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**Singular Eigenfunctions for Shearing Fluids I**

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## Abstract

We construct singular eigenfunctions corresponding to the continuous spectrum of eigenvalues for shear flow in a channel. These modes are irregular as a result of a singularity in the eigenvalue problem at the *critical layer* of each mode. We consider flows with monotonic shear, so there is only a single critical layer for each mode. We then solve the initial-value problem to establish that these continuum modes, together with any discrete, growing/decaying pairs of modes, comprise a complete basis. We also view the problem within the framework of Hamiltonian theory. In that context, the singular solutions can be viewed as the kernel of an integral, canonical transformation that allows us to write the fluid system, an infinite-dimensional Hamiltonian system, in action-angle form. This yields an expression for the energy in terms of the continuum modes and provides a means for attaching a characteristic signature (sign) to the energy associated with each eigenfunction. We follow on to consider shear-flow stability within the Hamiltonian framework. Next, we show the equivalence of integral superpositions of the singular eigenfunctions with the solution derived with Laplace transform techniques. In the long-time limit, such superpositions have decaying integral averages across the channel, revealing phase mixing or *continuum damping*. Under some conditions, this decay is exponential and is then the fluid analogue of Landau damping. Finally, we discuss the energetics of continuum damping.

## I. Introduction

The evolution of a shearing fluid is a subject to have captured the attention of a wide audience, from engineering to mathematics. Its beginnings emerged from the studies of Kelvin, Helmholtz and Rayleigh who considered the stability of ideal equilibria in linear theory. Since then, the subject has developed into both the nonlinear and viscous regimes, but here we reconsider the classical problem from a different viewpoint. In particular, we address the problem of the linear stability of an inviscid shear flow in a channel, Rayleigh's problem (Rayleigh 1880; Drazin & Howard, 1966).

The goal of linear stability theory is to determine the evolution of an infinitesimal disturbance from an arbitrary initial condition. In some situations, however, it suffices to establish whether the disturbance amplifies exponentially as a result of intrinsic instability, but this is not the complete story. Here we pursue a complete answer by solving the problem via a normal-mode expansion.

Normal-mode theory is a powerful technique for linear problems. It recasts the equations as an eigenvalue problem for the normal-mode frequencies, then constructs a complete basis set from the eigenfunctions that enable the representation of an arbitrary initial state. Moreover, for a large class of problems in which the eigenvalues arise as distinct points on the spectral plane, the normal modes are uncoupled for all time and so oscillate independently. Hence the evolution of the fluid is straightforward to determine.

Despite its advantages, difficulties can arise in the normal-mode approach if the eigenvalue problem is not of a regular form. By "regular eigenvalue problems" we mean problems consisting of regular differential equations solved on finite domains. Their eigenspectra are composed of an infinite number of discrete values with a single point of accumulation. Irregular problems, which in principle include everything else, need not have such a simple

form for the eigenspectrum. Those of relevance here are problems for which the eigenvalues populate a continuous interval on the spectral plane.

A continuous eigenvalue spectrum can arise in a variety of different physical contexts. In many cases, such a spectrum is associated with an infinite domain, for which the requirement of quantization places no constraints on the possible eigenvalues. In other cases, specifically those of immediate interest here, the spectrum is continuous as a result of irregular, eigenvalue-dependent coefficients in the differential equations of the eigenvalue problem. The critical difference between the two cases is that the continuum of the infinite domain can be thought of as the limit of the discrete set of eigenvalues of some finite domain, in which case there is an underlying dispersion relation. In the singular case, there is simply no dispersion relation that relates an eigenvalue of the continuous spectrum (frequency) to the mode number (wavenumber).

In the shear-flow problem, the differential equations for the normal modes contain a singular point. In itself, this feature is by no means unusual but what is important is that the location of the singular or *critical* point is determined by the eigenvalue. Moreover, although there is a regular solution in the vicinity of the singular point, that solution by itself cannot satisfy the boundary conditions. Normal modes of a regular form cannot therefore exist, but intrinsically they must contain the singular solution at the critical point. For this reason we call them *singular eigensolutions*. Intrinsically they are not continuously differentiable. In fact we can only impose continuity on as many derivatives of the eigenfunction as there can be defined on the singular solution at the critical point. In the shear-flow equations, we can enforce continuity on the streamfunction, but not on any of its derivatives. As a result, the eigenvalue problem is underconstrained and any eigenvalue that gives a critical point within the channel leads to a possible solution. Hence, following Rayleigh (1945), we may anticipate a continuous spectrum.

The problems associated with constructing singular eigensolutions to the normal-mode

equations have previously been effective deterrents to using such an approach to shear-flow stability, although Eliassen *et al.* (1953) considered the special case of a linear shear. An alternative approach was taken by Case (1960), Dikii (1960), and Engevik (1966) who addressed the problem using Laplace transforms, which avoids the need to find the singular solutions. They obtained the solution as an inverse Laplace transform involving the initial condition. However, it is only at asymptotically large times that characteristic features of the solution can be extracted from the inverse transform. The normal-mode expansion, on the other hand, is a general decomposition over all time. Consequently, we view this approach as providing more insight into the problem, even though the modes are singular in nature.

The singularity in the shear-flow equations arises at the point where the speed of the mode balances its rate of advection by the ambient flow. In other words, it is a kind of wave-mean flow resonance, a phenomenon which occurs in a variety of other fluid contexts. It is also closely related to wave-particle resonance in plasma physics (Van Kampen & Felderhof, 1967) and the Alfvén resonances of MHD (Ye *et al.* 1993); as a mathematical problem it shares many common features with scattering and transport theory (Kowalski and Feldman, 1966a and 1966b; Case and Zweifel, 1967). We use these analogies to employ methods developed in plasma physics and elsewhere to formulate the singular stability analysis for Rayleigh's problem (sections 2 and 3). Kamp (1991) has previously given a closely related formulation for rotating shear flow. We then present some numerically determined eigensolutions for a variety of shear flows (section 4; some analytical solutions for profiles with piece-wise constant vorticity are given in the Appendix). One important feature of these profiles is that they are all monotonic (the mean velocity downstream is a single-valued function of the coordinate across the channel). This leads to a significant simplification of the problem; the generalization to more complicated shears will be undertaken in a companion paper (paper II).

The success of the normal-mode expansion depends on whether the singular eigensolu-

tions are a complete set of basis functions. This fundamental question is addressed in section 5, where we establish completeness by solving the initial-value problem. Other properties of the singular eigensolutions are found in section 6. We then investigate the Hamiltonian structure of the shear-flow problem (section 7), which brings out the meaning of the singular eigensolutions as the fundamental degrees of freedom of the system, and allows us to coherently consider shear flow energetics. As a final piece of this work we expand arbitrary initial conditions in terms of the singular eigensolutions and study the temporal evolution. This allows us to make contact with the Laplace transform approach (section 8). There we also encounter the tendency of an ideal fluid to obliterate an integral average by phase mixing, a phenomenon we term *continuum damping*.

## II. Formulation of the problem

The two-dimensional Euler equations for the incompressible fluid take the form,

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p, \quad (2.1)$$

$$\nabla \cdot \mathbf{v} = 0 \quad (2.2)$$

(with density scaled to unity). Using the vorticity,  $\tilde{\omega}$ , and streamfunction,  $\tilde{\psi}$ , as dependent variables, we rewrite these equations as

$$\frac{\partial \tilde{\omega}}{\partial t} + [\tilde{\omega}, \tilde{\psi}] = 0 \quad (2.3)$$

and

$$\nabla^2 \tilde{\psi} = -\tilde{\omega}, \quad (2.4)$$

where

$$[\tilde{\omega}, \tilde{\psi}] = \frac{\partial \tilde{\omega}}{\partial x} \frac{\partial \tilde{\psi}}{\partial y} - \frac{\partial \tilde{\psi}}{\partial x} \frac{\partial \tilde{\omega}}{\partial y}. \quad (2.5)$$

Written in this way, we draw parallels with theory of the Vlasov-Poisson equation, and, inspired by Van Kampen's treatment of plasma oscillations (Van Kampen, 1955; Case, 1959), we construct the singular eigensolutions of the continuous spectrum.

The Poisson equation can be rewritten using Green's function,

$$\tilde{\psi}(\mathbf{x}) = - \int_D G(\mathbf{x}, \mathbf{x}') \tilde{\omega}(\mathbf{x}') d\mathbf{x}', \quad (2.6)$$

where  $D$  is the domain of the shear, namely an infinitely long channel of fixed width. Equations (2.3) and (2.6) are the central equations of our study. They must be solved subject to the condition that there be no flow normal to the boundaries, which we locate at  $y = \pm 1$ .

Configurations characterized by  $\psi_0$  and  $\omega_0$  exist as equilibrium solutions to (2.3) and (2.4), provided

$$[\omega_0, \psi_0] = 0, \quad \nabla^2 \psi_0 = -\omega_0. \quad (2.7)$$

The subset of solutions of particular interest are determined by

$$\psi_0 = \int U(y) dy, \quad \omega_0 = -U'(y), \quad (2.8)$$

where  $U(y)$ , the "flow profile," is an arbitrary function.

In what follows we consider normal modes that are characterized by exponential time dependence, or a downstream pattern or wave speed,  $u$ . This speed is an eigenvalue labeling each normal mode. In addition, when the equilibrium is independent of the coordinate  $x$ , the perturbations can be Fourier transformed in that coordinate. We can then consider each Fourier mode independently and write  $\tilde{\psi} - \psi_0 = \psi_k(y) \exp[ik(x - ut)]$  and  $\tilde{\omega} - \omega_0 = \omega_k(y) \exp[ik(x - ut)]$ , for the perturbations about equilibrium, where  $k$  is the streamwise wavenumber. The quantities  $\psi_k$  and  $\omega_k$  then satisfy the relations,

$$(U - u)\omega_k + U''\psi_k = 0 \quad (2.9)$$

and

$$\psi_k = - \int_{-1}^1 \mathcal{K}_k(y, y') \omega_k(y') dy', \quad (2.10)$$

where we have reduced the Green function to

$$\mathcal{K}_k(y, y') = \begin{cases} -\sinh k(1-y) \sinh k(1+y')/k \sinh 2k & \text{for } y > y', \\ -\sinh k(1-y') \sinh k(1+y)/k \sinh 2k & \text{for } y \leq y'. \end{cases} \quad (2.11)$$

The boundary conditions are that each  $\psi_k$  vanish at  $y = \pm 1$ .

The channel is infinite in  $x$ , so  $k$  is really a continuous variable, even though it appears above as a subscript (it corresponds to the eigenvalue of a continuous spectrum associated with a regular problem in an infinite domain). In sections to follow, we drop the subscript for convenience.

### III. Regularizing the problem

#### A. Complex modes, critical layers and singular solutions

The basic equation of our study, then, is Rayleigh's equation,

$$(U - u)\omega = -U''\psi, \quad (3.1)$$

or,

$$(U - u)(\psi'' - k^2\psi) = U''\psi. \quad (3.2)$$

This second-order differential equation is manifestly singular at the point for which  $u = U(y)$ . The crucial issue surrounding the singular modes is the presence of this so-called critical layer. This is the locus of points for which the mode travels locally with the speed of the mean flow. For general flow profiles, there may be more than one such layer for a given mode, and this introduces the complications that we address in paper II. But for now, our assumption of a monotonic velocity profile means that  $u = U(y)$  has a single solution,  $y = y_*$ .

Rayleigh's equation is singular provided the mode is both neutral and possesses a critical layer. We will show next that all neutral modes have critical layers and are generally singular there, but Rayleigh's equation also admits complex solutions. These are the growing and decaying mode pairs with complex eigenvalue,  $u$ , and so they are always regular and therefore



discrete. According to Rayleigh's (1880) theorem, complex modes may only appear when the velocity profile contains an inflection point,  $y_I$ :  $U''(y_I) = 0$ . Moreover, for any given  $k$ , there are only ever as many complex pairs as there are inflection points (Howard, 1964).

At the critical layer, we can analyze the solution to Rayleigh's equation by Frobenius expansion. This reveals there to be two nonsingular solutions for  $\psi$  at  $y = y_*$ . One is purely regular and vanishes at the critical point. The other is in leading order constant but has a discontinuous derivative at the critical layer unless  $U''$  vanishes there. We can now ask the question as to whether the boundary conditions imposed on  $\psi$  permit the neutral modes to consist of only the purely regular solution at the critical layer. The following argument (due in spirit to Barston, 1964; see also Rosencrans & Sattinger, 1966) indicates that this is not the case.

First we introduce the displacement,  $\xi$ , defined such that  $(U - u)\xi = \psi$ . Then  $\xi$  satisfies the differential equation,

$$\frac{d}{dy} \left[ (U - u)^2 \frac{d\xi}{dy} \right] - k^2 (U - u)^2 \xi = 0, \quad (3.3)$$

which has both a regular and a singular solution at  $y = y_*$ . If we multiply this equation by  $\xi^*$  and integrate over the interval  $[y_1, y_2]$ , then we obtain,

$$\left[ (U - u)^2 \xi^* \frac{d\xi}{dy} \right]_{y_1}^{y_2} = \int_{y_1}^{y_2} [U(y) - u]^2 \left[ \left| \frac{d\xi}{dy} \right|^2 + k^2 |\xi|^2 \right] dy. \quad (3.4)$$

If we select  $y_1$  and  $y_2$  to be the boundaries of the domain, then it is straightforward to see that only trivial solutions exist if the critical layer lies outside the channel. This establishes that only neutral modes with critical layers exist. If we now select  $y_1 = -1$  and  $y_2 = y_*$  or  $y_1 = y_*$  and  $y_2 = 1$ , then we observe that, unless  $\xi$  is singular at the critical point, the solution is again trivial. This indicates that  $\xi$  must be intrinsically singular, or equivalently, that the streamfunction has a discontinuous derivative (unless, perhaps,  $U''(y_*) = 0$ ).

Consequently, the neutral modes generally have irregular shapes. In order to bring this

out, we follow van Kampen (1955), and rewrite (3.1) in the *distributional* form,

$$\omega = -\mathcal{P} \left( \frac{U''\psi}{U-u} \right) - \mathcal{C}\delta(y-y_*), \quad (3.5)$$

where  $\mathcal{P}$  signifies the Cauchy principal value,  $\mathcal{C}$  is (as yet) arbitrary, and  $\delta$  is Dirac's delta function. (Van Kampen considered a solution in the form of (3.5) in the context of plasma oscillations; the solution was previously used in quantum scattering theory and perhaps appeared first in a paper by Rice (1929), as interpreted by Van Kampen (1951).)

By writing the vorticity of the eigenfunction in this way, we explicitly reveal the non-analytic, singular nature of the mode. Its vorticity is comprised of two pieces. There is a divergent, global component that corresponds to advection of the underlying flow by the perturbation, and relies on the presence of a mean vorticity gradient. For a linear velocity profile only the second term is present, and this takes the form of a line vortex positioned at the critical layer.

## B. Reduction to a regular Fredholm problem

We now eliminate  $\omega$  between equations (2.10) and (3.1), yielding the inhomogeneous, integral equation,

$$\psi(y) = \mathcal{P} \int_{-1}^1 \mathcal{K}(y, y') \frac{U''(y')\psi(y')}{U(y')-u} dy' + \mathcal{C}\mathcal{K}(y, y_*), \quad (3.6)$$

with  $\mathcal{P}$  now to be interpreted as meaning the principal value of the integral. The kernel of this integral equation is singular at the critical layer, but the inhomogeneous term contains an arbitrary constant,  $\mathcal{C}$ . We do, however, have the freedom of a normalization condition in order to fix  $\mathcal{C}$ ; we select one that simplifies the problem. Specifically, we choose a normalization that allows us to deal with the singularity. This amounts to a selection for  $\mathcal{C}$  that regularizes the divergent kernel (Kamp, 1991; Kowalski and Feldman, 1966a and 1966b; Sattinger, 1966a and 1966b; Sedláček, 1971b and 1972). It is equivalent to the condition,

$$1 = - \int_{-1}^1 \omega(y) dy$$

$$= \mathcal{C} + \mathcal{P} \int_{-1}^1 \frac{U''(y')\psi(y')}{U(y') - U(y_*)} dy'. \quad (3.7)$$

With such a selection for  $\mathcal{C}$ , we rewrite the integral problem in the form,

$$\psi(y) = f(y; y_*) + \lambda \int_{-1}^1 \mathcal{F}(y, y'; y_*)\psi(y') dy', \quad (3.8)$$

where the inhomogeneous term is

$$f(y; y_*) = \mathcal{K}(y, y_*), \quad (3.9)$$

formally we have  $\lambda = 1$ , and the kernel is

$$\mathcal{F}(y, y'; y_*) = \left[ \frac{\mathcal{K}(y, y') - \mathcal{K}(y, y_*)}{U(y) - U(y_*)} \right] U''(y'). \quad (3.10)$$

This kernel is regular at the critical layer (accordingly we have omitted the principal-value symbol), and depends continuously on a “parameter,”  $y_*$ . For convenience we have suppressed the dependence of  $\psi$  on  $y_*$ , though we will explicitly reinstate it later.

## IV. The regular Fredholm problem

### A. Fredholm theory

The integral equation (3.8) is an inhomogeneous Fredholm equation of the second kind (*e.g.* Tricomi 1985). Guided by Fredholm theory, we anticipate two kinds of solutions. One is a particular solution,  $\tilde{\psi}(y)$ ; the others are homogeneous solutions,  $\chi(y)$ , that satisfy

$$\chi(y) = \lambda \int_{-1}^1 \mathcal{F}(y, y'; y_*)\chi(y') dy'. \quad (4.1)$$

In general, the homogeneous solutions exist only for certain values of the Fredholm eigenvalue,  $\lambda$ . These particular values,  $\lambda_j$ ,  $j = 1, 2, \dots, J$  (with  $J$  being the total number, which may be infinite) smoothly depend upon the parameter  $y_*$ . Hence, should there be an eigenvalue,  $\lambda_j(y_*)$ , that equals unity for a particular critical layer,  $y_h$  (so  $\lambda_j(y_h) = 1$ ), then we can say that there is a homogeneous solution to our original problem.

If there are no homogeneous solutions, Fredholm theory establishes that there exists a unique, particular solution,  $\psi = \tilde{\psi}$ . Furthermore, it can be written in the form,

$$\tilde{\psi}(y) = f(y) + \int_{-1}^1 \mathcal{R}(y, y'; y_*) f(y') dy', \quad (4.2)$$

where  $\mathcal{R}$  is the *resolvent kernel* of the integral equation. It is a meromorphic function of the Fredholm eigenvalue, and is usually written as the ratio of two infinite determinants whose entries are convolutions of the kernel  $\mathcal{F}$ :

$$\mathcal{R}(y, y'; y_*) = \frac{\tilde{\mathcal{D}}_\lambda(y, y')}{\tilde{\mathcal{D}}(\lambda)} \quad (4.3)$$

(an explicit statement of the form of the two determinants is provided by Tricomi (1985)). Since the determinants depend smoothly on critical layer position,  $y_*$ , we can set  $\lambda = 1$  for our shear-flow problem and rewrite this as

$$\mathcal{R}(y, y'; y_*) = \frac{\mathcal{D}_{y_*}(y, y')}{\mathcal{D}(y_*)}, \quad (4.4)$$

a meromorphic function of  $y_*$ . Provided we have no homogeneous solutions, we therefore have a unique solution for every  $y_* \in [-1, 1]$ .

When a homogeneous solution,  $\chi$ , exists the situation is more complicated. The appearance of a homogeneous solution is signified by a zero of the Fredholm determinant ( $\mathcal{D}(y_*) = 0$  is a “dispersion relation” for the Fredholm eigenvalues); it creates a pole in the resolvent kernel. The particular solution is then bounded only if the inhomogeneous term,  $f$ , satisfies an additional constraint, namely the Fredholm Alternative, which can be expressed as

$$\int_{-1}^1 \chi^\dagger(y) f(y) dy = 0, \quad (4.5)$$

where  $\chi^\dagger$  is the solution to the adjoint problem:

$$\chi^\dagger(y') = \int_{-1}^1 \chi^\dagger(y) \mathcal{F}(y, y'; y_*) dy. \quad (4.6)$$

Should this constraint be satisfied, we have a solution,  $\psi(y) = \tilde{\psi}(y) + AX(y)$ , that is determined only up to the constant  $A$ .

For Rayleigh's equation, the constraint represented by (4.5) poses an embarrassing problem. It is equivalent to

$$\int_{-1}^1 \chi^\dagger(y) \mathcal{K}(y, y_*) dy = 0. \quad (4.7)$$

In other words, the adjoint eigenvectors of the homogeneous problem must be the null vectors of the integral operator with kernel  $\mathcal{K}(y, y_*)$ . By definition, this operator is the inverse of the Laplacian, and in the domain defined by the channel it is invertible, at least when acting on a class of well-defined, continuous functions.

This observation suggests that we may never have a bounded particular solution should a homogeneous solution exist. In addition, even if we could satisfy the Fredholm Alternative, the general solution is still not unique. It would then appear that the utility of the regularization hinges on whether there are homogeneous solutions.

From a physical point of view, the homogeneous solutions are not purely mathematical curiosities. They satisfy the condition,

$$\int_{-1}^1 \omega(y) dy = 0, \quad (4.8)$$

which follows if the ambient vorticity is rearranged, and are therefore physically meaningful. In fact, since the condition (3.7) precludes a solution of this kind, it suggests that the problem is more a result of that normalization than a real issue. Consequently we can circumvent the problem according to the following argument.

In principle, our normalization (3.7) should contain an arbitrary amplitude, so that

$$\int_{-1}^1 \omega(y) dy = -\Lambda(y_*) \quad (4.9)$$

(we shall, in fact, use this amplitude in subsequent sections). With this alternative normal-

ization, the value for  $C$  becomes,

$$C = \Lambda(y_*) - \mathcal{P} \int_{-1}^1 \frac{U''(y)\psi(y)}{U(y) - U(y_*)} dy, \quad (4.10)$$

and the inhomogeneous term is  $f(y) = \Lambda\mathcal{K}(y, y_*)$ . If we now rescale the amplitude of the continuum mode such that  $\Lambda = \tilde{\Lambda}\mathcal{D}(y_*)$ , with  $\tilde{\Lambda}(y_*)$  bounded, we observe that the inhomogeneous term vanishes in the limit  $y_* \rightarrow y_h$ . At that point, there is consequently *only* the homogeneous solution,  $\psi(y) = \Lambda_h\chi(y)$ , and so the general solution is unique to within the arbitrary amplitude  $\Lambda_h$ .

We can, therefore, suitably rescale the amplitudes of the continuum modes, so that when homogeneous solutions appear, the solution can be made unique. Thus, with or without homogeneous solutions, we establish that there is a unique solution for every  $y_* \in [-1, 1]$ . In other words, by using Fredholm theory, we can verify the existence of the continuous spectrum and show that its representation in terms of the singular eigenmodes (3.5) is unique (we have not, however, been completely rigorous and defined the function space in which the solutions reside).

The properties of the homogeneous solutions suggest that they are distinctive modes. In fact, the condition  $\mathcal{D}(y_h) = 0$  could be viewed as a dispersion relation for these solutions, which sets them apart from the continuum, which has no such relation. This argument suggests that the homogeneous solutions are a kind of discrete mode, and are more than simply a normalization issue. However, it is important to realize that, after the rescaling (4.9), the homogeneous solution is the limiting eigenfunction of the continuum modes as  $y_* \rightarrow y_h$ . This follows from Fredholm theory which indicates that the residues of the resolvent kernel at its poles,  $y_* = y_h$ , are just the homogeneous solutions.

In practical applications, we find that continuous velocity profiles typically do not appear to have any homogeneous Fredholm solutions. To establish this with some degree of certainty we can numerically construct the Fredholm determinant,  $\mathcal{D}(y_*)$ , in any given situation. That

is the direction taken in section 4.3. This feature of Rayleigh’s problem is at least partly the result of the finite width of the channel. Sedláček (1971b) presents a related example from plasma physics in which the integral problem analogous to our Fredholm equation is defined on an infinite domain and possesses a continuous spectrum of homogeneous solutions. In other words, there is a homogeneous solution for every value of  $y_*$ . The particular solution therefore never exists and  $\Lambda(y_*)$  must be made to vanish identically.

In general situations, then, we cannot exclude the possibility that there are homogeneous solutions. However, when they do exist, the representation of the singular spectrum by the eigenmodes (3.5) is slightly more complicated. Since they appear to be absent for the continuous velocity profiles we have studied, when we turn to considering the initial-value problem and other applications of the singular spectral expansion, we shall ignore them for brevity.

## B. Sample numerical solutions

Except in some special cases (Appendix A), the solution of the Fredholm problem cannot be given in a closed form. For general velocity profiles, we resort to numerical techniques to construct the singular eigenfunctions. To do this we first divide the range of integration at the critical point. We then evaluate each piece of the integral using Gauss-Legendre quadrature (*e.g.* Press *et al.* 1992). This reduces the integral equation to

$$\psi(y) = \mathcal{K}(y, y_*) + \sum_{m=1}^2 \sum_{j=1}^{n_m} w_{m,j} \mathcal{F}(y, y_{m,j}; y_*) \psi(y_{m,j}), \quad (4.11)$$

where  $(y_{m,j})$  and  $n_m$  are the locations and number of quadrature points, and  $(w_{m,j})$  are the weights. Gauss-Legendre quadrature is suited to continuous functions, which motivates the split of the integration range at the point at which the integrand has either a corner or a discontinuity. The total number of quadrature points used was typically about 400, with a division into the two intervals taken according to the location of the critical layer.

Equation (4.11) can be solved by matrix inversion. This allows us to construct various eigenfunctions. With it, we can also approximately examine the spectrum of the Fredholm operator, and estimate the Fredholm Determinant for families of flow profiles. As illustration, we take the family,

$$U(y) = y + \alpha y^3, \quad (4.12)$$

with  $\alpha > 0$  to ensure monotonicity. In figure 1 we display the shapes of eigenfunctions for various values of  $y_*$  for flows with  $\alpha = 0.1$  and  $\alpha = 1$  (we set  $k = 1$  for the illustrations). The first picture (a) shows eigenfunctions for  $\alpha = 0.1$  that are little different from their Couette (linear-shear) counterparts. The profile with a stronger background vorticity gradient in panel (b) appears to have smoother eigenfunctions, and their maxima do not always lie at the critical layer. In both cases, the critical amplitude,  $\psi_* = \psi(y_*)$ , falls on a smooth curve which never crosses the axis (a feature required by Barston's argument).

A more compact way to display the solutions is as the density,  $\psi(y; y_*)$ , where we have introduced the dependence on eigenvalue,  $y_*$ , as a second argument. Such densities are displayed as contour plots on the  $(y, y_*)$  plane in figure 2. The horizontal cuts indicated on those pictures correspond to the eigenfunctions of figure 1. We write the streamfunction in this fashion for reasons other than just as a convenient way to display the eigenfunctions,  $\psi(y; y_*)$ . In following sections we treat  $\psi(y; y_*)$  explicitly as a function of two variables, and its similarity to a density or kernel becomes more apparent.

The cubic profile (4.12) is always stable according to Fjørtoft's (1950) extension of Rayleigh's theorem (*e.g.* Drazin & Howard, 1966). As a result, the equilibria never support any discrete modes. Another family of profiles for which complex pairs do exist is given by

$$U(y) = \tanh \beta y. \quad (4.13)$$

Singular eigenfunctions for these equilibria are shown in figure 3. The first panel shows various critical modes with  $k = 1$  and  $\beta = \beta_c \simeq 1.462$ . The second picture shows the



behaviour of the eigenfunction with  $y_* = y_I = 0$  as  $\beta$  varies near  $\beta_c$  (again with  $k = 1$ ). Even though the critical layer for this eigenfunction always lies at the inflection point (hence there is no singularity in Rayleigh's equation), it is smooth only when  $\beta = \beta_c$ . This reflects the fact that the delta-function piece of the singular eigenfunction (3.5) is still needed in general in order to satisfy the boundary conditions.

For  $\beta = \beta_c$ , we have an eigenfunction given by  $u = 0$  and

$$\omega = -\frac{U''\psi}{U-u} \quad \text{and} \quad \psi = \int_{-1}^1 \mathcal{K}(y, y') \frac{U''(y')\psi(y')}{U(y')-u} dy', \quad (4.14)$$

which constitutes another eigenvalue problem for  $\beta_c$ . It is clear from figure 3 that the smooth solution is the limiting eigenfunction as  $\beta \rightarrow \beta_c$  and  $y_* \rightarrow 0$ .

The smooth solution, or *inflection-point mode*, is discrete in the sense that it arises from the eigenvalue problem (4.14). It has a dispersion relation  $\mathcal{C}(y_*) = 0$ , which can be seen by integrating the first piece of (4.14) and using the normalization (3.7). In the sense that the inflection-point mode is the limit of the singular solutions, however, it is not distinct from the continuum.

The inflection-point mode occurs at the bifurcation point of the complex pair from the continuum; for  $\beta > \beta_c$ , we also have a discrete, complex pair. This pair shares the dispersion relation  $\mathcal{C}(y_*) = 0$ , where now we must view  $y_*$  as a *complex* critical layer (provided the inverse  $U^{-1}$  is well defined). For either the inflection-point mode or the complex pair, Rayleigh's equation is not singular, and we write it as the inhomogeneous, regular equation,

$$\psi'' - k^2\psi - \frac{U''\psi}{U-u} = \mathcal{C}\delta(y - y_*). \quad (4.15)$$

This equation is self adjoint and has a bounded solution provided it satisfies a solvability condition. That condition is automatically satisfied as a result of the dispersion relation  $\mathcal{C} = 0$ , which indicates that the normalization (3.7) consistently treats both the singular solutions and any discrete modes.

For profiles like (4.12) and (4.13), there are no homogeneous solutions, as we verify by constructing Fredholm determinants (figure 4). The lack of homogeneous solutions can sometimes be verified more directly by considering the inequality,

$$\sum_{j=1}^J |\lambda_j|^{-2} \leq \frac{1}{4} \int_{-1}^1 \int_{-1}^1 |\mathcal{F}(y, y'; y_*)|^2 dy dy'. \quad (4.16)$$

If we replace the integrand by its maximal value, we can extract some analytical bounds on the modulus of the smallest eigenvalue,  $\lambda_1$ . For flows with monotonic velocity profile, this often turns out to be the point for which  $y = y' = y_*$ , and so

$$|\lambda_1| \geq \left| \frac{U'(y_*)}{U''(y_*)} \right|. \quad (4.17)$$

This bound is a shear length, and, when exceeding unity, rules out the possibility of homogeneous solutions. However, it is sometimes too weak and gives inaccurate predictions when the profile becomes nearly flat.

## V. The initial-value problem and completeness

### A. The amplitude distribution

In this section we examine the completeness of the spectrum of singular eigensolutions by representing an arbitrary initial condition in terms of them. This procedure amounts to the solution of a related Riemann-Hilbert problem (*e.g.* Gakhov, 1990).

With a distribution of amplitudes,  $\Lambda(y_*)$ , for the eigenfunctions with critical layers at  $y = y_*$ , the perturbation vorticity can be represented by

$$\begin{aligned} \omega(y, t) = & -\mathcal{P}U''(y) \int_{-1}^1 \frac{\Lambda(y_*)\psi(y; y_*)}{U(y) - U(y_*)} e^{-ikU(y_*)t} dy_* \\ & -\Lambda(y)e^{-ikU(y)t} \left[ 1 - \mathcal{P} \int_{-1}^1 \frac{U''(y')\psi(y'; y)}{U(y') - U(y)} dy' \right]. \end{aligned} \quad (5.1)$$

We now solve (5.1) for  $\Lambda$  in terms of  $\omega(y, t = 0) =: Q(y)$ , which amounts to solving yet another singular integral equation.

By extending the variable  $y$  off the real axis and into the complex plane, we can make use of formulae arising in the complex analysis of singular integral problems. For the present problem the relevant formulae are (Gakhov, 1990),

$$\Phi^\pm(y) = \frac{1}{2}\psi_*(y)\frac{U''(y)}{U'(y)} \pm \frac{\mathcal{P}}{2\pi i} \int_{-1}^1 \frac{U''(y)\psi(y';y)}{U(y')-U(y)} dy' \quad (5.2)$$

and

$$\Psi^\pm(y) = \frac{1}{2}\psi_*(y)\frac{\Lambda(y)}{U'(y)} \mp \frac{\mathcal{P}}{2\pi i} \int_{-1}^1 \frac{\Lambda(y_*)\psi(y;y_*)}{U(y)-U(y_*)} dy_*, \quad (5.3)$$

where  $\psi_*(y) := \psi(y; y)$  is the critical-layer amplitude of the streamfunction. These formulae define two pairs of functions,  $\Phi^\pm$  and  $\Psi^\pm$ , that are analytic in the upper and lower (respectively) halves of the strip in the complex  $y$ -plane with  $-1 < y < 1$ . As  $y$  tends to the real axis from above and below within this strip,  $\Phi^\pm$  and  $\Psi^\pm$  are the limiting values of the integrals in (5.2) and (5.3). On that axis, the integrals undergo jumps of  $U''\psi_*/U'$  and  $\Lambda\psi_*/U'$ , respectively. At the boundaries of the channel,  $y = \pm 1$ , we should, in principle, modify the formulae to account for the termination of the contour of integration (Gakhov, 1990). However, since the integrands always vanish there because  $\psi(\pm 1; y_*) = \psi(y; \pm 1) = 0$ , we need not treat the end points separately. We can then take the strip in which the functions are analytic to be  $-1 \leq y \leq 1$ , and use analytic continuation to extend those functions into the rest of the complex plane should that be necessary.

In terms of our analytic pairs, the arbitrary initial condition,  $Q(y)$ , can be written as,

$$Q = - \left[ (1 - 2\pi i \Phi^+) \Psi^+ + (1 + 2\pi i \Phi^-) \Psi^- \right] U' \quad (5.4)$$

$$=: -(\epsilon \Psi^+ + \epsilon^* \Psi^-) U'. \quad (5.5)$$

The function,  $\epsilon(y)$ , is the fluid analogue of the plasma dielectric function (*e.g.* Morrison & Pfirsch, 1992).

We now introduce the pair  $Q^\pm$  according to the relations,

$$Q^\pm(y) = \frac{1}{2}\psi_*(y)\frac{Q(y)}{U'(y)} \pm \frac{\mathcal{P}}{2\pi i} \int_{-1}^1 \frac{\psi(y';y)Q(y')}{U(y')-U(y)} dy', \quad (5.6)$$

which have the same analyticity properties as  $\Phi^\pm$  and  $\Psi^\pm$ . Then the initial condition becomes,

$$Q^+ = -\epsilon\Psi^+, \quad Q^- = -\epsilon^*\Psi^-. \quad (5.7)$$

Thus,

$$\psi_*\Lambda = -\left(\frac{Q^+}{\epsilon} + \frac{Q^-}{\epsilon^*}\right)U'. \quad (5.8)$$

Or,

$$\psi_*\Lambda = -\frac{1}{\epsilon_R^2 + \epsilon_I^2} \left[ \epsilon_R\psi_*Q - \frac{\epsilon_I U'}{\pi} \mathcal{P} \int_{-1}^1 \frac{\psi(y'; y)Q(y')}{U(y') - U(y)} dy' \right], \quad (5.9)$$

where

$$\epsilon_R(y) = \frac{1}{2}(\epsilon + \epsilon^*) \equiv \mathcal{C} = 1 - \mathcal{P} \int_{-1}^1 \frac{U''(y')\psi(y'; y)}{U(y') - U(y)} dy', \quad (5.10)$$

and

$$\epsilon_I(y) = \frac{1}{2i}(\epsilon - \epsilon^*) \equiv -\pi\psi_*(y) \frac{U''(y)}{U'(y)}. \quad (5.11)$$

If we introduce the explicit forms for  $\epsilon_R$  and  $\epsilon_I$ , this becomes,

$$\Lambda = -\frac{1}{\epsilon_R^2 + \epsilon_I^2} \left[ \left( 1 - \mathcal{P} \int_{-1}^1 \frac{U''(y')\psi(y'; y)}{U(y') - U(y)} dy' \right) Q + U'' \mathcal{P} \int_{-1}^1 \frac{\psi(y'; y)Q(y')}{U(y') - U(y)} dy' \right]. \quad (5.12)$$

In order for us to employ the formulae (5.2) and (5.3), the functions  $\epsilon$ ,  $\epsilon^*$ ,  $\Psi^\pm$  and  $Q^\pm$  must be analytic. The pairs  $\epsilon$ ,  $\epsilon^*$  and  $Q^\pm$  are analytic if the flow profile,  $U(y)$ , and the initial distribution of vorticity,  $Q$ , satisfy a certain continuity condition (the Hölder condition – *e.g.* Gakhov, 1990). Provided we meet this requirement, the pair  $\Psi^\pm$  can then be made analytic in their respective halves of the strip,  $S$ , if the functions  $\epsilon_R \pm i\epsilon_I$  have no zeros there. Thus, unless  $\epsilon_R$  and  $\epsilon_I$  simultaneously vanish, we can construct an arbitrary initial disturbance with the singular eigenfunctions, and so the spectrum is therefore complete.

## B. Zeros of $\epsilon$ and discrete modes

Our recipe for reconstructing the initial condition from the singular eigensolutions fails when there exists a point  $y = y_D$  in our strip  $S$  for which

$$\epsilon_R = \epsilon_I = 0. \quad (5.13)$$

When written out explicitly, these conditions are

$$1 + \mathcal{P} \int \frac{U''(y')\psi(y'; y_D)}{U(y') - U(y_D)} dy' = 0 \quad (5.14)$$

and

$$i\pi\psi_*(y_D) \frac{U''(y_D)}{U'(y_D)} = 0. \quad (5.15)$$

The origin of the zeros represented by (5.14) and (5.15) can be traced by re-examining the regularized eigenvalue equation, which we write in the alternative form,

$$\psi(y; y_*) = \mathcal{P} \int_{-1}^1 \mathcal{K}(y, y') \frac{\psi(y'; y_*)U''(y')}{U(y') - u} dy' + \epsilon_R \mathcal{K}(y, y_*), \quad (5.16)$$

and the expression for the vorticity fluctuation,

$$\omega = -\mathcal{P} \left( \frac{U''\psi}{U - u} \right) - \epsilon_R \delta(u - U)U'. \quad (5.17)$$

To account for the complex zeros (5.13), we regard  $u$  and  $y_*$  as complex variables. At these zeros, the linear equations then reduce to

$$\psi(y; y_D) = \int_{-1}^1 \mathcal{K}(y, y') \frac{\psi(y'; y_D)U''(y')}{U(y') - u} dy', \quad (5.18)$$

and

$$\omega = -\frac{U''\psi}{U - u}, \quad (5.19)$$

which are the eigenvalue equations of the discrete modes of the system (*cf.* equation (4.14)).

The zeros of  $\epsilon$  off the real axis correspond to the complex, discrete modes (the growing/decaying pairs). Zeros lying on the real axis are equivalent to the inflection-point modes.

The most important feature of an inflection-point mode is that it marks the bifurcation point of a complex pair from the continuum. This observation has mistakenly led to the belief that the mode is the coalescence of the complex pair, thence degenerate. In fact, the complex pair probably emerges from two continuum modes moving off two different Riemann sheets of the branch cut in the spectral plane that represents the continuum (Crawford and Hislop, 1992). The inflection-point mode is not therefore a degenerate point, and it would seem natural to treat it as part of the continuum.

We emphasize this point because, if we are to establish completeness of the normal modes then we should include any discrete ones. This requires us to consider the inflection point modes explicitly, which necessitates an understanding of their origin.

### C. Completeness with discrete modes

The complex discrete modes are incorporated into the superposition (5.1) simply by adding arbitrary multiples of their vorticity:

$$Q(y) = \omega_s(y) + \omega_d(y) + \omega_d^*(y), \quad (5.20)$$

where  $\omega_s$  is the contribution of the continuum given by (5.1),

$$\omega_d(y) = - \sum_{l=1}^I A_l \frac{U''(y)\varphi_l(y)}{U(y) - v_l} \quad (5.21)$$

and

$$\omega_d^*(y) = - \sum_{l=1}^I A_l^* \frac{U''(y)\varphi_l^*(y)}{U(y) - v_l^*}, \quad (5.22)$$

if there are  $I$  pairs of complex conjugates,  $\varphi_l$ , with eigenvalue  $v_l$  and amplitude  $A_l$ . When we once more employ the formulae (5.2) and (5.3), the analytic pairs become related by,

$$\Psi^\pm = \frac{Q^\pm}{\epsilon_R \pm i\epsilon_I}, \quad (5.23)$$

where

$$Q^\pm = \frac{\psi_*}{2U'}q \pm \frac{\mathcal{P}}{2\pi i} \int_{-1}^1 \frac{\psi(y'; y)q(y')}{U(y') - U(y)} dy' \quad (5.24)$$

and  $q = Q - \omega_d - \omega_d^*$ . The denominators of (5.23) vanish at points  $y = y_l$ ,  $l = 1, 2, \dots, I$ , lying off the real axis; these are the complex critical layers of the discrete pairs. In order to avoid the poles in  $\Psi^\pm$  that then emerge, we must therefore choose the amplitudes  $A_l$  such that the numerators both vanish at those points. Specifically, we require that  $Q^\pm = 0$ , or, equivalently,

$$Q(y_l) = \omega_d(y_l) + \omega_d^*(y_l) \quad (5.25)$$

and

$$\mathcal{P} \int_{-1}^1 q(y) \frac{\psi(y; y_l) dy}{U(y) - v_l} = 0 \quad (5.26)$$

(*cf.* Case, 1978). For every zero of  $\epsilon$  there are two conditions that must be met, but since we may select two free constants (the equivalent of an amplitude and a phase of each complex mode), we can in general satisfy (5.25) and (5.26).

The removal of the poles of  $1/\epsilon$  on the real axis is a little more difficult. Case (1978) treats them by considering the inflection-point modes to be discrete. That approach runs into difficulty because there is then only a single free constant (namely the amplitude of the inflection-point mode), and it is not in general possible to choose it to satisfy a pair of conditions like (5.25) and (5.26). Case surmounts this difficulty by finding other kinds of modes (“Siewert solutions”) that can be added to the superposition. The justification for such a procedure is that the inflection-point mode is a degenerate coalescence of a complex pair, hence it must be possible to add another, generalized eigensolution. However, the inflection-point mode is not really degenerate.

Here, we choose to consider the inflection-point modes as part of the continuum. However, because they are distinguished by the need to solve the initial-value problem, we are forced to suitably modify the amplitude distribution  $\Lambda(y_*)$ . First we rewrite the superposition (5.1) in the form,

$$Q(y) = -U''(y) \mathcal{P} \int_{-1}^1 \frac{\Lambda(y_*) \psi(y; y_*)}{U(y) - U(y_*)} dy_* - \epsilon_R(y) \Lambda(y). \quad (5.27)$$

The origin of our difficulty associated with the zeros of  $\epsilon$  on the real axis lies in the fact that there is a point  $y = y_I$  for which  $U'' = \epsilon_R = 0$ . There, the right-hand side of (5.27) vanishes, but our initial vorticity distribution,  $Q(y)$ , need not be so constrained.

The way around this difficulty is to realize that we have assumed that the amplitude distribution is a continuous function (in the Hölder sense), since that is what is normally required in order to construct the analytic pairs of the formulae (5.3). However, all we really require is that we can perform the singular integrations and that the function  $Q(y)$  that results from equation (5.27) satisfies the continuity requirements. The condition that  $\Lambda$  be a continuous function is then a little restrictive. In particular, if there is a point for which  $U'' = \epsilon_R = 0$ , then we need not require  $\Lambda$  to remain regular there. For example, we may consider a distributional form for  $\Lambda$ ,

$$\Lambda(y_*) = \widehat{\Lambda}(y_*) + B_1 \delta(y_* - y_I) + \mathcal{P} \frac{B_2}{U(y_*) - U(y_I)}, \quad (5.28)$$

where  $\widehat{\Lambda}(y_*)$  is regular and  $B_1$  and  $B_2$  are arbitrary constants. This leads to an initial function,

$$Q(y) = \widehat{Q}(y) - B_1 \frac{U''(y)\psi(y; y_I)}{U(y) - U(y_I)} - \frac{B_2 \epsilon_R(y)}{U(y) - U(y_I)} - \frac{B_2 U''(y)}{U(y) - U(y_I)} \left[ \mathcal{P} \int_{-1}^1 \frac{\psi(y; y_*)}{U(y) - U(y_*)} dy_* + \mathcal{P} \int_{-1}^1 \frac{\psi(y; y_*)}{U(y_*) - U(y_I)} dy_* \right], \quad (5.29)$$

where  $\widehat{Q}$  satisfies (5.27) with  $\widehat{\Lambda}$ . The right-hand side of this equation is a regular function, but does not vanish at  $y = y_I$ . Moreover, we now have two free constants that can be fixed to satisfy the invertibility conditions analogous to (5.25) and (5.26).

In retrospect we observe that the distributional form we introduced for  $\Lambda$  has the following properties. The regular piece,  $\widehat{\Lambda}$ , is like the original continuum contribution. The delta-function part,  $B_1 \delta(y_* - y_I)$ , extracts the inflection-point mode from the rest of the continuum; hence adding that term is like superposing that mode as though it were discrete. The final, principal-part piece of (5.28) plays the role of Case's Siewert solution. We have, however,



not considered the inflection-point mode to be degenerate, nor used the counterpart of the Siewert solution. (In Rayleigh's problem that solution is in fact trivial since any solution proportional to  $\exp ik[x - U(y)t]$  must vanish identically.)

We can, therefore, either add in the complex, discrete pairs or modify the distribution of amplitudes in order to remove the poles in  $Q^\pm$ . We have established this essentially by inspection, although (5.28) is consistent with the Laurent expansion of (5.12). It would be better, and certainly more rigorous, to explicitly treat the construction of  $\Lambda$  as the inverse of the transformation defined by (5.27). The function space of both the forward and inverse transformations could then be determined to establish exactly what the functional forms of  $\Lambda$  and  $Q$  should be. The extra terms added into the amplitude  $\Lambda$  should then arise as permissible parts of the function space, and the necessity to include them as a requirement to regularize the inverse transformation (see Morrison & Shadwick, 1994; Shadwick, 1995): We go further along these lines in paper II. In any event, we have established the completeness of the combination of the continuum and discrete pairs.

## VI. Eigenfunction relations

### A. Completeness relations and other orthogonality conditions

The solution of the initial-value problem indicates that the spectrum of continuum and complex modes is complete. For simplicity we now concentrate solely on the continuum and assume that there are no complex pairs in the spectrum. We can then concisely write the results of §5 in terms of a pair of integral transforms:

$$Q_k(y) = \int_{-1}^1 \omega_k(y; y_*) \Lambda_k(y_*) dy_* =: G_k[\Lambda_k] \quad (6.1)$$

and

$$\Lambda_k(y) = \int_{-1}^1 \bar{\omega}_k(y_*; y) Q_k(y) dy =: \bar{G}_k[Q_k], \quad (6.2)$$

with

$$\omega_k(y; y_*) = -\epsilon_{k,R}(y)\delta(y - y_*) - \mathcal{P} \frac{U''(y)\psi_k(y; y_*)}{U(y) - U(y_*)} \quad (6.3)$$

and

$$\bar{\omega}_k(y_*; y) = -\frac{1}{|\epsilon_k(y_*)|^2} \left[ \epsilon_{k,R}(y_*)\delta(y - y_*) + \mathcal{P} \frac{U''(y_*)\psi_k(y; y_*)}{U(y) - U(y_*)} \right]. \quad (6.4)$$

Here we have restored the implicit dependence on the wavenumber  $k$ , and written  $\omega_k(y)$  as the superposition of continuum modes with wavenumber  $k$ . Hence the singular mode is  $\omega_k(y; y_*)$  and appears as the kernel of the forward transform.

By definition,  $\bar{G}_k$  is the inverse of the transform  $G_k$ , and so

$$\int_{-1}^1 \bar{\omega}_k(y_*; y)\omega_k(y; y_*) dy = \delta(y_* - y'_*) \quad (6.5)$$

is the completeness relation. We do not, however, need to use singular integral methods to verify this relation, and instead we can establish (6.5) more directly. If we substitute the explicit forms of  $\bar{\omega}(y_*; y)$  and  $\omega(y; y'_*)$  into the integral, we find

$$\begin{aligned} & \frac{1}{|\epsilon_k(y'_*)|^2} \left\{ \epsilon_{k,R}(y_*)^2 \delta(y_* - y'_*) + \frac{U''(y'_*)}{U(y_*) - U(y'_*)} [\epsilon_{k,R}(y_*)\psi_k(y_*; y'_*) - \epsilon_{k,R}(y'_*)\psi_k(y'_*; y_*)] \right. \\ & \left. + U''(y'_*) \mathcal{P} \int_{-1}^1 \frac{U''(y)\psi_k(y; y_*)\psi_k(y; y'_*)}{[U(y) - U(y_*)][U(y) - U(y'_*)]} dy \right\}. \end{aligned} \quad (6.6)$$

To evaluate the final integral we use the relation,

$$\begin{aligned} & U''(y'_*) \mathcal{P} \int_{-1}^1 \frac{U''(y)\psi_k(y; y'_*)\psi_k(y; y_*)}{[U(y) - U(y'_*)][U(y) - U(y_*)]} dy = \epsilon_{k,I}(y_*)^2 \delta(y_* - y'_*) + \\ & \frac{U''(y'_*)}{U(y_*) - U(y'_*)} \left[ \mathcal{P} \int_{-1}^1 \frac{U''(y)\psi_k(y; y'_*)\psi_k(y; y_*)}{U(y) - U(y_*)} dy - \mathcal{P} \int_{-1}^1 \frac{U''(y)\psi_k(y; y'_*)\psi_k(y; y_*)}{U(y) - U(y'_*)} dy \right], \end{aligned} \quad (6.7)$$

which is a form of the Poincaré-Bertrand transposition formula (Gakhov, 1990).

If we collect together the various terms, our integral can be written as

$$\delta(y_* - y'_*) + \frac{U''(y'_*)}{U(y_*) - U(y'_*)} \mathcal{J}_k(y_*, y'_*), \quad (6.8)$$

where

$$\begin{aligned} \mathcal{J}_k(y_*, y'_*) &= \epsilon_{k,R}(y_*)\psi_k(y_*; y'_*) + \mathcal{P} \int_{-1}^1 \frac{U''(y)\psi_k(y; y'_*)\psi_k(y; y_*)}{U(y) - U(y_*)} dy \\ &\quad - \epsilon_{k,R}(y'_*)\psi_k(y'_*; y_*) - \mathcal{P} \int_{-1}^1 \frac{U''(y)\psi_k(y; y'_*)\psi_k(y; y_*)}{U(y) - U(y'_*)} dy. \end{aligned} \quad (6.9)$$

We now observe that

$$\mathcal{J}_k(y_*, y'_*) = \int_{-1}^1 [\psi_k(y; y'_*)\omega_k(y; y_*) - \psi_k(y; y_*)\omega_k(y; y'_*)] dy = 0, \quad (6.10)$$

by virtue of the symmetrical, Poisson relationship between  $\psi_k(y; y_*)$  and  $\omega_k(y; y_*)$ .

Hence we can directly verify the completeness relation (6.5) without solving the initial-value problem. In a similar fashion we can derive two further orthogonality relations that will prove useful in following sections. These relations are namely,

$$\int_{-1}^1 \bar{\omega}_k(y_*; y)\bar{\omega}_k(y'_*; y)U''(y) dy = \frac{U''(y_*)}{|\epsilon_k(y_*)|^2}\delta(y_* - y'_*) \quad (6.11)$$

and

$$\int_{-1}^1 \left[ \frac{U(y)}{U''(y)}\omega_k(y; y_*) + \psi_k(y; y_*) \right] \omega_k(y; y'_*) dy = \frac{U(y_*)}{U''(y_*)}|\epsilon_k(y_*)|^2\delta(y_* - y'_*). \quad (6.12)$$

Relations (6.11) and (6.12) are analogous to the canonical transformation relations obtained by Morrison & Pfirsch (1992) in the context of the Vlasov equation.

## B. Adjoints

The singular vorticity perturbations,  $\omega_k(y; y_*)$ , arise as solutions of the equation,

$$[U(y_*) - \mathcal{O}]\omega_k(y; y_*) = 0, \quad (6.13)$$

where

$$\mathcal{O}\omega = U(y)\omega(y) - U''(y) \int_{-1}^1 \mathcal{K}_k(y, y')\omega(y') dy'. \quad (6.14)$$

The adjoints satisfy

$$[U(y_*) - \mathcal{O}^\dagger]\omega_k^\dagger(y; y_*) = 0, \quad (6.15)$$

with

$$\mathcal{O}\omega^\dagger = U(y)\omega^\dagger(y) - \int_{-1}^1 \mathcal{K}_k(y, y')U''(y')\omega^\dagger(y') dy'. \quad (6.16)$$

Multiplication by  $U''(y)$  turns equation (6.15) into (6.13) with eigenfunction  $U''(y)\omega_k^\dagger(y; y_*)$ .

The adjoint is therefore  $\omega_k(y; y_*)/U''(y)$ , from which we can derive the relation,

$$\int_{-1}^1 \omega_k^\dagger(y; y_*)\omega_k(y; y'_*) dy = \frac{|\epsilon_k(y_*)|^2}{U''(y_*)}\delta(y_* - y'_*), \quad (6.17)$$

which follows from manipulations like those used earlier. This relation shows that the continuum eigenfunctions are normalized by  $|\epsilon_k(y_*)|^2/U''(y_*)$ .

There are two problems associated with the normalization present in (6.17) (and also in (6.11) and (6.12)). Firstly, the normalization vanishes at points for which  $\epsilon = \epsilon^* = 0$ , which indicates a failure of completeness. These points correspond to the complex pairs and inflection-point modes, and we must resort to the recipe of §5.3 in order to complete the basis. A second problem in the normalization arises at the inflection points, at which, unless there is an inflection-point mode, the normalization diverges. The completeness proof given in §5 holds in this case, and there is some subtlety in interpreting the divergence. We discuss this issue more thoroughly in §7.5. Until then, we imagine that the flow profile has no inflection points, and so there are also no complex pairs.

## VII. Shear flow as a Hamiltonian theory

### A. Hamiltonian structure

The Euler equations for the incompressible (two-dimensional) fluid can be cast into a form that reveals an underlying Hamiltonian structure. In this section we investigate the relevance of the singular eigensolutions to this formalism, but first we sketch out the structure itself.

The Euler equations are an infinite-dimensional Hamiltonian system or field theory (e.g. Morrison, 1982 and 1993; Salmon, 1988). That system can be characterized by a *noncanon-*

ical Poisson bracket of two quantities  $f$  and  $g$ , written as  $\{f, g\}$ , and a Hamiltonian,  $H$ . Accordingly, the evolution of a quantity like  $f$  is generated by the relation,

$$\frac{\partial f}{\partial t} = \{f, H\}. \quad (7.1)$$

For the two-dimensional shear-flow problem, we can linearize the general form of the non-canonical bracket to find

$$\{f, g\} = \int_{-\infty}^{\infty} \int_{-1}^1 U''(y) \left[ \partial_x \frac{\delta f}{\delta \omega} \partial_y \frac{\delta g}{\delta \omega} - \partial_y \frac{\delta f}{\delta \omega} \partial_x \frac{\delta g}{\delta \omega} \right] dy dx, \quad (7.2)$$

where  $\delta f/\delta \omega$  and  $\delta g/\delta \omega$  are functional derivatives with respect to the vorticity perturbation,  $\omega(x, y, t)$ , which are defined by

$$\delta f[\omega; \delta \omega] := \left. \frac{d}{d\eta} f[\omega + \eta \delta \omega] \right|_{\eta=0} \quad (7.3)$$

$$=: \int_{-\infty}^{\infty} \int_{-1}^1 \delta \omega \frac{\delta f}{\delta \omega} dy dx. \quad (7.4)$$

In comparison to a 2+1 canonical field theory, the bracket (7.2) is manifestly noncanonical because it contains functional derivatives with respect to only the single variable  $\omega$ , rather than derivatives with respect to a canonically conjugate pair, and contains  $U''$  and the operators  $\partial_x$  and  $\partial_y$ . This noncanonical formulation of the bracket arises principally because the vorticity is an Eulerian variable; any such variable, in contrast to a pair with Lagrangian or material character, cannot constitute a canonically conjugate set of variables (Morrison, 1993).

We can further Fourier transform the expression (7.2) in  $x$ , to arrive at the bracket,

$$\{f, g\} = \frac{i}{\pi} \int_0^{\infty} \int_{-1}^1 k U''(y) \left( \frac{\delta f}{\delta \omega_k} \frac{\delta g}{\delta \omega_{-k}} - \frac{\delta g}{\delta \omega_k} \frac{\delta f}{\delta \omega_{-k}} \right) dy dk, \quad (7.5)$$

where  $\omega_k(y, t)$  is the Fourier transform of  $\omega(x, y, t)$ , in contrast to the earlier definition as the amplitude of the singular eigenfunction, and so an exponential time dependence is not assumed.

The Hamiltonian of the system follows in part from the kinetic energy functional,

$$H = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-1}^1 |\nabla \tilde{\psi}|^2 dy dx \quad (7.6)$$

(the tildes again signify the total vorticity or streamfunction, ambient plus perturbation), but the structure of the problem is complicated by the presence of an infinite number of other conserved quantities. These quantities, which are integrals over functions of the vorticity, are built into the bracket of (7.1) and are a consequence of the noncanonical vorticity variable. Their presence means that the dynamics of the system takes place on surfaces in phase space whereon these quantities are conserved (*symplectic leaves*). Such constraints on the accessible parts of phase space constitute a degeneracy in the formulation of the bracket. Their existence forces us to reject the kinetic energy functional,  $H$ , as a Hamiltonian alone, and instead build one out of both  $H$  and an arbitrary functional of the vorticity (a *Casimir*). Thus,

$$F[\omega] = \int_{-\infty}^{\infty} \int_{-1}^1 \left[ \frac{1}{2} |\nabla \tilde{\psi}|^2 + C(\tilde{\omega}) \right] dy dx, \quad (7.7)$$

In linear theory, we expand all quantities about the equilibrium state; the arbitrary function,  $C(\tilde{\omega})$ , is determined at leading order by requiring the first variation of  $F$  to vanish, which corresponds to locating the equilibrium state,  $\omega_0$  and  $\psi_0$ , as an extremal point. This operation yields

$$C'(\omega_0) = -\psi_0. \quad (7.8)$$

The second variation of  $F$  determines the energy of perturbations about the equilibrium, subject to the constraint imposed by the conservation of vorticity. In other words, it gives the linear Hamiltonian,

$$\delta^2 F = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-1}^1 \left( \omega \psi + \frac{U}{U''} \omega^2 \right) dy dx, \quad (7.9)$$

where  $C''$  has been eliminated by differentiating (7.8) with respect to  $y$ . If we Fourier

transform in  $x$ , this Hamiltonian becomes

$$\delta^2 F = \frac{1}{\pi} \int_0^\infty \int_{-1}^1 \left( \psi_k + \frac{U}{U''} \omega_k \right) \omega_{-k} dy dk. \quad (7.10)$$

Equation (7.2) with (7.9) or (7.5) with (7.10) casts the linear shear-flow system in the infinite-dimensional, Hamiltonian form of (7.1). Our goal is to write this structure in a canonical, diagonal (action-angle) form. In order to achieve that aim, we use the singular eigenfunctions. The exponential time dependence that we assumed for linear perturbations then emerges naturally, and the Hamiltonian equations of motion are trivial. In order to effect the transformation to action-angle form, we employ the integral transform pair (6.1) and (6.2). Consistent with our redefinition of the Fourier vorticity amplitude,  $\omega_k$ , we then interpret these relations as general coordinate transformations in which time appears as a parameter; that is,  $\omega_k(y, t) = G_k[\Lambda_k]$  and  $\Lambda_k(y, t) = \bar{G}_k[\omega_k]$ .

## B. Canonizing the Poisson bracket

In the Poisson bracket (7.5), the vorticity perturbations,  $\omega_k$ , appear as the coordinates of the Hamiltonian system. Equation (6.1) can be viewed as a canonical transformation that rewrites the system in terms of the coordinates,  $\Lambda_k$ , the amplitudes of the singular eigenmodes. In order to examine the form in which this places the Poisson bracket, we need to evaluate the functional derivative,  $\delta f / \delta \omega_k$ , in terms of  $\delta f / \delta \Lambda_k$ . This can be achieved as follows. We observe that,

$$\delta f = \int_0^\infty \int_{-1}^1 \frac{\delta f}{\delta \omega_k} \delta \omega_k(y) dy dk \quad (7.11)$$

$$= \int_0^\infty \int_{-1}^1 \frac{\delta f}{\delta \Lambda_k} \delta \Lambda_k(y_*) dy_* dk = \int_0^\infty \int_{-1}^1 \int_{-1}^1 \frac{\delta f}{\delta \Lambda_k} \bar{\omega}_k(y_*; y) \delta \omega_k(y) dy dy_* dk. \quad (7.12)$$

Since, after the integration over  $y$ , that in  $y_*$  is regular, we can straightforwardly interchange the order of integration in the final multiple integral. Comparing (7.11) with (7.12) then leads us to the identity,

$$\frac{\delta f}{\delta \omega_k} = \int_{-1}^1 \frac{\delta f}{\delta \Lambda_k} \bar{\omega}_k(y_*; y) dy_*. \quad (7.13)$$

The linearized Poisson bracket can be written in the form,

$$\begin{aligned} \{f, g\} &= \frac{i}{\pi} \int_0^\infty \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 k \bar{\omega}_k(y_*; y) \bar{\omega}_k(y'_*; y) \\ &\times \left[ \frac{\delta f}{\delta \Lambda_k(y_*)} \frac{\delta g}{\delta \Lambda_{-k}(y'_*)} - \frac{\delta g}{\delta \Lambda_k(y'_*)} \frac{\delta f}{\delta \Lambda_{-k}(y_*)} \right] U''(y) dy'_* dy_* dy dk, \end{aligned} \quad (7.14)$$

which follows from (7.5). We now use the orthogonality relation (6.11); this leaves the bracket,

$$\{f, g\} = i\pi \int_0^\infty \int_{-1}^1 \frac{k U''(y_*)}{|\epsilon_k(y_*)|^2} \left[ \frac{\delta f}{\delta \Lambda_k(y_*)} \frac{\delta g}{\delta \Lambda_{-k}(y_*)} - \frac{\delta g}{\delta \Lambda_k(y_*)} \frac{\delta f}{\delta \Lambda_{-k}(y_*)} \right] dy_* dk. \quad (7.15)$$

We now introduce action variables,  $J_k$  and  $\vartheta_k$ , defined according to

$$\Lambda_{\pm k}(y_*) = \sqrt{\frac{\pi k |U''_*| J_k}{|\epsilon_k|^2}} \exp(\mp i \vartheta_k \text{sign } U''_*), \quad (7.16)$$

where  $U''_* = U''(y_*)$ . In terms of these variables, the bracket is,

$$\{f, g\} = \int_0^\infty \int_{-1}^1 \left( \frac{\delta f}{\delta \vartheta_k} \frac{\delta g}{\delta J_k} - \frac{\delta g}{\delta \vartheta_k} \frac{\delta f}{\delta J_k} \right) dy_* dk, \quad (7.17)$$

which is in canonical form.

### C. Diagonalizing the Hamiltonian

The diagonalization of the Hamiltonian proceeds in a similar fashion. We introduce the singular eigenfunctions into equation (7.10):

$$\begin{aligned} \delta^2 F &= \frac{1}{\pi} \int_0^\infty \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \left[ \frac{U(y)}{U''(y)} \omega_k(y; y_*) + \psi_k(y; y_*) \right] \\ &\times \omega_k(y; y'_*) \Lambda_k(y_*) \Lambda_{-k}(y'_*) dy'_* dy_* dy dk. \end{aligned} \quad (7.18)$$

If we use the orthogonality condition (6.12), our Hamiltonian reduces to the form:

$$\delta^2 F = \frac{1}{\pi} \int_0^\infty \int_{-1}^1 \frac{|\epsilon_k|^2 U}{U''_*} |\Lambda_k|^2 dy_* dk \quad (7.19)$$



or

$$\delta^2 F = \int_0^\infty \int_{-1}^1 \nu_k J_k dy_* dk, \quad (7.20)$$

where the frequencies,

$$\nu_k = |kU(y_*)| \operatorname{sign}[U_*''/U(y_*)] = ku \operatorname{sign}(U_*''). \quad (7.21)$$

The Hamiltonian (7.20) is of the required diagonal form; it displays the distribution of energy among the various continuum eigenfunctions, namely,

$$\mathcal{E}_k(y_*) = \nu_k(y_*) J_k(y_*) = \frac{1}{\pi} |\epsilon_k|^2 |\Lambda_k|^2 U(y_*) \operatorname{sign}(U_*''). \quad (7.22)$$

Hence the energy of a perturbation can be unambiguously defined. This is in contrast with the original formulation of the system. There, the need to formulate a Hamiltonian that incorporates a Casimir reflects the ambiguity in the energy of the system when there are constraints on motion imposed by the accessibility of phase space. In other words, a global minimum of the kinetic energy functional,  $H$  (the motionless state), is not the actual minimum energy of any one realization of the system because the dynamics preserves the Casimir functionals of the vorticity.

Once the system is in diagonal canonical form, Hamilton's equations become trivial,

$$\frac{\partial \Lambda_k}{\partial t} = \{\Lambda_k, H\} = \nu_k \Lambda_k, \quad (7.23)$$

and the exponential time dependence of the singular modes follows naturally, as we advertised earlier.

## D. Modal signature

The diagonalization of the Hamiltonian reveals a characteristic *signature* of the singular modes. Specifically, this is the sign of the frequencies  $\nu_k$ , which is given by  $U_*''/u$ . Signature is a critically important concept for two reasons. Firstly, as is clear from (7.22), it determines

the intrinsic sense of the energy of the mode. Thus depending on signature, we establish the existence of a continuum of *positive* or *negative* energy modes. Cairns (1979) has previously pointed out the possibility of finding linear solutions with either sign of energy in shears, but he considered discrete modes. Secondly, the signature of the modes can have important repercussions on the possible bifurcation sequences of the continuum modes. In particular, in systems of finite dimension, Krein (1950) and Moser (1958) have established that the Hamiltonian Hopf bifurcation is only possible when pairs of modes of opposite signature collide on the real axis of the spectral plane. In a fluid dynamical context, MacKay and Saffman (1986) have previously considered the role of signature in bifurcations of water waves. For infinite-dimensional systems with a continuous spectrum there is an analogue of Krein's theorem, but this is beyond our present scope.

The importance of the quantity leads one to ask the question as to whether signature is invariant under coordinate changes. For transformations involving only the dependent variables, Sylvester's theorem (Whittaker, 1937) guarantees that this is the case. However, for time-dependent transformations, energy is not a covariant quantity and signature can change. An important example, which is relevant here, is afforded by Galilean transformations to frames moving at a constant velocity in the streamwise direction. This is a time-dependent canonical transformation on the Lagrangian variable level.

Normally, equilibria that are equivalent up to Galilean transformations are obtained as extremal points of the Hamiltonian plus a constant multiple of the conserved, total momentum. In noncanonical variables this leads us to extremize

$$F_c[\omega] := F + cP \quad , \quad (7.24)$$

where  $c$  is the Galilean boost, and

$$P := - \int_{-\infty}^{\infty} \int_{-1}^1 y \tilde{\omega}(x, y, t) dy dx \quad (7.25)$$

is the total streamwise momentum. If we insist that the first variation of  $F_c$  vanish, we find, in place of (7.8),

$$C''(\omega_0) = -\psi_0 + cy. \quad (7.26)$$

The second variation then yields, in place of (7.9),

$$\delta^2 F_c = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-1}^1 \left( |\nabla \psi|^2 + \frac{(U-c)}{U''} \omega^2 \right) dy dx. \quad (7.27)$$

If we now introduce the singular eigenfunctions to evaluate the integral, we find the diagonal Hamiltonian (7.20), with the usual action coordinate

$$J_k = \frac{1}{\pi k} |\epsilon_k|^2 |\Lambda_k|^2, \quad (7.28)$$

but with the shifted frequency

$$\nu_k = k(u-c) \text{sign}(U_*''). \quad (7.29)$$

Thus the signature in the new frame is given by  $(u-c) \text{sign}(U_*''')$ .

## E. Dynamical accessibility

The transformation to action-angle form described above can only be performed when the singular eigenfunctions are appropriately normalized. The theory therefore breaks down whenever there is an inflection point in the profile. Part of the problem centres around the possible existence of complex pairs and inflection-point modes. These correspond to zeros in the normalization, or, equivalently, in  $|\epsilon|^2$ . At those points the bracket (7.15) is evidently not defined. In order to correct this deficiency, we need to include the eigenfunctions of the distinguished modes in the canonical transformation. We pursue this further in paper II.

The other reason why we have ignored the inflection points is that the normalization of the singular modes diverges at those points. This irregularity is intimately connected with the dynamical accessibility of phase space, and we now interpret this apparently irregular behaviour.

According to the Hamiltonian formulation of the shear-flow problem, dynamics is generated by the Poisson bracket (7.2). Because it contains the factor  $U''$ , that bracket cannot generate dynamics at the inflection point unless the Hamiltonian is in some sense singular there. This accounts for the apparent divergence of the perturbation energy expressed in either equation (7.5) or (7.27). In spite of this divergence, the energy and Casimir functionals are well-behaved, and

$$\delta F := F[\omega_0 + \delta\omega] - F[\omega] \quad (7.30)$$

is regular for a large class of reasonable functions  $\delta\omega$ . It is by constructing the second variation as a formal expansion in  $\delta\omega$  that we introduce the apparent divergence; the Hamiltonian  $F$  is not sufficiently functionally differentiable at the inflection point. Nevertheless, the compensating zero in the Poisson bracket produces a regular Rayleigh equation, which is solvable for all initial conditions.

If we restrict the initial conditions to be of the form  $Q = U''(y)\check{Q}$ , where  $\check{Q}$  is arbitrary but bounded, then the singularities in both the normalization of the eigenfunctions and the Hamiltonian are removed. Such perturbations are what we call *dynamically accessible* (Morrison & Pfirsch, 1992), since they project initial conditions into the surfaces of constant Casimir constraints. If we assume this type of initial condition in the construction of the canonical variables  $(\vartheta_k, J_k)$  of §7.2, then that derivation can be extended to include profiles for which there are inflection points, but no complex pairs and inflection-point modes.

The restriction to dynamically accessibility perturbations is not severe. At an inflection point, the linear dynamics is governed by

$$\left( \frac{\partial}{\partial t} + U(y_I) \frac{\partial}{\partial x} \right) \omega(x, y_I, t) = 0. \quad (7.31)$$

This equation describes “free-streaming” motion along the line  $y = y_I$ , and is decoupled from motion at other values of  $y$ . A dynamically accessible initial condition removes such streaming and the solution remains dynamically accessible for all time. Thus, motion that

is lost in assuming dynamical accessibility is trivial and easily reproduced.

Physically, dynamically accessible perturbations rearrange the local vorticity distribution. Since this is a feature of the dynamics of an incompressible inviscid fluid, other kinds of perturbations must be created by a process lying beyond the simple physics of Rayleigh's problem. In any given problem of interest one should address, on physical grounds, whether or not the initial conditions are appropriate. Fluid systems are generally excited by nonideal means, although the resulting dynamics can be practically inviscid. Consequently, an inaccessible initial perturbation is not unreasonable. (Further discussion of this issue is given by Morrison & Pfirsch, (1992) and Morrison (1993).)

## **F. Energy and stability**

Perturbation energy is a crucial concept with regard to the stability of a system. If a perturbation always increases the total energy of the system, then that configuration is stable. This is a basic means for ascertaining stability in Hamiltonian systems, which is sometimes called Dirichlet's theorem, and is the essence of the energy-Casimir method of Kruskal and Oberman (1958) and Arnold (1966). In infinite-dimensional Hamiltonian systems, a positive or negative definite form for the perturbation energy guarantees linear (and is suggestive of nonlinear) stability. Such energy principles, along with other sufficient conditions for stability such as Rayleigh's inflection point criterion and Fjrtft's generalization, arise naturally within the Hamiltonian context; in fact they all are different views of Dirichlet's theorem, as will be described presently.

Normally in Hamiltonian systems, equilibria are obtained as extremal points of the Hamiltonian and the perturbation energy is obtained by expanding to second order. In noncanonical systems, it is necessary to extremize the energy subject to the constraint of constant Casimir in order to obtain the perturbation energy,  $\delta^2 F$ . The energy-Casimir method is simply Dirichlet's theorem in noncanonical variables.

In the current context, Dirichlet's theorem therefore states that linear stability is assured by the definiteness of either  $\delta^2 F$  or  $\delta^2 F_c$ . We prefer the latter since it contains an arbitrary constant and is therefore more powerful for ascertaining stability. This quantity is positive definite if

$$(U - c)U'' > 0 \tag{7.32}$$

throughout the flow. If we select  $c$  such that  $U - c$  is everywhere positive or negative, this reduces to the usual inflection-point criterion; the flow is stable if there is no inflection point. If there is an inflection point, then when we take  $c$  to be the flow speed at  $y = y_I$ , we arrive at Fjørtoft's sufficient condition for stability.

In other words, both Rayleigh's theorem and Fjørtoft's condition are simply statements of Dirichlet's theorem. In fact, we can interpret these conditions in a slightly different manner. Rayleigh's sufficient condition for stability implies that there is no inflection point; hence it indicates that there is solely a continuum of singular modes, and by a suitable Galilean boost, these can be made to be of one signature. Fjørtoft's condition indicates that the flow may contain an inflection point, and in some frames the continuum may therefore contain modes of both signatures. However, it also ensures that there is a frame (namely that containing a stationary inflection point) in which the continuum has purely one signature. The stability criteria therefore imply that instabilities cannot emerge from a continuum with purely positive or negative energy. Hence, classical shear instability can be viewed as the linear interaction between positive and negative energy modes.

These statements seem directed only towards ideal linear theory, particularly the generalization of Krein's theorem, but the assignment of signature has other implications regarding dissipative and finite-amplitude instability. Systems that are linearly stable but possess negative energy modes are generically destabilized by the addition of dissipation, the so-called Thompson-Tait theorem (Poincaré, 1885; Lamb 1907, Thompson & Tait, 1921; Greene &

Coppi, 1965). Linearly stable systems which have both positive and negative energy modes (in all frames of reference) can be destabilized by the inclusion of nonlinearity, a feature common in the interactions between multiplets of discrete modes (Cherry, 1925; Davidson, 1972; Weiland & Wilhelmsson, 1977; Kueny & Morrison, 1995). Thus, we may conjecture that profiles with negative-energy continua can be destabilized by the addition of dissipation, and that linearly stable flows that violate Fjørtoft's criterion for stability may be nonlinearly unstable.

### VIII. Superpositions of singular eigensolutions

We now illustrate the utility of the singular eigensolutions by studying the temporal behaviour of a superposition of singular eigenmodes, in the case for which there are no discrete modes. This allows us to make contact with the Laplace transform method (Case, 1960; Dikii, 1960; Engevik, 1966; Rosencrans and Sattinger, 1966; Briggs *et al.*, 1970), and permits us to derive the fluid analogue of Landau damping in the long-time limit.

For an initial vorticity distribution,  $Q$ , the initial-value problem indicates that the amplitudes of the singular eigensolutions are given by the function,

$$\Lambda = -\frac{U'}{\psi_*} (\Psi^+ + \Psi^-) = -\frac{U'}{\psi_*} \left( \frac{Q^+}{\epsilon} + \frac{Q^-}{\epsilon^*} \right), \quad (8.1)$$

for each Fourier component in  $x$  (we will treat these components separately once more, and so we have again dropped the subscript  $k$ ). The vorticity perturbation is then given by the normal-mode solution,

$$\omega(y) = \int_{-1}^1 \Lambda(y_*) \omega(y; y_*) e^{-ikU(y_*)t} dy_*. \quad (8.2)$$

#### A. Equivalence with Laplace transform theory

In order to compare the normal mode theory with Laplace transform analysis we first need to sketch the derivation of the solution with that technique. Rayleigh's equation in

terms of the vorticity perturbation is

$$\frac{\partial \omega}{\partial t} + ik\mathcal{O}\omega = 0, \quad (8.3)$$

with the operator  $\mathcal{O}$  defined in equation (6.14). We take the Laplace transform of this equation,

$$P\hat{\omega}_P + ik\mathcal{O}\hat{\omega}_P = Q, \quad (8.4)$$

where

$$\hat{\omega}_P(y) = \int_0^\infty e^{-Pt} \omega(y, t) dt \quad (8.5)$$

is the transformed vorticity fluctuation.

To solve equation (8.4), we could formally write  $\hat{\omega}_P$  in terms of the Green function of the operator  $P+ik\mathcal{O}$ . We could then compute the inverse Laplace transform in order to construct the solution. However, it is simpler to relate the solution to the singular eigenfunctions if we follow a different route. We multiply (8.4) by the adjoint  $\omega^\dagger(y; y_*) = \omega(y; y_*)/U''(y)$ , and then integrate over position. This yields

$$\int_{-1}^1 \hat{\omega}_P(y') \omega^\dagger(y'; y_*) dy' = \frac{1}{P + ikU(y_*)} \int_{-1}^1 Q(y') \omega^\dagger(y'; y_*) dy'. \quad (8.6)$$

If we now take the product of the equation with  $U''(y)\bar{\omega}(y_*; y)$ , integrate over critical-layer position, and use completeness, we obtain

$$\hat{\omega}_P(y) = \int_{-1}^1 \int_{-1}^1 U''(y) \bar{\omega}(y_*; y) \omega^\dagger(y'; y_*) \frac{Q(y')}{P + ikU(y_*)} dy' dy_* \quad (8.7)$$

$$= \int_{-1}^1 \int_{-1}^1 \frac{U''(y_*)}{|\epsilon(y_*)|^2} \omega(y, y_*) \omega^\dagger(y'; y_*) \frac{Q(y')}{P + ikU(y_*)} dy' dy_*, \quad (8.8)$$

which uses the identity,

$$\bar{\omega}(y_*; y) = \frac{U''(y_*) \omega(y, y_*)}{U''(y) |\epsilon(y_*)|^2}. \quad (8.9)$$

Equation (8.8) indicates that we have successfully inverted the operator of (8.4) and written  $\hat{\omega}_P$  in terms of  $Q$ . In other words, we have found an explicit representation of the



Green function of this operator in terms of the orthogonal continuum eigenfunctions, namely

$$\int_{-1}^1 \frac{U''(y_*)}{|\epsilon(y_*)|^2} \frac{\omega(y, y_*) \omega^\dagger(y'; y_*)}{P + ikU(y_*)} dy_*. \quad (8.10)$$

It is more common to solve Rayleigh's equation with Laplace transform theory in terms of the transformed streamfunction,  $\hat{\psi}_P$ . We quote the result,

$$\begin{aligned} \hat{\psi}_P(y) &= \int_{-1}^1 \int_{-1}^1 \frac{U''(y_*)}{|\epsilon(y_*)|^2} \psi(y, y_*) \omega^\dagger(y'; y_*) \frac{Q(y')}{P + ikU(y_*)} dy' dy_* \\ &=: \int_{-1}^1 G_P(y, y') \frac{Q(y')}{P + ikU(y')} dy', \end{aligned} \quad (8.11)$$

which follows from (8.8) and defines the Green function,  $G_P(y, y')$ , of the equation for  $\hat{\psi}_P$ .

We now compute the inverse Laplace transform of (8.8),

$$\omega(y, t) = \frac{1}{2\pi i} \int_C \int_{-1}^1 \int_{-1}^1 \frac{e^{Pt}}{P + ikU(y_*)} \frac{U''(y_*)}{|\epsilon(y_*)|^2} \omega(y, y_*) \omega^\dagger(y'; y_*) Q(y') dy' dy_* dP, \quad (8.12)$$

where  $C$  is, as usual, the Bromwich contour; a vertical line on the complex  $P$  plane lying to the right of any singularities in the integrand. We can add to this path of integration a semicircle of infinite radius lying to the left of  $C$ , since on that contour the integrand vanishes. We can then compute the integral over  $P$  as the residues of the simple pole in the integrand at  $P = -ikU(y_*)$ . Hence,

$$\omega(y, t) = \int_{-1}^1 \int_{-1}^1 e^{-ikU(y_*)t} \frac{U''(y_*)}{|\epsilon(y_*)|^2} \omega(y, y_*) \omega^\dagger(y'; y_*) Q(y') dy' dy_* dP. \quad (8.13)$$

If we introduce the explicit form for the adjoint,  $\omega^\dagger(y'; y_*)$ , we observe that

$$\omega(y, t) = \int_{-1}^1 e^{-ikU(y_*)t} \frac{\omega(y, y_*)}{|\epsilon(y_*)|^2} \left[ \epsilon_R(y_*) Q(y_*) + U''(y_*) \int_{-1}^1 \frac{\psi(y'; y_*) Q(y')}{U(y') - U(y_*)} dy' \right] dy_*. \quad (8.14)$$

If we now recall the form of the analytic pair  $Q^\pm$ , we see that

$$\omega(y, t) = \int_{-1}^1 e^{-ikU(y_*)t} \omega(y, y_*) \frac{U'(y_*)}{\psi_*(y_*)} \left[ \frac{Q^+(y_*)}{\epsilon(y_*)} + \frac{Q^-(y_*)}{\epsilon^*(y_*)} \right] dy_*. \quad (8.15)$$

Since (8.15) is just the superposition (8.2), we have established the equivalence of the Laplace transform solution with the normal-mode superposition.

## B. Long-time evolution and continuum damping

We now calculate the long-time evolution of the superposition by studying the total vorticity contained in the perturbation across the channel,

$$W(t) = \int_{-1}^1 \Lambda(y_*) e^{-ikU(y_*)t} dy_* \quad (8.16)$$

(by virtue of the normalization (3.7), the dependence on the cross-stream position integrates to unity). This quantity is the analogue of the electric field in the Vlasov-Poisson system, which is commonly used to study Landau damping (Van Kampen & Felderhof, 1967). If we introduce the explicit form (8.1), this becomes

$$W(t) = \int_{-1}^1 \frac{U'(y_*)}{\psi_*(y_*)} \left( \frac{Q^+}{\epsilon} + \frac{Q^-}{\epsilon^*} \right) e^{-ikU(y_*)t} dy_*. \quad (8.17)$$

Our aim is to turn the integral in (8.17) into one over a closed contour in the complex  $y_*$  plane, then use standard techniques to evaluate it. Since the function  $U(y)$  is invertible, we change the integration variable from  $y_*$  to  $u$ , so that

$$W(t) = \int_{u_1}^{u_2} \frac{1}{\psi_*} \left( \frac{Q^+}{\epsilon} + \frac{Q^-}{\epsilon^*} \right) e^{-ikut} du, \quad (8.18)$$

where  $u_1 := U(-1)$  and  $u_2 := U(1)$ , and the integrand is to be interpreted as a function of  $u$ .

The amplitude distribution,  $\Lambda$ , is given by (5.12) as the sum of various principal-value integrals. These integrals are analytic in the upper and lower halves of the strip  $S$ , but experience jumps across the real axis. The amplitude  $\Lambda$  can be viewed as the sum of the limiting values of the integrals from above and below. The two terms in the integrand of (8.18) describe those limits. In other words, the integrand is a function that possesses a branch cut on the real axis of the complex  $u$  plane coinciding with the continuous spectrum. We can therefore divide  $W(t)$  into two pieces,

$$W^\pm(t) = \int_{u_1 \pm i\mu}^{u_2 \pm i\mu} \frac{\Psi^\pm}{\psi_*} e^{-ikut} du, \quad (8.19)$$

with  $\mu \rightarrow 0$ . Each piece consists of an integral running either just above or just below the branch cut.

Along the real axis outside the strip  $S$ , we can analytically continue the functions,  $\Psi^\pm$ . In fact, there we must have  $\Psi^+ = -\Psi^-$ . Hence we may extend the integrations in (8.19) over the whole of the real axis. To this path we then add sections enclosing the entire lower half plane. Along these secondary paths, for  $t > 0$  the exponential in the integrands of (8.19) vanish, and so both integrals can be turned into integrations over closed contours. These contours,  $C^\pm$ , are illustrated in figure 5a.

By construction, the integrand of the contribution,  $W^-(t)$ , is analytic in the lower half of the strip  $S$ . With continuation, we can therefore make that integrand analytic throughout the region enclosed by the contour  $C^-$ . The integral,  $W^-(t)$ , must therefore vanish identically. The remaining piece can be written as

$$W(t) = W^+(t) = \int_{C^+} \frac{Q^+}{\psi_* \epsilon} e^{-ikut} du. \quad (8.20)$$

According to the Riemann-Lebesgue lemma, this integral must vanish as  $t \rightarrow \infty$ . It may do this algebraically or exponentially, and in order to find its true asymptotic behaviour, we must analyze the integrand in more detail.

The contributions to the integral arise from two sources. Firstly, the contour  $C^+$  encloses the branch cut along the real axis, and so we expect a contribution directly from integrating around that cut. Also, in the lower half of the strip  $S$ , it is no longer necessarily true that  $\epsilon$  has no zeros, and so there may be additional contributions from the resulting poles in the integrand. To make the contributions apparent, we deform the path of integration to encircle both the branch cut and the poles in the lower half plane. Our path of integration then becomes a contour  $C'$ , which is also shown in figure 5b.

The contribution from the branch cut can be written as

$$\int_{u_1}^{u_2} \frac{1}{\psi_*} [\Psi^+(u + i\mu) - \Psi^+(u - i\mu)] e^{-ikut} du \quad (8.21)$$

for  $\mu \rightarrow 0$ . This integral has an algebraic, asymptotic time-dependence (*cf.* Case, 1960), which can be found by integrating by parts. The index of the algebraic decay depends on the form of initial condition (Weitzner, 1963). This can be seen particularly clearly for a linear flow profile,  $U(y) = y$ . In this ‘‘Couette’’ case, we have a simple solution to the initial-value problem,  $Q = \Lambda$  (Eliassen *et al.*, 1953), and so

$$W(t) = \int_{-1}^1 Q(u) e^{-ikut} du. \quad (8.22)$$

The contribution to  $W$  arises entirely from the branch cut ( $\epsilon = 1$  throughout the complex plane). If  $Q$  is infinitely differentiable, we find,

$$W(t) = \sum_{n=1}^{\infty} \left[ \frac{Q^{(n)}(u) e^{-ikut}}{(ikt)^n} \right]_{-1}^1. \quad (8.23)$$

Evidently, we can suitably select  $Q$  such that the integral average decays by any inverse power of time. For a general flow profile, this presumably remains true.

The contributions to the integral that arise from the poles of  $1/\epsilon$ , are given by the residues of the integrand at those points. In general we cannot find these explicitly, but if there is a zero in the continuation of  $\epsilon$  close to the real axis, we can approximately construct the contribution. Specifically, we have

$$\epsilon(y_*) = 1 - \mathcal{P} \int_{-1}^1 \frac{U''(y) \psi(y; y_*)}{U(y) - U(y_*)} dy + i\pi \psi_*(y_*) \frac{U''(y_*)}{U'(y_*)}. \quad (8.24)$$

If the leading zero is close to the real axis at  $U(y_*) = u = u_0 - i\gamma$ , we have the approximation,

$$1 \simeq \mathcal{P} \int_{-1}^1 \frac{U'' \psi(y; y_0)}{U(y) - u_0} dy \quad (8.25)$$

and

$$\gamma \mathcal{P} \int_{-1}^1 \frac{U'' \psi(y; y_0)}{[U(y) - u_0]^2} dy \simeq \pi \psi_*(y_0) \frac{U''(y_0)}{U'(y_0)}, \quad (8.26)$$

where  $U(y_0) := u_0$ . In addition to the zero represented by (8.25) and (8.26),  $\epsilon$  may have further zeros in the lower half plane (*e.g.* Sedláček, 1971a), and all contribute to the integral.

For sufficiently large times, however, the leading pole dominates the integral. Provided the approximation (8.25) and (8.26) remains accurate, we have the contribution,

$$\frac{Q^+(u_0)}{\psi_*(u_0)\epsilon'_R(u_0)} e^{-iku_0t - \gamma kt}. \quad (8.27)$$

This has the form of a damped wave, and it signifies the analogue of Landau damping.

The decay of  $W(t)$  over asymptotically large times reflects phase mixing of the initial perturbation. The decay is typically dominated by the contribution from the branch cut, (8.21), which is algebraic. The exponential analogue of Landau damping is much quicker, but is the only part to be manifest as a coherent, decaying oscillation, or a *quasi-mode* (cf. Sedláček, 1971a; Kamp, 1991).

The timescale associated with continuum damping, for either algebraic or Landau damping, is of the order of the advection time of the fluid, namely the ratio of channel width to typical flow speed. In comparison to the timescale characteristic of viscous decay when the fluid is not ideal, this ratio may be short. Ideal, continuum-damping physics may therefore dominate the linear dynamics of a shear flow at high Reynolds number.

### C. Energy transfer during continuum damping

Our calculations show that phase mixing eventually obliterates the total vorticity across the channel. This feature remains true of any weighted integral average across the channel. In particular, it indicates that the streamfunction, given by

$$\psi(y) = \int_{-1}^1 \int_{-1}^1 \mathcal{K}(y, y') \Lambda(y_*) \omega(y'; y_*) e^{-ikU(y_*)t} dy_* dy'. \quad (8.28)$$

will be obliterated over the course of time. For Couette flow, it is established that  $\psi \sim t^{-2}$  as  $t \rightarrow \infty$  (Case, 1960; Engevik, 1966), for finite initial vorticity. The velocity field therefore decays pointwise as  $t^{-1}$ . This observation is seemingly difficult to reconcile with energy conservation, and could mistakenly lead to the belief that continuum damping dissipates the kinetic energy of a perturbation. However, this is not the case.

The exact energy difference between a perturbed state and a shear flow equilibrium is given by

$$\begin{aligned}\delta H[\omega; \delta\omega] &:= H[\omega_0 + \delta\omega] - H[\omega_0] \\ &= \frac{1}{2} \int_{-\infty}^{\infty} \int_{-1}^1 [\delta\omega\delta\psi + \omega_0\delta\psi + \psi_0\delta\omega] dydx,\end{aligned}\tag{8.29}$$

where  $H[\omega]$  is defined by (7.6) and  $\delta$  is used to denote a finite perturbation. It is important to recall that energy difference is a second-order quantity in perturbation amplitude. Thus, if we expand perturbations according to the series,  $\delta\omega = \delta\omega + \delta^2\omega + \dots$ , with a similar expression for  $\delta\psi$  (where  $\delta\omega \equiv \omega$  and  $\delta\psi \equiv \psi$ ), we obtain the following formula for the energy difference to second order,

$$\delta^2 H = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-1}^1 [\delta\omega\delta\psi + \omega_0\delta^2\psi + \psi_0\delta^2\omega] dydx.\tag{8.30}$$

For general canonical Hamiltonian systems, second-order variations of the dynamical variables do not contribute to the energy when expanding about an equilibrium point. However, for a system written in terms of noncanonical variables, such as the Eulerian fluid variables, second-order variations do contribute. Moreover, the energy can be completely expressed as a quadratic form in terms of first-order variations. If  $\delta^2 H$  is rewritten accordingly, we then arrive at the constant of motion,  $\delta^2 F$ , of (7.9). This quantity therefore provides the physically relevant kinetic energy content of a perturbation (Morrison, 1993).

From  $\delta^2 F$  we observe that the kinetic energy contribution arising from the term  $\omega\psi$  asymptotically decays. In order to preserve the energy balance of  $\delta^2 F$ , the other terms, arising from the second-order variations, simultaneously increase. Continuum damping therefore represents a phenomenon of *energy transfer*, rather than dissipation.

As an illustration of continuum damping we take a Couette profile, and add an initial perturbation with a single Fourier component in  $x$  of wave number  $k = 1$ , and which is

independent of  $y$ . Thus,

$$\psi(y, t) = \frac{1}{1+t^2} e^{i(x-ty)} - \frac{\sinh(1+y)}{\sinh 2} e^{i(x-t)} - \frac{\sinh(1-y)}{\sinh 2} e^{i(x+t)} \quad (8.31)$$

and

$$\omega(y, t) = e^{i(x-ty)}. \quad (8.32)$$

Illustrations of the perturbations to the streamfunction and vorticity at various times are displayed in figure 6. Shown are contours of constant value of the real parts of  $\psi$  and  $\omega$ . The mean flow differentially advects the vorticity; its contours continually tilt downstream and remain linear. In streamfunction, the evolution proceeds somewhat differently. The initial condition is a periodic array of closed streamlines. The ambient flow first shears these structures out and elongates them. Subsequently the tips of the elongated structures etch into the background shear; this induces an additional, local swirl, and the tips emerge as new regions with closed streamlines. The newly formed elliptical structures are again sheared out, generate further structures, and the cycle continues indefinitely. Eventually, the perturbation in the velocity field becomes ever finely fluted, but its amplitude damps away.

## IX. Further remarks

In this discussion of Rayleigh's problem, we have established the existence of the continuous eigenvalue spectrum, and found a unique set of singular eigensolutions to represent it. Together with any discrete, complex eigenmodes, the continuum forms a complete set of functions. We specialized immediately to consider the case in which the velocity profile is monotonic. We can, however, generalize the problem to permit an arbitrary velocity profile, for which there may be multiple critical layers for each wavespeed (we will do this in the subsequent communication, paper II), or, in fact, compressible shears in general geometries (Balmforth & Morrison, 1995).

The Hamiltonian formulation of the problem indicates that the singular eigensolutions are a kind of normal coordinate; in terms of the amplitudes of the singular modes, the Poisson bracket and Hamiltonian are of a special, action-angle form. This allows us to unambiguously discuss the energy of an infinitesimal perturbation. We therefore have some grounds for claiming that these modes are the intrinsic degrees of freedom of the fluid, much as the modes of vibration of the classical tri-atomic molecule.

In physical terms, the addition of a particular continuum eigenfunction creates a singular alteration of the fluid vorticity. An integral superposition is needed to regularize such a contribution. The eigenmode itself is given by

$$\omega = -C\delta(y - y_*) - \mathcal{P} \frac{U''\psi}{U - u}. \quad (9.1)$$

The first term on the right-hand side of this formula is what we might call a linear mode of advection; it is the Fourier-transformed solution to the Liouville equation governing the advection of a scalar contaminant,

$$\frac{\partial s}{\partial t} + U \frac{\partial s}{\partial x} = 0 \quad (9.2)$$

(*cf.* equation (7.31)). A delta-function distribution of contaminant is the only periodic structure that can be sustained against the shear. When the contaminant is vorticity, the advective mode is a line vortex, and there is also interaction with the background shear gradient. This interaction leads to the second term in the perturbed vorticity, and we can think of it as a source term driving the advective mode. Hence we arrive at the form of the Van Kampen mode (9.1). It contains a “pure mode” and a forced response which resonates at the critical layer.

This way of thinking of the Van Kampen mode is motivated by the plasma analogue of Rayleigh’s equation. There the Liouville equation describes the orbits of the unperturbed plasma, and the source term is the collisionless, electrostatic interaction. In the general



context, the solutions to Liouville's equation (9.2) are dependent on the functional form of  $U(y)$ , and may not be single delta-functions (Koopman, 1931; Sedláček, 1972). When,  $U(y)$  is multi-valued, as it is for jets and general, nonmonotonic shears, the advective modes are superpositions of delta-functions. Incidentally, the Liouville modes provide an alternative way to formulate the problem to construct the singular eigenmodes (Symon, Seyler & Lewis, 1982).

By establishing that the singular eigenmodes and complex pairs are complete, we can represent an arbitrary initial condition in terms of an appropriate superposition. This allows us to consider the evolution over asymptotically large times, and make contact with the Laplace transform approach (Case, 1960; Dikii, 1960). Moreover, integral spatial averages of singular-mode superpositions become vanishingly small as a result of phase mixing, a feature characteristic of perturbations to an ideal system when there is no dispersion relation.

The singular eigenmodes also allow to do more than simply solve the initial-value problem; they open an avenue to various perturbation theories. Specifically, the singular eigensolutions provide a convenient (well-defined) basis set with which we can begin asymptotic expansions. With them we can therefore advance to consider weakly nonlinear theory, and attempt to make contact with the ideal limit of dissipative theory (*cf.* Case, 1961; Lin, 1961). Moreover, as canonical Hamiltonian coordinates, the amplitudes of the singular eigenfunctions allow us to use Hamiltonian perturbation theory. These directions constitute future work.

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## Appendix A.

Much as Rayleigh first solved the problem for the instabilities of a shear flow, the continuous spectrum can be straightforwardly constructed for profiles with piece-wise constant vorticity. We can characterize such equilibria as follows. If the flow is composed of  $n + 1$  distinct layers, then there are that many pieces of linear shear to the profile. When we label the vorticity interfaces by  $y_1, y_2, \dots, y_n$ , then we can write the profile as  $U(y) = U_j + (y - y_j)U'_j$  for  $y \in [y_j, y_{j+1}]$ ,  $j = 0, 1, \dots, n$ , where  $U_j$  and  $U'_j$  are constant, and  $y_0$  and  $y_{n+1}$  delineate the walls of the channel. We have the freedom to set  $U_0 = 0$  and  $U_{n+1} = 1$ , but in order for the profile to be monotonic, we must restrict the shear constants,  $U'_j$ , to all be of one sign.

Within each of the layers,  $U''$  vanishes. At the interfacial points, however, it is ill-defined, and we write,

$$U''(y) = \sum_{j=1}^n (U'_{j+1} - U'_j) \delta(y - y_j) =: \sum_{j=1}^n \Delta_j \delta(y - y_j). \quad (A.1)$$

Strictly speaking, we cannot refer to Fredholm theory in cases such as these because  $\mathcal{F}(y, y')$  itself must be interpreted in a distributional sense. Fortunately, however, because  $U''$  has the form of a sum of delta functions, we can immediately perform all of the various integrals appearing in the problem.

For completeness, we begin with the calculation of the discrete modes, which satisfy,

$$\begin{aligned} \psi(y) &= \int_{-1}^1 \mathcal{K}(y, y') \frac{U''(y') \psi(y')}{U(y') - u} dy' \\ &= \sum_{j=1}^n \mathcal{K}(y, y_j) \frac{\Delta_j \psi_j}{U_j - u}, \end{aligned} \quad (A.2)$$

where  $\psi_j = \psi(y_j)$ . By taking  $y = y_j$ , we reduce this equation to an  $n \times n$  matrix problem:

$$\psi = K\psi, \quad (A.3)$$

where  $\psi = (\psi_j)$  and

$$K_{ij} = \mathcal{K}(y_i, y_j) \frac{\Delta_j}{U_j - u}. \quad (\text{A.4})$$

The eigenvalue problem for the discrete modes therefore reduces to finding the characteristic roots,  $u$ , of the equation,

$$\det [I - K(u)] = 0. \quad (\text{A.5})$$

Once we solve this equation for  $c$ , we construct the solution over the whole domain using the relation (A.2).

The continuous spectrum can be found by introducing the singular eigenmodes given by (3.5). Although it is not strictly necessary to regularize the singular term at internal points of the various layers, we shall nevertheless proceed as dictated by the method outlined in the main text. This yields the Fredholm problem (3.8), which now simplifies to the algebraic problem,

$$\psi(y) = \Lambda(y_*) \mathcal{K}(y, y_*) + \sum_{j=1}^n \frac{\mathcal{K}(y, y_j) - \mathcal{K}(y, y_*)}{U(y_j) - U(y_*)} \Delta_j \psi_j, \quad (\text{A.6})$$

where we have used the normalization (4.9) rather than (3.7). If we again set  $y = y_i$ , then the values of the eigenfunction at the interfaces can be found from the matrix relation,

$$(I - F)\psi = \mathbf{k}, \quad (\text{A.7})$$

where

$$F_{ij} = \frac{\mathcal{K}(y_i, y_j) - \mathcal{K}(y_i, y_*)}{U(y_j) - U(y_*)} \Delta_j \quad (\text{A.8})$$

and

$$f_j = \Lambda(y_*) \mathcal{K}(y_j, y_*) \quad (\text{A.9})$$

is the  $j$ -th component of  $\mathbf{f}$ . The homogeneous solutions to the Fredholm problem are the null vectors of the matrix  $I - F$ . Should these exist, the Fredholm alternative is equivalent to the requirement that  $\mathbf{k}$  be orthogonal to all of the null vectors, a situation that is likely

impossible. In that circumstance, we need to scale the inhomogeneous term to be zero, and so  $\Lambda = 0$ .

For values of  $y_*$  that do not give homogeneous solutions, we can invert the matrix  $I - F$ , to give,

$$\psi = (I - F)^{-1}\mathbf{k}. \quad (\text{A.10})$$

The complete solution for the singular eigenmode is then be constructed from (A.6).

## 1. Couette flow

When the flow is purely linear (Couette flow),  $U''$  vanishes everywhere and there are no discrete modes. The kernel of the Fredholm problem also vanishes identically and we only have the particular solution,

$$\psi(y) = \mathcal{K}(y, y_*). \quad (\text{A.11})$$

The associated vorticity distribution is

$$\omega(y) = \delta(y - y_*), \quad (\text{A.12})$$

that is, a line vortex. In other words, in the absence of any ambient vorticity, the singular modes are simply vortex lines or velocity jumps. Such solutions were previously found by Fjørtoft and Høiland (see Eliassen *et al.*, 1953, and Drazin and Howard, 1965).

## 2. Multi-layer profiles

The simplest case for which the equations for the discrete and singular modes are non-trivial is when there is a single vorticity interface. The eigenvalue problem for the discrete modes reduces to

$$u = U_1 - \Delta\mathcal{K}(y_1, y_1). \quad (\text{A.13})$$

That is, there is a single discrete mode which is always neutral.

The singular modes satisfy the equation,

$$\left[1 - \frac{\mathcal{K}(y_1, y_1) - \mathcal{K}(y_1, y_*)}{U_1 - U(y_*)} \Delta\right] \psi_1 = \Lambda \mathcal{K}(y_1, y_*). \quad (\text{A.14})$$

If there is a homogeneous solution, it must occur for a critical layer  $y_* = y_h$ , with

$$1 = \frac{\mathcal{K}(y_1, y_1) - \mathcal{K}(y_1, y_h)}{U_1 - U(y_h)} \Delta. \quad (\text{A.15})$$

This can be shown to be impossible unless  $y_1 = y_h = \pm 1$ , which indicates that there are no nontrivial homogeneous solutions, and so the eigenfunction of the continuum mode can always be determined from (A.14). These modes have two singularities in vorticity; one lies at the critical layer and the other at the velocity interface. When  $U(y_*)$  satisfies (A.13), the singularity arising at the critical layer disappears, and so the continuum modes smoothly limit to the neutral, discrete mode.

If the equilibrium consists of more than two layers, the behaviour of the eigenfunctions is somewhat similar; there are vorticity singularities at the critical layer and the various interfaces. Some solutions for the streamfunction are shown in figure 7 for an anti-symmetrical arrangement of three layers. The streamfunctions of the singular modes have a jump in derivative at the three distinctive locations. When the critical layer approaches one of the interfaces, the streamfunction develops a discontinuity.

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## FIGURE CAPTIONS

FIG. 1. Shapes of singular eigenfunctions for the shear flow given by  $U(y) = y + \alpha y^3$ . The streamfunction of the singular eigensolution is plotted for various critical layers,  $y_*$ , between 0 and 0.9 in steps of 0.1. The critical amplitudes,  $\psi(y_*; y_*) = \psi_*(y_*)$  are indicated by stars. Panel (a) shows eigenfunctions for  $\alpha = 0.1$ , panel (b) for  $\alpha = 1$ .

FIG. 2. Contours of the singular eigensolutions as the density,  $\psi(y; y_*)$ , on the  $(y, y_*)$  plane. The two panels show solutions for  $U(y) = y + \alpha y^3$  and (a)  $\alpha = 0.1$  and (b)  $\alpha = 1$ . The eigenfunctions pictured in figure 1 are given as the horizontal slices at fixed values of  $y_*$ . The diagonal, dashed line shows the location of the critical layer.

FIG. 3. Shapes of singular eigenfunctions for the shear flow given by  $U(y) = \tanh \beta y$ . In panel (a), the streamfunction of the singular eigensolution is plotted for  $\beta = 1.462$  and for various critical layers,  $y_*$ , between 0. and 0.9 in steps of 0.1. The critical amplitudes,  $\psi(y_*; y_*) = \psi_*(y_*)$  are indicated by stars. In panel (b), eigenfunctions are displayed for  $y_* = 0$  and various values of  $\beta$  surrounding  $\beta = \beta_c \simeq 1.462$  (the values of  $\beta$  increase from 0.3 and decrease from 2.6 in steps of 0.2).

FIG. 4. The Fredholm determinant,  $\mathcal{D}(y_*)$ , for various flow profiles. In panel (a) we display  $\mathcal{D}(y_*)$  for various cubical profiles given by equation (4.12) and  $\alpha$  from 0.1 to 1.9 in steps of 0.2. In (b) we show the determinant for the hyperbolic tangent profile (4.13), with  $\beta$  running from 0.3 to 2.3 in steps of 0.2.

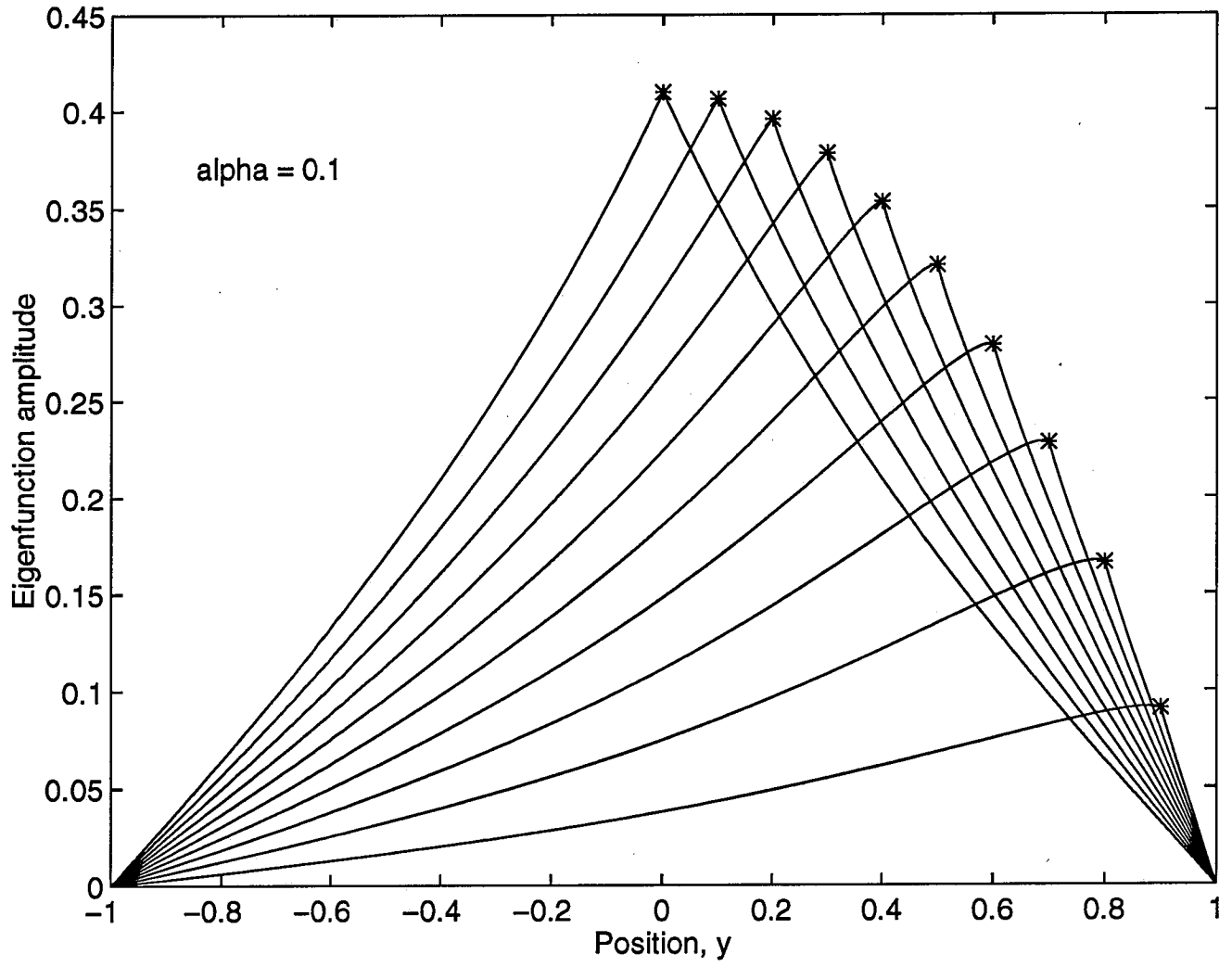
FIG. 5. Illustration of the contour  $C$  used in the evaluation of the integral (8.17) over asymptotically long times. In panel (a) we show the contours  $C^\pm$ , described in the text. In panel (b) we show both  $C^+$ , and its deformation,  $C'$ , which encircles the branch cut along the piece of the real axis corresponding to the channel, and various poles in the

integrand arising from zeros of the analytic continuation of  $\epsilon$  in the lower half of the strip  $S$  (three poles are shown for illustration).

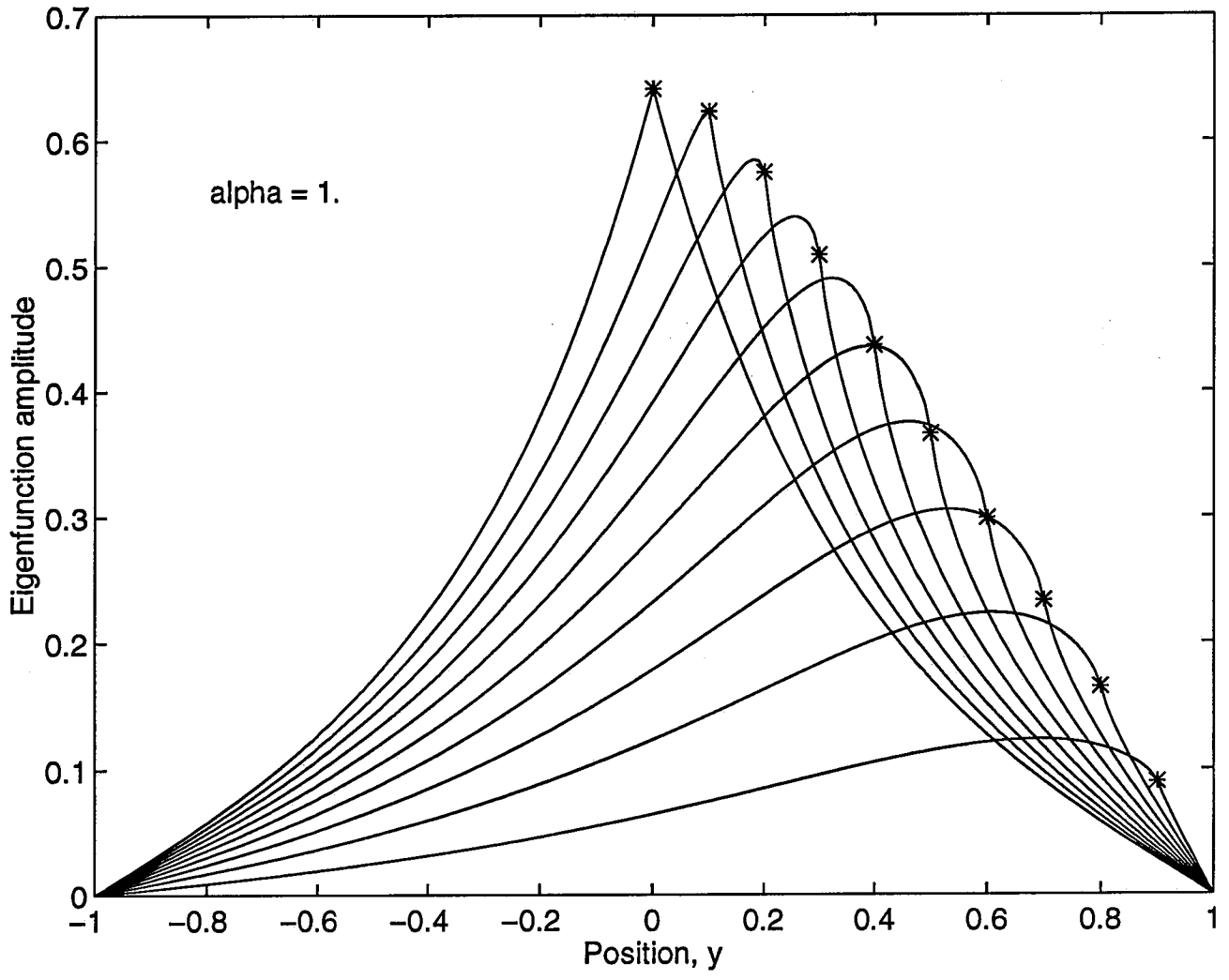
FIG. 6. Illustration of the evolving streamfunction and vorticity in Couette flow. The four pairs of panels show the perturbation in the streamfunction and vorticity at  $t = 0, 3, 4.5$  and  $20$ . Contours of constant  $\psi$  and  $\omega$ , as given by equations (8.31) and (8.32), are shown for sections of the channel. The contouring of vorticity is the same in each of the respective panels; that for the streamfunction is not, since the maximum amplitude is decreasing as  $1/(1 + t^2)$ .

FIG. 7. Shapes of singular eigenfunctions for shear flow with piece-wise constant vorticity. In panel (a), the streamfunctions of the singular eigensolutions are plotted for  $U_1 = 0.4 = 1 - U_2$ ; in panel (b) for  $U_1 = 0.1 = 1 - U_2$ . Also,  $y_2 = -y_1 = 0.25$ . Solutions with various critical layers,  $y_*$ , between 0. and 0.9 in steps of 0.1 are shown. The critical amplitudes,  $\psi(y_*; y_*) = \psi_*(y_*)$  are indicated by stars. The dashed lines illustrate the velocity profile in each case.

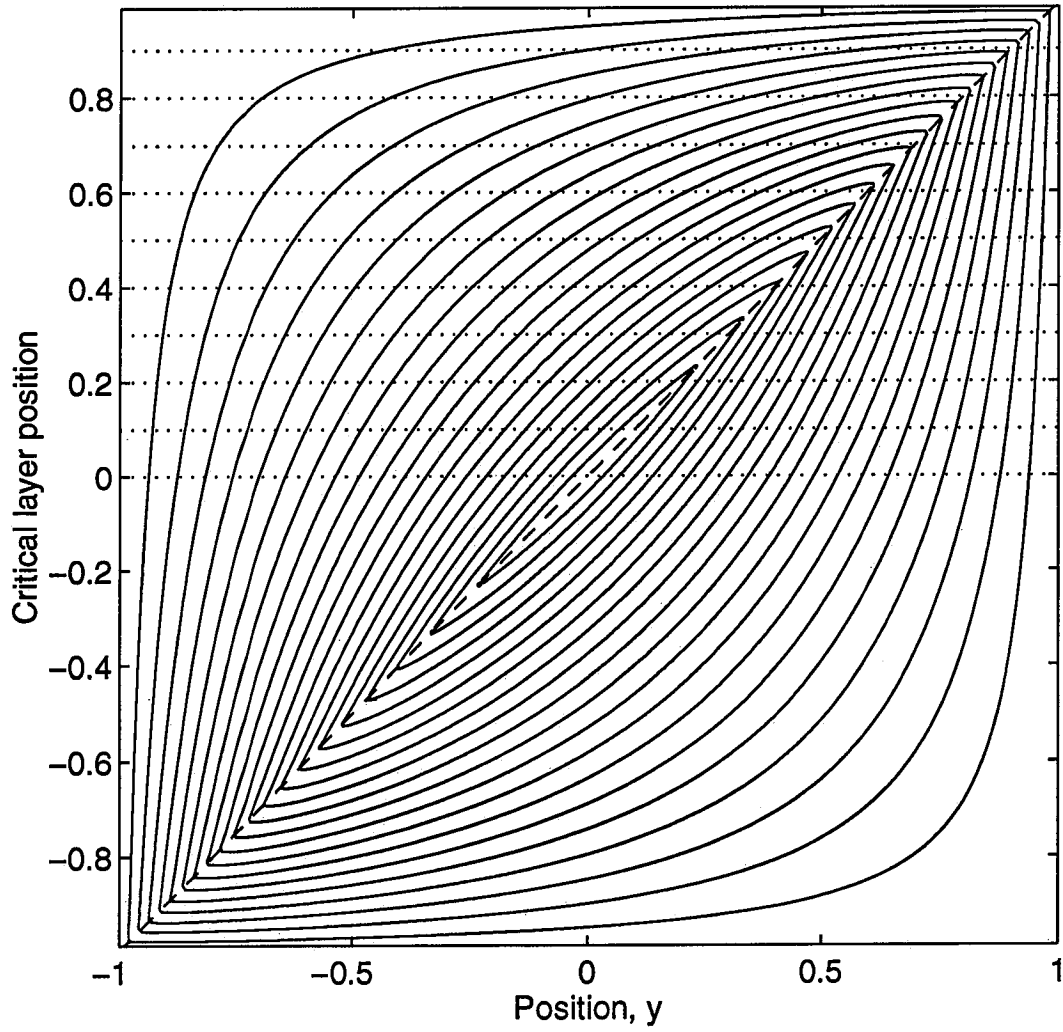
(a) Shapes of singular eigenfunctions



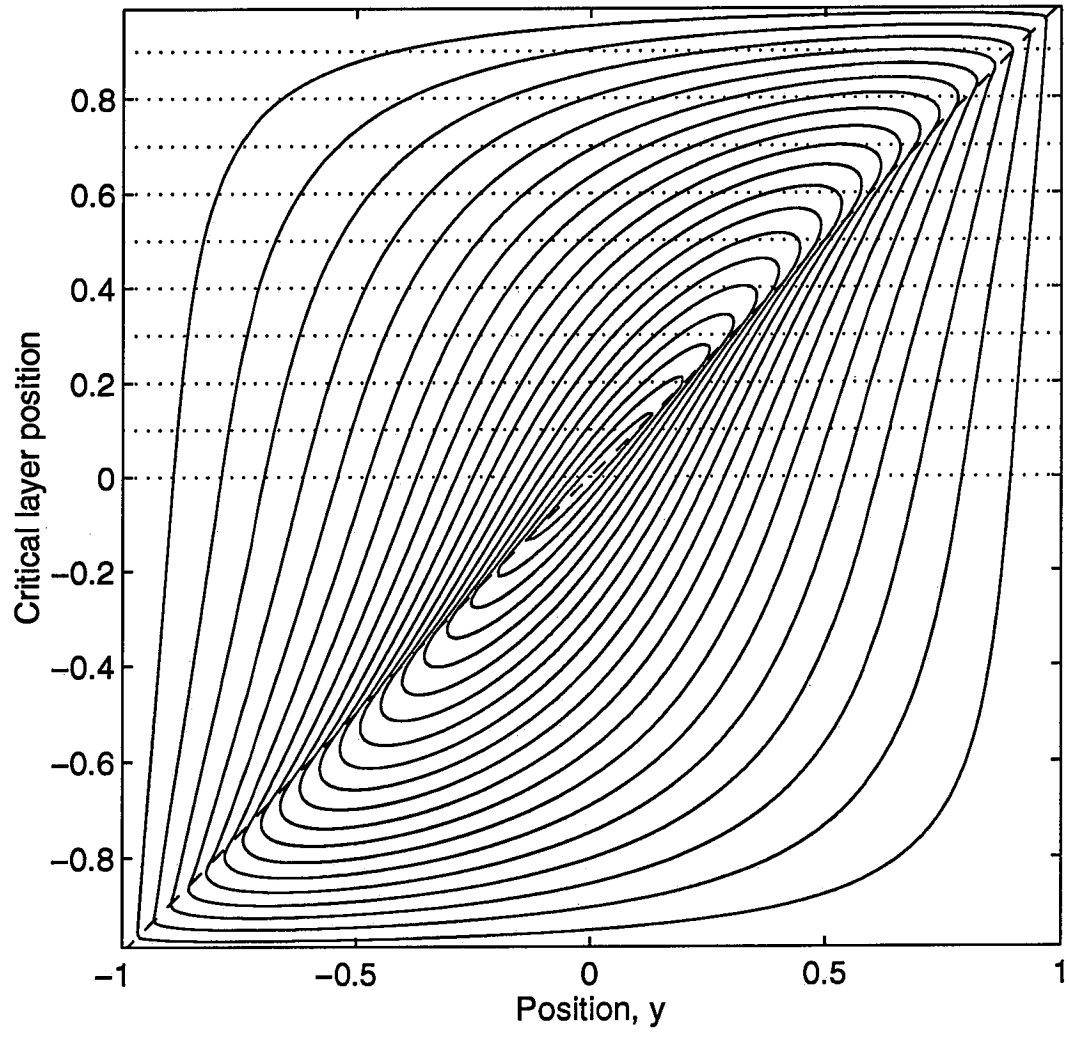
(b) Shapes of singular eigenfunctions



(a) Contouring of singular eigenfunctions

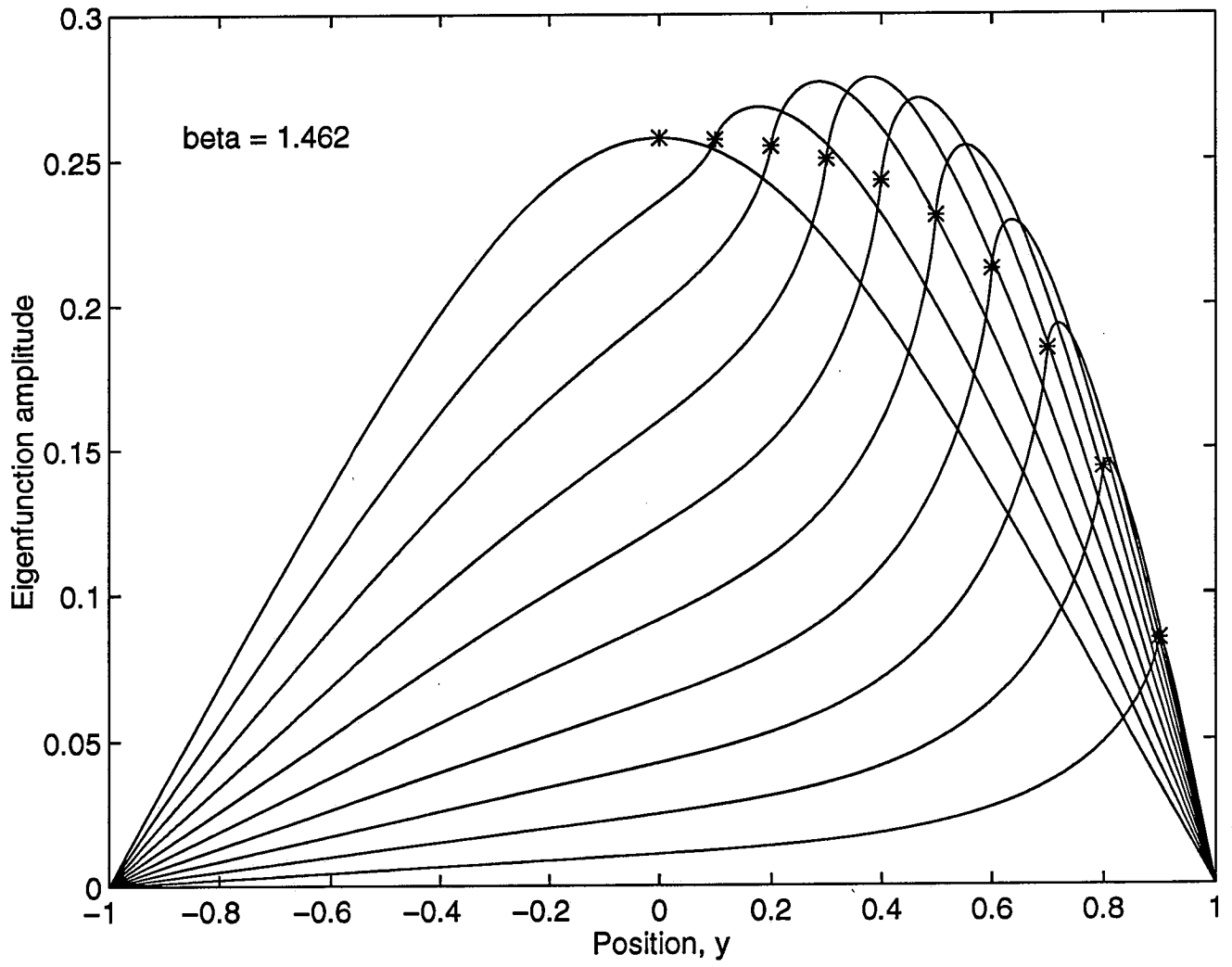


(b) Contouring of singular eigenfunctions

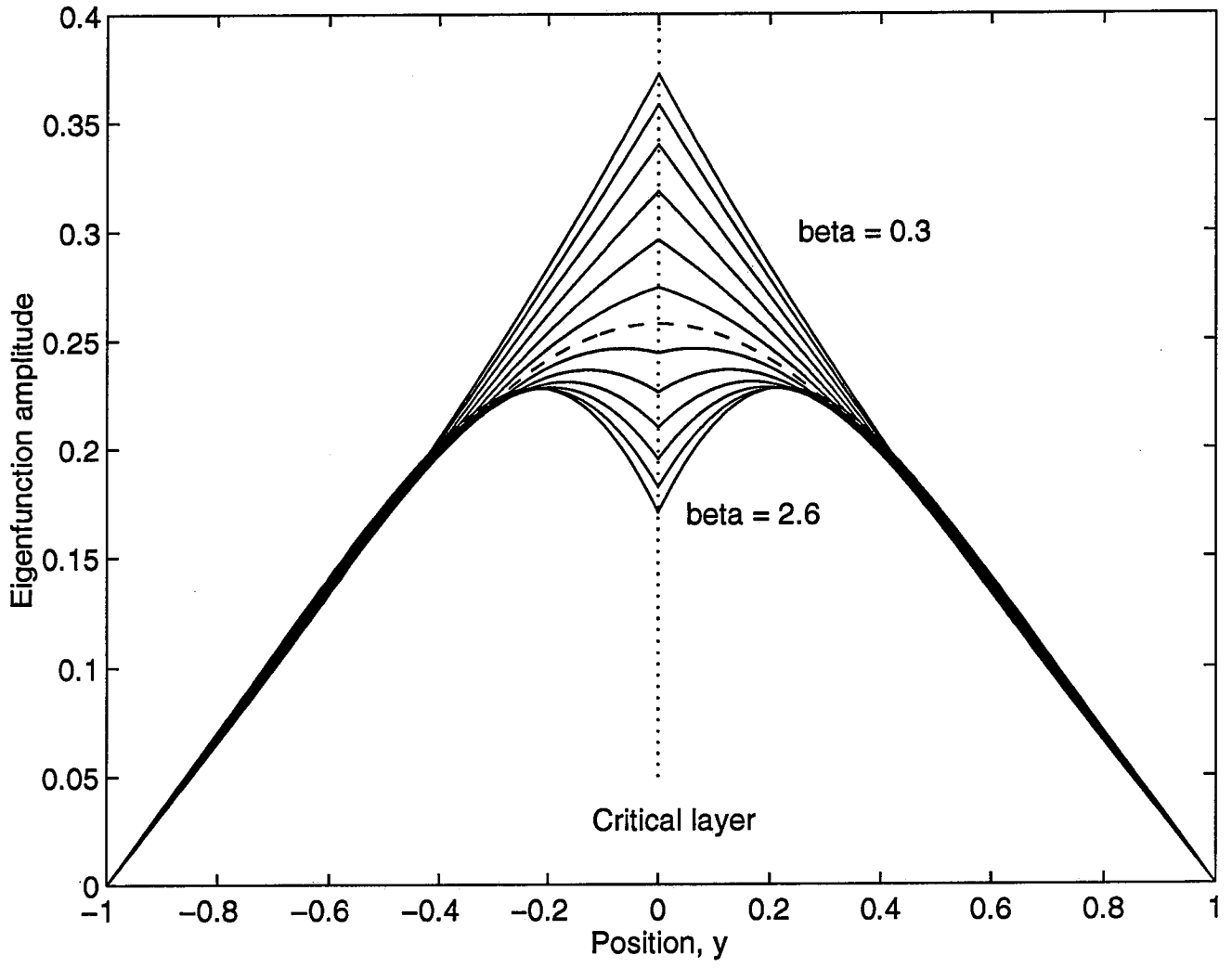




(a) Hyperbolic tangent profile

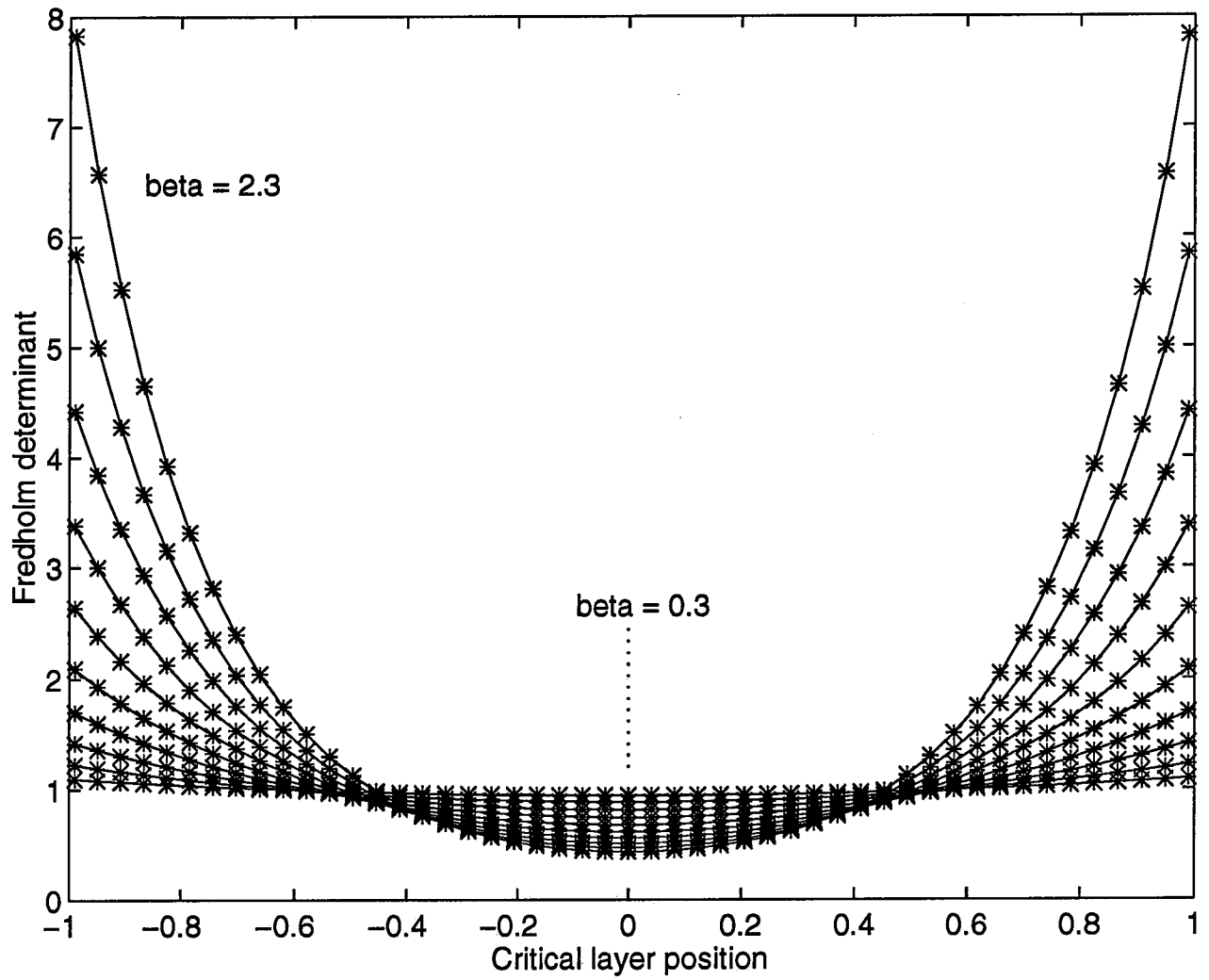


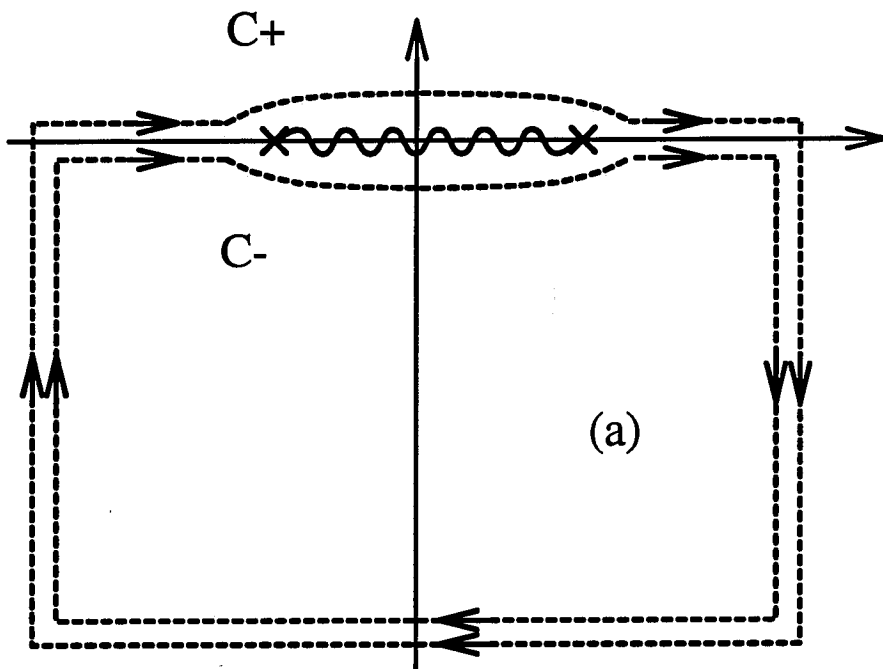
(b) Hyperbolic tangent profile

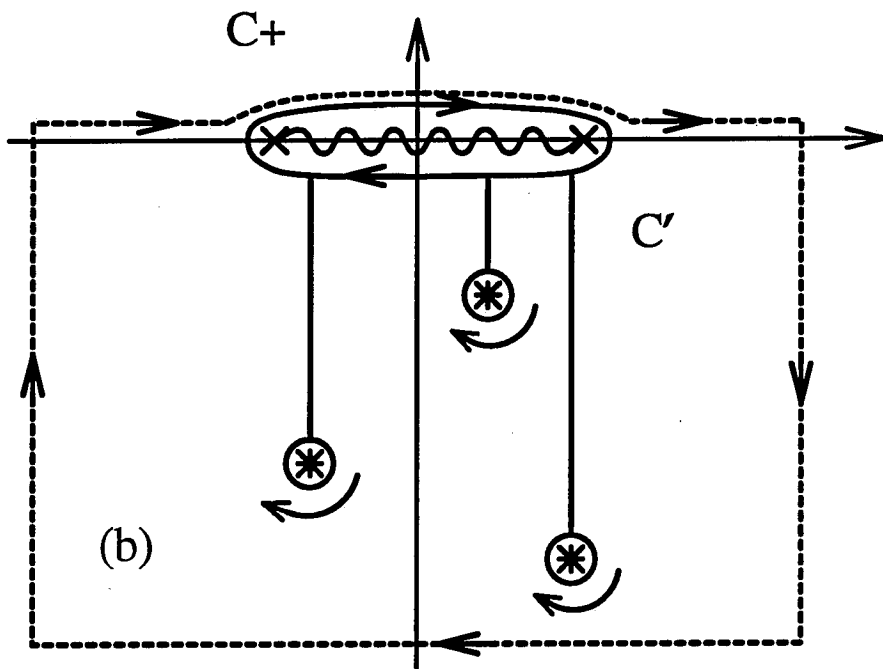




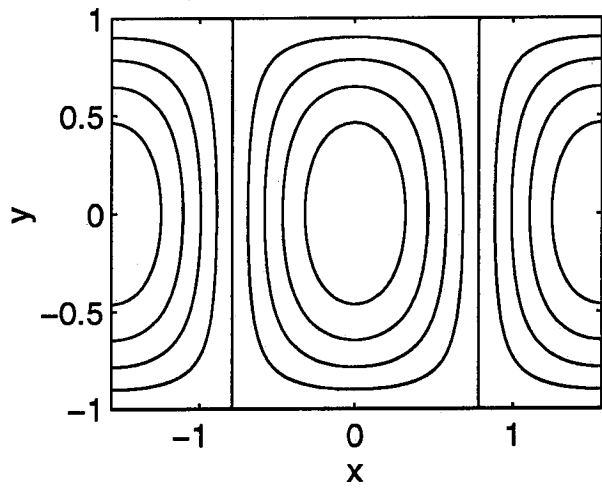
(b) The Fredholm determinant for hyperbolic tangent profiles



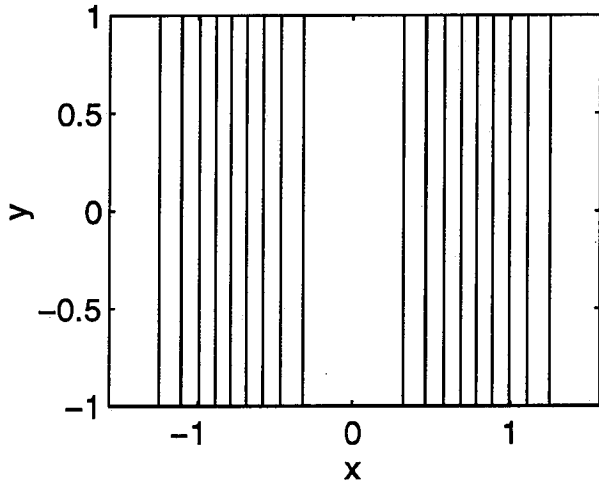




(a) Streamfunction at  $t=0$

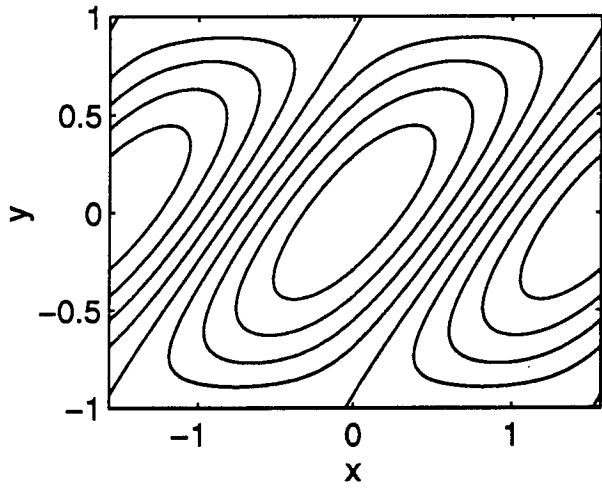


(b) Vorticity at  $t=0$

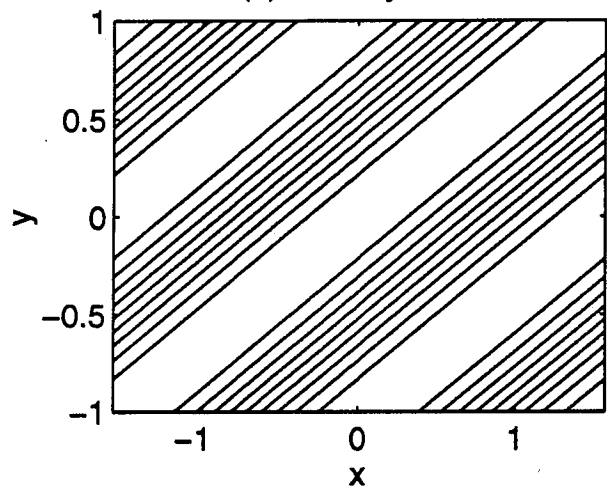




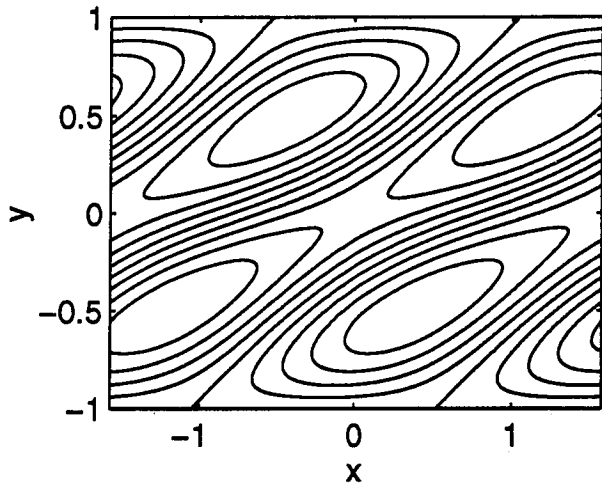
(c) Streamfunction at t=3



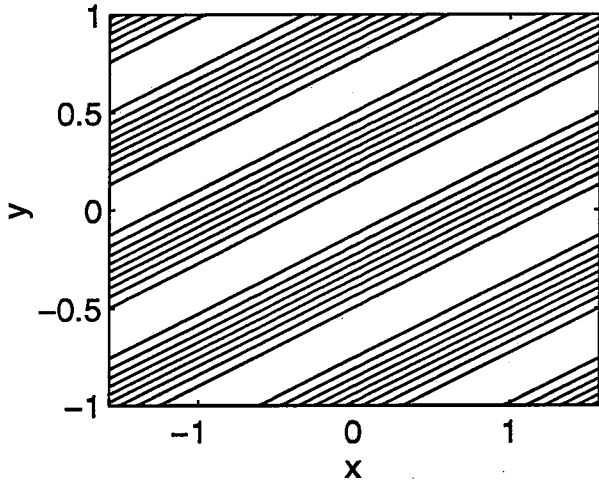
(d) Vorticity at  $t=3$



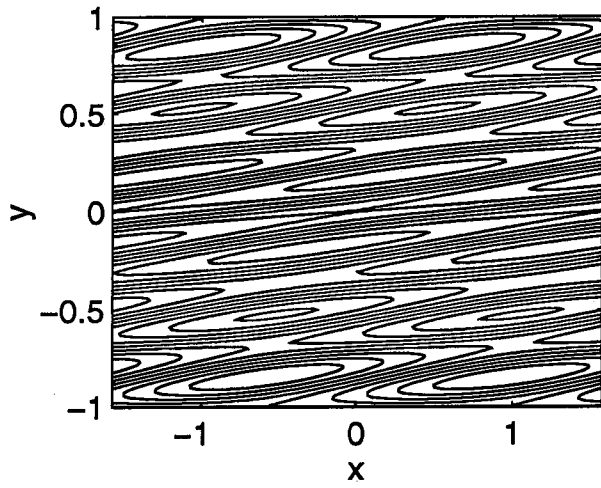
(e) Streamfunction at t=5



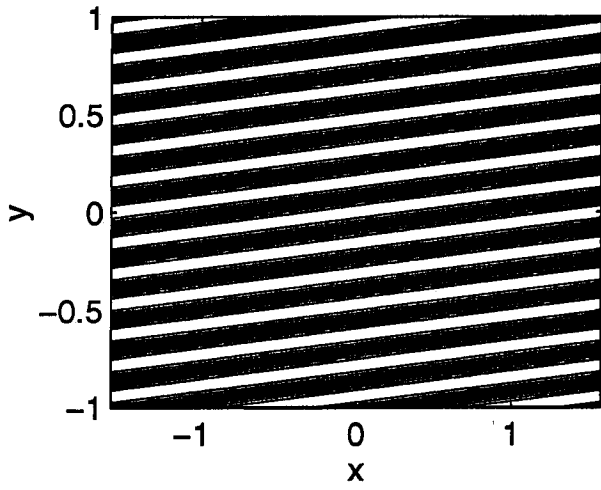
(f) Vorticity at  $t=5$



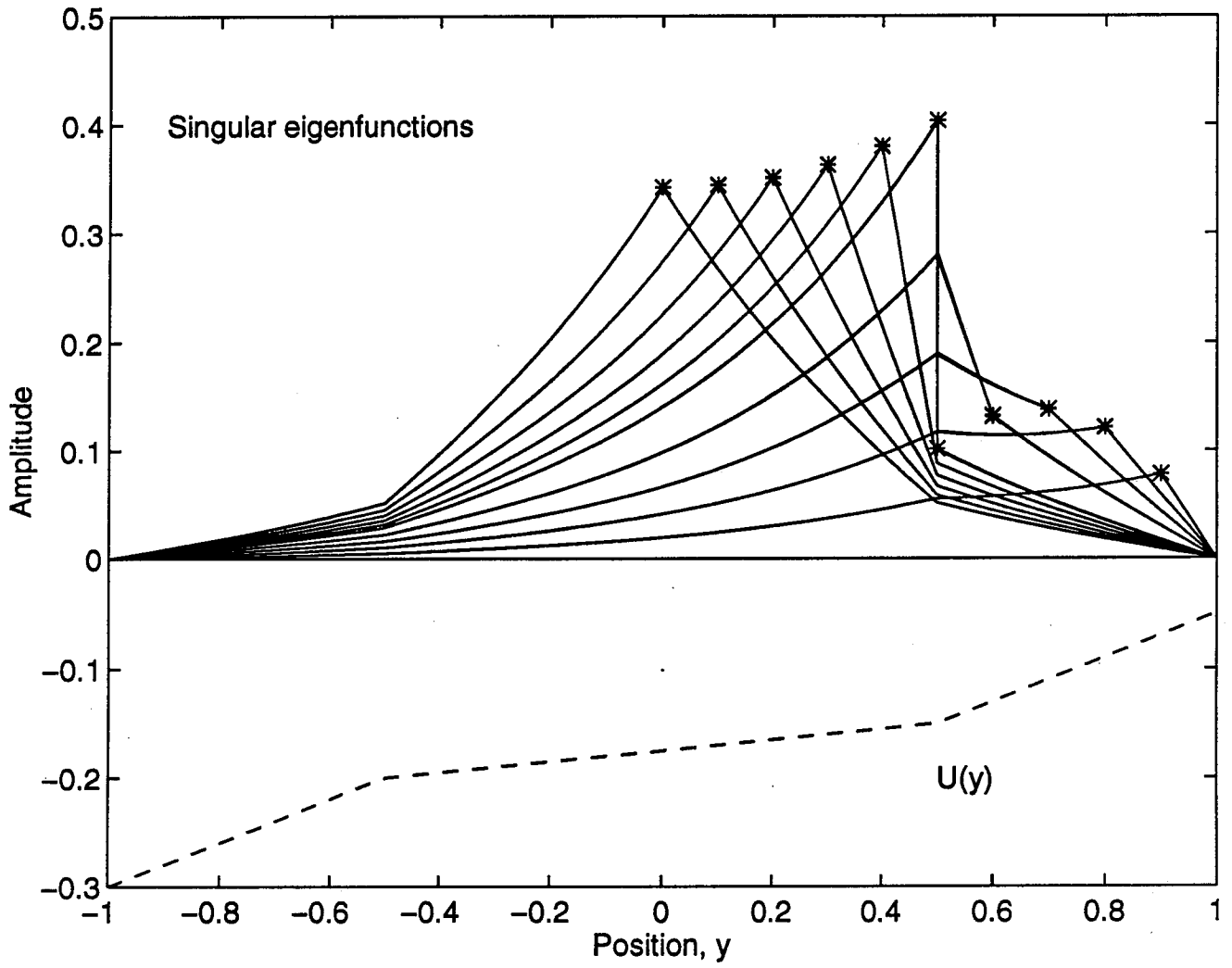
(g) Streamfunction at  $t=20$



(h) Vorticity at t=20



(a) Three-layer profiles



(b) Three-layer profiles

