_\perp v_\perp/\omega_c$, $J'_\ell = dJ_\ell/dz$, $L_\ell = (\underset{\sim}{k}_\perp \times \underset{\sim}{v} \cdot \underset{\sim}{b})/\omega_c - \ell\zeta$, $\omega_c = eB/mc$, and ζ is the particle gyro-phase angle between $\underset{\sim}{v}_\perp$ and $\underset{\sim}{k}_\perp$. The function g_ℓ satisfies the equation[15,16]

$$\left(\omega + iv_\parallel \frac{\partial}{\partial s} - \omega_d - \ell\omega_c\right) g_\ell$$

$$= \frac{eF}{T}\left(\tilde{\omega}_\ell - \omega_*^T\right)\left[\left(\phi - \frac{v_\parallel A_\parallel}{c}\right) J_\ell - \frac{k_\perp v_\perp}{c} \mathcal{A}_\ell J_\ell'\right] \qquad (7)$$

where

$$\omega_d = \omega_b\left(\frac{m\mu B}{T}\right) + \omega_\kappa\left(\frac{mv_\parallel^2}{T}\right) + \omega_E$$

$$\omega_b = \left(\frac{cT}{eB^2}\right) \underset{\sim}{k}_\perp \cdot \underset{\sim}{b} \times \underset{\sim}{\nabla} B$$

$$\omega_\kappa = \left(\frac{cT}{eB}\right) \underset{\sim}{k}_\perp \cdot \underset{\sim}{b} \times \underset{\sim}{\kappa}$$

$$\omega_E = \frac{c}{B} \underset{\sim}{k}_\perp \cdot \underset{\sim}{b} \times \underset{\sim}{\nabla}\Phi$$

$$\tilde{\omega}_\ell = -\left(\frac{T}{m}\right)\left(\omega \frac{\partial}{\partial E} + \frac{\ell\omega_c}{B}\frac{\partial}{\partial \mu}\right) \ln F$$

$$\omega_*^T = \frac{cT}{eB} \underset{\sim}{k}_\perp \cdot \underset{\sim}{b} \times \underset{\sim}{\nabla} \ln F \quad . \qquad (8)$$

The parameter T is a scaling factor introduced arbitrarily to serve as a rough measure of the temperature of a given species.

The gyro-kinetic equation is coupled with Maxwell's equations to form the basic integro-differential equations for the system. After applying the standard approximations of quasi-neutrality and the neglect of displacement current, the forms of the three equations we use are:

(1) Quasi-neutrality Condition:

$$\sum_j \int d^3v\,ef = \sum_j e \int d^3v \left(\frac{e}{m} \frac{\partial F}{\partial E} \phi + \sum_\ell q_\ell g_\ell J_\ell \right) = 0 \qquad (9)$$

where q_ℓ is an operator of the form,

$$q_\ell = 1 + \ell\, \underset{\sim}{k}_\perp \times \underset{\sim}{b} \cdot \underset{\sim}{\nabla} \frac{1}{k_\perp^2} ,$$

The term q_ℓ is a correction[18] needed to account for the difference between guiding-center position and real particle position, and the $\underset{\sim}{\nabla}$ operator in q_ℓ operates only on equilibrium parameters. The modification of q_ℓ from unity is important for describing waves with $\omega \gtrsim \omega_c$. To obtain Eq. (9) we have used $\sum J_\ell^2 = 1$ and $\sum J_\ell J_\ell' = 0$.

(2) Perpendicular Current Equation:

$$\underset{\sim}{b} \cdot \underset{\sim}{\nabla} \times (\underset{\sim}{\nabla} \times \underset{\sim}{B}) = \frac{4\pi}{c} \underset{\sim}{b} \cdot \underset{\sim}{\nabla} \times \underset{\sim}{j} \equiv \frac{4\pi i}{c} \sum_j e \int d^3v (\underset{\sim}{k}_\perp \times \underset{\sim}{v}) \cdot \underset{\sim}{b} f \quad .$$

In terms of our variables, this equation can be written as

$$k_\perp^4 \mathscr{A} = \frac{4\pi}{c} \sum_j e \int d^3v k_\perp v_\perp \left(\frac{e}{2mB} \frac{\partial F}{\partial \mu} \frac{k_\perp v_\perp}{c} \mathscr{A} + \sum_\ell q_\ell g_\ell J_\ell' \right) \qquad (10)$$

where several Bessel function summations were used, including $\sum J_\ell'^2 = 1/2$.

(3) Parallel Current Equation:

$$\underset{\sim}{\nabla}\cdot\left(\underset{\sim}{b}\underset{\sim}{b}\cdot\underset{\sim}{\nabla}\times\underset{\sim}{B}\right) = \frac{4\pi}{c}\,\underset{\sim}{\nabla}\cdot\left(j_\parallel \underset{\sim}{b}\right) = \frac{4\pi}{c}\,\underset{\sim}{B}\cdot\underset{\sim}{\nabla}\left(\frac{1}{B}\sum_j e\int d^3v v_\parallel \sum_\ell q_\ell g_\ell J_\ell\right). \tag{11}$$

Instead of evaluating the right-hand side of Eq. (11) directly, it is convenient to construct the parallel current by taking the $\exp(-iL_\ell)$ moment of Eq. (7), the equation of motion for g_ℓ . Then, after we use the quasi-neutrality condition, Eq. (9), and the fact that the equilibrium distribution function is even in its $v_\parallel$ dependence we find

$$\frac{B}{4\pi}\frac{d}{ds}\frac{k_\perp^2 c^2}{B\omega^2}\frac{d}{ds}\chi = \sum_j \int d^3v\left[-\omega_*^T\frac{e^2F\phi}{T} + \sum_\ell e q_\ell\left(\ell\omega_c + \omega_d\right)g_\ell J_\ell\right.$$
$$\left. - \frac{e^2}{m}\sum_\ell\left(q_\ell - 1\right)\frac{\ell\omega_c}{B}\frac{\partial F}{\partial\mu}\left(\phi J_\ell^2 - \frac{v_\perp}{k_\perp c}B_1 J_\ell J_\ell'\right)\right] \tag{12}$$

where $d\chi/ds = i\omega A_\parallel/c$ and $k_\perp^2\mathcal{A} = B_1$ is the perturbed magnetic field parallel to the equilibrium field.

On a given field line, the equilibrium is periodic within each mirror cell. Hence, it follows from Floquet's theorem that all perturbed quantities will vary as $\xi(s) = \hat{\xi}(s)\exp\left(ik_\parallel s\right)$, where $\hat{\xi}(s)$ is periodic in each cell (the superscript carot will refer to functions that are periodic over a single cell), and $k_\parallel = 2\pi n/ML$, where L is length of the field line in one mirror cell, M is the number of cells, and n is an integer such that $-M/2 \leqq n < M/2$. The quantization condition on $k_\parallel$ insures single-valuedness of the solution.

We now proceed to solve the kinetic equation for g_ℓ, with the solution being valid in either of the following two limits:

$$\mathrm{Max}(\omega, \omega_d, \omega_c) \gg \omega_\parallel \sim v_\parallel \frac{\partial}{\partial s},$$

$$\mathrm{Max}(\omega, \omega_d) \ll \omega_\parallel \sim v_\parallel \frac{\partial}{\partial s},$$

where $\omega_\parallel$ is the transit (or bounce) frequency of a particle in a given cell.

For $\ell \neq 0$ we readily obtain

$$\hat{g}_\ell = \left(1 + \frac{k_\parallel v_\parallel - i v_\parallel \partial/\partial s}{\omega - \omega_d - \ell\omega_c}\right) \frac{eF}{T} \left(\frac{\tilde{\omega}_\ell - \omega_*^T}{\omega - \omega_d - \ell\omega_c}\right)$$

$$\times \left\{ \left[\hat{\phi} - \left(\frac{k_\parallel v_\parallel}{\omega} - \frac{i v_\parallel}{\omega} \frac{\partial}{\partial s}\right)\hat{\chi}\right] J_\ell - \frac{v_\perp \hat{B}_1}{k_\perp c} J_\ell' \right\} + \mathcal{O}\left(\frac{\omega_b}{\omega - \omega_d - \ell\omega_c}\right)^2 . \qquad (13)$$

For g_0 we find two possible results:

(1) High frequency: $\mathrm{Max}(\omega, \omega_d) \gg \omega_\parallel$

$$\hat{g}_0 \equiv \hat{g}_0^H \approx \frac{eF_0}{T} \left(\frac{\tilde{\omega}_0 - \omega_*^T}{\omega - \omega_d - k_\parallel v_\parallel}\right) \left[J_0\left(\hat{\phi} - \frac{k_\parallel v_\parallel}{\omega} \hat{\chi}\right) + J_1 \frac{v_\perp \hat{B}_1}{k_\perp c}\right] \qquad (14)$$

(2) Low frequency: $\omega_\parallel \gg \mathrm{Max}(\omega, \omega_d, k_\parallel v_\parallel)$

$$\hat{g}_0 \equiv \hat{g}_0^L \approx \frac{eF_0}{T} \frac{(\tilde{\omega}_0 - \omega_*^T)}{\omega} J_0 \hat{\chi}$$

$$+ \frac{eF_0}{T}(\tilde{\omega}_0 - \omega_*^T) \frac{\left\langle J_0\left[(\hat{\phi} - \hat{\chi}) + (\omega_d/\omega)\hat{\chi}\right] + J_1 v_\perp \hat{B}_1/k_\perp c\right\rangle}{\omega - \langle\omega_d\rangle - k_\parallel \langle v_\parallel\rangle} \quad . \tag{15}$$

Here we have assumed that the argument of J_ℓ does not vary with position, an assumption strictly valid only in the long-thin approximation where $k_\perp^2/B$ depends only on χ. The condition $\omega_\parallel \gg k_\parallel v_\parallel$ is satisfied if $n \ll M$. Furthermore, we have defined

$$\langle\alpha\rangle = \frac{\oint ds\alpha/v_\parallel}{\oint ds/v_\parallel} \quad , \tag{16}$$

where for a passing particle the orbit average integral is over a single transit through the cell, and for a trapped particle the orbit integral is over a full bounce period. Note that $\langle v_\parallel\rangle = 0$ for trapped particles. We can use the $\langle\ \rangle$ symbol in both the high and low frequency regimes if we define $\langle\alpha\rangle = \alpha$ when $\mathrm{Max}(\omega, \omega_d) \gg \omega_\parallel$, but continue to use $\langle\alpha\rangle$ to denote the definition in Eq. (16) when $\mathrm{Max}(\omega, \omega_d, k_\parallel v_\parallel) \ll \omega_\parallel$. With this understanding, Eq. (15) reduces to Eq. (14) in the high-frequency limit.

If we now substitute Eqs. (13) and (15) into Eqs. (9), (10), and (12), and use the Bessel function identities stated previously, we obtain

(a) Quasi-neutrality Condition:

$$\sum_j e^2 \int d^3v \left(\frac{F}{T} \left\{ -\frac{\tilde{\omega}_0}{\omega}\phi + (\tilde{\omega}_0 - \omega_*^T) J_0^2 \frac{\chi}{\omega} \right.\right.$$

$$\left. + \frac{(\tilde{\omega}_0 - \omega_*^T)\left\langle \phi J_0^2 - \chi[1 - (\omega_d/\omega)] J_0^2 + J_1 J_0 v_\perp B_1/k_\perp c \right\rangle}{\omega - \langle\omega_d\rangle - k_\parallel \langle v_\parallel \rangle} \right\} .$$

$$+ \frac{\partial F}{B\partial\mu}\left[(1 - J_0^2)\phi - J_0 J_1 \frac{v_\perp B_1}{k_\perp c}\right] + \sum_{\ell\neq 0} \frac{q_\ell(\omega - \omega_d)}{(\omega - \omega_d - \ell\omega_c)}\left[\frac{F}{T}\frac{(\tilde{\omega}_0 - \omega_*^T)}{(\omega - \omega_d)} - \frac{\partial F}{mB\partial\mu}\right]$$

$$\left. \times \left(J_\ell^2\phi - J_\ell J_\ell' \frac{v_\perp B_1}{k_\perp c}\right)\right) = 0 \tag{17}$$

(b) Perpendicular Current Equation:

$$B_1 = -\sum_j 4\pi e^2 \int d^3v \frac{v_\perp}{k_\perp c}$$

$$\left(\frac{F}{T}(\tilde{\omega}_0 - \omega_T^*) \left\{ \frac{\left\langle J_1^2(v_\perp B_1/k_\perp c) + [\phi - \chi + (\omega_d/\omega)\chi] J_0 J_1 \right\rangle}{\omega - \langle\omega_d\rangle - k_\parallel\langle v_\parallel\rangle} \right.\right.$$

$$\left. + \chi \frac{J_0 J_1}{\omega} \right\} - \frac{1}{m}\frac{\partial F}{B\partial\mu}\left(J_1^2 \frac{v_\perp B_1}{k_\perp c} + J_0 J_1 \phi\right)$$

$$\left. - \sum_{\ell\neq 0} \frac{q_\ell(\omega - \omega_d)}{(\omega - \omega_d - \ell\omega_c)}\left[\frac{F}{T}\frac{(\tilde{\omega}_0 - \omega_*^T)}{(\omega - \omega_d)} - \frac{1}{mB}\frac{\partial F}{\partial\mu}\right]\left(J_\ell J_\ell'\phi - J_\ell'^2 \frac{v_\perp B_1}{k_\perp c}\right)\right) \tag{18}$$

(c) Parallel Current Equation:

$$B\left(\frac{\partial}{\partial s} + ik_\parallel\right)\left[\frac{k_\perp^2 c^2 \sigma}{\omega^2 B}\left(\frac{\partial}{\partial s} + ik_\parallel\right)\chi\right]$$

$$= 4\pi \sum_j e^2 \int d^3v \left\{ -\frac{F}{T}\left[\phi\,\frac{\tilde{\omega}_0}{\omega} - \frac{(\tilde{\omega}_0 - \omega_*^T)}{\omega}\left(\phi J_0^2 + J_0 J_1 \frac{v_\perp}{k_\perp c} B_1 + \frac{\omega_d}{\omega}\chi J_0^2\right)\right]\right.$$

$$- (\tilde{\omega}_0 - \omega_*^T)\,\frac{\omega_d}{\omega}\,\frac{\langle \phi J_0^2 - \chi[1 - (\omega_d/\omega)]J_0^2 + J_0 J_1 B_1 v_\perp / k_\perp c\rangle}{\omega - \langle\omega_d\rangle - k_\parallel \langle v_\parallel\rangle} \Bigg]$$

$$+ \frac{\partial F}{mB\partial\mu}\left[(1 - J_0^2)\phi - J_0 J_1 \frac{v_\perp B_1}{k_\perp c}\right]$$

$$\left. + \sum_{\ell\neq 0} \frac{q_\ell(\omega - \omega_d)}{\omega - \omega_d - \ell\omega_c}\left[\frac{F}{T}\frac{(\tilde{\omega}_0 - \omega_*^T)}{(\omega - \omega_d)} - \frac{\partial F}{mB\partial\mu}\right]\left(J_\ell^2 \phi - J_\ell J_\ell' \frac{v_\perp}{k_\perp c} B_1\right)\right\} \qquad (19)$$

where the superscript carets have been suppressed on all perturbed quantities, since all perturbed quantities are now to be taken as periodic functions in a single mirror cell. We have defined

$$\sigma = 1 + \frac{4\pi}{k_\perp^2 c^2} \sum_j e^2 \int d^3v\, v_\parallel^2 \left\{ \frac{1}{B}\frac{\partial F}{\partial \mu}(1 - J_0^2)\right.$$

$$\left. + \sum_{\ell\neq 0} q_\ell \left[\frac{(F/T)(\tilde{\omega}_0 - \omega_*^T) - (\omega - \omega_d)(\partial F/\partial\mu)/m}{\omega - \omega_d - \ell\omega_c}\right] J_\ell^2 \right\} . \qquad (20)$$

To lowest-order in Larmor radius, $\sigma \to 1 + 4\pi(P_\perp - P_\parallel)/B^2$ when $\omega_{ci} > \omega, \omega_d$.

These are the eigenmode equations for our system which apply when, for each species, the following frequency orderings hold:

(a) $\mathrm{Max}(\omega, \omega_d, \omega_c) > \omega_{\parallel}$

and

(b) either $\mathrm{Max}(\omega, \omega_d) \gg \omega_{\parallel}$ or $\mathrm{Max}(\omega, \omega_d, k_{\parallel} v_{\parallel}) \ll \omega_{\parallel}$.

(Recall that the meaning of the symbol, $\langle\ \rangle$, depends on which of the approximations (b) applies to a given species). These equations are integro-differential equations that include the kinetic effects due to finite Larmor radius, magnetic drift resonance, trapped particles, and arbitrary frequency. The general solution of these equations requires extensive numerical work. Hence, for the detailed analysis in the remainder of this paper, we will invoke the approximation $E_{\parallel} = 0$ (i.e. $\phi = \chi$) and use Eqs. (18) and (19) as the governing equations. Furthermore, we will neglect the equilibrium electric field in the subsequent analysis and also assume $k_{\perp}^2 v_{\perp}^2/\omega_c^2 \ll 1$, although relevant small gyroradius correction terms will be kept.

V. LOCAL STABILITY ANALYSIS

A common procedure in past work has been to treat ϕ and B_1 as constant and then formally solve Eqs. (18) and (19) in the limit $\omega \ll \omega_{\parallel}, \omega_d$ for all species. Although quite crude, this procedure has the benefit of isolating the types of flute modes that can arise in an EBT plasma. For the hot electrons we will use the symbol "h" , and for the background electrons and ions we will use the symbols "e" and "i" .

A. Low Frequency Analysis $(\omega \ll \omega_{ci};\ \omega_{de,i} \ll \omega < \omega_{dh};\ \kappa \ll r\partial B/\partial\psi)$

We will first assume $\omega \ll \omega_{ci}$ so that we can neglect the higher harmonic terms in Eqs. (18) and (19) except for $n = \pm 1$ contributions from the ion polarization term in Eq. (18). For electrons and ions, we use the approximation

$$\frac{1}{\omega - \omega_d} \cong \frac{1}{\omega} + \frac{\omega_d}{\omega^2}$$

whereas for the hot electrons we use

$$\frac{1}{\omega - \omega_d} \cong -\frac{1}{\omega_{db}} - \frac{(\omega - \omega_\kappa)}{\omega_{db}^2} \quad ,$$

where ω_{db} is the magnetic drift frequency. The Bessel functions are approximated as $J_0(z) = 1 - z^2/4$ and $J_1 = (z/2)(1 - z^2/8)$. Then the velocity integrals of each species can be expressed as simple moments of the equilibrium distribution. The local dispersion relation (with $P_c = P_e + P_i$ taken to be isotropic) is then found to be

$$D_{es}D_{em} + D_{ct}^2 = 0 \quad , \tag{21}$$

where

$$D_{es} = \frac{k_\perp^2 c^2}{v_A^2}\left(1 - \frac{\omega_i^*}{\omega}\right) + 4\pi\,\frac{k_\theta^2}{\omega^2}\left(B\,\frac{dB}{d\psi} + \frac{\kappa B}{r}\right)\frac{d}{d\psi}\left(\frac{P_c}{B^2}\right) - \frac{4\pi e_h}{c\omega}\,k_\theta B\,\frac{\partial}{\partial\psi}\left(\frac{N_h}{B}\right) ,$$

$$D_{em} = \frac{\kappa\left(1 + P'_{\parallel h}/P'_\perp\right)}{rB'} + \frac{4\pi\omega e_h B\left(N_h/B\right)'}{ck_\theta B'^{\,2}} - \frac{4\pi B\left(P_c/B^2\right)'}{B'}$$

$$- \frac{4\pi\left(B^2 P_\perp k_\perp^2\rho_h^2\right)'}{B^3 B'} - \frac{\omega^2}{k_\perp^2 v_A^2}$$

$$D_{ct} = 4\pi\left\{\frac{k_\theta}{\omega}\left[B\frac{d}{d\psi}\left(\frac{P_c}{B^2}\right) - \frac{\kappa}{r}\,\frac{P_c}{B^2}\right] - \frac{e_h B}{c\left(dB/d\psi\right)}\,\frac{d}{d\psi}\left(\frac{N_h}{B}\right)\right\} \qquad (22)$$

where we have defined

$$\omega_{*i} = \frac{ck_\theta B^2}{N_i e_i}\,\frac{d\left(P_{\perp i}/B^2\right)}{d\psi} ,$$

$$k_\perp^2\rho_h^2 = \frac{k_\perp^2 m_h \int d^3v\ \left(v_\perp^4/8\right) F_h}{\omega_{ch}^2 P_{\perp h}}$$

$$\frac{d}{d\psi} = \left.\frac{\partial}{\partial\psi}\right|_B + \frac{\partial B}{\partial\psi}\,\frac{\partial}{\partial B}$$

$$P' = \frac{dP}{d\psi}$$

$$\underset{\sim}{k}_\perp = k_\theta \underset{\sim}{\nabla}\theta + k_\psi \underset{\sim}{\nabla}\psi$$

$$P_c = P_e + P_i$$

$$P_\perp = P_{\perp h} + P_c$$

$$v_A^2 = \frac{B^2}{4\pi N_i M_i} \quad .$$

If we seek a high-frequency solution of Eq. (21), viz., the magnetic compressional mode, the root is approximately given by setting $D_{em} = 0$. This yields the dispersion relation

$$\frac{\omega^2}{k_\perp^2 v_A^2} + \frac{\omega}{\bar{\omega}_{db}} + D_1 = 0 \tag{23}$$

where

$$D_1 = -\frac{B}{dB/d\psi}\left[\frac{\kappa}{rB}\left(1 + \frac{P_{\parallel h}'}{P_\perp'}\right) - 4\pi\frac{d}{d\psi}\left(\frac{P_c}{B^2}\right) - \frac{4\pi}{B^4}\frac{d}{d\psi}\left(B^2 P_{\perp h} k_\perp^2 \rho_h^2\right)\right]$$

$$\frac{1}{\bar{\omega}_{db}} = -\frac{4\pi e_h B}{ck_\theta (dB/d\psi)^2}\frac{d}{d\psi}\left(\frac{N_h}{B}\right) \quad . \tag{24}$$

The stability condition is then

$$\left(\frac{k_\perp v_A}{2\bar{\omega}_{db}}\right)^2 \geqslant D_1 \approx \left[\frac{\Delta_b}{R}\left(1 - \frac{P_{\parallel h}'}{P_{\perp h}'}\right) - \frac{\Delta_b}{2\Delta}\beta_c - k_\perp^2 \tilde{\rho}_h^2\right] \tag{25}$$

where

$$\Delta_b^{-1} = r\frac{dB}{d\psi}$$

$$R = -\frac{1}{\kappa} \equiv \text{radius of curvature}$$

$$\Delta^{-1} = -\frac{rBP'}{P}$$

$$\beta_c = \frac{8\pi P_c}{B^2}$$

$$k_\perp^2 \tilde{\rho}_h^2 = -\frac{\left(P_{\perp h} k_\perp^2 \rho_h^2 B^2\right)'}{B^3 B'} .$$

For purposes of estimation, all species are assumed to have the same pressure scale length on the outer side of the hot electron ring. We note that instability can arise only if D_1 is positive. Hence, the hot electron finite Larmor radius effects will always stabilize the magnetic compressional mode if

$$D_1 < 0 , \tag{26}$$

or roughly if

$$\frac{\beta_c}{\beta_{\perp h}} + k_\perp^2 \tilde{\rho}_h^2 > \frac{2\Delta}{\beta_{\perp h} R} \left(1 + \frac{P_\parallel'}{P_\perp'}\right) . \tag{27}$$

We now investigate the interchange modes and assume $\omega \ll k_\perp v_A, \omega_{ci}$. Then, using Eqs. (21) and (22) leads to the relation

$$\omega\left(\omega - \omega_i^*\right) + \Lambda = 0 , \tag{28}$$

with

$$\Lambda = \frac{4\pi k_\theta^2 v_A^2 \kappa}{rBk_\perp^2} \left(P_c' + \right.$$

$$\left. \frac{\left[(P_c/B^2)' + (B'\omega/B\bar{\omega}_{db})\right]\left[P_\perp' + P_{\parallel h}' + 4\pi rB'(P_{\perp h}k_\perp^2\rho_h^2B^2)'/\kappa B^3\right]}{\left\{(P_c/B^2)' + (\omega B'/\bar{\omega}_{db}B) + \left[(P_{\perp h}k_\perp^2\rho_h^2B^2)'/B^4\right] - \kappa(1 + P_{\parallel h}'/P_\perp')/4\pi\right\}} \right) \qquad (29)$$

where $P_\perp' = P_{\perp h}' + P_c'$.

Equations (28) and (29) contain the various stability limits for the interchange mode. If $4\pi\omega/\bar{\omega}_{db} \gg P_c'/P_h'$, D_1 , the equation reduces to the standard MHD interchange mode (with FLR corrections). The opposite condition, $4\pi\omega/\bar{\omega}_{db} \ll D_1$, is the basic decoupling condition for the hot species, which was previously found to be satisfied at sufficiently low β_c if[11]

$$\left[\left(1 + \frac{P_{\parallel h}'}{P_{\perp h}'}\right)\bar{\omega}_{db}\right]^2 > \frac{4rk_\theta^2 v_A^2 B'^2}{\kappa k_\perp^2 B} \frac{d}{d\psi}\left(P_{\perp h} + P_{\parallel h}\right) \quad . \qquad (30)$$

We shall now assume that Eq. (30) is well satisfied and neglect the $\omega/\bar{\omega}_{db}$ in Eq. (29). We then note that Eq. (28) will always be stable if $\Lambda < 0$. Assuming $(P_c'/B^2) < 0$, we can obtain $\Lambda < 0$ when

$$D_1\left[P_\perp' + P_{\parallel h}' + \frac{r(P_{\perp h}k_\perp^2\rho_h^2B^2)'}{\kappa B^3}\right] < 0 \quad . \qquad (31)$$

Without the electron FLR term, Eq. (31) leads to the Lee-Van Dam[2] and Nelson[3] β_c limit; i.e., when we satisfy Eq. (25), we obtain stability with a diamagnetic well if

$$\left|4\pi\left(\frac{P_c}{B^2}\right)'\right| < \left|\frac{\kappa}{rB}\left(1+\frac{P'_{\parallel h}}{P'_{\perp}}\right)\right| \quad . \tag{32}$$

Taking the hot electron Larmor radius into account can change the sign of the two terms in Eq. (31). As $k_{\perp}^2\rho^2$ increases, D_1 first changes sign, so that moderate electron finite Larmor radius effects actually lower the β_c limit. For somewhat larger $k_{\perp}^2\rho^2$, the bracketed term in Eq. (31) then changes sign as finite Larmor radius stabilization of the interchange mode is achieved. Hence, there is a possible window of instability for the interchange mode that is given by

$$-\frac{\kappa}{rB}\left(1+\frac{P'_{\parallel}}{P'_{\perp}}\right) > -\frac{4\pi}{B^4}\left(B^2P_{\perp}k_{\perp}^2\rho_h^2\right)' > -\frac{\kappa}{rB}\left(1+\frac{P'_{\parallel h}}{P'_{\perp}}\right) + 4\pi\left(\frac{P_c}{B}\right)' \quad . \tag{33}$$

If the right-hand inequality is not satisfied, then the interchange instability is stabilized by the condition

$$-4\pi\left(\frac{P_c}{B^2}\right)' < -\frac{\kappa}{rB}\left(1+\frac{P'_{\parallel h}}{P'_{\perp h}}\right) - \frac{4\pi}{B^4}\left(B^2P_{\perp h}k_{\perp}^2\rho_h^2\right)' \quad , \tag{34}$$

but as this is the condition that $D_1 > 0$, we are susceptible to the magnetic compressional instability unless Eq. (25) is stabilized (a condition that is not too restrictive). When electron FLR effects are sufficiently large so that

$$\frac{4\pi}{B^4}\left(B^2 P_\perp k_\perp^2 \rho_h^2\right)' > -\frac{\kappa}{rB}\left(1 + \frac{P'_{\parallel h}}{P'_{\perp h}}\right) , \tag{35}$$

then the interchange and the magnetic compressional mode are both stabilized and there is no β_c limit (for a given $k_\perp$). For modes with $k_\perp \Delta \gtrsim 1$, we appear to have found a reasonable stabilization condition with no core beta limit. However, the interchange modes can also be long wavelength, $k_\perp \Delta \ll 1$, as in the layer mode of the z-pinch model.[11] Then it is necessary to perform a more exact calculation that does not invoke the eikonal approximation.

We also note that ion FLR effects can produce stabilization if

$$\omega_i^{*2} > 4\Lambda(\omega = 0) = \frac{4k_\theta^2 v_A^2 \kappa}{rBk_\perp^2}\left\{P'_c + \frac{\left[P'_\perp + P'_{\parallel h} + rB'\left(B^2 P_{\perp h} k_\perp^2 \rho_h^2\right)'/\kappa B^2\right]}{1 - \left[\kappa\left(1 + P'_{\parallel h}/P'_\perp\right)/rB - 4\pi\left(B^2 P_{\perp h} k_\perp^2 \rho_h^2\right)'/B^4\right]/4\pi\left(P_c/B^2\right)'}\right\} . \tag{36}$$

This is a moderately strong stabilization effect except for those values of $k_\perp$ that have a β_c threshold where Eq. (34) is just violated; then Eq. (36) cannot be satisfied since its right-hand side is arbitrarily large. Hence, for those $k_\perp$ values that cannot satisfy Eq. (35), the ion FLR term cannot modify the β_c threshold. Furthermore, for layer modes which tend to have constant electric field perturbations, the stabilization formula of Eq. (36) is not applicable, and further investigation is needed.

B. High-Frequency Analysis $(\omega_{ce} \gg \omega \gtrsim \omega_{ci})$

We now investigate the stability of the hot electron ring to high frequency modes, which are of interest when $\omega_{cv}, k_\perp v_A \gtrsim \omega_{ci}$. For simplicity, we will take the pressure of the ions and background electrons to be as much smaller than the hot electron pressure and neglect their contribution in D_{es}. If we now include contributions from the $q_{\pm 1}$ terms in the small Larmor radius limit, we find that the quantities in the dispersion relation of Eq. (21) alter to

$$D_{es} = \frac{k_\perp^2}{v_A^2[1 - (\omega^2/\omega_{ci}^2)]} + Bk_\theta \left[\frac{\omega}{v_A^2(1 - \omega^2/\omega_{ci}^2)\omega_{ci}}\right]' + \frac{k_\theta^2 B'^2}{\bar{\omega}_{db}\omega} \tag{37}$$

$$D_{em} = -\frac{\omega^2}{k_\perp^2 v_A^2}\frac{1}{(1 - \omega^2/\omega_{ci}^2)} - \frac{Bk_\theta}{k_\perp^4}\left[\frac{\omega^3}{v_A^2(1 - \omega^2/\omega_{ci}^2)\omega_{ci}}\right]' - 4\pi B\frac{(P_c/B^2)'}{B'}$$

$$+ \frac{\kappa}{rB'}\left(1 + \frac{P'_{\parallel h}}{P'_\perp}\right) - \frac{\omega}{\bar{\omega}_{db}} - \frac{4\pi}{B^3 B'}\left(B^2 P_{\perp h} k_\perp^2 \rho_h^2\right)' \tag{38}$$

$$D_c = -\frac{\omega^2}{v_A^2\omega_{ci}(1 - \omega^2/\omega_{ci}^2)} - \frac{Bk_\theta}{k_\perp^2}\left[\frac{\omega}{v_A^2(1 - \omega^2/\omega_{ci}^2)}\right]' + \frac{k_\theta B'}{\bar{\omega}_{db}} \ . \tag{39}$$

Combining these terms we find a cubic equation for ω,

$$A\omega^3 + B\omega^2 + C\omega + D = 0 \quad , \tag{40}$$

where we have used the assumption $\beta_h \ll 1$ and

$$A = \bar{\omega}_{db} + \frac{k_\theta v_A^2}{r\omega_{ci}\Delta}$$

$$B = -k_\perp^2 v_A^2 + \frac{k_\theta v_A^2 \bar{\omega}_{db}}{r\omega_{ci}\Delta} DR + \frac{k_\theta^2 v_A^4}{r^2\omega_{ci}^2\Delta_b^2} DR$$

$$C = -k_\perp^2 v_A^2 \bar{\omega}_{db}\, DR$$

$$D = -\frac{k_\theta^2 v_A^4}{r^2\Delta_b^2} DR$$

where $DR = -\kappa\Delta_b\left(1 + P'_{\parallel h}/P'_\perp\right) + r\Delta_b B\beta'_c/2 - k_\perp^2\rho_h^2$.

Equation (40) is nearly identical to the one obtained in Reference (11) when $1 \gg \beta_h \gg \kappa\Delta$ ($\Delta^{-1} = -P'_h rB/P_h$ with all pressure gradients taken to be equal to each other), $\beta_c \ll \beta_h$, and if one replaces the quantity q in Ref. 11 by the new quantity

$$q' = \frac{k^2}{k_\perp^2} q_0\left[1 - \tilde{\beta}_c - \frac{2k_\perp^2\rho_h^2 R\beta_\perp}{\left(1 + P'_{\parallel h}/P'_\perp\right)\Delta}\right] \tag{41}$$

where $k = k_\theta/r$, $q_0 = \bar{\omega}_{db}\kappa\Delta_b/\left(k\omega_{ci}\Delta\right)$, $R^{-1} = -\kappa$, $\tilde{\beta}_c = -\beta'_c rBR\Delta/\left[2\left(1 + P'_{\parallel h}/P'_{\perp h}\right)\right]$.

In Ref. (11), it was established that if $p/q_0\left(k_\perp^2\Delta^2\right) \ll 1$ (where $p = n_h/n_i$) , we can neglect the D term of Eq. (40). Then, in regions where q' is not close to unity, this leads to the two stability criteria

$$p < p_1 \equiv \left(1 - \frac{1}{q'^{1/2}}\right)^2 \tag{42a}$$

$$p > p_2 \equiv \frac{\beta_\perp^2 q'}{(1 - q'^2)} \quad . \tag{42b}$$

The first condition, Eq. (42a), is that there be enough background density to stablize the high frequency hot electron mode. This mode will always be stable if $q' < 1/4$, or

$$FR \equiv \frac{2k_\perp^2 \rho_h^2 R \beta_\perp}{(1 + P_\parallel'/P_\perp')\Delta} > 1 - \frac{1}{4q_0} \frac{k_\perp^2}{k^2(1 - \tilde{\beta}_c)} \quad . \tag{43}$$

The second condition, Eq. (42b), is the criterion that there not be too much background plasma to excite the magnetic compressional mode. It is completely stable when $FR > 1$, as was also found for the low frequency case.

If $q' \approx 1$, then without the finite Larmor radius term, there can occur instability for all p, which exists in a bandwidth determined by

$$\left|1 - \frac{1}{q'}\right| < 2\sqrt{2}\beta_\perp{}^{1/2} \quad . \tag{44}$$

However, with the finite Larmor radius term this instability band can be prevented if

$$FR > 1 - \frac{k_\perp^2}{q_0 k^2(1 - \tilde{\beta}_c)} \quad . \tag{45}$$

V. HOT ELECTRON BOUNCE AVERAGE ANALYSIS

We now solve Eqs. (18) and (19) in the limit of weak curvature and small but finite β, assuming

$$\frac{\beta R}{\Delta} \gg 1\,, \qquad \beta \ll 1\,, \qquad \frac{\beta_h}{\beta_c} \gg 1\,, \quad \text{and} \quad \omega \ll \omega_{ci}$$

where R is the radius of curvature, Δ is the ring annulus thickness, and β_c is the beta of the background plasma. We shall assume that the hot electrons are trapped and restricted relatively close to the mid-plane of each mirror cell. In the region of the hot electrons, ϕ is nearly constant, a consequence that follows from Eq. (19) if one is to balance the first term with the remaining ones. In the hot electron region we have

$$\frac{1}{\omega - \langle\omega_{dh}\rangle} \cong$$

$$\frac{e}{Mck_\theta\mu}\left(\left\langle\frac{dP_{\perp h}}{Bd\psi}\right\rangle\right)^{-1}\left[1 + \frac{\left\langle\kappa\left(\sigma\mu B + v_\parallel^2\right)/rB\right\rangle - \mu\left\langle B^{-1}dP_c/d\psi\right\rangle - e\omega/ck_\theta M}{\mu\left\langle B^{-1}dP_{\perp h}/d\psi\right\rangle}\right] + \ldots.$$

Then neglecting $\mathcal{O}(\omega_{dc}/\omega)$ terms, taking an isotropic background pressure, and setting $B_1 = 4\pi ck_\theta B\left(P_c/B^2\right)'\phi/\omega + \tilde{B}_1$ (note that $\tilde{B}_1$ vanishes wherever $P_h = 0$), we find that Eq. (18) can be written as

$$\tilde{B}_1 - m_h \int_h d^3v\mu \; \frac{\langle \tilde{B}_1 \rangle}{\langle P'_{\perp h}/B \rangle} \frac{\partial F}{\partial \psi} - \frac{dB/d\psi}{dP_h/d\psi} \mu \frac{\partial F}{\partial \mu} \tilde{B}_1$$

$$= \left(\frac{\omega^2}{k_\perp^2 v_A^2} - \frac{8\pi P_c}{B^2} \right) \tilde{B}_1$$

$$+ \int_h d^3v \left(\frac{m_h \langle \tilde{B}_1 \rangle}{\langle P'_{\perp h}/B \rangle} \frac{\partial F}{\partial \psi} \left\{ \left[\frac{\langle \kappa(\sigma\mu + v_\parallel^2/B)/r \rangle - 4\pi\mu \langle P'_c/B \rangle - (e\omega/k_\theta m_h)}{4\pi \langle P'_{\perp h}/B \rangle} \right] \right. \right.$$

$$\left. - \frac{\mu B k_\perp^2}{2\omega_c^2} \right\} + \frac{e\omega\mu(\partial F/\partial E)\langle \tilde{B}_1 \rangle}{c k_\theta \langle P'_{\perp h}/B \rangle} + \frac{m\mu^2}{B} \frac{\partial F}{\partial \mu} \left(- \frac{k_\perp^2 \mu B}{2\omega_c^2} + \frac{-4\pi \, dP_c/d\psi + \kappa\sigma B/r}{dP_{\perp h}/d\psi} \right) \tilde{B}_1$$

$$+ \phi \frac{4\pi k_\theta c\mu}{\omega} \left[\frac{\langle B(P_c/B^2)' \rangle}{\langle P'_{\perp h}/B \rangle} \frac{\partial F}{\partial \psi} + \mu \frac{\partial F}{\partial \mu} \left(\frac{P_c}{B^2} \right)' \right]$$

$$\left. + e_h \phi \frac{1}{\langle P'_{\perp h}/B \rangle} \left(\frac{\partial F}{\partial \psi} - 4\pi\mu \left\langle \frac{dP_{\perp h}}{B d\psi} \right\rangle \frac{\partial F}{\partial E} \right) \right) \tag{46}$$

where the left-hand side of the equation is zeroth order and the right-hand side is first order. In Eq. (46) we have assumed that for the background plasma, $\omega \gg \omega_\parallel$. However, it can be shown that the final results of the following calculation are independent of the bounce frequency ordering of the background plasma.

The solution of the zeroth-order part of Eq. (46) is

$$\tilde{B}_1 = \frac{C(\psi)}{B} \frac{dP_{\perp h}}{d\psi} \;.$$

The coefficient $C(\psi)$ is determined by the solubility condition that is obtained by multiplying Eq. (46) by $P'_{\perp h}/B$ and integrating by ds/B. In this way the left-hand side of Eq. (46) is annihilated and only the right-hand side contributes to the integral. We then find the $C(\psi)$ can be expressed in terms of ϕ as

$$C(\psi)\left(\int\frac{ds}{B}\left[-\left(\frac{P'_{\perp h}}{B}\right)^2\frac{\omega^2}{k_\perp^2 v_A^2}+\frac{dP_{\perp h}}{d\psi}\frac{d}{d\psi}\left(\frac{P_c}{B^2}\right)+\frac{\omega e_h}{4\pi c k_\theta}B\frac{d}{d\psi}\left(\frac{N_h}{B}\right)\right.\right.$$

$$\left.\left.-\frac{\kappa}{4\pi rB}\frac{d}{d\psi}\left(P_{\perp h}+P_{\parallel h}\right)+B^{-4}\frac{dP_{\perp h}}{d\psi}\frac{d}{d\psi}\left(P_{\perp h}k_\perp^2\rho_h^2B^2\right)\right]\right)$$

$$=4\pi\phi\int\frac{ds}{B}\left[\frac{k_\theta}{\omega}\frac{d}{d\psi}\left(\frac{P_c}{B^2}\right)\frac{dP_{\perp h}}{d\psi}+\frac{e_hB}{4\pi c}\frac{d}{d\psi}\left(\frac{N_h}{B}\right)\right] \qquad (47)$$

where $k_\perp^2\rho_h^2$ is defined after Eq. (22).

To proceed further, we substitute $B_1 = 4\pi c k_\theta B\left(P_c/B^2\right)'\phi/\omega + CP'_{\perp h}/B$ into Eq. (19). To the appropriate order of our expansion, we find

$$\mathscr{L}_s\phi \equiv \left(\frac{d}{ds} + ik_\parallel\right) \frac{k_\perp^2\sigma}{B}\left(\frac{d}{ds} + ik_\parallel\right)\phi + \frac{k_\perp^2\omega\left(\omega - \omega_i^*\right)}{Bv_A^2}\phi + \frac{8\pi k_\theta^2 \kappa P_c'}{rB^2}\phi$$

$$= \frac{4\pi}{B}\left[4\pi k_\theta^2 \frac{dP_{\perp h}}{d\psi}\frac{d}{d\psi}\left(\frac{P_c}{B^2}\right) + \omega k_\theta B e_h \frac{d}{d\psi}\left(\frac{N_h}{B}\right)\right]\phi$$

$$- \frac{4\pi\omega}{Bc^2}\left[ck_\theta B \frac{d}{d\psi}\left(\frac{P_c}{B^2}\right)\frac{1}{B}\frac{d}{d\psi}\left(P_{\perp h}\right) + \frac{\omega e_h}{4\pi} B \frac{d}{d\psi}\left(\frac{N_h}{B}\right)\right] C$$

$$+ \frac{4\pi\omega^2}{c}\sum_\sigma \int \frac{dE d\mu}{|v_\parallel|}\left[\frac{k_\theta\mu}{\omega}\left(\tilde{B}_\parallel - \frac{\omega_d}{\langle\omega_d\rangle}\langle\tilde{B}_\parallel\rangle\right)\frac{\partial F_h}{\partial\psi}\right.$$

$$\left. + \frac{e_h}{c} C \frac{\partial F_h}{\partial\psi} \frac{\left(P_{\perp h}'/B - \left\langle P_{\perp h}'/B\right\rangle\right)}{4\pi\left\langle P_{\perp h}'/B\right\rangle}\right] \quad . \tag{48}$$

where we note that the last term vanishes when spatially averaged over the hot electrons.

With the assumption that ϕ is nearly constant in the region containing hot electrons, an approximate solution of Eq. (48) can be obtained. Integrating Eq. (48) along a field line through the hot electron region and using Eq. (47) for C, we can find the change of $k_\perp^2\sigma d\phi/ds$ across a hot electron region. The effective differential equation satisfied by ϕ is then

$$\mathscr{L}_s\phi + G(\omega)\delta\left(s - s_n\right)\phi = 0 \tag{49}$$

where s_n is the center of a hot electron region (i.e., the mid-plane of the n^{th} mirror sector) and $G(\omega)$ is given by

$$G(\omega) = \frac{\langle M(\omega)\rangle\langle N(\omega)\rangle}{\langle D(\omega)\rangle} \tag{50}$$

where now $\langle\alpha\rangle \equiv \int ds\alpha/B$ and

$$M(\omega) = (4\pi k_\theta)^2 \left[\frac{dP_{\perp h}}{d\psi}\frac{d}{d\psi}\left(\frac{P_c}{B^2}\right) + \frac{e_h\omega}{4\pi k_\theta c}\frac{d}{d\psi}\left(\frac{N_h}{B}\right)\right] ,$$

$$N(\omega) = \frac{\kappa}{4\pi rB}\frac{d}{d\psi}(P_{\perp h} + P_{\parallel h}) - \frac{dP_{\perp h}}{B^4 d\psi}\frac{d}{d\psi}(P_{\perp h}k_\perp^2\rho_h^2B^2) - \frac{\omega^2}{k_\perp^2 v_A^2}\left(\frac{dP_{\perp h}}{d\psi}\right)^2$$

$$D(\omega) = \frac{dP_{\perp h}}{d\psi}\frac{d}{d\psi}\left(\frac{P_c}{B^2}\right) + \frac{e_h\omega B}{4\pi k_\theta c}\frac{d}{d\psi}\left(\frac{N_h}{B}\right) - \frac{\kappa}{4\pi rB}\frac{d}{d\psi}(P_{\perp h} + P_{\parallel h})$$
$$+ \frac{dP_{\perp h}}{B^4 d\psi}\frac{d}{d\psi}(P_{\perp h}k_\perp^2\rho_h^2B^2) - \frac{\omega^2}{k_\perp^2 v_A^2}\left(\frac{1}{B}\frac{dP_{\perp h}}{d\psi}\right)^2 .$$

Alternatively, if ϕ does not vary appreciably in the hot electron region, the differential equation satisfied by ϕ can be written as

$$\mathscr{L}_s\phi + Q(\omega)\phi = 0 , \tag{51}$$

where

$$Q(\omega) = \frac{\langle M(\omega)\rangle N(\omega)}{B\langle D(\omega)\rangle} . \tag{52}$$

If the first two terms in the expression for $D(\omega)$ is dominant, then Eq. (51) reduces to the standard MHD ballooning mode equation, with important FLR corrections.

To analyze Eq. (51), we order the equation so that its zeroth, first, and second order forms are:

$$\frac{d}{ds}\frac{k_\perp^2\sigma}{B}\frac{d\phi_0}{ds} = 0$$

$$\frac{d}{ds}\frac{k_\perp^2\sigma}{B}\frac{d\phi_1}{ds} = -ik_\parallel \frac{k_\perp^2\sigma}{B}\frac{d\phi_0}{ds} - ik_\parallel \frac{d}{ds}\left(\frac{k_\perp^2\sigma\phi_0}{B}\right)$$

$$\frac{d}{ds}\frac{k_\perp^2\sigma}{B}\frac{d\phi_2}{ds} = -ik_\parallel \frac{k_\perp^2\sigma}{B}\frac{d\phi_1}{ds} - ik_\parallel \frac{d}{ds}\left(\frac{k_\perp^2\sigma\phi_1}{B}\right) - \frac{\langle M(\omega)\rangle N(\omega)\phi_0}{B\langle D(\omega)\rangle} + \frac{k_\perp^2\sigma k_\parallel^2\phi_0}{B}$$

$$- \frac{k_\perp^2\omega\left(\omega - \omega_i^*\right)}{Bv_A^2}\phi_0 - \frac{8\pi k_\theta^2\kappa P_c'\phi_0}{rB^2} \quad . \qquad (53)$$

With the constraint that ϕ be periodic over each cell, we find the solution

$$\phi_0 = 1, \quad \frac{d\phi_1}{ds} = -ik_\parallel\left[1 - BL\left(k_\perp^2\sigma\int_{-L/2}^{L/2}\frac{dsB}{k_\perp^2\sigma}\right)^{-1}\right] ,$$

where L is the length of a field line in each mirror cell. By demanding that $d\phi_2/ds$ be periodic over a cell, we find the solubility condition, which leads to the dispersion relation

$$k_\parallel^2L^2\left(\int_{-L/2}^{L/2}\frac{dsB}{k_\perp^2\sigma}\right)^{-1} = \int_{-L/2}^{L/2}\frac{ds}{B}\left[\frac{k_\perp^2\omega\left(\omega - \omega_i^*\right)}{v_A^2} + \frac{8\pi k_\theta^2\kappa P_c'}{rB}\right] + G(\omega) \quad .$$

(54)

The case of $k_\parallel = 0$ constitutes the field line averaged generalization of Sec. V. The generalization of Eq. (23) for the magnetic compressional mode with field line averaging is obtained by setting the denominator in Eq. (50) equal to zero. We then obtain

$$\omega^2 (\overline{\overline{k_\perp v_A}})^{-2} + \omega \bar{\bar{\omega}}_{db}^{-1} + \bar{\bar{D}}_1 = 0 \tag{55}$$

with

$$(\overline{\overline{k_\perp v_A}})^{-2} = \frac{\int ds \left[(P'_{\perp h}/B)^2/(B k_\perp^2 v_A^2) \right]}{\int ds (P'_{\perp h}/B)^2/B}$$

$$\bar{\bar{\omega}}_{db}^{-1} = \frac{-e_h}{4\pi k_\theta c} \frac{\int ds (N_h/B)'}{\int ds (P'_{\perp h}/B)^2/B}$$

$$\bar{\bar{D}}_1 = \frac{\int \frac{ds}{B} \left[\frac{\kappa}{4\pi r B} \frac{d}{d\psi} (P_{\perp h} + P_{\parallel h}) - \frac{dP_{\perp h}}{d\psi} \frac{d}{d\psi} \left(\frac{P_c}{B^2} \right) - \frac{dP_{\perp h}}{B^4 d\psi} \frac{d}{d\psi} (P_{\perp h} k_\perp^2 \rho_h^2 B^2) \right]}{\int \frac{ds}{B} \left(\frac{1}{B} \frac{dP_{\perp h}}{d\psi} \right)^2} .$$

The stability condition is

$$(\bar{\bar{\omega}}_{db}^{-1})^2 \geq 4 \bar{\bar{D}}_1 (\overline{\overline{k_\perp v_A}})^{-2} ,$$

which is the line-weighted generalization of Eq. (25). Because the line average heavily weights the parameters where $P_{\perp h}$ is large, reasonably accurate stability criterion are obtained by using Eqs. (23) and (25) with parameters evaluated at the midplane of a mirror.

Now, to examine interchange stability, we assume $\omega \ll k_\perp v_A(\Delta_b/R)^{1/2}$. The line-averaged generalization of Eq. (28) becomes

$$\omega^2 - \omega\omega_i^* + \bar{\bar{\Lambda}}(\omega) = 0 \quad , \tag{56}$$

where we have assumed that ω_i^* is independent of position along a field line and

$$\bar{\bar{\Lambda}}(\omega) = \frac{\langle N(\omega=0)\rangle\langle M(\omega)\rangle}{\langle W\rangle\left\langle k_\perp^2/v_A^2\right\rangle} + \frac{\langle A\rangle}{\left\langle k_\perp^2/v_A^2\right\rangle}$$

where

$$A = 8\pi k_\theta^2 \frac{\kappa}{rB}\frac{dP_c}{d\psi}$$

$$W = \left[\frac{dP_{\perp h}}{d\psi}\frac{d}{d\psi}\left(\frac{P_c}{B^2}\right) + \frac{e_h\omega}{4\pi k_\theta}\frac{d}{d\psi}\left(\frac{N_h}{B}\right) - \frac{\kappa}{4\pi rB}\frac{d}{d\psi}\left(P_{\perp h} - P_{\parallel h}\right)\right.$$
$$\left. + \frac{dP_{\perp h}}{Bd\psi}\frac{d}{d\psi}\left(P_{\perp h}k_\perp^2\rho_h^2B^2\right)\right] \quad .$$

The structure of Eq. (56) is identical to that of Eq. (28), except for the appearance of appropriate line averages. When FLR terms are ignored, the decoupling condition for the interchange mode is

$$\left[\int_{-L/2}^{L/2}\frac{ds}{B}\frac{\kappa}{rB}\frac{d}{d\psi}\left(P_{\perp h} + P_{\parallel c}\right)\right] > \frac{4\left[\int_{-L/2}^{L/2} ds\, e_h\left(N_h/B\right)\right]^2}{4\pi m_i\int ds\, k_\perp^2 N_i/B^3} \quad . \tag{57}$$

A significant feature of this condition is that the decoupling condition is improved over that of Eq. (30) because the core density now includes contributions from the entire field line with a B^{-3} weighting.

If Eq. (57) is well satisfied, we can set $\omega = 0$ in $\bar{\bar{\Lambda}}(\omega)$ of Eq. (56) in the further analysis of Eq. (55). Without FLR effects the Lee-Van Dam condition then becomes

$$\int \frac{ds}{B} \left[\frac{dP_{\perp h}}{d\psi} \frac{d}{d\psi} \left(\frac{P_c}{B^2} \right) \right] > \int \frac{ds}{B} \left[\frac{\kappa}{4\pi r B} \frac{d}{d\psi} \left(P_{\perp h} + P_{\parallel h} \right) \right] \tag{58}$$

which is not significantly different from Eq. (32).

When we consider the FLR terms we find, as in the previous local analysis, that the threshold for instability is reduced. A possible window of instability exists for the interchange mode when

$$\int \frac{ds}{B} \frac{\kappa}{4\pi r B} \frac{d}{d\psi} \left(P_{\perp h} + P_{\parallel h} \right) > \int \frac{ds}{B^5} \left(P_h k_\perp^2 \rho_h^2 B^2 \right)$$
$$> \int \frac{ds}{B} \left[\frac{\kappa}{4\pi r B} \frac{d}{d\psi} \left(P_{\perp h} + P_{\parallel h} \right) - \frac{dP_{\perp h}}{d\psi} \frac{d}{d\psi} \left(\frac{P_c}{B^2} \right) \right] . \tag{59}$$

When the left-hand inequality is reversed, the interchange is stabilized.

Finally, when we consider the line-averaged modification of Eq. (36), i.e., the condition for stabilization with ion and electron FLR effects, we find

$$\omega_i^{*2} > 4\langle \Lambda(\omega=0) \rangle . \tag{60}$$

Likewise, this yields a relatively optimistic stabilization criterion except in the region where the right-hand inequality of Eq. (59) is just satisfied, when $\Lambda(\omega=0)$ becomes very large.

VI. SUMMARY

We have developed an eikonal ballooning mode theory to describe the curvature-driven modes of a hot electron plasma. The unique feature of our method is to include the extra terms that arise for modes where $\omega \gtrsim \omega_{ci}$. These terms are particularly important for high frequency modes that arise in EBT-S, the EBT device that is presently operating. The modes have been analyzed both roughly in a local approximation and in the high hot electron bounce frequency limit of a single mirror cell, where weightings along a field line can be properly accounted for. We find that there is a fundamental finite Larmor radius parameter FR [see Eq. (43) for its definition]. At low frequencies ($\omega \ll \omega_{ci}$), the interchange mode is unstable if $1 < \mathrm{FR} < 1 - \tilde{\beta}_c$ with $\tilde{\beta}_c \approx \beta_c R/[2\Delta(1 + P_\parallel'/P_\perp')]$, whereas complete stability is achieved for the low frequency interchange, as well as all other modes, if $\mathrm{FR} > 1$. For unstable modes, we observe that if $\mathrm{FR} = 0$, instability arises when the Lee-Van Dam[2], Nelson[3] limit of Eq. (32) is exceeded. However, with non-zero FR, this limit is lowered. Hence, moderate FLR effects introduce a band of instability which may cause enhanced diffusion.

Short wavelength instability can be avoided by having $\mathrm{FR} > 1$ for all wavenumbers. Studies in the z-pinch model indicate that for WKB-like modes the radial wavenumber k_r is approximately given by $\mathrm{Min}(k_r\Delta) \sim 2$. Hence, for

$$\frac{\rho_h^2}{\Delta^2} \gtrsim \frac{\Delta}{R}\,\beta_h\left(1 + \frac{P_\parallel'}{P_\perp'}\right) \quad , \tag{61}$$

one may expect to stabilize all WKB-like short wavelength modes. This appears a likely situation with present-day EBT-S experiments. However, in experiments with larger scale lengths, such as in the proposed EBT-P experiments and in conceptual reactor devices, Eq. (61) cannot be satisfied with hot electrons except at extremely high energies. However, one should keep in mind that hot ion rings, whose low frequency stability is substantially the same as that of hot electron rings, can be designed to satisfy Eq. (61).

If Eq. (61) is satisfied, the core beta limit, $\tilde{\beta}_c < 1$, is still likely to be a fundamental core beta limit. This is because there still exist displacement-like modes in the ring region. A zero FLR theory[11] for this mode shows that the core beta is limited by $\tilde{\beta}_c = 1$ and may even be less. FLR effects on such a mode are not expected to alter the core beta limit appreciably, although the proper theory still needs to be developed to confirm this speculation.

Acknowledgment

We are indebted to T. M. Antonsen, Jr., for showing how to derive Eq. (54) from Eq. (51). Work on this paper was supported by United States Department of Energy contract DE-FG05-80ET-53088.

REFERENCES

1. D. B. Nelson and C. L. Hedrick, Nucl. Fusion 19, 238 (1979).

2. J. W. Van Dam and Y. C. Lee in Proc. EBT Ring Workshop (ORNL, Oak Ridge, 1979), Conf. 791288, p. 471.

3. D. B. Nelson, Phys. Fluids 23, 1850 (1980).

4. A. M. El Nadi, to be published in The Physics of Fluids.

5. J. W. Van Dam, H. L. Berk, M. N. Rosenbluth, D. A. Spong, Proc. EBT Stability Workshop (ORNL, Oak Ridge, 1981), Conf. 810512, p. 97.

6. C. Z. Cheng, K. Tsang, op. cit. 4, p. 161.

7. K. T. Tsang and P. Catto, SAI Report PRI-34 (Science Applications, Inc., Boulder, Colorado, 1981).

8. D. M. Kruskal and C. Oberman, Phys. Fluids 1, 275 (1958).

9. T. M. Antonsen, Jr., B. Lane, and J. J. Ramos, Phys. Fluids 24, 1465 (1981).

10. J. W. Van Dam, M. N. Rosenbluth, and Y. C. Lee, Phys. Fluids 25, 1349 (1982).

11. H. L. Berk, J. W. Van Dam, M. N. Rosenbluth, D. A. Spong, to be published in Phys. Fluids.

12. R. R. Dominguez and H. L. Berk, Phys. Fluids 21, 827 (1978).

13. C. Z. Cheng, Princeton Plasma Physics Laboratory PPPL Report 1782 and PPPL Report 1841 (Princeton University, New Jersey, 1981).

14. H. L. Berk, M. N. Rosenbluth, H. V. Wong, T. M. Antonsen Jr., and D. E. Baldwin, Institute for Fusion Studies Report 59 (University of Texas at Austin, 1982), submitted to Soviet Journal of Plasma Physics.