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Lie Group Analysis of Plasma-Fluid Equations
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To my wonderful daughter
Ary Jael Acevedo-Fillat
who has been my source of inspiration and
strength during good and bad times.

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Lie group methods for nonlinear partial differential equations are implemented to study, analytically, a subset of the full solution space of a family of plasma-fluid models. The solutions obtained by this method are known as group invariant solutions. The basic set of equations considered comprise the three-field fluid model due to Hazeltine (HTFM), which was obtained to describe nonlinear large aspect ratio tokamak physics. This model contains as particular limits the physics of the Charney-Hasegawa-Mima equation (CHM) and reduced magnetohydrodynamics (RMHD), which are two other models known to describe some features of nonlinear behavior of tokamak plasmas.

Lie's method requires a large number of systematic calculations to determine the Lie point symmetries of the system of differential equations.

These symmetries form a Lie group and describe the geometrical invariance of the equations. The Lie symmetries have been calculated for the systems mentioned above by using a symbolic manipulation program. A detailed analysis of the physical meaning of these symmetries is given. Using the Lie algebraic properties of the generators of the symmetries, a reduction of the number of independent variables for the full nonlinear systems of equations is calculated, which in turn yields simplified equations that sometimes can be solved analytically. A discussion of some of the reductions and solutions generated by this technique is presented. The results show the feasibility of using Lie methods to obtain analytical results for complicated nonlinear systems of partial differential equations that describe physically interesting situations.

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Chapter 1

Introduction

Over the last thirty years, the scientific community in almost any major field of study has come to accept and confront one of the crucial issues inherent of real phenomena in the physical universe; namely its pervasive nonlinear behavior. The need for a better understanding of this complicated behavior in nature has constituted a major driving force behind new and powerful methods developed explicitly to extract information characteristic of the nonlinear regime. In particular, in many branches of the physical sciences, from basic to applied physics, nonlinear phenomena are described by partial differential equations (PDE's), generating naturally an increased interest in their systematic solution.

Many impressive advances have been accomplished in dealing with nonlinear PDE's by recent developments in two specific areas that can be consider the opposite ends of the whole spectrum of possible cases: a) The area of completely integrable equations, which yields closed analytical solutions and is associated with methods such as the inverse spectral transform, Hamiltonian structures, recursion operators, Lax pairs, etc., and b) The area of chaotic dynamical systems, which is of a more qualitative basis, stresses the nonintegrable nature of the system, deals with models of observed physical phenomena, and is associated with notions such as Poincaré sections, transition to chaos, turbulence, intermittency, etc.

However, in between these two extreme cases there are nonlinear systems that do not belong to either one, but with the use of specific mathematical methods, one can get exact particular solutions, or conservation laws, that provide physically meaningful information. These are called *partially integrable* systems. The mathematical methods used to study these systems represent a subset of the more general methods dealing with integrability.

Among the methods developed to study the solution space of a PDE regardless of whether it is integrable or not, Lie group analysis constitutes one of the most powerful analytical techniques for obtaining particular solutions of PDE's, specially in the nonlinear case. The main idea of this method, developed by S. Lie in the late part of last century, consists of an integration procedure based on the invariance of the differential equation under a continuous group of symmetries. This observation inspired Lie to further develop his theory of continuous groups, now known as Lie groups. However, although the application of Lie groups to the study of differential equations has a long history, as long as the history of Lie groups themselves, the applications to physical systems remained mostly dormant. Early research by G. Birkhoff and later Ovsiannikov pertaining to physically interesting applications revealed the difficulty in implementing this method. However, more recently, the development of symbolic manipulation programs has contributed to an explosion of new activity in the field, allowing applications of Lie group methods to growingly complicated systems of PDE's.

In the description for the dynamics of a plasma we find systems with the features mentioned above. First of all, the evolution of a real (lab or space) plasma is necessarily nonlinear, which has been long recognized in the plasma community, and sometimes investigation has been reduced to numerical simulations because of the complicated nature of the problem. Second, an approximated description can be derived such that it contains some of the most important effects of nonlinear plasma evolution. Such models have proved to be predictive and quantitatively accurate in some instances, demonstrating that they describe a good amount of physics. These are the so-called plasma-fluid models, and are given generally as sets of nonlinear PDE's. Also their simple fluid limit generally concides with interesting hydrodynamical models. But aside from some early efforts by Rosenau and Schwarzmeier with MHD and the most recent activity of the group from Braunschweig (Richter, Fuchs and Galas), the use of Lie group methods for plasma-fluid models has not been thoroughly explored.

The use of Lie group theory makes it possible to elucidate features of the structure of differential equations and their solution sets, and to construct some exact particular solutions. It is the intention of the present work to explore these applications for a significant fluid model of plasma evolution.

In chapter 2 the derivation that proceeds from a kinetic integro-differential equation that describes a plasma to a closed system of fluid PDE's is explained in some detail, emphasizing physical assumptions and regions of validity for the ensuing truncation that is used to close the system of equations. Next, a reduction scheme based on the smallness of the inverse aspect ratio is adopted to model some interesting features of tokamak dynamics. We end the chapter with a plausible derivation of Hazeltine's three-field model (HTFM) as a further approximation of a reduced fluid model. This model contains in appropiate limits the physics of two well-known but less complete plasma-fluid models the Charney-Hasegawa-Mima equation (CHM) and reduced magnetohydrodynamics (RMHD). HTFM constitutes the basic fluid model of our Lie group implementation.

In chapter 3 a concise introduction to the basic ideas of Lie group methods for differential equations is presented. The emphasis is on establishing the basic results of the theory that will allow immediate computation of symmetries and interpretation of results.

In chapter 4 we implement the calculation of symmetries for the simplest limit of HTFM, CHM. Using this model as a working example we calculate the symmetries of the system, give a physical interpretation of the symmetries, and use the symmetries to generate exact solutions for the CHM equation. In order to do so we develop the concept of the adjoint representation of an algebra and calculate the optimal system of first and second orders.

In chapter 5 we consider the full three-dimensional HTFM and calculate the symmetry algebra. Based on earlier results, we calculate some interesting reductions of the equations and some exact solutions. Then we consider the two-dimensional limit and the RMHD limit of HTFM, obtaining some new reductions and solutions.

We finish with conclusions and indicate directions of future work in chapter 6.

Chapter 2

Nonlinear Plasma-Fluid Models

2.1 Introduction

The description of magnetized plasmas presents a highly challenging problem from the theoretical point of view. It encompasses collective phenomena from the many-body problem, fluid-like behavior from classical field theory, and charged particle motion from single particle dynamics (among other detailed features), under a common nonlinear environment. The resulting perspective for the theoretical physicist trying to derive a model of such a complicated system is, at best, to come up with an approximate mathematical description for a particular regime, emphasizing specific scales in space and time, that will allow some physical interpretation and basic understanding of the real phenomena. In this regard, there is no 'perfect' model that will describe the global behavior of a plasma, however within the limitations of a systematic approximation and explicit understanding of the physical picture it represents, there is a wide variety of simplified models that can be derived to study particular phenomena in plasmas with different ranges of applicability.

In this chapter, I embark on the task of deriving, from first principles, some fluid models for magnetized plasma. They will provide an interesting arena to apply the mathematical techniques known as Lie group analysis, which will be introduced in the next chapter. Here, I will emphasize the physical assumptions and approximations involved in the derivation of the fluid models.

Starting from a kinetic equation for the distribution function $f(\mathbf{x}, \mathbf{v}, t)$, I obtain moment equations for the first few velocity moments of f (the fluid variables), and through a systematic ordering and fluid truncation approach, I arrive at a closed set of PDE's containing the basic features of finite Larmor radius (FLR) physics for a magnetized plasma. By studying some of this effects separately, as closed subsystems, one can define a family of fluid models describing interesting nonlinear plasma behavior that have resisted nonlinear analytical treatment.

In the remainder of the present chapter, I will review the principal features of this family of nonlinear, plasma-fluid models, which share in common their MHD like structure. This type of models have proven to be of immense value in broadening our understanding of a wide range of nonlinear plasma phenomena.

2.2 From Kinetic to Fluid Equations

2.2.1 Kinetic Description of a Plasma

A fundamental description for a large collection of interacting charged particles starts with a kinetic equation for the *distribution function* $f(\mathbf{x}, \mathbf{v}, t)$, defined such that $f(\mathbf{x}, \mathbf{v}, t)d\mathbf{x}d\mathbf{v}$ is the number of particles in phase space volume $d\mathbf{x}d\mathbf{v}$ around position $\mathbf{x}$, with velocity $\mathbf{v}$, at time t. The distribution function f contains the detailed information about the physical system, and it is the purpose of kinetic theory to study the properties and solutions of the kinetic equation for f. The generic form of such an equation, averaged over an ensemble of macroscopically equivalent plasma systems, is

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \mathbf{a} \cdot \frac{\partial f}{\partial \mathbf{v}} = C(f)\,, \tag{2.1}$$

where C denotes the collision operator that accounts for effects of particle correlations, and $\mathbf{a}$ and f denote, respectively, the force per unit mass and the distribution function averaged over an ensemble. I will consider eq.(2.1) as the basic kinetic equation that contains the relevant physics governing the dynamical behavior of the plasma.

While kinetic theory is a basic tool for studying central issues of theoretical plasma physics, the fact that it is set in six dimensional phase space makes it more difficult to handle, compared to the lower dimensional configuration space, where fluid variables exist. What I would like to do instead, is to adopt a practical closure strategy based on some physical properties of the plasma, and use moments of the kinetic equation as the fundamental entities to be analyzed. This procedure allows easier access to physical insight, and sidesteps the kinetic equation , reducing the burden. The closure strategy provides explicit limits of validity for the approximations involved and allows for higher order corrections as needed.

A basic concept related to motion of charged particles in magnetic fields is their gyration around the field lines. The thermal gyroradius is defined as $\rho_s := \frac{v_{ts}}{\Omega_s}$, where s refers to the plasma species, $v_t := \left(\frac{2T}{m}\right)^{1/2}$ is the thermal speed, and $\Omega_s \equiv \frac{e_s B}{m_s c}$ is the gyrofrequency. This simple concept, the magnetically induced gyration, leads naturally to a fundamental small parameter for magnetized plasma that can be used as a basis of an approximation scheme for dealing with the kinetic equation. By a magnetized plasma I mean one of a size much larger than the gyroradii of its constituent charged particles. Thus if L is the scale length characterizing the plasma, and ρ the thermal gyroradius, then the plasma is magnetized if the parameter

$$\delta \equiv \frac{\rho}{L} \tag{2.2}$$

is much less than one. More specifically, if one assumes that all species have equal temperatures, then the gyroradii of all ion species may be pressumed comparable while the electron gyroradius is smaller according to

$$\rho_e \sim \left(\frac{m_e}{m_i}\right)^{1/2} \rho_i \,. \tag{2.3}$$

Evidently the same kind of relation holds for the parameter δ; therefore I will call a plasma magnetized only if its ions are magnetized,

$$\delta_i \sim \left(\frac{m_i}{m_e}\right)^{1/2} \delta_e \ll 1 \,. \tag{2.4}$$

Under this condition on the smallness of the gyroradius, one can estimate the relevance of each term in the kinetic equation. For the sake of generality I will consider an arbitrary collision operator and three dimensional geometry. However, later on I will revert to the collisionless case, where one can exploit the Hamiltonian nature of the system, and also restrict to two dimensions (2-D), as simplifications in the context of symmetry analysis.

The first term of the kinetic equation (2.1) will be assumed small in the sense

$$\frac{\partial f}{\partial t} \sim \delta \Omega f \,. \tag{2.5}$$

This ordering describes most plasma motions of interest. It is chosen so that typical long-wavelength instabilities can be treated.

The convective term, $\mathbf{v} \cdot \nabla f$, involves different scales, since plasma motions can happen on very different lengthscales. Therefore two scale lengths are distinguished: a slow scale length L, typically measuring a density gradient scale-length, and a fast scale-length λ, referring to the thickness of some boundary layer or the wave length of a linear disturbance. I will assume in the remaining present analysis that the plasma equilibrium varies exclusively on the slow scale L. Thus decomposing the distribution function into terms varying slowly, f_{slow}, and rapidly, f_{fast} one has

$$f = f_{slow} + \Delta f_{fast} \tag{2.6}$$

where Δ is a parameter that measures the amplitude of the rapidly varying perturbation. For a magnetized plasma the condition $\Delta \ll 1$ must be satisfied. Thus

$$\mathbf{v} \cdot \nabla f \sim \delta \Omega f_{slow} + \Delta f_{fast} \tag{2.7}$$

and the two terms in (2.7) are small of order δ and Δ respectively.

The acceleration, third term on the left hand side of eq.(2.1), is given by the Lorentz force

$$\mathbf{a}(\mathbf{x}, \mathbf{v}, t) = \frac{e}{m} \left[\mathbf{E}(\mathbf{x}, t) + \frac{1}{c} \mathbf{v} \times \mathbf{B}(\mathbf{x}, t) \right] , \tag{2.8}$$

where e and m are the particle charge and mass, respectively. The acceleration due to $\mathbf{E}$ is decomposed into its components parallel and perpendicular to $\mathbf{B}$

$$\mathbf{E} = \mathbf{E}_{\parallel} + \mathbf{E}_{\perp} . \tag{2.9}$$

The contribution from $\mathbf{E}_{\parallel}$ is estimated by

$$\frac{e}{m} \mathbf{E}_{\parallel} \cdot \frac{\partial f}{\partial \mathbf{v}} \sim \frac{e}{m} \frac{E_{\parallel}}{v_t} f := \nu_E f , \tag{2.10}$$

where the natural ordering $\frac{v_E}{\Omega} \sim \delta$ is mandatory to treat situations near equilibrium.

The perpendicular components, are expressed in terms of the $\mathbf{E} \times \mathbf{B}$ drift

$$\mathbf{V}_E = c\frac{\mathbf{E} \times \mathbf{B}}{B^2}\,, \tag{2.11}$$

yielding the estimate

$$\frac{e}{m}\mathbf{E}_\perp \cdot \frac{\partial f}{\partial \mathbf{v}} = \frac{v_E}{v_t}\Omega f\,, \tag{2.12}$$

where clearly two cases of interest can be distinguished: the MHD ordering,

$$\frac{V_E}{v_t} \sim 1 \tag{2.13}$$

in which case electric drifts dominate the dynamics and the most violent perturbations are described (fast phenomena); and the drift ordering,

$$\frac{V_E}{v_t} \sim \delta \tag{2.14}$$

in which case the electric drift enters the picture only in conjunction with other slow motions, such as gradient-B drifts or curvature drifts. Although MHD-ordered fluid theory is well recognized for its usefulness and has led to enormous advances in our understanding of plasma phenomena, I will be interested in deriving drift-ordered fluid models, motivated in part by modern confinement physics, where most of the prevalent perturbations are consistent with the drift ordering.

The largest term in eq.(2.1) is that corresponding to Larmor gyration

$$\frac{e}{mc}\mathbf{v} \times \mathbf{B} \cdot \frac{\partial f}{\partial \mathbf{v}} \sim \Omega f\,; \tag{2.15}$$

this is a zeroth order term with respect to δ.

The collision operator C, is ordered such that the collision frequency ν is small compared to the gyrofrequency Ω:

$$C \sim \nu \sim \delta\Omega\,. \tag{2.16}$$

This analysis allows for different approximations to the general kinetic equation (2.1) depending on the particular ordering chosen for the parameters defined above. Specifically, as I said before, my main interest is to study an ordering that forbids rapid $\mathbf{E} \times \mathbf{B}$ motion and fast variation of all perturbations, i.e. $V_E \sim \delta v_t$ and $\Delta = 0$. This defines the *drift ordering*, where all terms in the kinetic equation are of order δ compared to gyration. The corresponding approximation of the general kinetic equation (2.1) is the so-called *drift-kinetic equation* (DKE). It describes a wide variety of plasma instabilities, transport processes, and confined plasma equilibria. In particular, it has been studied in the pariaxial limit, where Newcomb has implemented a scheme that involves moment equations coupled with the DKE in the collisionless regime [Newcomb 85]. What I will do instead is to study moments of the general kinetic equation, then introduce the drift ordering at the moment equation level, and finally make some assumptions that will allow a fluid closure. What I will obtain is a typical FLR model for plasma dynamics which consists of a fluid model of drift-ordered motion whose main effects will include slow evolution of instabilities, diamagnetic drifts and gyroviscosity.

2.2.2 Fluid Closure

As was mentioned before, in a fluid description of a plasma one works with a small set of velocity moments of the distribution function. These are the so-called fluid variables which represent the quantities commonly measured in experiments. The first few, most relevant, velocity moments of f are defined as follows. The density of species α is

$$n_\alpha \equiv \int d^3v f_\alpha\,, \tag{2.17}$$

and the flow velocity, $\mathbf{V}_\alpha$, is given by

$$n_\alpha \mathbf{V}_\alpha \equiv \int d^3v f_\alpha \mathbf{v}\,. \tag{2.18}$$

For higher order velocity moments I use well-known conventions (see e.g. [Braginskii 65]), which give for the stress tensor

$$\boldsymbol{\mathcal{P}}_\alpha := \int d^3v f_\alpha m_\alpha \mathbf{v}\mathbf{v} \tag{2.19}$$

and for the closely related pressure tensor

$$\mathbf{p}_\alpha \equiv \int d^3v f_\alpha m_\alpha (\mathbf{v} - \mathbf{V}_\alpha)(\mathbf{v} - \mathbf{V}_\alpha)\,. \tag{2.20}$$

The trace of the latter defines the scalar pressure p_α as follows

$$p_\alpha \equiv \frac{1}{3} Tr(\mathbf{p}_\alpha)\,, \tag{2.21}$$

and the temperature of species α is defined as

$$T_\alpha \equiv \frac{p_\alpha}{n_\alpha}\,. \tag{2.22}$$

In the same context, one can write the definition for the energy flux

$$\mathbf{Q}_\alpha \equiv \int d^3v f_\alpha \frac{1}{2} m_\alpha v^2 \mathbf{v} \tag{2.23}$$

and the energy-weighted stress,

$$\Re_\alpha \equiv \int d^3v f_\alpha \frac{1}{2} m v^2 \mathbf{v}\mathbf{v}\,. \tag{2.24}$$

These are the basic fluid variables needed to write down the first three moment equations of the general kinetic equation, (2.1). They are obtained by multiplying (2.1) by appropriate powers of $\mathbf{v}$ and integrating over velocity space. Note that this procedure does not introduce any approximation, and therefore the moment equations are exact relations and do not depend on particular properties of the collision operator.

Following the approach depicted above I start with the most basic moment equation, the conservation of the number of particles, obtained simply by integrating eq.(2.1) over velocity space, yielding

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{V}) = 0\,. \tag{2.25}$$

The next moment equation is obtained by multiplying eq.(2.1) by $m\mathbf{v}$ and integrating over velocity. The resulting equation describes the momentum evolution, and it takes the form

$$\frac{\partial}{\partial t} mn\mathbf{V} + \nabla \cdot \boldsymbol{\mathcal{P}} - en\left(\mathbf{E} + \frac{1}{c}\mathbf{V} \times \mathbf{B}\right) = \int d^3v\, m\mathbf{v}C\,. \tag{2.26}$$

This equation of motion can be cast in a convenient form in terms of the pressure tensor $\mathbf{p} = \boldsymbol{\mathcal{P}} - mn\mathbf{V}\mathbf{V}$

$$mn\left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V}\right) + \nabla \cdot \mathbf{p} - en\left(\mathbf{E} + \frac{1}{c}\mathbf{V} \times \mathbf{B}\right) = \int d^3v\, m\mathbf{v}C\,, \tag{2.27}$$

where I have used (2.25) to express $\partial n/\partial t$ in terms of $\mathbf{V}$.

For the moment equation describing the evolution of pressure, or total fluid energy, one has to multiply the general kinetic equation by $mv^2/2$ and integrate over velocity, yielding after some manipulation the following form

$$\frac{3}{2}\frac{\partial}{\partial t}\left[p+\frac{1}{2}mnV^2\right]+\nabla\cdot\mathbf{Q}=en\mathbf{V}\cdot\mathbf{E}+W+\mathbf{V}\cdot\mathbf{F}\,, \tag{2.28}$$

where the first term on the right hand side represents electromagnetic work and the next two terms account for collisional exchange. Since I will be dealing with fluid models in the collisionless limit these terms will be dropped. Of course this is a specifically contracted moment equation, I could have multiplied by $\mathbf{vv}$ rather than v^2 obtaining an equation for the evolution of $\boldsymbol{\mathcal{P}}$, but for the present purpose eq.(2.28) will be enough.

At this point I would like to emphasize that one could keep calculating higher order moments and in principle recover all the information contained in the original kinetic equation. However, from a practical point of view, by displaying a finite number of these moment equations one can recognize the basic drawback of the procedure: Each moment equation, given in conservation equation form, relates a fluid density with its corresponding flux, which in turn is determined by the next moment equation in terms of a higher order flux, etc., generating an infinite set of coupled equations that cannot be rigorously closed with any finite number of moments. This defines the fundamental problem of closure for a fluid description, and there are essentially two ways to deal with it: a) Asymptotic Closure, and b) Truncation.

The first approach, which is based upon solving the kinetic equation perturbatively with respect to a physically motivated small parameter ϵ, and using the corresponding distribution function (accurate to some order in the perturbation parameter ϵ) to evaluate certain moments, is rigorous and has been used with some success in the past (see [Newcomb 85] for a particular quasi-three dimensional application of this method). However, this method requires solving a non-trivial kinetic equation and after that using this information to supplement the fluid equations that have to be simultaneously solved. This could be a very complicated problem. Alternatively, one has the second approach, truncation, which involves crude approximation but provides the quickest route to physical interpretation, as has been proved by various versions of MHD, best seen as truncation theories that nonetheless have lead the way in understanding basic principles of plasma phenomena.

In this context I will derive below a closed system of fluid equations containing the physics consistent with the drift ordering described before. The ensuing approximations involved in this derivation are not completely systematic, as with any truncated system, but on the positive side, the result provides easy access to FLR physics, a very simple form closely resembling MHD, and is very accurate. Also this system defines a whole family of plasma-fluid models, obtained as particular limits of a general model.

In order to implement a particular fluid closure, we need to study the effects of the drift ordering in the fluid variables of the system. As a first step, consistent with the small gyro-radius approximation for magnetized plasmas, one can average the orbit of a gyrating particle over the short scale-length of gyration, obtaining as an approximation the picture of a drifting magnetic dipole, the so-called guiding center. Let the distribution function of guiding centers be denoted by $\bar{f}$, then f is expanded as

$$f = \bar{f} + \mathcal{O}(\delta)\,, \tag{2.29}$$

where $\bar{f}$ denotes a gyrophase averaged distribution and the small terms of order δ represent the gyrophase dependent part of the distribution. The guiding center drift, up to first order in δ is $\mathbf{v}_{gc} = v_{\parallel}\mathbf{b} + \mathbf{v}_d + \mathcal{O}(\delta^2)$, where $\mathbf{v}_d$ includes the $\mathbf{E}\times\mathbf{B}$ drift, the ∇B and curvature drifts. If we define the mean flow of guiding centers as

$$n\mathbf{v}_{gc} \equiv \int d^3v\bar{f}(v_{\parallel} + \mathbf{v}_d) \tag{2.30}$$

and the plasma magnetization as

$$e\mathbf{M} \equiv \int d^3v\bar{f}\mu \tag{2.31}$$

then one can prove [Hazeltine-Meiss 92] the so-called "magnetization law" for the plasma flow (2.18) in terms of eqs.(2.30) and (2.31) as follows:

$$n\mathbf{V} = n\mathbf{V}_{gc} + c\nabla\times\mathbf{M}\,, \tag{2.32}$$

which states the physical distinction between plasma motion and guiding center motion due to the additional current generated by the curl of $\mathbf{M}$, the so-called magnetization current. In order to evaluate the plasma flow to first order in δ one needs to specify the distribution function $\bar{f}$. It can be shown (see for example [Hazeltine-Meiss 92]) that for confined plasmas the equilibrium distribution function is nearly a Maxwellian, therefore it is customary to evaluate the plasma flow for $\bar{f}$ being Maxwellian: $\bar{f} = f_M$, where

$$f_M = \frac{1}{\pi^{1/2}v_t} n e^{-v^2/v_t^2}\,. \tag{2.33}$$

Using this Maxwellian distribution it is straightforward to calculate the first order plasma flow consistent with the magnetization law, eq.(2.32), yielding

$$\mathbf{V} = V_{\|}\mathbf{b} + \mathbf{V}_E + \mathbf{V}_p + \mathcal{O}(\delta^2)\,, \tag{2.34}$$

where $\mathbf{V}_E$ is the $\mathbf{E} \times \mathbf{B}$ drift and $\mathbf{V}_p$ is the diamagnetic drift given by

$$\mathbf{V}_p = \frac{1}{mn\Omega}\mathbf{b} \times \nabla p\,. \tag{2.35}$$

One of the most important consequences of the drift ordering is that the diamagnetic drift, $\mathbf{V}_p$, and the $\mathbf{E} \times \mathbf{B}$ drift are both small, of the same order δ, compared to the thermal velocity v_t.

In order to derive a closed fluid model, let us consider the species sums of the moment equations (2.25)-(2.28) in exact form. Starting with the density evolution equation, its species sum becomes

$$\frac{d\rho_m}{dt} + \rho_m \nabla \cdot \mathbf{V} = 0\,, \tag{2.36}$$

where d/dt is the advective derivative

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla\,. \tag{2.37}$$

Next consider the equation of motion in the form of (2.27). By taking into account collisional momentum conservation, $\sum_s \mathbf{F}_s = 0$, and quasineutrality, $\sum_s e_s n_s = 0$, its species sum simplifies to

$$\rho_m \left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} \right) + \nabla \cdot \mathbf{p}_T = \frac{1}{c} \mathbf{J} \times \mathbf{B}\,, \tag{2.38}$$

where $\mathbf{p}_T$ is the total pressure tensor. Finally, let us consider the species sum of the pressure evolution equation (2.28) which reduces after taking into account collisional energy conservation, $\sum_s(W_s + \mathbf{V}_s \cdot \mathbf{F}_s) = 0$, to the form

$$\frac{\partial}{\partial t}\left(\frac{3}{2}P + \frac{1}{2}\sum_s m_s n_s V_s^2\right) + \nabla \cdot \mathbf{Q} = \mathbf{J} \cdot \mathbf{E}\,. \tag{2.39}$$

The three equations (2.36), (2.38), and (2.39), are exact relations, and will represent the basic fluid model upon closure. In order to do this, expressions for the pressure tensor and the energy flux in terms of lower order moments are necessary. This will be achieved by imposing the drift ordering and using the Maxwellian distribution as the lowest order approximation for $\bar{f}$, as was done before with the guiding center flow.

For the stress tensor, a simple form is assumed that effectively truncates the moment equations by neglecting anisotropy of the Chew-Goldberger-Low form [Chew et al. 56]. This assumption is given by

$$\boldsymbol{\mathcal{P}} = \mathbf{I}p + mn\mathbf{V}\mathbf{V} + \mathbf{\Pi}_g + \mathcal{O}(\delta^3)\,, \tag{2.40}$$

where $\mathbf{\Pi}_g$ denotes the gyroviscosity tensor, which represents a nondissipative transport of momentum due to spatial variation of the density and energy of magnetic moments. An important remark is that according to the drift ordering both $mnVV$ and Π_g are of the same order δ^2, and therefore for consistency one has to retain gyroviscosity when keeping advective inertia. This is a typical feature of FLR theory. The expression for the pressure tensor is trivially obtained from the relation

$$\mathbf{p} = \boldsymbol{\mathcal{P}} - mn\mathbf{V}\mathbf{V}\,. \tag{2.41}$$

Now, recall eq.(2.23), the definition of the energy flux, which is related to the heat flux $\mathbf{q}$ through the relationship

$$\mathbf{Q} = \mathbf{q} + \frac{3}{2}p\mathbf{V} + \mathbf{p}\cdot\mathbf{V} + \frac{1}{2}mnV^2\mathbf{V}\,, \tag{2.42}$$

and therefore can be approximated to first order in δ as

$$\mathbf{Q} = \mathbf{q} + \frac{5}{2} p\mathbf{V} + \mathcal{O}(\delta^2)\,, \tag{2.43}$$

where the statement of the drift ordering $V = \mathcal{O}(\delta v_t)$ has been used explicitly and the fact that the lowest order stress is isotropic and proportional to the scalar pressure. The first order expression for the heat flux becomes

$$\mathbf{q} = \frac{5}{2} \frac{p}{m\Omega} \mathbf{b} \times \nabla T + \mathcal{O}(\delta^2)\,. \tag{2.44}$$

Notice that under the drift ordering one can neglect $\mathcal{O}(\delta^2)$-terms for the continuity and energy equations, (2.36) and (2.39), while neglecting only $\mathcal{O}(\delta^3)$-terms in the equation of motion, (2.38).

These drift ordered expressions for the stress tensor, particle and energy flows, can be used in conjunction with the exact moment equations, or the equivalent species-summed equations (2.36), (2.38), and (2.39), and Maxwell's equations, to provide a closed FLR fluid description of the plasma that generalizes the MHD model. However, following this program in the most general way produces a very complicated system of fluid equations. Therefore, in order to facilitate the analytical treatment of the resulting equations, as well as the comparison with known models in some particular limit, I will treat a simplified FLR model that, nevertheless, contains a good deal of interesting physics.

As a first simplification, consider a two species plasma with equal species temperatures:

$$T_i = T_e \tag{2.45}$$

This together with quasineutrality makes both pressures the same, $p_i = p_e = p$, and the total pressure, denoted by P, will be given by

$$P \equiv p_i + p_e = 2p\,. \tag{2.46}$$

The center of mass velocity $\mathbf{V}$ will be approximated by the ion velocity $\mathbf{V}_i$ by freely neglecting terms of the order of the small mass ratio m_e/m_i. Then one will have, recalling the drift ordered plasma flow (2.34), the following approximate relation

$$\mathbf{V} \cong \mathbf{V}_i \cong V_{\|}\mathbf{b} + \mathbf{V}_E + \mathbf{V}_p\,. \tag{2.47}$$

In general the variable $\mathbf{V}$ must be determined from solution of the model equations.

The zero order moment equation (2.25), expressing conservation of the number of particles, is used in exact form and constitutes the first equation of the closed fluid system. Next I consider the species-summed equation of motion (2.38), together with eqs. (2.40) and (2.41) for the ion stress tensor. For electrons I keep only the first, scalar pressure term in (2.40) because gyroviscous stress is proportional to mass and we are neglecting terms $\mathcal{O}(m_e/m_i)$. Then

$$\boldsymbol{\mathcal{P}}_e \cong \mathbf{I}p_e \tag{2.48}$$

defines the contribution of electrons to the total stress. Taking this into account one can write down, in compact notation, the FLR equation of motion

$$m_i n \frac{d\mathbf{V}}{dt} + \nabla \cdot \mathbf{\Pi}_{gi} - \frac{1}{c}\mathbf{J} \times \mathbf{B} + \nabla P = 0\,. \tag{2.49}$$

The only term that has not been specified is the gyroviscosity tensor $\mathbf{\Pi}_g$, which in general is a very complicated object. However, under the assumption of a uniform magnetic field and the fact that the divergence of $\mathbf{V}$ is small, i.e., the plasma is nearly incompressible, one can prove the so-called "gyroviscous cancellation" (for a detailed proof see [Hazeltine-Meiss 85]) which is a general feature of FLR acceleration and consists of the following result:

$$m_i n \frac{d\mathbf{V}_{pi}}{dt} + \nabla \cdot \mathbf{\Pi}_{gi} =$$
$$-\nabla \left[\frac{p_i}{2\Omega_i}\mathbf{b} \cdot \nabla \times (\mathbf{V}_E + \mathbf{V}_{pi})\right] - \mathbf{b} m_i n (\mathbf{V}_{pi} \cdot \nabla) V_\parallel + \mathcal{O}(\delta^3)\,, \quad (2.50)$$

which shows how gyroviscosity serves to simplify the nonlinear equation of motion by cancelling various terms due to diamagnetic acceleration. The surviving terms involving the gradient of the quantity $\mathbf{b} \cdot \nabla \times (\mathbf{V}_E + \mathbf{V}_{pi})$, the parallel vorticity, provide a small correction to the pressure

$$p_i \left[1 - \frac{\mathbf{b} \cdot \nabla \times \mathbf{V}_{\perp i}}{2\Omega_i}\right] = p_i[1 + \mathcal{O}(\delta)]\,, \quad (2.51)$$

because $V_{\perp i} = \mathcal{O}(\delta)$. Therefore they are neglected in the lowest order approximation. This implies that the FLR equation of motion can be written simply as

$$m_i n \left[\frac{d\mathbf{V}_E}{dt} + \frac{d}{dt}\Big|_{MHD} (\mathbf{b} V_\parallel)\right] + \nabla P - \frac{1}{c}\mathbf{J} \times \mathbf{B} = 0\,, \quad (2.52)$$

where d/dt is the ordinary advective derivative (2.37), and $d/dt|_{MHD}$ corresponds to the MHD version

$$\frac{d}{dt}\Big|_{MHD} \equiv \frac{\partial}{\partial t} + \mathbf{V}_{MHD} \cdot \nabla \equiv \frac{\partial}{\partial t} + (\mathbf{V} - \mathbf{V}_{pi}) \cdot \nabla\,. \quad (2.53)$$

The relatively simple form of the equation of motion (2.49) is due in part, as was mentioned before, to the gyroviscous cancellation and the neglect of the parallel vorticity terms, but also because of the particular representation chosen. Recall that $\mathbf{V}$ is the physical fluid velocity and if one decides to write the equation of motion as an evolution equation for $\mathbf{V}$ instead of $\mathbf{V}_E$, then the form of the equation would be transformed according to (2.47), which can be seen as a coordinate transformation between a fixed (lab) frame and one moving with the $\mathbf{E}\times\mathbf{B}$ drift velocity. By doing this transformation of coordinates one would be forced to deal with terms containing the diamagnetic acceleration in the lab frame. I will come back to this point in later sections.

In order to complete the closed fluid description we need to consider the pressure evolution equation in its species summed form (2.39). By applying the drift ordering and keeping only first order terms in δ we obtain

$$\frac{3}{2}\frac{\partial P}{\partial t} + \nabla\cdot(\mathbf{Q}_i + \mathbf{Q}_e) = \mathbf{V}\cdot\nabla P\,, \tag{2.54}$$

where the drift-ordered equation of motion, conservation of mass, and Ohm's law have been used to eliminate $\mathbf{J}\cdot\mathbf{E}$ in favor of $\mathbf{V}\cdot\nabla P$. If we now use (2.43) and (2.44) to find the energy flux of species s, we get

$$\mathbf{Q}_s = \frac{5}{2}p_s\left[\frac{c}{e_sB}\mathbf{b}\times\left(\frac{1}{n}\nabla p_s + \nabla T_s\right) + \mathbf{V}_E + \mathbf{b}V_{\|}\right] + \mathcal{O}(\delta^2)\,. \tag{2.55}$$

Since we need to add the contribution of each species, and we assume equal temperatures (2.45); the terms involving gradients will cancel out because they are equal and opposite for the two plasma species. This implies for the total energy flux

$$\mathbf{Q} = \mathbf{Q}_i + \mathbf{Q}_e = \frac{5}{2}P(\mathbf{V}_E + \mathbf{b}V_{\|}) + \mathcal{O}(\delta^2) \cong \frac{5}{2}P\mathbf{V}_{MHD}\,. \tag{2.56}$$

Finally, recall (2.35), which implies

$$\mathbf{V}\cdot\nabla P = \mathbf{V}_{MHD}\cdot\nabla P \tag{2.57}$$

and therefore we can write (2.54) in its final form

$$\frac{d}{dt}|_{MHD}\, P + \frac{5}{3} P \nabla \cdot \mathbf{V}_{MHD} = 0\,, \tag{2.58}$$

which together with (2.25) and (2.52) represent the first three moment equations, closed under the truncation procedure presented above.

A constitutive relation between the electric and magnetic fields, and some of the flow variables is needed. This is typically achieved using Ohm's law. Here we will adopt a generalization of Ohm's law that is plausibly consistent with the approximations done so far, althought not a systematic consequence of drift-ordered kinetic theory. Starting with the electron version of the equation of motion (2.26) and neglecting the acceleration term and nonscalar stresses, as in (2.48), due to the smallness of the electron mass, we find that

$$\nabla p_e + en\left(\mathbf{E} + \frac{1}{c}\mathbf{V}_e \times \mathbf{B}\right) = \mathbf{F}_e\,, \tag{2.59}$$

where $\mathbf{F}_e$ is the friction force. Using the definition for the current density

$$\mathbf{J} = en(\mathbf{V}_i - \mathbf{V}_e) \cong en(\mathbf{V} - \mathbf{V}_e) \tag{2.60}$$

we can express (2.59) in terms of $\mathbf{V}$ and $\mathbf{J}$ as follows:

$$\mathbf{E} + \frac{1}{c}\mathbf{V} \times \mathbf{B} + \frac{1}{en}\left(\nabla p_e - \frac{1}{c}\mathbf{J} \times \mathbf{B}\right) = \frac{1}{en}\mathbf{F}_e\,. \tag{2.61}$$

The term on the right hand side leads to the usual Ohmic resistivity, but we will consider the ideal case from here on and therefore it will be dropped. The ∇p_e term on the left hand side is the so-called Hall term.

The three moment equations (2.25), (2.52), (2.58), and the generalized Ohm's law (2.61), together with Maxwell's equations for the electromagnetic field, constitute a closed fluid model known in the literature as the *Drift*

Model [Hazeltine-Meiss 92]. It is based on early work by Roberts and Taylor [Roberts-Taylor 62] and Rosenbluth and Simon [Rosenbluth-Simon 65]. In the following sections it is shown to contain as particular cases a family of plasma fluid models of general interest.

2.3 Reduced Fluid Models

The derivation of the drift model in the preceding section was carried out in some detail for pedagogical reasons, it shows the vast spectrum of nonlinear effects that this kind of fluid model describes and also the limitations of such an approximate approach to the exact kinetic equations. On the other hand, even though a series of systematic approximations were made to facilitate physical insight, few results based on analytical analysis have been obtained, because of the obvious complicated nature of systems of coupled nonlinear partial differential equations, leaving hope for further understanding to numerical simulations. Here is where novel techniques, such as Lie's analysis, can open new horizons of understanding for the set of solutions and general behavior of plasma evolution, approximately modeled by this fluid description.

As was emphasized during its derivation, the drift model describes a wide range of physical phenomena for confined magnetized plasmas. The most important effects taken into account by the model are slow evolution, diamagnetic drifts, gyroviscosity and some effects of electron dynamics. For the sake of simplicity, sometimes it is useful to pinpoint some of this phenomena and derive a model, much simpler than the original, that emphasizes specific nonlinear behavior at the cost of neglecting other important effects. I will show in the present section how the drift model generalizes some well known models, obtained as extensions of MHD, containing some of the above mentioned effects.

In order to do this we have to adopt a simplification scheme devised to select dominant effects by the systematic use of a scale-length ordering. Note that this approximation is taken in addition to the "fluid closure" already explained in the previous section, and therefore may appear rather crude. When such geometrical simplification is applied to the full nonlinear model, the resulting dynamical system is said to be "reduced". In spite of the "double" approximation involved, reduced fluid models have been found to be remarkably predictive for the dominant nonlinear effects of tokamak physics. In particular I will present a reduction based on a large aspect ratio expansion, which is traditionally used to simulate conditions in tokamaks and other confinement devices.

The aspect ratio of a circular cross-section tokamak is R_0/a, where R_0 is the major radius of the magnetic axis and a is the minor radius of the confining vessel. In general, the aspect ratio measures the ratio of cylindrical curvature to toroidal curvature for any device. At sufficiently large aspect ratio, a tokamak appears nearly cylindrical with respect to curvature, although it remains topologically closed. Thus, the large aspect ratio approximation treats toroidal curvature effects perturbatively by considering an expansion in the inverse aspect ratio ε

$$\varepsilon := a/R_o \ll 1\,. \tag{2.62}$$

The implementation of the ordering procedure is clarified by means of dimensionless field variables and coordinates, chosen to make the various powers of

ε explicit. A convenient set of dimensionless coordinates (x, y, z) is defined in terms of cylindrical coordinates (R, θ, Z), centered on the symmetry axis of the tokamak, by

$$x = (R - R_o)/a; \qquad y = Z/a; \qquad z = -\theta. \tag{2.63}$$

where R measures radial displacement away from the symmetry axis, θ is the toroidal angle, and Z varies along the symmetry axis. Hence, (x, y, z) is a right-handed set of effectively Cartesian coordinates. The basic geometric assumption of this ordering is to distinguish the transverse (poloidal) and longitudinal (toroidal) scale lengths by taking the latter relatively large: $\partial/\partial z = \mathcal{O}(\epsilon)$. This result is complementary to the conventional assumption that the vacuum magnetic field B_T, is purely toroidal, and therefore the confining magnetic fields follow the ordering $B_P/B_T \sim \varepsilon$, comparable to a flute-like ordering where perturbations are characterized by $k_\|$ being small,

$$k_\| \ll k_\perp, \tag{2.64}$$

and giving as a consequence the distinction between two time scales for electromagnetic disturbances. It turns out that the most important toroidal instabilities belong to the slow time scale. Thus, we define the Alfvén time τ_A, as

$$\tau_A \equiv a/v_A\,, \tag{2.65}$$

where v_A is the Alfvén speed

$$v_A^2 = B_T^2/(4\pi n_0 m_i) \tag{2.66}$$

with m_i the ion mass and n_0 a constant measure of the plasma density. Taking this into account, we introduce a dimensionless time variable τ, such that

$$\tau = \varepsilon t/\tau_A = \varepsilon(tv_A/a)\,, \tag{2.67}$$

which is appropiate for the shear-Alfvén fluid motions of interest.

Having defined the relevant scales in time and space, we can study the scaling of the fields in terms of the natural units B_T, v_A and a. The magnetic field is written as

$$\mathbf{B} = B_T\hat{z}/(1+\varepsilon x) + \nabla \times \mathbf{A}\,, \tag{2.68}$$

where B_T is constant and $\mathbf{A}$ is the vector potential. The first term in (2.68) is the dominant vacuum toroidal field, consequently we write for the vector potential

$$\mathbf{A} = \varepsilon B_T a \hat{\mathbf{A}}\,, \tag{2.69}$$

where the hat denotes a dimensionless variable. Splitting $\mathbf{A}$ into its components parallel and perpendicular to the vacuum field: $\mathbf{A} = \mathbf{A}_\perp + \hat{z}A_z$, with $\mathbf{A}_\perp \equiv \hat{x}A_x + \hat{y}A_y$, we can rewrite (2.68) using B_T to normalize as

$$\hat{\mathbf{B}} = \mathbf{B}/B_T = \hat{z} + \varepsilon(-\hat{z}x + \hat{z}B_\parallel - \hat{z}\times\nabla_\perp\psi) + \mathcal{O}(\varepsilon^2)\,. \tag{2.70}$$

Here, I have introduced the useful abbreviation

$$\psi(\mathbf{x},t) = \hat{\mathbf{A}}_z(\mathbf{x},t)\,. \tag{2.71}$$

It can be seen that $(-\psi)$ is proportional to the poloidal flux. Also, in (2.70), I have used $\nabla_\perp$, the normalized gradient in the poloidal plane, given by

$$\nabla_\perp \equiv \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}\,. \tag{2.72}$$

Similarly a dimensionless electrostatic potential can be introduced as

$$\varphi(\mathbf{x}, t) = (c/\varepsilon B_T a v_A)\Phi(\mathbf{x}, t)\,, \tag{2.73}$$

where Φ is the ordinary potential.

If we exploit the fact that the longitudinal scale length is relatively large, then for any scalar function f, we would get $\nabla f = \nabla_\perp f + \mathcal{O}(\varepsilon)$, which combined with (2.70) yields

$$\mathbf{B}\cdot\nabla f = \varepsilon\left(\frac{\partial f}{\partial z} - \hat{z}\cdot\nabla_\perp\psi\times\nabla_\perp f\right) + \mathcal{O}(\varepsilon^2)\,. \tag{2.74}$$

We will use a more compact expression of (2.74) in terms of the conventional Poisson bracket

$$[f, g] = \hat{z}\cdot\nabla_\perp f\times\nabla_\perp g\,, \tag{2.75}$$

and the definition of the nonlinear parallel gradient

$$\nabla_\| f \equiv \frac{\partial f}{\partial z} - [\psi, f]\,, \tag{2.76}$$

resulting in the following expression for the normalized directional derivative along $\mathbf{B}$:

$$\mathbf{B}\cdot\nabla f = \varepsilon\nabla_\| f + \mathcal{O}(\varepsilon^2)\,. \tag{2.77}$$

As for the electric field, using Faraday's law, we get the dimensionless definition

$$\mathbf{E} = -\varepsilon(\nabla\varphi + \varepsilon\frac{\partial\mathbf{A}}{\partial\tau})\,, \tag{2.78}$$

where the shear-Alfvén time scale (2.67) has been used. Note that, as a consequence of this shear-Alfvén time scale ordering, the lowest order transverse field is electrostatic. However, for the parallel component $E_\|$, the electrostatic and electromagnetic terms make comparable contributions, namely

$$E_\| = -\varepsilon^2 \left(\nabla_\| \varphi + \frac{\partial \psi}{\partial t} \right) + \mathcal{O}(\varepsilon^3) \,. \tag{2.79}$$

These geometrical approximations define the basic structure of the reduction in terms of a large aspect ratio ordering.

2.3.1 The Reduced Drift Model

Now, we are ready to tackle the task of "reducing" the simplified moment equations derived in the last section, by the systematic ordering scheme just explained above. In particular, let us consider the plasma equation of motion (2.49), under the time scale relevant to this ordering. Recall that we are considering relatively slow evolution on the scale of compressional Alfvén waves, described by $\omega \simeq \omega_{CA} \equiv kv_A$; while shear-Alfvén waves have the frequency $\omega_{SA} \equiv k_\| v_A$, comparable to the frequencies of interest. Evidently,

$$\omega_{SA} \ll \omega_{CA} \tag{2.80}$$

in the case of a long toroidal scale length, where (2.64) holds. Plasma acceleration is measured by the competition between the forces represented by the last two terms in (2.49). A simple estimate from Ampere's law shows that the $\mathbf{J} \times \mathbf{B}$ force yields acceleration on the fast scale, ω_{CA} [Hazeltine-Meiss 85]. Therefore, a description of slower evolution, proceeding through a sequence of near-equilibrium states, requires that the dominant part of the $\mathbf{J} \times \mathbf{B}$ force be annihilated. A simple way to achieve this simplification is by taking the parallel component of the curl of the plasma equation of motion, (2.49). The resulting equation, a vorticity equation, is called the "*shear-Alfvén law*". It is the plasma equation of motion, in which the relevant electromagnetic driving force is to be extracted by the operation $\mathbf{B}\cdot\nabla(\mathbf{J}\times\mathbf{B})$, when $\omega \le \omega_{SA}$. Then, following this procedure, we can obtain the general form of the shear-Alfvén law

$$\mathbf{B}\cdot(\nabla\times\mathbf{f} - 2\boldsymbol{\kappa}\times\mathbf{f}) = c^{-1}B^2\mathbf{B}\cdot\nabla(J_{\|}/B) + 2\mathbf{B}\times\boldsymbol{\kappa}\cdot\nabla P\,, \qquad (2.81)$$

where $\mathbf{f}$ is a shorthand for the acceleration terms in the equation of motion

$$\mathbf{f} \equiv m_i n\frac{d\mathbf{V}}{dt} + \nabla\cdot\mathbf{\Pi}\,, \qquad (2.82)$$

and $\boldsymbol{\kappa}$ denotes the magnetic field curvature $\boldsymbol{\kappa} := (\mathbf{b}\cdot\nabla)\mathbf{b}$. The "reduction" of the shear-Alfvén law, consistent with the large aspect ratio ordering, will be carried out by considering the dimensionless form of the different terms in (2.81). In particular, we will start analizing the terms on the right-hand side, representing the "kink" and interchange instabilities respectively. For the first term, using (2.77) and the definition of the parallel current $J_{\|} = \mathbf{B}\cdot\nabla\times\mathbf{B}/B$, we obtain

$$\mathbf{B}\cdot\nabla(J_{\|}/B) = \varepsilon^2(\nabla_{\|}J) + \mathcal{O}(\varepsilon^3)\,, \qquad (2.83)$$

where J, proportional to the negative of the parallel current, is defined as

$$J \equiv \nabla_{\perp}^2\psi\,. \qquad (2.84)$$

For the second term, involving the curvature $\boldsymbol{\kappa} = -\varepsilon\nabla_{\perp}x$, we can use the lowest order field to obtain the interchange term up to order ε^2

$$\varepsilon\mathbf{B}\times\boldsymbol{\kappa}\cdot\nabla p = -\varepsilon^2\hat{\mathbf{z}}\times\nabla_{\perp}x\cdot\nabla_{\perp}p = -\varepsilon^2[x,p]\,. \qquad (2.85)$$

In order to consider the reduced form of the terms in the left hand side of (2.81), we need to use the gyroviscous cancellation shown in the previous section, (2.50). The main result is contained in (2.52), where we can identify the function $\mathbf{f}$ given by

$$\mathbf{f} = m_i n \frac{d\mathbf{V}_E}{dt} + m_i n \frac{d}{dt}\,|_{MHD}\,(\mathbf{b}V_\parallel)\,. \tag{2.86}$$

This result can be expressed in a more convenient form, by explicitly using (2.53), as

$$\mathbf{f} = m_i n \left[\left(\frac{\partial}{\partial t} + \mathbf{V}_E\cdot\nabla + \mathbf{V}_{pi}\cdot\nabla\right)\mathbf{V}_E + \left(\frac{\partial}{\partial t} + \mathbf{V}_E\cdot\nabla\right)\mathbf{b}V_\parallel\right]\,. \tag{2.87}$$

The velocity is naturally expressed in units of v_A, and it is assumed of order ε:

$$\mathbf{V}/v_A = \varepsilon\mathbf{u}\,. \tag{2.88}$$

Then we write $\mathbf{V}_E = \varepsilon v_A \mathbf{u}_E$, $\mathbf{V}_{pi} = \varepsilon v_A \mathbf{u}_{pi}$, and $\mathbf{u} = \mathbf{u}_E + \mathbf{u}_{pi}$, to obtain, by making use of (2.35), (2.78), the following

$$\mathbf{u} = \hat{z}\times\nabla_\perp[\varphi + \hat{\delta}p]\,, \tag{2.89}$$

where the usual Boussinesq approximation of fluid dynamics has been used to avoid higher order nonlinearities, and $\hat{\delta}$ is the gyroradius parameter that measures FLR effects entering the dynamical equations. It is defined as:

$$\hat{\delta} \equiv (2\Omega\tau_A)^{-1}\,, \tag{2.90}$$

which differs from the gyroradius parameter, δ, defined by (2.2), directly proportional to the gyroradius. In fact, the gyroradius is measured by the product of $\hat{\delta}$ and a special version of an electron toroidal beta $\beta_e = 8\pi n_c T_e / B_T^2$ as follows

$$\hat{\delta}^2\beta_e \sim \rho_i^2/a^2 \sim \rho_i^2\nabla_\perp^2\,, \tag{2.91}$$

where ρ_i is the ion gyroradius. Hence restricting our attention to the small-gyroradius case, we assume $\hat{\delta}^2\beta_e \sim \varepsilon$. Then, consistent with the low-frequency case that we have been emphasizing, the basic ordering assumed would be

$$\hat{\delta} \sim 1 \qquad \beta_e \sim \varepsilon\,. \tag{2.92}$$

The reduced shear-Alfvén law is obtained by substituting (2.83), - (2.85), and (2.89) into (2.81), yielding

$$\hat{z} \cdot \nabla_\perp \times \left[\frac{n}{n_c}\left(\frac{\partial \mathbf{u}_E}{\partial \tau} + \mathbf{u} \cdot \nabla \mathbf{u}_E\right)\right] = -\nabla_\| J - 2[x, p]\,. \tag{2.93}$$

The left-hand side of this equation can be written more explicitly by noting that

$$\hat{z} \cdot \nabla_\perp \times \mathbf{u}_E = \nabla_\perp^2 \phi := U\,, \tag{2.94}$$

where U denotes the vorticity, and using the following identity

$$\begin{aligned} \hat{z} \cdot \nabla \times [\mathbf{u}_E \cdot \nabla_\perp \mathbf{u}_E + \mathbf{u}_{pi} \cdot \nabla_\perp \mathbf{u}_E] &= [\varphi, U] \\ &+ \frac{\hat{\delta}}{2}\left\{[p, U] + [\varphi, \nabla_\perp^2 p] + \nabla_\perp^2 [p, \varphi]\right\}\,, \end{aligned} \tag{2.95}$$

which can be straightforwardly verified using (2.89) and the cartesian nature of ∇. Also, we use the Boussinesq approximation to replace the n/n_c factor by unity. Then (2.93) becomes

$$\begin{aligned} \frac{\partial U}{\partial \tau} + \left[\varphi + \hat{\delta} p, U\right] + \nabla_\| J + 2[x, p] & \\ = -\frac{\hat{\delta}}{2}\left\{[\varphi, \nabla_\perp^2 p] + \nabla_\perp^2 [p, \varphi]\right\}\,. \end{aligned} \tag{2.96}$$

This is the shear-Alfvén law describing the perpendicular evolution for ions. The second term on the left-hand side shows that U is advected by the total drift **u**, while the terms on the right-hand side reflect spatial variation of the diamagnetic drift.

For electrons we need to consider only the parallel dynamics. This is accomplished by considering the parallel component of the acceleration law for electrons, i. e. , the generalized Ohm's law (2.61). The parallel electron speed is deduced from the definition of the current density (2.60), and the normalization of the parallel velocity given by $v = (\varepsilon v_A)^{-1} V_\parallel$, yielding

$$V_{\parallel e} = \varepsilon v_A (v + 2\hat{\delta} J) + \mathcal{O}(\varepsilon^2) \,. \tag{2.97}$$

If we consider for the parallel electric field the form given by (2.79), and take the normalization suggested by (2.97), we find

$$\frac{\partial \psi}{\partial \tau} + \nabla_\parallel \varphi = \eta J + \hat{\delta} \nabla_\parallel p \,, \tag{2.98}$$

which is the generalized, reduced Ohm's law, where the $\mathcal{O}(m_e/m_i)$ electron inertia terms have been neglected.

The final equation to close the system is the equation of pressure evolution (2.58). Its normalized form reduces to simple $\mathbf{E} \times \mathbf{B}$ convection

$$\frac{\partial p}{\partial \tau} + [\varphi, p] = 0 \,. \tag{2.99}$$

As was mentioned before, compressibility has been consistently neglected by dropping terms $\mathcal{O}(\varepsilon^3)$.

The coupled equations, (2.96), (2.98), and (2.99), represent a closed reduced model for the three independent fields: φ, ψ, and p, which measure respectively, the electrostatic potential, the poloidal magnetic flux, and the single specie pressure. The vorticity U and current J are given in terms of the potentials as (2.94) and (2.84) respectively. This model generalizes reduced MHD [Strauss 76-77], which can be recovered in the limit $\hat{\delta} \to 0$. Terms involving $\hat{\delta}$ enter the model equations as finite gyroradius corrections in a conventional sense. Neglecting $\hat{\delta}$ is equivalent to assuming the MHD ordering, (2.13).

2.3.2 A Simplified Model and Compressibility

In this section I will discuss briefly the consequences of retaining the higher-order terms due to compressibility in the reduced model described above, even though it implies departing from the systematic ε ordering. Such an extension of the theory is justifiable if one recognizes the qualitative importance of compressibility in certain contexts. Therefore, I will rederive the pressure evolution equation (2.99), but this time keeping terms proportional to $\nabla \cdot \mathbf{u}$. Later, I will derive a simplified reduced model that keeps effects of parallel compressibility.

For an isothermal plasma, pressure evolution is determined by density evolution, i. e. , by the particle conservation law, and in particular, for quasineutral plasmas, by simply considering electron conservation

$$\frac{\partial n}{\partial t} + \mathbf{V}_e \cdot \nabla n = -n \nabla \cdot \mathbf{V}_e \, . \tag{2.100}$$

A straightforward normalization of this equation yields

$$\frac{\partial p}{\partial \tau} + [\varphi, p] = -\frac{\beta_e}{\varepsilon} \nabla \cdot \mathbf{u}_e \, , \tag{2.101}$$

where it is clear that the right-hand side of (2.101) is $\mathcal{O}(\beta_e) = \mathcal{O}(\varepsilon)$ and therefore consistently neglected. But in order to include the lowest order contribution to $\nabla \cdot \mathbf{u}_e$, we will treat β_e as $\mathcal{O}(1)$, unlike the basic ordering assumed so far (2.92), allowing us to retain terms $\mathcal{O}(\beta_e \varepsilon^2)$ but neglecting $\mathcal{O}(\varepsilon^3)$. Then the parallel contribution is easily written down as

$$\frac{\beta_e}{\varepsilon} \nabla \cdot \mathbf{b} u_{\| e} = \beta_e \nabla_{\|} (v + 2\hat{\delta} J) \,, \tag{2.102}$$

where v is the ion parallel flow velocity, which will be neglected in order to study effects of electron parallel mobility. This implies that the parallel current will be approximated by

$$J_{\|} = -en V_{\| e} \,. \tag{2.103}$$

The perpendicular terms can be easily calculated, see [Hazeltine-Meiss 85], to finally give, for the pressure evolution equation with lowest order compressibility effects, the following

$$\frac{\partial p}{\partial \tau} + [\varphi, p] = \beta \{ 2[x, \varphi - \hat{\delta} p] - \nabla_{\|}(2\hat{\delta} J) + \eta \nabla_{\perp}^2 p \} \,. \tag{2.104}$$

The first term on the right-hand side measures perpendicular compressibility due to toroidal curvature. This term is similar in form and significance to the interchange term in (2.96). The term proportional to J reflects parallel compressibility, as was noted above. The final term corresponds to resistive diffusion. The parameter $\beta = [1 + \beta_e]^{-1} \beta_e$ is a convenient new measure of beta. Equation (2.104) together with (2.96) and (2.98) form a reduced drift model with compressibility effects. When β and $\hat{\delta}$ are both neglected, one obtains the high-beta RMHD equations of Strauss [Strauss 76-77], [Morrison-Hazeltine 84].

The reduced drift model and the compressible version derived above contain a significant amount of physical effects in addition to the basic fluid-like behavior described by RMHD. For the sake of analytical tractability and clarity of the forthcoming Lie group analysis, I would like to further simplify the compressible reduced model, by emphasizing electron parallel flow and neglecting the additional physics contained in the other terms proportional to $\hat{\delta}$ and β. To achieve this goal, first I consider the low-beta ordering, which implies $\beta_T \sim \varepsilon^2$, getting rid of all toroidal curvature effects. This will allow us to omit the interchange term in the shear-Alfvén law (2.96), as well as all the $\hat{\delta}$ terms that involve the pressure, obtaining the final form of the equation of motion

$$\frac{\partial U}{\partial \tau} + [\varphi, U] + \nabla_\| J = 0\,, \tag{2.105}$$

which corresponds exactly to the one used for low-beta RMHD. Next, we consider the density evolution equation, (2.100), where the density is treated as only mildly perturbed from a constant n_c

$$n(\mathbf{x}, t) = n_c + \tilde{n}(\mathbf{x}, t)\,, \qquad \tilde{n} \ll n_c\,, \tag{2.106}$$

and where, for low-beta, the transverse electron velocity is approximated by the first term of (2.89), while the electron parallel flow is given by (2.103), yielding an equation for the density perturbation $\tilde{n}/n_c$ of the form

$$\frac{\partial}{\partial \tau}\frac{\tilde{n}}{n_c} + \left[\varphi, \frac{\tilde{n}}{n_c}\right] + \frac{2\varepsilon\alpha}{\Delta}\nabla_\| J = 0\,. \tag{2.107}$$

The last term of (2.106) has been ordered by assuming $\Delta \sim \varepsilon$, in such a way that we will obtain the simplest nonlinear system containing the effects of electron parallel compressibility. The parameters $\alpha = \rho_s^2/a^2$, and $\Delta = \beta_e/\Omega_T\tau_A$, with $\rho_s^2 = (T_e/m_i)\Omega_T^{-2}$ and Ω_T the ion gyrofrequency, define an appropriate normalization for $\tilde{n}$ as follows

$$\frac{\tilde{n}}{n_c} = \frac{2\varepsilon\alpha}{\Delta}\chi\,, \tag{2.108}$$

where χ is the dimensionless field variable that represents density perturbations, and satisfies

$$\frac{\partial\chi}{\partial\tau} + [\varphi, \chi] + \nabla_{\parallel} J = 0\,. \tag{2.109}$$

The final equation is obtained from reduced Ohm's law, (2.98), by retaining the parallel pressure gradient term, Hall term, which represents the fluid manifestation of electron parallel mobility, and writing it in terms of χ as

$$\frac{\partial\psi}{\partial\tau} + \nabla_{\parallel}\varphi = \eta J + \alpha\nabla_{\parallel}\chi\,. \tag{2.110}$$

The reduced fluid system given by equations (2.105), (2.109), and (2.110), was first derived by Hazeltine [Hazeltine 83] as a generalization of both RMHD and the Charney-Hasegawa-Mima (CHM) models. The first model, RMHD in its resistive, low-beta version, describes the nonlinear dynamics of several instabilities in large aspect-ratio tokamak geometry. The second model, constructed by Hasegawa and Mima [Hasegawa-Mima 78] to describe electrostatic plasma turbulence in slab geometry, was first derived by Charney in the context of planetary atmosphere investigations.

The two models differ sharply in their intentions, but they can be shown to appear as special limits of the inclusive reduced model derived above. When the parameter α is negligibly small, χ is decoupled from the evolution of φ and ψ. In this case, (2.105) and (2.110) show that φ and ψ evolve according to RMHD dynamics. Thus, the RMHD limit is obtained when the fields are assumed to vary on scale lengths large compared to ρ_s. On the opposite end, for short scale length variation we have $\alpha \sim 1$, and the system of equations can be interpreted as an electromagnetic generalization of CHM. In the special case: $\varphi = \alpha\chi$, which corresponds to adiabatic electrons, i. e. ,

$$\frac{\tilde{n}}{n_c} = e\frac{\Phi}{T_e}, \tag{2.111}$$

substituting into (2.109) implies

$$\nabla_{\|} J = -\frac{1}{\alpha}\frac{\partial \varphi}{\partial \tau}, \tag{2.112}$$

which together with (2.105) and taking $\alpha = 1$, yields

$$\frac{\partial U}{\partial \tau} + [\varphi, U] - \frac{\partial \varphi}{\partial \tau} = 0. \tag{2.113}$$

This equation coupled with the relation $U = \nabla_{\perp}^2 \varphi$, constitute the intensively studied CHM equation. Note that in this case, the generalized Ohm's law (2.110) describes ordinary resistive diffusion.

Besides the fact that Hazeltine's inclusive nonlinear model includes and generalizes the physics of RMHD and CHM, as shown above, it possesses intrinsic interests, particularly in regard to nonlinear applications, where it has been shown to provide electromagnetic generalizations of drift-solitary waves [Hazeltine et al. 85]. Also, because it requires an additional field than RMHD, it is sometimes called the *Three-Field Model* (HTFM). This system will constitute the base for our implementation of Lie group techniques for plasma fluid models in chapter 4.

Chapter 3

Lie Groups and Differential Equations

3.1 Introduction

Solving differential equations has been one of the most important problems in mathematical physics since Newton and Leibniz introduced the basic concepts of what we know as differential and integral calculus. It turns out that most of the physical laws and models that we use to describe physical phenomena are written as differential (or integral) equations. From the earlier days up until now there have been developed several integration methods for special cases, ad-hoc techniques for particular classes of differential equations that have to be applied on a case by case basis; unfortunately no sytematic way to solve a given equation exists. This is particularly true for nonlinear equations where our knowledge of "solvable" cases is very limited.

This was the state of affairs at the end of last century when Sophus Lie started his research on solution of differential equations motivated by Abel's work for polynomial equations. Lie recognized that for ordinary differential equations (ODE's) there is a systematic method that places under the same construct all the previously developed integration techniques, e.g. those for differential equations that are homogeneous, linear, and separable; the use of integrating factors and reduction of order techniques; undetermined coefficient methods; transform techniques such as Laplace transform; etc. The basis of

Lie's method is the concept of an *infinitesimal transformation* and the closely related concept of a *one-parameter group* [Lie 1891], which together constitute the building blocks for the so-called *symmetry group analysis* of differential equations, which I will be discussing in the present chapter. Lie's idea represented an impressive discovery and was accompanied by his developing of the theory of continuous groups, now known as Lie Groups. However, this powerful method did not receive proper attention for a long time, due to the fact that in order to find the symmetry group of a differential equation one has to perform a very large number of algebraic computations and solve the resulting system of coupled differential equations. This task is difficult or impossible to do with pencil and paper. With the recent development of symbolic manipulation programs we have discovered many of the implications of Lie's methods for differential equations. There has been a growing interest in the last two decades to expand and augment Lie's mathematical formalism, and there have been numerous research papers applying these techniques to physically interesting problems (see [Olver 93], [Ibragimov et al. 93] and references therein).

My main concern in this study is systems of partial differential equations (PDE's), like the ones we encountered in the previous chapter. Lie group techniques have proven to be an effective tool for finding particular solutions (the so-called similarity type) for the full nonlinear problem, for reducing the number of independent variables of the system, and for studying conservation laws. The emphasis throughout this work is on the novel use of analytical methods for finding solutions of nonlinear fluid equations rather than the further refinement of approximate models.

In the remainder of this chapter I will develop the necessary mathematical tools to study the symmetry group of a system of nonlinear PDE's in a condensed and self-contained exposition, stressing their applicability rather than mathematical rigor. I refer the interested reader to the excellent general texts on the subject [Olver 93], [Bluman-Kumei 89], [Ibragimov 85], [Stephani 89], [Ovsiannikov 82], for proofs of the most important results, further details, and possible applications. Examples pertaining to plasma-fluid equations will be treated in later chapters.

3.2 Classical Lie Point Symmetries

The fundamental concept behind the symmetry group of a system of differential equations is that of a transformation acting on the space of dependent and independent variables, with the property that it will transform solutions of the system to other solutions. The simplest form of such a transformation is when the mapping is of a purely geometrical nature, the so-called point transformations.

To make this concept more precise let us consider a system S of n-th order differential equations that involves p independent variables $\mathbf{x} = (x^1, \ldots, x^p)$, q dependent variables $\mathbf{u} = (u^1, \ldots, u^q)$ and the derivatives of $\mathbf{u}$ with respect to $\mathbf{x}$ up to order n: $\mathbf{u}^{(n)}$, given as a system of equations

$$\Delta_\nu(\mathbf{x}, \mathbf{u}^{(n)}) = 0\,, \qquad \nu = 1, \ldots, \ell\,, \tag{3.1}$$

whose solutions will be of the form $u^\alpha = f^\alpha(x^1, \ldots, x^p)$, $\alpha = 1, \ldots, q$. Let $\mathbf{X} = R^p$ be the space representing the independent variables and let $\mathbf{U} = R^q$ represent the dependent variables. A symmetry group of the system S is a local group of transformations G acting on an open subset M of the base space $\mathbf{X}\times\mathbf{U}$ with the property that whenever $\mathbf{u} = f(\mathbf{x})$ is a solution of S, and whenever $g \circ f$ is defined for $g \in G$, then $\tilde{\mathbf{u}} = g \circ f(\mathbf{x})$ is also a solution of the system. The one-parameter group of point transformations will depend explicitly on a continuous parameter ϵ as follows:

$$\tilde{x}^i = \tilde{x}^i(x, u; \epsilon)\,,$$

$$\tilde{u}^\alpha = \tilde{u}^\alpha(x, u; \epsilon)\,, \tag{3.2}$$

with the usual properties of closure, inverse, identity $\epsilon = 0$, etc., which guarantee that the transformations form a one-parameter Lie group. Note that this definition of a point symmetry allows for arbitrary nonlinear transformations of both the independent and dependent variables but does not involve derivatives of $\mathbf{u}$ the dependent variables.

From a more geometrical viewpoint, the one-parameter group given by (3.2) and its action can be visualized as motion in the base space $\mathbf{X}\times\mathbf{U}$. For simplicity lets consider one independent variable x and one dependent variable u. Therefore, the base space would be defined by the $x - u$ plane. Take an arbitrary point (x_0, u_0) in that plane for $\epsilon = 0$. When the parameter ϵ varies, the images $(\tilde{x}, \tilde{u})$ will move along some line. If we repeat this procedure for different initial points, we will obtain the picture given in figure 3.1, where each curve represents points that can be transformed into each other under the group action. They are called the *orbits of the group.*

Now, as I mentioned before, one of the key elements in S. Lie's discoveries is the use of infinitesimal methods; to introduce them in our treatment

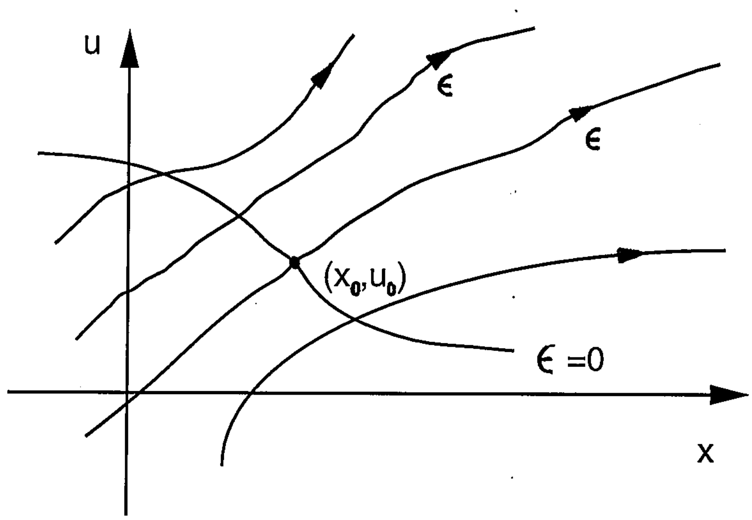

Figure 3.1: Action of a one-parameter group of transformations.

of symmetries we need to consider the expansion of (3.2) about $\epsilon = 0$:

$$\tilde{\mathbf{x}} = \mathbf{x} + \epsilon \left(\frac{\partial \tilde{\mathbf{x}}}{\partial \epsilon}(\mathbf{x}, \mathbf{u}; \epsilon) \mid_{\epsilon=0} \right) + O(\epsilon^2)$$

$$\tilde{\mathbf{u}} = \mathbf{u} + \epsilon \left(\frac{\partial \tilde{\mathbf{u}}}{\partial \epsilon}(\mathbf{x}, \mathbf{u}; \epsilon) \mid_{\epsilon=0} \right) + O(\epsilon^2) . \tag{3.3}$$

Let

$$\boldsymbol{\xi}(\mathbf{x}, \mathbf{u}) := \frac{\partial \tilde{\mathbf{x}}}{\partial \epsilon}(\mathbf{x}, \mathbf{u}; \epsilon)|_{\epsilon=0} \tag{3.4}$$

$$\boldsymbol{\phi}(\mathbf{x}, \mathbf{u}) := \frac{\partial \tilde{\mathbf{u}}}{\partial \epsilon}(\mathbf{x}, \mathbf{u}; \epsilon)|_{\epsilon=0} . \tag{3.5}$$

The transformation: $\tilde{\mathbf{x}} = \mathbf{x} + \epsilon\boldsymbol{\xi}$, $\tilde{\mathbf{u}} = \mathbf{u} + \epsilon\boldsymbol{\phi}$ is called the infinitesimal transformation of the Lie group of transformations (3.2); the components of $\boldsymbol{\xi}(\mathbf{x}, \mathbf{u})$ and $\boldsymbol{\phi}(\mathbf{x}, \mathbf{u})$ are called the infinitesimals of the transformation. This implies that the Lie group of transformations (3.2) is equivalent to the solution of the initial value problem for the following system of first order differential equations:

$$\frac{d\tilde{\mathbf{x}}}{d\epsilon} = \boldsymbol{\xi}(\tilde{\mathbf{x}}, \tilde{\mathbf{u}}) \tag{3.6}$$

$$\frac{d\tilde{\mathbf{u}}}{d\epsilon} = \boldsymbol{\phi}(\tilde{\mathbf{x}}, \tilde{\mathbf{u}}) , \tag{3.7}$$

with $\tilde{\mathbf{x}} = \mathbf{x}$ and $\tilde{\mathbf{u}} = \mathbf{u}$ when $\epsilon = 0$. This is the so-called *First Fundamental Theorem of Lie*, and it shows that the infinitesimal transformation contains the essential information determining a one-parameter Lie group of transformations. Moreover, we define the infinitesimal generator of the one-parameter Lie group of transformations (3.2), as the vector field (or differential operator)

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi^\alpha(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial u^\alpha}, \tag{3.8}$$

which can be shown to be equivalent to the infinitesimal transformations, and therefore also equivalent to the finite Lie group of transformation through the following relation:

$$\begin{aligned} \tilde{\mathbf{x}} = e^{\epsilon \mathbf{v}} \mathbf{x} &= \mathbf{x} + \epsilon \mathbf{v} \mathbf{x} + \frac{\epsilon^2}{2} \mathbf{v}^2 \mathbf{x} + \ldots \\ &= (1 + \epsilon \mathbf{v} + \frac{\epsilon^2}{2} \mathbf{v}^2 + \ldots) \mathbf{x} \\ &= \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \mathbf{v}^k \mathbf{x}, \end{aligned} \tag{3.9}$$

where the operator $\mathbf{v}^k F(\mathbf{x}) = \mathbf{v}\mathbf{v}^{k-1} F(\mathbf{x})$ with $\mathbf{v}^0 F(\mathbf{x}) = F(\mathbf{x})$ for any differentiable function $\mathrm{F}(\mathbf{x})$.

Related to our geometrical picture, the infinitesimal generator (3.8) corresponds to a different representation of the transformation group; the set of curves given in figure 3.1 (group orbits) is completely characterized by the field of its tangent vectors and viceversa. This field of tangent vectors is precisely the one defined by $\mathbf{v}$ given in (3.8). Therefore, the study of multi-parameter Lie groups of transformations effectively reduces to the study of infinitesimal generators of one-parameter subgroups.

The infinitesimal generators form a vector space called a *Lie algebra*, which is closed under an additional structure, the *commutator.* The commutator of two generators: $\mathbf{v}_\alpha$ and $\mathbf{v}_\beta$ is another first order operator defined as follows:

$$[\mathbf{v}_\alpha, \mathbf{v}_\beta] = \mathbf{v}_\alpha \mathbf{v}_\beta - \mathbf{v}_\beta \mathbf{v}_\alpha \,. \tag{3.10}$$

The general properties of closure under commutation, antisymmetry, and Jacobi's identity for the commutator of elements of the Lie algebra are summarized in *Lie's Second and Third Fundamental Theorems*, which I briefly paraphrase here: The commuator of any two infinitesimal generators of an r-parameter Lie group of transformations is also an infinitesimal generator,

$$[\mathbf{v}_\alpha, \mathbf{v}_\beta] = C^\gamma_{\alpha\beta} \mathbf{v}_\gamma \,, \tag{3.11}$$

where the coefficients $C^\gamma_{\alpha\beta}$ are constants called *structure constants.* (In 3.11 I am assuming the usual convention of summation over a repeated index). The structure constants satisfy the relations

$$C^\gamma_{\alpha\beta} = -C^\gamma_{\beta\alpha}, \tag{3.12}$$

$$C^\rho_{\alpha\beta} C^\delta_{\rho\gamma} + C^\rho_{\beta\gamma} C^\delta_{\rho\alpha} + C^\rho_{\gamma\alpha} C^\delta_{\rho\beta} = 0 \,, \tag{3.13}$$

which come out as an immediate consequence of the basic properties of the commutator mentioned above. Knowledge of the structure constants amounts to a complete picture of the structure of the algebra. This algebraic properties will be of crucial importance when reducing the number of independent variables of a PDE. I will discuss additional properties of Lie algebras in the next chapter where explicit symmetry groups will be calculated.

In order to implement the infinitesimal criterion of invariance to calculate the symmetry group of a differential equation, we have to realize that the differential equation represents a hypersurface in a high dimensional "extended" space, one that includes as additional coordinates derivatives of the dependent variables with respect to independent variables. This means that we need to "prolong" the basic space $\mathbf{X} \times \mathbf{U}$, representing the independent and dependent variables, to a space which also represents the various partial derivatives occurring in the system. Given a smooth function $\mathbf{u} = f(\mathbf{x}) = f(x^1, \ldots, x^p)$ of p independent variables, there is an induced function $\mathbf{u}^{(n)} = pr^{(n)} f(\mathbf{x})$ called the nth prolongation of f, which is defined by the equations,

$$u_J^\alpha = \partial_J f^\alpha(x) \tag{3.14}$$

giving the numbers needed to represent all the different kth order derivatives of the components of f at a point x. In this compact notation

$$\partial_J f^\alpha(\mathbf{x}) = \frac{\partial^k f^\alpha(\mathbf{x})}{\partial x^{j_1} \partial x^{j_2} \cdots \partial x^{j_k}}$$

with $J = (j_1, \ldots, j_k)$ and $1 \leq j_k \leq p$. There are $q \cdot p_k$ such derivatives for each k, where p_k is the binomial coefficient

$$\begin{pmatrix} p+k-1 \\ k \end{pmatrix}.$$

Thus $pr^{(n)} f$ is a vector whose entries represent the values of f and all its derivatives up to order n at the point x. The total space $\mathbf{X} \times \mathbf{U}^{(n)}$, whose coordinates represent the independent variables, the dependent variables, and the derivatives of the dependent variables up to order n, is called the nth order jet space of the underlying space $\mathbf{X} \times \mathbf{U}$. In the same way, there is an induced local action of G, a local group of transformations, on the n-jet space denoted by $pr^{(n)}G$, and if we take the corresponding infinitesimal generators associated with the group transformations we can also define their prolongation on the n-jet space, denoted by $pr^{(n)}\mathbf{v}$, and explicitly given as

$$pr^{(n)}\mathbf{v} = \mathbf{v} + \sum_{\alpha=1}^{q} \sum_{J} \phi_J^\alpha(\mathbf{x}, \mathbf{u}^{(n)}) \frac{\partial}{\partial u_J^\alpha}\,, \tag{3.15}$$

where the second summation runs over all multi-indices $J = (j_1, \ldots, j_k)$, with $1 \leq j_k \leq p$, $1 \leq k \leq n$; and the coefficient functions ϕ_J^α are given by the following formula:

$$\phi_J^\alpha(\mathbf{x}, \mathbf{u}^{(n)}) = D_J \left(\phi^\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha \right) + \sum_{i=1}^{p} \xi^i u_{J,i}^\alpha\,, \tag{3.16}$$

where $u_i^\alpha = \frac{\partial u^\alpha}{\partial x^i}$, and $u_{J,i}^\alpha = \frac{\partial u_J^\alpha}{\partial x^i}$, and D_J stands for the total derivative operator, which for a given function $P(\mathbf{x}, \mathbf{u}^{(n)})$ has the general form

$$D_i P = \frac{\partial P}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{J} u_{J,i}^\alpha \frac{\partial P}{\partial u_J^\alpha}\;. \tag{3.17}$$

The sum is over all J's of order $0 \leq J \leq n$, with n the highest order derivative appearing in P.

The prolonged vector field (3.15) and the defining relation for the coefficients ϕ_J^α given by (3.16) have a relatively simple, easily computable expression. But this is a misleading impression. Since the derivatives in (3.16) are to be taken with respect to all arguments, using the chain rule, the explicit expressions for the prolongation coefficients become enormously large with increasing values of n, p, and q. This is the main reason why the use of symbolic manipulation programs is so important, to overcome the large number of calculations that appear in Lie group analysis. On the positive side, these calculations are highly algorithmic, and therefore relatively easy to implement. For references to the existing programs for calculating Lie groups of differential equations I refer the reader to the recent review by W. Hereman [Hereman 93].

Now, using the concepts of this chapter, I present without proof, the important infinitesimal condition of invariance for a system of differential equations.

Suppose

$$\Delta_\nu(\mathbf{x}, \mathbf{u}^{(n)}) = 0, \qquad \nu = 1, \ldots, \ell$$

is a system of differential equations defined over $M \subset X \times U$. If G is a local group of transformations acting on M, and

$$pr^{(n)}\mathbf{v}[\Delta_\nu(\mathbf{x}, \mathbf{u}^{(n)})] = 0\ , \qquad \nu = 1, \ldots, \ell \tag{3.18}$$

whenever $\Delta(\mathbf{x}, \mathbf{u}^{(n)}) = 0$, for every infinitesimal generator $\mathbf{v}$ of G, then G is a symmetry group of the system.

The meaning of condition (3.18) is that $pr^{(n)}\mathbf{v}$ vanishes on the solution set of the system of equations (3.1). This condition assures that $\mathbf{v}$ is an infinitesimal symmetry generator of the transformation (3.2), i.e. that $\mathbf{u}(\mathbf{x})$ is a solution of (3.1) whenever $\tilde{\mathbf{u}}(\tilde{\mathbf{x}})$ is one.

The explicit expression for the infinitesimal condition given above, (3.18), implies a set of equations for the infinitesimals ξ^i and ϕ^α. These are the so-called *determining equations*, which consist of an overdetermined system of linear PDE's whose solutions provide the infinitesimal generators of the symmetries and therefore the complete symmetry group allowed by the system of PDE's under consideration. The fact that the determining equations are linear is one of the main reasons why Lie's infinitesimal techniques are so appealing, even though the number of equations and the number of terms in them could be very large.

When solving the system of determining equations, one of the following three possibilities may ocurr: (1) The system of equations has only a trivial solution. In this case $\xi^i = 0$ and $\phi^\alpha = 0$ for $i = 1, \ldots, p$; $\alpha = 1, \ldots, q$, and the symmetry group is trivial. (2) The solution of the system depends on r significant arbitrary constants. Therefore one obtains an r-dimensional symmetry group of Lie point symmetries. (3) The solution depends on arbitrary functions of some (or all) of the independent variables x_i. This would mean an infinite dimensional symmetry algebra and symmetry group for the original system of PDE's.

In practice, most systems of differential equations modelling physical phenomena, do have a symmetry group different from the trivial one. In the following chapter I will present examples of symmetry groups corresponding to cases (2) and (3) above.

3.3 Generalized Symmetries

In this section I will explore the basic consequences of the natural generalization of the concept of Lie point symmetry discussed above. So far all the symmetry groups of differential equations considered have been local transformation groups acting geometrically on the space of independent and dependent variables. This means that for a vector field of the form

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi^\alpha(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial u^\alpha} \tag{3.19}$$

defined on the space of independent and dependent variables $\mathbf{X} \times \mathbf{U}$, the coefficient functions ξ^i and ϕ^α depend only on x and u, and therefore $\mathbf{v}$ will generate a local one-parameter group of transformations $\exp(\epsilon\mathbf{v})$ acting pointwise on the underlying space.

A natural generalization of this notion of symmetry group is obtained by relaxing the geometrical assumption, and allowing the coefficient functions ξ^i and ϕ^α to also depend on derivatives of u. E. Noether was the first to recognize this significant extension of the application of symmetry group methods by including derivatives of the dependent variables in the infinitesimal generators of the transformations in her trascendental paper [Noether 18]. More recently, these generalized symmetries have proved to be of importance in the study of nonlinear wave equations, where it appears that the possession of an infinite number of such symmetries is a characterizing property of "solvable" equations.

Following the generalization described above we define a generalized vector field in a form analogous to (3.19) as

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(\mathbf{x}, \mathbf{u}^{(n)}) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi^\alpha(\mathbf{x}, \mathbf{u}^{(n)}) \frac{\partial}{\partial u^\alpha} \tag{3.20}$$

in which ξ^i and ϕ^α are smooth differential functions of their arguments. For simplicity we will denote this argument with a square bracket $[u]$, meaning for an arbitrary function $F[u]$ that F depends on $\mathbf{x}$, $\mathbf{u}$ and derivatives of $\mathbf{u}$ up to an order n. A generalized vector field can be treated as if it were an ordinary vector field. Thus, we can define the prolonged generalized vector field

$$pr^{(n)}\mathbf{v} = \mathbf{v} + \sum_{\alpha=1}^{q} \sum_{J \leq n} \phi_J^\alpha[\mathbf{u}] \frac{\partial}{\partial u_J^\alpha} \tag{3.21}$$

whose coefficients are determined by the formula

$$\phi_J^\alpha = D_J \left(\phi^\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha \right) + \sum_{i=1}^{p} \xi^i u_{J,i}^\alpha \,, \tag{3.22}$$

in complete analogy with (3.15) and (3.16).

Now I introduce another fundamental concept that is extremely useful when dealing with generalized symmetries. This is the concept of the *characteristic* of a symmetry. It is defined as follows: Given $\mathbf{v}$ as in (3.19), let

$$Q^\alpha(\mathbf{x}, \mathbf{u}^{(1)}) = \phi^\alpha(\mathbf{x}, \mathbf{u}) - \sum_{i=1}^{p} \xi^i(\mathbf{x}, \mathbf{u}) u_i^\alpha \,, \quad \alpha = 1, \ldots, q, \tag{3.23}$$

then the q-tuple $Q(\mathbf{x}, \mathbf{u}^{(1)}) = (Q_1, \ldots, Q_q)$ is referred to as the *characteristic* of the vector field $\mathbf{v}$. Of course this concept can be trivially extended to generalized vector fields, and among all of them, we will be studying those in which the coefficients $\xi^i[u]$ of the $\partial/\partial x^i$ are zero. These will play a distinguished role and therefore will be defined separately as follows. The generalized vector field of the form

$$\mathbf{v}_Q = \sum_{\alpha=1}^{q} Q^\alpha[u] \frac{\partial}{\partial u^\alpha} \tag{3.24}$$

is called an *evolutionary vector field*, and the q-tuple $Q[u] = (Q_1[u], \ldots, Q_q[u])$ is called its *characteristic*.

Note that according to (3.21) and (3.22), the prolongation of an evolutionary vector field takes a particularly simple form:

$$pr\ \mathbf{v}_Q = \sum_{\alpha,J} D_J Q^\alpha \frac{\partial}{\partial u_J^\alpha} \,. \tag{3.25}$$

Then we can conclude that any generalized vector field $\mathbf{v}$ as in (3.20) has an associated evolutionary representative $\mathbf{v}_Q$ in which the characteristic Q is given by (3.23), and these two generalized vector fields determine essentially the same symmetry.

It is pertinent to distinguish between the point symmetries, introduced in the previous section, and the true generalized symmetries, presently being discussed, by referring to the former as geometric symmetries since they act geometrically on the underlying space $\mathbf{X} \times \mathbf{U}$. Notice that every geometric symmetry also has an evolutionary representative with characteristic depending on at most first order derivatives, as in (3.23). However, not every first order evolutionary symmetry comes from a geometrical group of transformations; the characteristic, in this particular case, must be of the specific form

$$Q^\alpha = \phi^\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha \qquad (3.26)$$

with ξ^i and ϕ^α depending only on x and u, which of course is a very specialized case.

In principle, the computation of generalized symmetries of a given system of differential equations proceeds in the same way as the earlier computations of geometric symmetries. In particular, we have a direct analogue of the infinitesimal criterion given in (3.18) that can be stated as follows: A generalized vector field $\mathbf{v}$ is a generalized infinitesimal symmetry of a system of differential equations

$$\Delta_\nu[u] = \Delta_\nu(\mathbf{x}, \mathbf{u}^{(n)}) = 0, \qquad \nu = 1, \ldots, l,$$

if and only if

$$pr\ \mathbf{v}[\Delta_\nu] = 0, \qquad \nu = 1, \ldots, l, \qquad (3.27)$$

for every smooth solution $\mathbf{u} = f(\mathbf{x})$.

The simplest method for calculating generalized symmetries is based on the idea of displaying the symmetry in evolutionary form $\mathbf{v}_Q$, effectively reducing the number of unknown functions from $p+q$ to just q, while simultaneously simplifying the computation of the prolongation $pr\ \mathbf{v}_Q$, given by (3.25). Next, one must fix *a priori* the order of derivatives on which the characteristic $Q(\mathbf{x}, \mathbf{u}^{(m)})$ may depend. Of course, there is a basic trade-off in this regard due to practical limitations, on the one hand the more derivatives of u that Q depends on, the greater the possibility for finding generalized symmetries. On the other hand, the larger we take m the more tedious and time-consuming the solution of the ensuing symmetry equations becomes. Therefore, one usually starts with a small value of m and tries to obtain information about the general form of the symmetries. Evidently, such an approach cannot find all generalized symmetries, but the knowledge of a few generalized symmetries can be of tremendous value for reducing the dynamics of a nonlinear system when they are related to conservation laws through Noether's theorem.

Above we have reviewed a topic that is the subject of current research, with a growing number of papers in the literature (see [Vinogradov 84], [Olver 93], and the latest review [Ibragimov 94]). I have only presented the basic ideas leading to practical tools necessary for computations of symmetries for physical systems. In the following chapter I will present explicit symmetry groups for the plasma-fluid models discussed before. Later, I will get back to generalized symmetries in chapter 6.

Chapter 4

Symmetries of Plasma-Fluid Models

In this chapter I will present symmetry groups admitted by Hazeltine's inclusive nonlinear system HTFM and by the CHM equation. Besides the fundamental physical relation between the two models, CHM being the electrostatic, adiabatic limit of the inclusive system, they can be seen as prototypical examples of the two types of Lie groups that one usually encounters when calculating the symmetries of systems of nonlinear PDE's. On the one side, Hazeltine's inclusive model admits an infinite dimensional group of point transformations, characterized by a set of arbitrary functions of time and space. On the other side, the more restricted, single nonlinear equation of CHM, admits a finite six parameter group of point transformations. These two cases cover a wide spectrum of possibilities for symmetry reduction, and therefore, will serve the purpose of showing the power of Lie's analytical methods to determine and classify invariant solutions for systems of PDE's. I will use the simpler group for CHM as a working example throughout this chapter, as a vehicle for developing some general results in the theory of Lie algebras. Exploiting the algebraic structure of the symmetry generators constitutes the strongest point of the theory, which will be done in the remainder of the chapter.

4.1 Lie Point Symmetries for CHM

The Charney-Hasegawa-Mima equation (CHM) has been intensively studied, both in the context of geophysical fluid dynamics (GFD) and plasma physics, as a prime example of a nonlinear system that describes large scale, coherent structures of the solitary wave type (for an updated review see [Horton-Hasegawa 94]). Unlike the family of fully integrable equations, the KdV equation and the like, the CHM equation has eluded extensive analytical analysis due in part to the nature of its nonlinear term: the so-called Jacobian (or Poisson bracket) nonlinearity, that is typical of fluid descriptions of continuous media. To see this more clearly, let us take a look at the general form of the CHM equation

$$\frac{\partial}{\partial t}(\nabla_\perp^2 \varphi - \varphi) - v_d \frac{\partial \varphi}{\partial y} + [\varphi, \nabla_\perp^2 \varphi] = 0\,. \tag{4.1}$$

Note that the the last term of the equation describes the only nonlinearity, the Poisson bracket defined in (2.75). These kind of nonlinear equations are in general nonintegrable, and have been solved only under very special conditions. Lie group analysis will allow us to study a number of systematic symmetry reductions, from the original nonlinear PDE to a PDE with a reduced number of independent variables, and also to nonlinear and linear ODE's.

To begin the calculation of symmetries for CHM, we need to establish a practical form for it. Note that (4.1) does not have the exact form of the equation derived as the limit of Hazeltine's inclusive system, recall (2.113). Actually, their relation is very simple. It can be understood as a change of reference frame. If we change frames, from the natural ("lab") coordinate system of (4.1), to a frame moving with the constant velocity v_d, we would get a modified version of (4.1), in the new coordinate system which is exactly (2.113). The net effect of this transformation is to hide the inhomogeneous term, proportional to v_d, in the new definition of the coordinate along the line of motion of the moving frame. More precisely, we effect a Galilean transformation of the form

$$y' = y - v_d t\,, \qquad \varphi' = \varphi - v_d x\,. \tag{4.2}$$

This transformation implies, using $x' = x$, and $t' = t$,

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - v_d \frac{\partial}{\partial y'}\,; \tag{4.3}$$

therefore (4.1) reduces to the more symmetric form

$$\frac{\partial}{\partial t'}(\nabla_\perp^2 \varphi' - \varphi') + [\varphi', \nabla_\perp^2 \varphi'] = 0\,, \tag{4.4}$$

where ∇ and the bracket are defined in terms of the $x' - y'$ coordinates. This is the form of the CHM equation that we will be studying, and therefore, from now on, we will drop the primes, keeping in mind the meaning of the new coordinates.

Now we turn our attention to the actual calculation of the Lie point symmetries admitted by the CHM equation using the ideas exposed in the previous chapter. First, consider (4.4). It is a nonlinear, third order, PDE, with a three dimensional space, $\mathbf{X}$, representing the independent variables: $\mathbf{x} = (x, y, t)$, and a one dimensional space, $\mathbf{U}$, for the single dependent variable: $\mathbf{u} = (\varphi)$. These two spaces define the underlying base space $\mathbf{X} \times \mathbf{U}$, upon which the infinitesimal transformations act. I would like to emphasize that throughout the symmetry group calculation, the PDE will be taken as an isolated mathematical object, independent of any particular physical situation that, typically, enters in through initial and/or boundary conditions. These will be consider *aposteriori*; the symmetry group will first be used to reduce the equation by adjusting the free parameters or functions left in the problem. I will come back to this point in chapter 5.

The strategy consists of finding transformations of the dependent and independent variables for which the CHM equation remains invariant. As was shown in the last chapter, this can be done by considering the infinitesimal version of the transformation, recall (3.3)

$$\begin{aligned}
\tilde{x} &= x + \epsilon\xi^x(x, y, t, \varphi) \\
\tilde{y} &= y + \epsilon\xi^y(x, y, t, \varphi) \\
\tilde{t} &= t + \epsilon\xi^t(x, y, t, \varphi) \\
\tilde{\varphi} &= \varphi + \epsilon\phi^\varphi(x, y, t, \varphi)\,,
\end{aligned} \tag{4.5}$$

which is generated by the vector field

$$\mathbf{v} = \xi^x\frac{\partial}{\partial x} + \xi^y\frac{\partial}{\partial y} + \xi^t\frac{\partial}{\partial t} + \phi^\varphi\frac{\partial}{\partial\varphi}\,. \tag{4.6}$$

This infinitesimal generator was shown to be equivalent to the finite Lie group of transformations through the exponential relation $\tilde{\mathbf{x}} = e^{\epsilon\mathbf{v}}\mathbf{x}$. Given (4.4) in the abstract condensed form

$$\Delta(x, y, t, \varphi, \varphi_x, \varphi_y, \varphi_t, \ldots, \varphi_{yyt}) = 0\,, \tag{4.7}$$

we apply the infinitesimal condition of invariance, (3.18). For $\mathbf{v}$ to be an infinitesimal symmetry generator of the equation (4.7) it must satisfy

$$pr^{(3)}\mathbf{v}[\Delta(x, y, t, \ldots, \varphi_{yyt})] = 0, \tag{4.8}$$

$$\text{on} \qquad \Delta(x, y, t, \ldots, \varphi_{yyt}) = 0\,, \tag{4.9}$$

where $pr^{(3)}\mathbf{v}$ is the third prolongation of the vector field $\mathbf{v}$, consistent with the order of the equation.

The explicit calculation of the determining equations for ξ and ϕ, which are an immediate consequence of expanding (4.8) subject to (4.9), was carried away using a symbolic manipulation program, called SYMMGRP.MAX, developed for MACSYMA by B. Champagne, W. Hereman and P. Winternitz (see [Champagne et al. 91]). This program calculates the determining equations, a very large set of *linear* PDE's, and can be used interactively to solve them explicitly, step by step on the computer, by means of a feedback mechanism.

For the CHM equation (4.4), the general solution of the determining system is

$$\xi^x(x,y,t,\varphi) = c_1 + c_5 y \tag{4.10}$$

$$\xi^y(x,y,t,\varphi) = c_2 - c_5 x \tag{4.11}$$

$$\xi^t(x,y,t,\varphi) = c_3 + c_6 t \tag{4.12}$$

$$\phi^\varphi(x,y,t,\varphi) = c_4 - c_6 \varphi \tag{4.13}$$

where the c's are arbitrary constants. Thus, the CHM equation is invariant under a six parameter group of transformations. Each of the six infinitesimal generators will be obtained from (4.10)-(4.13), by taking one of the constants equal to unity and all the others equal to zero, together with the definition of $\mathbf{v}$ (4.6), yielding

$$\begin{gathered} \mathbf{v}_1 = \frac{\partial}{\partial x}, \qquad \mathbf{v}_2 = \frac{\partial}{\partial y}, \qquad \mathbf{v}_3 = \frac{\partial}{\partial t}, \\ \mathbf{v}_4 = \frac{\partial}{\partial \varphi}, \qquad \mathbf{v}_5 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \qquad \mathbf{v}_6 = t\frac{\partial}{\partial t} - \varphi\frac{\partial}{\partial \varphi}. \end{gathered} \tag{4.14}$$

Each one of these infinitesimal generators corresponds to a finite transformation that leaves (4.4) invariant. This can be easily seen by solving the initial value problem referred to as Lie's First Fundamental Theorem (see (3.6) and (3.7)). Following this procedure, and writing only the variables that are changed by the transformation, we obtain

$$
\begin{aligned}
\mathbf{v}_1 &= \frac{\partial}{\partial x} &&\Longrightarrow\quad \tilde{x} = x + \epsilon\\
\mathbf{v}_2 &= \frac{\partial}{\partial y} &&\Longrightarrow\quad \tilde{y} = y + \epsilon\\
\mathbf{v}_3 &= \frac{\partial}{\partial t} &&\Longrightarrow\quad \tilde{t} = t + \epsilon\\
\mathbf{v}_4 &= \frac{\partial}{\partial \varphi} &&\Longrightarrow\quad \tilde{\varphi} = \varphi + \epsilon\\
\mathbf{v}_5 &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} &&\Longrightarrow\quad \begin{cases} \tilde{x} = x\cos\epsilon + y\sin\epsilon \\ \tilde{y} = -x\sin\epsilon + y\cos\epsilon \end{cases}\\
\mathbf{v}_6 &= t\frac{\partial}{\partial t} - \varphi\frac{\partial}{\partial \varphi} &&\Longrightarrow\quad \begin{cases} \tilde{t} = e^{\epsilon} t \\ \tilde{\varphi} = e^{-\epsilon}\varphi \end{cases}
\end{aligned}
$$

The meaning of the induced finite transformations is clear: the first two imply invariance of the CHM equation under space translation, of x and y, respectively. The third transformation implies invariance under time translation. The fourth, is a simple gauge transformation for the φ field, which although trivial from the point of view of the physics described by the CHM equation, is an important part of the complete group of symmetries, and should not be overlooked. The fifth transformation is rotational invariance in the $x-y$ plane. Finally, the sixth transformation implies invariance under a change of scale involving t and φ. I would like to emphasize that the six transformations shown above constitute the complete symmetry group of point transformations for the CHM equation, and in fact, most (perhaps all) of these transformations could have been guessed by inspection of (4.4). However, the fact that we can be sure none of the symmetries of the group have been missed, and that we know their exact form, proves the value of using the powerful symbolic manipulation program. This subject will be of critical importance when analyzing more complicated systems.

4.1.1 How to Use Symmetries

The first and probably most obvious application of symmetry group transformations is the mapping of solutions into other solutions of a PDE. The point is that, by construction, the symmetry transformations leave the equation invariant, and therefore the set of solutions remains invariant. Let an arbitrary solution of the CHM equation be denoted as

$$\varphi = \Phi(x, y, t) \tag{4.15}$$

and take, as an example, the transformation generated by $\mathbf{v}_1 = \partial_x$. According to the finite transformation shown above, we have

$$\tilde{x} = x + \epsilon\,, \qquad \tilde{y} = y\,, \qquad \tilde{t} = t\,, \qquad \tilde{\varphi} = \varphi\,. \tag{4.16}$$

Thus, if we transform a particular solution (4.15) by $\mathbf{v}_1$, as in (4.16), we will obtain

$$\tilde{\varphi} = \Phi(\tilde{x} - \epsilon, \tilde{y}, \tilde{t}). \tag{4.17}$$

But we know that, by the definition of symmetry, the new variables (tilde) also satisfy the CHM equation. We therefore conclude that if $\Phi(x, y, t)$ is a solution of CHM, then so must be $\Phi(x - \epsilon, y, t)$, for any real number ϵ. Repeating this calculation with the generators: $\mathbf{v}_2 = \partial_y$, $\mathbf{v}_3 = \partial_t$, $\mathbf{v}_4 = \partial_\varphi$, $\mathbf{v}_5 = y\partial_x - x\partial_y$ and $\mathbf{v}_6 = t\partial_t - \varphi\partial_\varphi$, we obtain that, if (4.15) is a solution of the CHM equation, then so are the functions

$$\begin{aligned}
\varphi^{(2)} &= \Phi(x, y-\epsilon, t), \\
\varphi^{(3)} &= \Phi(x, y, t-\epsilon), \\
\varphi^{(4)} &= \Phi(x, y, t) + \epsilon, \\
\varphi^{(5)} &= \Phi(x\cos\epsilon - y\sin\epsilon, x\sin\epsilon + y\cos\epsilon, t), \\
\varphi^{(6)} &= e^{-\epsilon}\Phi(x, y, e^{-\epsilon}t),
\end{aligned}$$

where ϵ is any real number. Of course, the most general one-parameter group of symmetries is obtained by considering a general linear combination $c_1\mathbf{v}_1 + \ldots + c_6\mathbf{v}_6$ of the given generators, (4.14). Then, an arbitrary group transformation g, can be represented in the form

$$g = exp(\epsilon_6\mathbf{v}_6) \cdot \ldots \cdot exp(\epsilon_1\mathbf{v}_1), \tag{4.18}$$

which implies that the most general solution obtainable from a given solution $\varphi = \Phi(x, y, t)$ by group transformations, takes the form

$$\varphi = e^{-\epsilon_6}\Phi(x\cos\epsilon_5 - y\sin\epsilon_5 - \epsilon_1, x\sin\epsilon_5 + y\cos\epsilon_5 - \epsilon_2, e^{-\epsilon_6}t - \epsilon_3) + \epsilon_4. \tag{4.19}$$

The ability to transform solutions into solutions is sometimes useful by itself. The transformation of even trivial solutions can yield nontrivial results. This concept will be used explicitly when treating the more complicated three field model.

Now, in order to make a systematic use of the algebraic structure associated with the infinitesimal generators, we turn our attention to the determination and classification of group invariant solutions, where the symmetry group methods show their full power. We start out by considering the Lie algebra $\mathcal{G}^6$, defined by the infinitesimal generators (4.14). It is a solvable Lie algebra, composed by the direct sum of the two-dimensional Euclidean Lie algebra (space translations and rotations), $E(2)$, and the algebra associated with time translation P_0, gauge of φ, P_φ, and the scaling (dilation) of t and φ, $D_{t\varphi}$. We can write $\mathcal{G}^6$ as:

$$\mathcal{G}^6 = E(2) \oplus \{\{P_0 \oplus P_\varphi\} \ominus D_{t\varphi}\}, \tag{4.20}$$

where $\oplus$ denotes the direct sum, and $\ominus$ a semi-direct sum. Recall that $E(2) = \{P_x, P_y\} \ominus SO(2)$. Some basic properties of the algebra $\mathcal{G}^6$ are contained in the commutation relations between the infinitesimal generators, which are compiled in table 4.1. One of the most important of these properties, is the closure under commutation, which is a consequence of the correspondence between generators and transformation groups. If $\mathbf{v}_i$ and $\mathbf{v}_j$ are any two generators, then their commutator, defined as

$$[\mathbf{v}_i, \mathbf{v}_j] = \mathbf{v}_i\mathbf{v}_j - \mathbf{v}_j\mathbf{v}_i = C_{ij}^k\mathbf{v}_k, \tag{4.21}$$

is a linear combination of all the generators. This result will be of utmost importance when calculating group invariant solutions.

We have shown above how to use a symmetry to calculate solutions for the differential equation from known solutions. But we can also use the algebraic properties of the generators $\mathbf{v}_i$, to calculate particular solutions of a PDE, called group invariant or "similarity" solutions. These are solutions that remain invariant under the action of a subgroup of the symmetry group of the equation. This condition of invariance imposes additional constraints on

	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$	$\mathbf{v}_6$
$\mathbf{v}_1$	0	0	0	0	$-\mathbf{v}_2$	0
$\mathbf{v}_2$	0	0	0	0	$\mathbf{v}_1$	0
$\mathbf{v}_3$	0	0	0	0	0	$\mathbf{v}_3$
$\mathbf{v}_4$	0	0	0	0	0	$-\mathbf{v}_4$
$\mathbf{v}_5$	$\mathbf{v}_2$	$-\mathbf{v}_1$	0	0	0	0
$\mathbf{v}_6$	0	0	$-\mathbf{v}_3$	$\mathbf{v}_4$	0	0

Table 4.1: Commutation relations for the algebra $\mathcal{G}^6$ of the CHM equation. The entry in row i and column j represents $[\mathbf{v}_i, \mathbf{v}_j]$.

the original equation, expressed as first order linear PDE's, and therefore, the group invariant solution will satisfy the original equation plus the additional constraints, leading to the so-called symmetry reduction. What this means is that the original equation will be expressed as a transformed equation with a reduced number of independent variables, defined by the equations of constraint. In order to study this systematic reduction we need to introduce some additional concepts from the theory of Lie algebras.

I will use the following example to motivate the concept of a group invariant solution. For simplicity, consider the simple case of translational invariance along the x-axis, given by the infinitesimal generator $\mathbf{v}_1 = \partial_x$. If Φ denotes a solution of the CHM equation, we would like to consider the special case for which $\mathbf{v}_1\Phi = 0$. Since $\mathbf{v}_1$ has a single component in the x-direction, then the condition of invariance under $\mathbf{v}_1$; i.e. $\mathbf{v}_1\Phi = 0$, will be satisfied only if $\Phi = \Phi(y, t, \varphi)$. That is, solutions invariant under the transformation generated by $\mathbf{v}_1$ are independent of x and take the form $\varphi = \Phi(y, t)$.

For an arbitrary generator $\mathbf{v}$, the invariant solutions are calculated by a method that amounts to finding the special coordinates in (x, y, t, φ) space for which $\mathbf{v}$ takes the *canonical form* of a simple translation in the new frame. This is easily done by the method of characteristics, where one determines the invariant functions of the generator $\mathbf{v}$, called differential invariants, and uses this information to reduce the number of independent variables of the equation. This yields an invariant solution under the generator $\mathbf{v}$.

To illustrate the method, lets calculate the invariant solutions corresponding to the generator $a_2\mathbf{v}_2+\mathbf{v}_3$ of the CHM equation, with a_2 an arbitrary constant. The generator has the form

$$a_2\mathbf{v}_2 + \mathbf{v}_3 = a_2\frac{\partial}{\partial y} + \frac{\partial}{\partial t}, \tag{4.22}$$

which corresponds to the translation group on the space of independent variables y and t. In general, we can expect travelling wave solutions to arise as a consequence of using a translation group on a given PDE. The form of the generator (4.22), implies the characteristic equations

$$\frac{dy}{a_2} = dt\,. \tag{4.23}$$

Integration of these equations yields the differential invariants

$$\eta_1 = x, \qquad \eta_2 = y - a_2 t, \qquad \eta_3 = \varphi. \tag{4.24}$$

If we choose η_3 as the new dependent variable ζ, and η_1 and η_2 as the new independent variables, then the invariant solution takes the form $\zeta = F(\eta_1, \eta_2)$, where F is a function to be determined by substitution into the transformed CHM equation, which depends on one less independent variable. In terms of the original variables the invariant solution takes the form

$$\varphi = F(x, y - a_2 t)\,. \tag{4.25}$$

Solving for the derivatives of φ with respect to x, y and t in terms of those of ζ with respect to η_1 and η_2, we find

$$\frac{\partial\varphi}{\partial t} = -a_2\frac{\partial\zeta}{\partial\eta_2}\,, \qquad \frac{\partial\varphi}{\partial x} = \frac{\partial\zeta}{\partial\eta_1}\,, \qquad \frac{\partial^2\varphi}{\partial x^2} = \frac{\partial^2\zeta}{\partial\eta_1^2}\,, \tag{4.26}$$

and so on. Substituting these expressions into the CHM equation, yields the reduced PDE for the special travelling wave solutions,

$$-\,a_2\frac{\partial}{\partial\eta_2}\left(\nabla_\eta^2\zeta\right) + a_2\frac{\partial\zeta}{\partial\eta_2} + [\zeta, \nabla_\eta^2\zeta]_\eta = 0\,, \tag{4.27}$$

where the subscript η implies that the operators ∇^2 and $[f, g]$ are taken in the space with η_1 and η_2 as independent variables. This nonlinear PDE can be rewritten, neglecting the subscript η, as

$$\Big[\zeta - a_2\eta_1\,,\ \nabla^2\zeta - a_2\eta_1\Big] = 0\,. \tag{4.28}$$

This single bracket equation has a general solution of the form

$$\nabla^2\zeta - a_2\eta_1 = f(\zeta - a_2\eta_1)\,, \tag{4.29}$$

where f is an arbitrary function. A solitary vortex or "modon" solution was first discovered by Stern [Stern 75] and independently by Larichev and Reznik [Larichev-Reznik 76], for an equation similar to (4.28) in the context of geophysical fluid dynamics, by imposing a condition for localized solutions to the general solution (4.29). This condition reduces the problem of finding a solution of (4.29) to a piecewise linear one, by choosing f as a linear function of its argument, with different coefficients inside and outside the circle of radius $r = a$. Introducing polar coordinates defined as: $r^2 = \eta_1^2 + \eta_2^2$ and $\tan\theta = \eta_1/\eta_2$, and taking the limit $\zeta \to 0$ as $y \to \infty$, for each x, which implies using a linear function for f, we obtain for (4.29) the form

$$\nabla^2\zeta = \zeta, \qquad r \geq a, \tag{4.30}$$

$$\nabla^2\zeta = -k^2\zeta + a_2(k^2+1)\eta_1, \qquad r \leq a, \tag{4.31}$$

where k is an arbitrary constant. The famous dipole solution of these equations is given by

$$\zeta = \begin{cases} a_2\left[a\frac{1}{k^2}\frac{J_1(kr)}{J_1(ka)} - r\left(1+\frac{1}{k^2}\right)\right]\sin\theta, & r \leq a, \\ -a_2 a\frac{K_1(r)}{K_1(a)}\sin\theta, & r \geq a, \end{cases} \tag{4.32}$$

where J_1 is a first-order Bessel function, and K_1 is a modified first-order Bessel function. This dipole vortex solution has properties that makes it somewhat similar to a soliton, for instance, its size depends on the translational speed, and it is stable with respect to collisions (see [Flierl et al. 80] and [Meiss-Horton 83]).

It is pertinent to make some comments on the procedure that we have followed to derive the group invariant solution of Larichev and Reznik (4.32), which, by the way, represents the only known analytical family of solutions for the CHM equation (The other members of the family are obtained from the same reduction of the equation (4.29), only imposing different conditions on f at infinity, see [Petviashvili-Pokhotelov 92]). First, notice that we have used a very simple subgroup (4.22), of the complete six-parameter group for the CHM equation. What this means is that in principle we have available a large number of potentially useful symmetry reductions to analyze the solution space of the CHM equation. Actually, as many as the number of different subgroups that we can construct by combining the six generators. However, we don't know if the group invariant solutions from different subgroups are all fundamentally different, i. e., invariant solutions which are not related by a transformation in the full symmetry group, as was shown before when transforming solutions into solutions. The answer to this question constitutes the so-called classification of group-invariant solutions, which is a systematic approach leading to an *optimal system* of group-invariant solutions, from which every other such solution can be derived. This implies that usually we won't need all subgroups of $\mathcal{G}$ to span the space of possible solutions, but only a selected "optimal system" of them. The crucial concept to generate this classification is the study of the adjoint representation of the symmetry group acting on its Lie algebra. In the next section I will develop this powerful classification scheme.

4.1.2 Classification of Solutions: The Optimal System

The basic concept underlying the classification scheme for group-invariant solutions can be simply stated as follows: Let G be the symmetry group of a system of differential equations Δ and let $H \subset G$ be an s-parameter subgroup. If $\varphi = f(\mathbf{x})$ is an H-invariant solution to Δ and $g \in G$ is any other group element, then the transformed function: $\varphi = \tilde{f}(\mathbf{x}) = g \cdot f(\mathbf{x})$ is a $\tilde{H}$-invariant solution, where $\tilde{H} = gHg^{-1}$ is the *conjugate* subgroup to H under g. What this means is that the problem of classifying invariant solutions reduces to the problem of classifying subgroups of the full symmetry group G under conjugation.

Then, for a Lie group G, we define group conjugation as $K_g(h) :=$

ghg^{-1}, for each $g \in G$, $h \in G$. The differential dK_g determines a linear map on the Lie algebra of G, called the *adjoint representation*

$$Ad \;\; g(\mathbf{v}) \equiv dK_g(\mathbf{v}), \qquad \mathbf{v} \in \mathcal{G}. \tag{4.33}$$

This definition has a simple interpretation: if $\mathbf{v} \in \mathcal{G}$ generates the one-parameter subgroup $H = \{\exp(\epsilon\mathbf{v})\}$, then $Ad \;\; g(\mathbf{v})$ is easily seen to generate the conjugate one-parameter subgroup $K_g(H) = gHg^{-1}$.

The problem of finding a classification of subgroups is equivalent to that of finding a classification of subalgebras. This statement can be paraphrased in a formal context by saying that if we take H and $\tilde{H}$ to be Lie subgroups of the Lie group G with corresponding Lie subalgebras $\mathcal{H}$ and $\tilde{\mathcal{H}}$ of the Lie algebra $\mathcal{G}$ of G, then $\tilde{H} = gHg^{-1}$ are conjugate subgroups, if and only if $\tilde{\mathcal{H}} = Ad \;\; g(\mathcal{H})$ are conjugate subalgebras.

In practice we deal with the Lie algebra of the generators, and therefore, we will take advantage of the fact that the adjoint representation of a Lie group on its Lie algebra is most easily constructed from its infinitesimal generators. If $\mathbf{v}$ generates the one-parameter subgroup $\{\exp \epsilon\mathbf{v}\}$, then we let $ad \;\; \mathbf{v}$ be the infinitesimal adjoint action generating the corresponding one-parameter group of adjoint transformations, defined by

$$ad \;\; \mathbf{v}\,|_{\mathbf{w}} := \frac{d}{d\epsilon}\,|_{\epsilon=0}\, Ad(\exp(\epsilon\mathbf{v}))\mathbf{w}, \qquad \mathbf{w} \in \mathcal{G}. \tag{4.34}$$

which can be shown to agree (up to a sign) with the Lie bracket on $\mathcal{G}$

$$ad \;\; \mathbf{v}\,|_{\mathbf{w}} = [\mathbf{w}, \mathbf{v}] = -[\mathbf{v}, \mathbf{w}]. \tag{4.35}$$

Conversely, if we know the infinitesimal adjoint action $ad\ \mathcal{G}$ of a Lie algebra $\mathcal{G}$ on itself, we can reconstruct the adjoint representation $Ad\ G$ of the underlying Lie group, by integrating the system of linear ODE's

$$\frac{d\mathbf{w}}{d\epsilon} = ad\ \mathbf{v}\,|_{\mathbf{w}} = -[\mathbf{v}, \mathbf{w}]\,, \qquad \mathbf{w}(0) = \mathbf{w}_0, \tag{4.36}$$

with solution

$$\mathbf{w}(\epsilon) = Ad(\exp(\epsilon \mathbf{v}))\mathbf{w}_0, \tag{4.37}$$

which can be represented by a matrix $\mathbf{A}$ such that $\mathbf{w}(\epsilon_i) = \mathbf{A}_i^T \mathbf{w}_0$ with $i = 1, \ldots, r$ and r being the number of parameters in the group. The general adjoint transformation is then

$$\tilde{\mathbf{w}} = \mathbf{A}^T \mathbf{w}, \tag{4.38}$$

where $\mathbf{A}^T$ is the transpose of the matrix: $\mathbf{A} = \mathbf{A}_1 \cdot \ldots \cdot \mathbf{A}_r$. This representation will be useful when we treat higher dimensional optimal systems.

Another, perhaps simpler way to reconstruct the adjoint representation is by summing the Lie series

$$\begin{aligned} Ad(\exp(\epsilon \mathbf{v}))\mathbf{w}_0 &= \sum_{n=0}^{\infty} \frac{\epsilon}{n!}(ad\ \mathbf{v}^n(\mathbf{w}_0) \\ &= \mathbf{w}_0 - \epsilon[\mathbf{v}, \mathbf{w}_0] + \frac{\epsilon^2}{2}[\mathbf{v}, [\mathbf{v}, \mathbf{w}_0]] - \ldots. \end{aligned} \tag{4.39}$$

The convergence of this series follows since (4.36) is a linear system of ODE's, for which (4.39) is the corresponding matrix exponential. Therefore, in order to compute the adjoint representation for the Lie algebra of the CHM equation, we use the Lie series (4.39) in conjunction with the commutator Table 4.1. For instance, we can calculate

$$\begin{aligned} Ad(\exp(\epsilon \mathbf{v}_1))\mathbf{v}_5 &= \mathbf{v}_5 - \epsilon[\mathbf{v}_1, \mathbf{v}_5] + \frac{1}{2}\epsilon^2[\mathbf{v}_1, [\mathbf{v}_1, \mathbf{v}_5]] - \cdots \\ &= \mathbf{v}_5 + \epsilon \mathbf{v}_2. \end{aligned} \tag{4.40}$$

A simple geometrical interpretation of the adjoint representation can be given as follows: If the trajectories tangent to $\mathbf{v}_5$ are subjected to a coordinate transformation corresponding to $\mathbf{v}_1$, then the transformed trajectories are tangent to a linear combination of $\mathbf{v}_5$ and $\mathbf{v}_2$. Now, we can regard the trajectories as material lines in a perfect fluid with velocity $\mathbf{v}_1$. Then, the material lines are carried along by the fluid, and the tangents to these lines are *Lie dragged* by the velocity field $\mathbf{v}_1$ in the same way as the magnetic field vector in a perfect fluid. Therefore, in analogy with the induction equation, the evolution for the tangent $\mathbf{w}$ is given by (4.36), whose Taylor-series solution is (4.39). Thus, in the previous example (4.40), an "advection" by $\mathbf{v}_1$ drags $\mathbf{v}_5$ into $\mathbf{v}_5 + \epsilon\mathbf{v}_2$. Following in the same manner, the adjoint operations for all generators of the CHM equation are calculated by summing up the Lie series (4.39). The results are given in Table 4.2.

Now we are ready to introduce the concept of an optimal system. We have already mentioned the equivalence between the classification of subgroups and subalgebras, and so we concentrate on the latter. Then, a list of s-parameter subalgebras forms an *optimal system*, if every s-parameter subalgebra of $\mathcal{G}$ is equivalent to a unique member of the list under some element of the adjoint representation: $\tilde{\mathcal{H}} = Ad\ g(\mathcal{H}),\quad g \in G$, and no two algebras in the list are conjugate to each other. This concept leads to a natural classification of the algebra into conjugacy classes, where the union of single representatives for each conjugacy class, of given dimensionality r, defines the optimal system of order r, denoted by the symbol Θ_r. Each member of the n-dimensional optimal system Θ_n is a collection of n linear combinations of generators of the algebra $\mathcal{G}$.

Ad	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$	$\mathbf{v}_6$
$\mathbf{v}_1$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5 + \epsilon\mathbf{v}_2$	$\mathbf{v}_6$
$\mathbf{v}_2$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5 - \epsilon\mathbf{v}_1$	$\mathbf{v}_6$
$\mathbf{v}_3$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$	$\mathbf{v}_6 - \epsilon\mathbf{v}_3$
$\mathbf{v}_4$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$	$\mathbf{v}_6 + \epsilon\mathbf{v}_4$
$\mathbf{v}_5$	$\mathbf{v}_1 \cos\epsilon$ $+\mathbf{v}_2 \sin\epsilon$	$\mathbf{v}_2 \cos\epsilon$ $-\mathbf{v}_1 \sin\epsilon$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$	$\mathbf{v}_6$
$\mathbf{v}_6$	$\mathbf{v}_1$	$\mathbf{v}_2$	$e^{\epsilon}\mathbf{v}_3$	$e^{-\epsilon}\mathbf{v}_4$	$\mathbf{v}_5$	$\mathbf{v}_6$

Table 4.2: Adjoint table for the algebra $\mathcal{G}^6$ of the CHM equation. The (i, j) entry represents $\mathrm{Ad}(\exp\,(\epsilon\mathbf{v}_i))\mathbf{v}_j$.

In general, the problem of finding an optimal system of subalgebras for a given dimensionality r can be quite complicated, although for small dimensionality $r \leq 3$, it can be done relatively easily. For Lie algebras with additional structure, like simple, semisimple, Levi decomposed algebras, etc., sophisticated techniques are available to find the appropriate classification method (see [Winternitz 90] and references therein). Here, I will be concerned with optimal systems of first and second order only, which will lead, respectively, to single and double reductions of the number of independent variables for the PDE in question.

In order to calculate the first order optimal system Θ_1 for the algebra of the CHM equation, I will use the simple (and naive) approach of taking a general element $\mathbf{v}$ in $\mathcal{G}$ given as a linear combination of all the generators of symmetries for the CHM equation, namely

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + a_4\mathbf{v}_4 + a_5\mathbf{v}_5 + a_6\mathbf{v}_6, \tag{4.41}$$

where the a_i's are arbitrary constants, and subjecting this element to judicious applications of adjoint transformations, defined in Table 4.2, so as to simplify it as much as possible. Using this procedure I present next the explicit calculation of the elements of the optimal system of first order Θ_1 for the algebra $\mathcal{G}$ of the CHM equation.

Suppose first that $a_6 \neq 0$ in (4.41). We can assume that $a_6 = 1$ without loss of generality (using a simple scaling of $\mathbf{v}$ if necessary). Referring to Table 4.2, if we act on $\mathbf{v}$ by $Ad(\exp(-a_4\mathbf{v}_4))$, we can make the coefficient of $\mathbf{v}_4$ vanish:

$$\mathbf{v}' = Ad(\exp(-a_4\mathbf{v}_4))\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + a_5\mathbf{v}_5 + \mathbf{v}_6. \tag{4.42}$$

Next we act on $\mathbf{v}'$ by $Ad(\exp(a_3\mathbf{v}_3))$ to cancel the coefficient of $\mathbf{v}_3$, leading to

$$\mathbf{v}'' = Ad(\exp(a_3\mathbf{v}_3))\mathbf{v}' = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_5\mathbf{v}_5 + \mathbf{v}_6. \tag{4.43}$$

Now we have to consider two possibilities: (i) $a_5 \neq 0$ or (ii) $a_5 = 0$. If we take case (i), $a_5 \neq 0$, the following further reductions take place. Upon acting on $\mathbf{v}''$ by $Ad(\exp(-a_2/a_5\mathbf{v}_1))$ the coefficient of $\mathbf{v}_2$ is seen to vanish

$$\mathbf{v}''' = Ad\left(\exp\left(-\frac{a_2}{a_5}\mathbf{v}_1\right)\right)\mathbf{v}'' = a_1\mathbf{v}_1 + a_5\mathbf{v}_5 + \mathbf{v}_6. \tag{4.44}$$

Finally, we act on $\mathbf{v}'''$ with $Ad(\exp(a_1/a_5\mathbf{v}_2))$ to cancel the coefficient of $\mathbf{v}_1$, so that $\mathbf{v}$ is equivalent to

$$\mathbf{v}^{(iv)'} = Ad\left(\exp\left(\frac{a_1}{a_5}\mathbf{v}_2\right)\right)\mathbf{v}''' = a_5\mathbf{v}_5 + \mathbf{v}_6, \tag{4.45}$$

under the adjoint transformation in case (i), and no further simplification is possible.

For case (ii), we have to go back to (4.43) and consider $a_5 = 0$. From Table 4.2 we see that by acting with $Ad(\exp(\epsilon \mathbf{v}_5))$ on $\mathbf{v}''$ we can either cancel the coefficient of $\mathbf{v}_1$ by choosing $\epsilon = \arctan(a_1/a_2)$, or the coefficient of $\mathbf{v}_2$ by choosing $\epsilon = \arctan(-a_2/a_1)$. We select this second form, leaving as a final reduction

$$\mathbf{v}''' = Ad\left(\exp\left(\arctan\left[-\frac{a_2}{a_1}\right]\mathbf{v}_5\right)\right)\mathbf{v}'' = a_1'\mathbf{v}_1 + \mathbf{v}_6, \tag{4.46}$$

where a_1' is a certain scalar depending on a_1 and a_2, and no other simplification is possible. Therefore, any one-dimensional subalgebra spanned by $\mathbf{v}$ with $a_6 \neq 0$ is equivalent to the subalgebra spanned by $a_5\mathbf{v}_5 + \mathbf{v}_6$ when $a_5 \neq 0$, or to one spanned by $a_1\mathbf{v}_1 + \mathbf{v}_6$ when $a_5 = 0$. These reduced subalgebras are elements of the one dimensional optimal system Θ_1.

The remaining one-dimensional subalgebras, elements of Θ_1, are obtained by following the same procedure as above with the constant $a_6 = 0$. If $a_5 \neq 0$, we scale to make $a_5 = 1$, and then act on $\mathbf{v}$ with $Ad(\exp(-a_2\mathbf{v}_1))$, cancelling the coefficient of $\mathbf{v}_2$, so that $\mathbf{v}$ is equivalent to

$$\mathbf{v}' = a_1\mathbf{v}_1 + a_3\mathbf{v}_3 + a_4\mathbf{v}_4 + \mathbf{v}_5\,. \tag{4.47}$$

We can further simplify $\mathbf{v}'$ by acting with $Ad(\exp(a_1\mathbf{v}_2))$, which cancels the coefficient of $\mathbf{v}_1$, yielding

$$\mathbf{v}'' = a_3\mathbf{v}_3 + a_4\mathbf{v}_4 + \mathbf{v}_5\,. \tag{4.48}$$

Now we use the action of the group generated by $\mathbf{v}_6$, which has the net effect of scaling the coefficients of $\mathbf{v}_3$ and $\mathbf{v}_4$, as follows

$$\mathbf{v}''' = Ad(\exp(\epsilon\mathbf{v}_6))\mathbf{v}'' = a_3e^{\epsilon}\mathbf{v}_3 + a_4e^{-\epsilon}\mathbf{v}_4 + \mathbf{v}_5\,. \tag{4.49}$$

We cannot scale out both coefficients at the same time, therefore we choose to rescale the coefficient of $\mathbf{v}_4$, which depending on the sign of a_4 can be made to take the values of either $+1$, -1 or 0. If a_3 and a_4 are both equal to zero, then $\mathbf{v}$ can be seen to be equivalent to $\mathbf{v}_5$. Thus any one-dimensional subalgebra spanned by $\mathbf{v}$ with $a_6 = 0$ and $a_5 \neq 0$ is equivalent to one spanned by either $a_3\mathbf{v}_3 \pm \mathbf{v}_4 + \mathbf{v}_5$ or $\mathbf{v}_3 + \mathbf{v}_5$ or $\mathbf{v}_5$.

Next we consider the case where $a_6 = a_5 = 0$ and $a_4 = 1$, which simplifies by acting with $Ad(\exp(\arctan[a_1/a_2]\mathbf{v}_5))$ on $\mathbf{v}$, yielding $a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \mathbf{v}_4$, as the irreducible element of Θ_1. Then we consider the case with $a_6 = a_5 = a_4 = 0$, and $a_3 = 1$, which is simplified again by acting with $Ad(\exp(\epsilon\mathbf{v}_5))$ on $\mathbf{v}$, resulting in the form $a_2\mathbf{v}_2 + \mathbf{v}_3$. For the remaining cases we take $a_6 = a_5 = a_4 = a_3 = 0$, which are similarly seen to be equivalent either to $\mathbf{v}_2$ or to $\mathbf{v}_1$.

Summarizing, we have found an optimal system of one-dimensional subalgebras of the CHM algebra $\mathcal{G}^6$, that is generated by

$$\begin{array}{ll} (a) & a_5\mathbf{v}_5 + \mathbf{v}_6 \\ (b) & a_1\mathbf{v}_1 + \mathbf{v}_6 \\ (c) & a_3\mathbf{v}_3 \pm \mathbf{v}_4 + \mathbf{v}_5 \\ (d) & \mathbf{v}_3 + \mathbf{v}_5 \\ (e) & \mathbf{v}_5 \\ (f) & a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \mathbf{v}_4 \\ (g) & a_2\mathbf{v}_2 + \mathbf{v}_3 \\ (h) & \mathbf{v}_2 \\ (i) & \mathbf{v}_1 \end{array} \tag{4.50}$$

This classification of one-dimensional subalgebras is now directly useable for the classification of group-invariant solutions. Actually, the collection of all group-invariant solutions, corresponding to subgroups of the optimal system, forms an optimal system of group-invariant solutions of the original equation. One of the elements of this optimal system of group-invariant solutions for the CHM equation has been already calculated in the previous section. Recall equations (4.22)-(4.32), which describe the calculation of the group-invariant solution corresponding to the subalgebra spanned by the generator (g) of the optimal system given above (4.50). This is the famous Larichev-Reznik dipole solution already discussed. A salient feature of such a symmetry reduction is how the dimensionality of the base space for the original equation is related to that of the reduced equation. The original equation possesses a base space composed of three independent and one dependent variables. By using a one-dimensional subalgebra, like the (g) element of Θ_1, we obtain a reduced equation, (4.28), with a reduced base space composed of two independent and one dependent variables. The symmetry reduction amounted to a reduction in the number of independent variables by one, resulting in a nonlinear PDE that we were able to solve. The obvious extension of this idea is to find optimal systems of higher dimensional subalgebras, which would allow, at once, a multiple reduction in the number of independent variables by means of symmetry reduction. For instance, for the CHM equation, a reduction by a two-dimensional subalgebra, an element of Θ_2, will reduce the original PDE to an ODE, which in principle is orders of magnitude easier to solve. This constitutes the most powerful application of the algebraic properties of the symmetry group, namely the successive application of symmetry reductions to transform a PDE into a more tractable and often solvable form. But before tackling the higher dimensional case, I would like to present some other single reductions afforded by the use of elements of Θ_1, where we can get some analytical insight into the solution space of the CHM equation.

It is easy to see that not all the symmetry reductions implied by the one-dimensional optimal system will lead to analytically tractable equations. The point is that the result of this single reduction of the CHM equation is a nonlinear PDE in two independent variables, which in general is an unsolvable problem by analytical methods. But in some cases, like the one presented before for the dipole solution, we can solve the resulting equation.

Another fully solvable example is given by the rotationally invariant solution, obtained from the subalgebra spanned by $\mathbf{v}_5$, corresponding to element (e) of the optimal system Θ_1 shown in (4.50). This element $\mathbf{v}_5$, is the generator of rotations in the $x - y$ plane with the obvious invariants

$$\eta_1 = t\,, \qquad \eta_2 = (x^2 + y^2)^{1/2}\,, \qquad \zeta = \varphi\,, \tag{4.51}$$

where $\zeta(\eta_1, \eta_2)$. The CHM equation is reduced upon substitution of (4.51) to the stationary form

$$\eta_2^2 \frac{\partial^2 \zeta}{\partial \eta_2^2} + \eta_2 \frac{\partial \zeta}{\partial \eta_2} - \eta_2^2 \zeta = f(\eta_2)\,, \tag{4.52}$$

with f an arbitrary function of its argument. The general solution of equation (4.52) is an arbitrary function $\mathcal{F}$ of η_2, which is the consequence of imposing rotational symmetry on solutions of the CHM equation. If we further require this function to be localized, then we would get an exact axisymmetric monopole solution, which consists of an isotropic function of r that decays rapidly when

$r \to \infty$. A typical form for this kind of stationary solution would be the well known soliton solution:

$$\varphi = A\mathrm{sech}^2(kr) = A\mathrm{sech}^2(k[x^2 + (y - v_d t)^2])\,, \tag{4.53}$$

where A is a constant and we have made explicit use of the relation: $y' = y - v_d t$, defined before, recall (4.2). In the literature the monopole solution has been defined by taking the particular choice $f(\eta_2) = 0$ in (4.52), and studying the remaining equation, which is a modified Bessel equation, with localized solutions

$$\zeta \sim K_0(\eta_2)\,, \tag{4.54}$$

where $K_0(\eta_2)$ is the modified Bessel function of the second kind of order zero. Therefore an axisymmetric, rotational invariant solution is necessarily a localized monopole of arbitrary shape.

A slight generalization of the previous example is given by element (d) of the optimal system (4.50), i.e. as

$$\mathbf{v}_3 + \mathbf{v}_5 = \frac{\partial}{\partial t} + y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}. \tag{4.55}$$

The differential invariants corresponding to this generator are

$$\eta_1 = x\sin t + y\cos t\,, \qquad \eta_2 = -x\cos t + y\sin t\,, \qquad \varphi = \zeta\,, \tag{4.56}$$

which describe a time dependent rotation of the coordinate system. Writing the CHM equation in this new coordinates yields the reduced form

$$\left[\nabla^2\zeta + \alpha\zeta + \frac{1+\alpha}{2}(\eta_1^2 + \eta_2^2),\ \zeta + \frac{1}{2}(\eta_1^2 + \eta_2^2)\right] = 0\,, \tag{4.57}$$

where α is an arbitrary constant, and the ∇ operator and the bracket are given in terms of η_1 and η_2 in the usual form. The general solution for the bracket equation (4.57) implies a relation between both terms in the bracket, given as follows:

$$\nabla^2\zeta + \alpha\zeta + \frac{1+\alpha}{2}(\eta_1^2+\eta_2^2) = F\left(\zeta + \frac{1}{2}(\eta_1^2+\eta_2^2)\right) . \tag{4.58}$$

The form of the free function F can be determined by imposing again the condition for a localized solution: ζ has to approach zero faster than $1/r$, which in this case means that $\zeta \to 0$ for $r = (\eta_1^2+\eta_2^2)^{1/2}$ large. This implies for (4.58) the condition

$$F\left(\frac{1}{2}(\eta_1^2+\eta_2^2)\right) = \left(\frac{1+\alpha}{2}\right)\left(\eta_1^2+\eta_2^2\right) , \tag{4.59}$$

and therefore the function F has to be linear in its argument, i.e.

$$F(Z) = (1+\alpha)Z . \tag{4.60}$$

Substituting this form of F back into equation (4.58), yields a simple equation for ζ which satisfies the condition for an exterior solution:

$$\nabla^2\zeta = \zeta . \tag{4.61}$$

This equation is formally the same as that for the Larichev-Reznik case, recall (4.30). The main difference is the interpretation of the coordinates, which in the present case depend on t in a fundamental way; recall the definitions of η_1 and η_2, (4.56). The solution can be given again in terms of modified Bessel functions K_n and a harmonic θ-dependence as follows:

$$\zeta = \sum_n D_n K_n(r)\sin(n\theta) , \tag{4.62}$$

where the D_n are constants, and r, θ are defined by

$$r = (\eta_1^2 + \eta_2^2)^{1/2} = (x^2 + y^2)^{1/2}\,, \tag{4.63}$$

$$\theta = \arctan\left(\frac{\eta_2}{\eta_1}\right) = t - \arctan\left(\frac{x}{y}\right)\,. \tag{4.64}$$

For simplicity we take only one mode out of the sum (4.62) and try to match with the interior solution. The only condition on F for the interior solution is that ζ should be finite at the origin. Therefore, we take a linear function of its argument as the simplest form of F:

$$F(Z) = AZ + Q\,, \tag{4.65}$$

where A and Q are constants that will be constrained by the matching conditions at the boundary $r = a$. This ansatz for F leads to the following PDE for the interior:

$$\nabla^2\zeta + \frac{k^2}{a^2}\zeta = Q - r^2\left(1 + \frac{k^2}{a^2}\right)\,, \tag{4.66}$$

where we have defined the interior wavenumber $k^2 = a^2(\alpha - A)$. Notice that the inhomogeneous terms on the RHS are independent of θ, therefore the matching solution will have to be proportional to the homogeneous solution of (4.66) in order to preserve the anisotropic terms proportional to $\sin\theta$, giving as a result a dipole like structure. Otherwise, we can consider the problem as one independent of the angle θ obtaining again an isotropic monopole solution similar to the one already discussed.

If we consider all the other reductions induced by the remaining elements of the optimal system of order one, Θ_1, we will get complicated equations in two independent variables that are not easily solvable by analytical methods.

However we can try to extend the concept of optimal system to higher dimensions and reduce the original equation to an ODE, as was mentioned before, yielding additional analytic solutions.

4.1.3 Two-Dimensional Optimal System

The method I will be using here to construct the two-dimensional optimal system was developed by F. Galas [Galas 88], which is based on a method introduced by Ovsiannikov [Ovsiannikov 82]. The basic idea is to construct a list of two-dimensional subalgebras $\mathcal{G}(u_1, u_2)$, where the first component u_1 is an element of the one-dimensional optimal system Θ_1, and the second component u_2 is chosen so that u_1 and u_2 form a closed two-dimensional subalgebra. Galas showed that a sufficient condition for this to happen is to choose u_2 as an element of the quotient algebra $\mathrm{Nor}(u_1)/u_1$, where $\mathrm{Nor}(u_1)$ is the normalizer of u_1. The normalizer $\mathrm{Nor}(\mathcal{H})$ of a subalgebra $\mathcal{H}$ of $\mathcal{G}$ is defined as the largest subalgebra of $\mathcal{G}$ such that $\mathcal{H}$ is an ideal of $\mathrm{Nor}(\mathcal{H})$. Therefore, we can easily calculate the normalizer of any subalgebra $\mathcal{H}$ of a given algebra $\mathcal{G}$, by using the properties given in the commutator Table 4.1, as follows:

$$\mathrm{Nor}(\mathcal{H}) = \{u \in \mathcal{G} : [u, w] \in \mathcal{H} \quad \forall \ w \in \mathcal{H}\}. \tag{4.67}$$

Finally, the list of two-dimensional subalgebras obtained by the above prescription, is separated into conjugacy classes under the adjoint transformation, which now will be taken in its more general matrix form given by (4.38), yielding as a result the elements of the two-dimensional optimal system Θ_2.

Continuing with the CHM equation as a working example, I will next calculate the 2-D optimal system Θ_2 for the symmetry algebra of the CHM equation. We start out by constructing a list of two-dimensional subalgebras, based on the result by Galas mentioned above. The normalizer is obtained by setting

$$[u_1, \sum_i \alpha_i v_i] = cu_1\,, \tag{4.68}$$

where c is some constant, and determining the possible nonzero α_i's. This together with the one-dimensional optimal system (4.50), generates the following list of two-dimensional subalgebras:

$$\mathcal{G}_1(a_5\mathbf{v}_5 + \mathbf{v}_6,\quad \alpha_5\mathbf{v}_5)$$

$$\mathcal{G}_2(\mathbf{v}_6,\quad \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_5\mathbf{v}_5)$$

$$\mathcal{G}_3(a_1\mathbf{v}_1 + \mathbf{v}_6,\quad \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2)$$

$$\mathcal{G}_4(a_3\mathbf{v}_3 + \mathbf{v}_4 + \mathbf{v}_5,\quad \alpha_3\mathbf{v}_3)$$

$$\mathcal{G}_5(\mathbf{v}_4 + \mathbf{v}_5,\quad \alpha_3\mathbf{v}_3)$$

$$\mathcal{G}_6(\mathbf{v}_3 + \mathbf{v}_5,\quad \alpha_4\mathbf{v}_4)$$

$$\mathcal{G}_7(\mathbf{v}_5,\quad \alpha_3\mathbf{v}_3 + \alpha_4\mathbf{v}_4 + \alpha_6\mathbf{v}_6)$$

$$\mathcal{G}_8(a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \mathbf{v}_4,\quad \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3)$$

$$\mathcal{G}_9(a_2\mathbf{v}_2 + \mathbf{v}_4,\quad \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3)$$

$$\mathcal{G}_{10}(a_3\mathbf{v}_3 + \mathbf{v}_4,\quad \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 + \alpha_5\mathbf{v}_5)$$

$$\mathcal{G}_{11}(\mathbf{v}_4,\quad \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 + \alpha_5\mathbf{v}_5)$$

$$\mathcal{G}_{12}(a_1\mathbf{v}_1 + \mathbf{v}_3,\quad \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_4\mathbf{v}_4)$$

$$\mathcal{G}_{13}(\mathbf{v}_3,\quad \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_4\mathbf{v}_4 + \alpha_5\mathbf{v}_5 + \alpha_6\mathbf{v}_6)$$

$$\mathcal{G}_{14}(\mathbf{v}_2,\quad \alpha_1\mathbf{v}_1 + \alpha_3\mathbf{v}_3 + \alpha_4\mathbf{v}_4 + \alpha_6\mathbf{v}_6)$$

$$\mathcal{G}_{15}(\mathbf{v}_1,\quad \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 + \alpha_4\mathbf{v}_4 + \alpha_6\mathbf{v}_6)\,.$$

Now, in order to span the space of possible reductions induced by Θ_2, we must simplify each pair of subalgebra elements as much as possible using the adjoint action. This is accomplished by noting that for each two-dimensional algebra $\mathcal{G}(u_1, u_2)$ in the list above, we seek an equivalent algebra $\mathcal{G}'(u_1', u_2')$ under the adjoint transformation. Since we are now dealing with a multidimensional case, the new elements u_i' can be formed as linear combinations of the transformed elements $\tilde{u}_i$. For the two-dimensional case we can write this statement as

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{pmatrix}, \tag{4.69}$$

where both constants of at least one of the pairs (c_{11}, c_{22}) or $(c_{12}, c_{21}) \neq 0$, and the $\tilde{u}_i$'s represent the adjoint transformation of the u_i's, $\tilde{u}_i = Ad(\exp(\epsilon \mathbf{v}))u_i$. As in the one-dimensional case, we wish to find the simplest representation of the u_i''s through judicious choices of the ϵ's in the adjoint transformation.

In practice we use the inverse of (4.69) as a simpler representation for the reduction of the u_i's, in the form

$$\begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix}. \tag{4.70}$$

We assume without loss of generality that both a_{11} and $a_{22} \neq 0$. This form is simpler than (4.69) because each separate equation implied by (4.70) contains only one adjoint transformation, which makes clearer which of the ϵ_i's must be zero.

For the general adjoint transformation (4.38), we need to calculate the adjoint transformation matrix $\mathbf{A}$. This matrix is defined as the product of the matrices of the separate adjoint actions for each element of the algebra:

$\mathbf{A} = \mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\mathbf{A}_4\mathbf{A}_5\mathbf{A}_6$. Each matrix $\mathbf{A}_i$ can be read off the adjoint Table 4.2 by the following construction [Coggeshall-Meyer 92]: Let each element in Table 4.2 be labeled $\mathcal{O}_{ij}$, with i and j being the rows and columns respectively, then we have by definition

$$\mathcal{O}_{ij} = Ad(\exp(\epsilon \mathbf{v}_i))\mathbf{v}_j \,. \tag{4.71}$$

Each element can be rewritten explicitly as:

$$\mathcal{O}_{ij} = g_{ij}(\epsilon)\mathbf{v}_j + \sum_k h^k_{ij}(\epsilon)\mathbf{v}_k \quad (\textit{no sum on } j)\,, \tag{4.72}$$

where $g_{ij}(\epsilon = 0) = 1$, $h^k_{ij}(\epsilon = 0) = 0$, and $h^j_{ij}(\epsilon) = 0$. Each row in $\mathcal{O}_{ij}$ makes a matrix $\mathbf{A}_i$ through

$$(\mathbf{A}_i)_{kj} = g_{ij}(\epsilon_i)\delta_{kj} + h^k_{ij}(\epsilon_i) \quad (\textit{no sum})\,. \tag{4.73}$$

Therefore, according to this construction, we obtain for each single element adjoint action matrix the following:

$$\mathbf{A}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \epsilon_1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{A}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & -\epsilon_2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{A}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\epsilon_3 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{A}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \epsilon_4 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{A}_5 = \begin{pmatrix} \cos\epsilon_5 & \sin\epsilon_5 & 0 & 0 & 0 & 0 \\ -\sin\epsilon_5 & \cos\epsilon_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{A}_6 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{\epsilon_6} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-\epsilon_6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The general adjoint transformation matrix **A** is equal to the product of all six matrices, given above, taken in any order. The final result is

$$\mathbf{A} = \begin{pmatrix} \cos\epsilon_5 & \sin\epsilon_5 & 0 & 0 & \epsilon_1\sin\epsilon_5 - \epsilon_2\cos\epsilon_5 & 0 \\ -\sin\epsilon_5 & \cos\epsilon_5 & 0 & 0 & \epsilon_1\cos\epsilon_5 + \epsilon_2\sin\epsilon_5 & 0 \\ 0 & 0 & e^{\epsilon_6} & 0 & 0 & -\epsilon_3 e^{\epsilon_6} \\ 0 & 0 & 0 & e^{-\epsilon_6} & 0 & \epsilon_4 e^{-\epsilon_6} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

This matrix **A** will be used to construct the two-dimensional optimal system, Θ_2, by separating the previous list of two-dimensional algebras $\mathcal{G}_i$ into equivalence classes under the adjoint action given by (4.70). **A** will yield the adjoint of an arbitrary element of the algebra, u, through the relation: $\tilde{\mathbf{u}} = \mathbf{A}^T\mathbf{u}$.

As an example, consider the case of $\mathcal{G}_2(\mathbf{v}_6, \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_5\mathbf{v}_5)$. First, we must consider the possibility that the constants α_1, α_2, and/or α_5 could vanish. If two of the three constants vanish we get the three irreducible cases $(\mathbf{v}_6, \mathbf{v}_1)$, $(\mathbf{v}_6, \mathbf{v}_2)$, and $(\mathbf{v}_6, \mathbf{v}_5)$. If we suppose that only one of the three constants vanishes at a time we will get $(\mathbf{v}_6, \alpha_2\mathbf{v}_2 + \alpha_5\mathbf{v}_5)$, $(\mathbf{v}_6, \alpha_1\mathbf{v}_1 + \alpha_5\mathbf{v}_5)$, and $(\mathbf{v}_6, \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2)$, for α_1, α_2 or $\alpha_5 = 0$, respectively. These three cases, together with the general case (with all the constants different from zero), must now be considered under the adjoint action. Taking (4.70) with the adjoint matrix defined above yields the following pair of equations:

$$-\epsilon_3 e^{\epsilon_6}\mathbf{v}_3 + \epsilon_4 e^{-\epsilon_6}\mathbf{v}_4 + \mathbf{v}_6 = a_{11}\mathbf{v}_6 + a_{12}(\alpha_1'\mathbf{v}_1 + \alpha_2'\mathbf{v}_2 + \alpha_5'\mathbf{v}_5)\,, \quad (4.74)$$

$$\alpha_1(\mathbf{v}_1 cos\epsilon_5 - \mathbf{v}_2 sin\epsilon_5) + \alpha_2(\mathbf{v}_1 sin\epsilon_5 + \mathbf{v}_2 cos\epsilon_5)$$
$$+\alpha_5\left[(\epsilon_1 sin\epsilon_5 - \epsilon_2 cos\epsilon_5)\mathbf{v}_1 + (\epsilon_1 cos\epsilon_5 + \epsilon_2 sin\epsilon_5)\mathbf{v}_2 + \mathbf{v}_5\right]$$
$$= a_{21}\mathbf{v}_6 + a_{22}(\alpha_1'\mathbf{v}_1 + \alpha_2'\mathbf{v}_2 + \alpha_5'\mathbf{v}_5)\,. \quad (4.75)$$

In these equations α_1, α_2 and α_5 are fixed constants, and the ϵ_i's and a_{ij}'s are free parameters that can be chosen in such a way as to simplify α_1', α_2' and α_5'; it is desirable to have one or more of these constants equal to zero.

In the case where all the α_i's are different from zero, we start by equating the coefficients of the $\mathbf{v}_i$'s in equation (4.74), yielding $\epsilon_3 = \epsilon_4 = a_{12} = 0$ and $a_{11} = 1$, which does not provide any helpful reduction. Following the same procedure for the second equation (4.75), we obtain: $a_{22}\alpha_5' = \alpha_5$, which means that we can scale α_5' to 1 by setting $a_{22} = \alpha_5$. The remaining conditions imply that $\alpha_1' = \alpha_2' = 0$ by means of the choice $\epsilon_1 = -\alpha_2/\alpha_5$ and $\epsilon_2 = \alpha_1/\alpha_5$. Therefore this case reduces to $(\mathbf{v}_6, \mathbf{v}_5)$, which is already known.

Next we consider the case with $\alpha_1 = 0$. From equation (4.74) we get the same result as before, $\epsilon_3 = \epsilon_4 = a_{12} = 0$, and from (4.75) we obtain $\alpha_5' = 1$ and α_2' remains arbitrary. Therefore, this case reduces to $(\mathbf{v}_6, \alpha_2\mathbf{v}_2 + \mathbf{v}_5)$.

Following exactly the same rationale for the case $\alpha_2 = 0$ we obtain the reduction $(\mathbf{v}_6, \alpha_1\mathbf{v}_1 + \mathbf{v}_5)$.

The last case to study is when $\alpha_5 = 0$. In this case equation (4.75) yields: $\alpha_1' = 0$ or $\alpha_2' = 0$, by using the free constant ϵ_5, and the remaining

α can be normalized to one using a_{22}. Finally we end up with the reductions $(\mathbf{v}_6, \mathbf{v}_1)$ or $(\mathbf{v}_6, \mathbf{v}_2)$, which are already known.

In conclusion, we have found that the two-dimensional algebra $\mathcal{G}_2(\mathbf{v}_6, \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_5\mathbf{v}_5)$ is equivalent, under the adjoint transformation (4.70), to one of the following algebras: $(\mathbf{v}_6, \mathbf{v}_1)$, $(\mathbf{v}_6, \mathbf{v}_2)$, $(\mathbf{v}_6, \alpha_1\mathbf{v}_1 + \mathbf{v}_5)$, $(\mathbf{v}_6, \alpha_2\mathbf{v}_2 + \mathbf{v}_5)$, where the α_i's are arbitrary real numbers (possibly zero).

This procedure has been done for each of the two-dimensional algebras $\mathcal{G}_i$, listed before, and has generated a collection of all unique reduced two-dimensional algebras Θ_2, the two-dimensional optimal system. The following is a complete list of the elements of Θ_2 for the CHM algebra:

$$
\begin{array}{rl}
\mathcal{H}_1(a_5\mathbf{v}_5 + \mathbf{v}_6, \mathbf{v}_5) & \mathcal{H}_2(\mathbf{v}_6, \mathbf{v}_1) \\
\mathcal{H}_3(\mathbf{v}_6, \mathbf{v}_2) & \mathcal{H}_4(\mathbf{v}_6, a_1\mathbf{v}_1 + \mathbf{v}_5) \\
\mathcal{H}_5(\mathbf{v}_6, a_2\mathbf{v}_2 + \mathbf{v}_5) & \mathcal{H}_6(a_1\mathbf{v}_1 + \mathbf{v}_6, \mathbf{v}_1) \\
\mathcal{H}_7(a_1\mathbf{v}_1 + \mathbf{v}_6, \mathbf{v}_2) & \mathcal{H}_8(a_3\mathbf{v}_3 + \mathbf{v}_4 + \mathbf{v}_5, \mathbf{v}_3) \\
\mathcal{H}_9(\mathbf{v}_3 + \mathbf{v}_5, \mathbf{v}_4) & \mathcal{H}_{10}(\mathbf{v}_5, \mathbf{v}_3) \\
\mathcal{H}_{11}(\mathbf{v}_5, \mathbf{v}_4) & \mathcal{H}_{12}(\mathbf{v}_5, \mathbf{v}_6) \\
\mathcal{H}_{13}(a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \mathbf{v}_4, \mathbf{v}_1) & \mathcal{H}_{14}(a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \mathbf{v}_4, \mathbf{v}_2) \\
\mathcal{H}_{15}(a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \mathbf{v}_4, \alpha_1\mathbf{v}_1 + \mathbf{v}_3) & \mathcal{H}_{16}(a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \mathbf{v}_4, \alpha_2\mathbf{v}_2 + \mathbf{v}_3) \\
\mathcal{H}_{17}(a_2\mathbf{v}_2 + \mathbf{v}_4, \mathbf{v}_1) & \mathcal{H}_{18}(a_2\mathbf{v}_2 + \mathbf{v}_4, \alpha_1\mathbf{v}_1 + \mathbf{v}_2) \\
\mathcal{H}_{19}(a_2\mathbf{v}_2 + \mathbf{v}_4, \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \mathbf{v}_3) & \mathcal{H}_{20}(a_3\mathbf{v}_3 + \mathbf{v}_4, \mathbf{v}_1) \\
\mathcal{H}_{21}(a_3\mathbf{v}_3 + \mathbf{v}_4, \mathbf{v}_2) & \mathcal{H}_{22}(a_3\mathbf{v}_3 + \mathbf{v}_4, \mathbf{v}_3) \\
\mathcal{H}_{23}(a_3\mathbf{v}_3 + \mathbf{v}_4, \alpha_3\mathbf{v}_3 + \mathbf{v}_5) & \mathcal{H}_{24}(a_1\mathbf{v}_1 + \mathbf{v}_3, \mathbf{v}_1) \\
\mathcal{H}_{25}(a_1\mathbf{v}_1 + \mathbf{v}_3, \alpha_1\mathbf{v}_1 + \mathbf{v}_2) & \mathcal{H}_{26}(a_1\mathbf{v}_1 + \mathbf{v}_3, \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \mathbf{v}_4)
\end{array}
$$

$$\mathcal{H}_{27}(\mathbf{v}_3, a_5\mathbf{v}_5 + \mathbf{v}_6) \qquad \mathcal{H}_{28}(\mathbf{v}_3, \alpha_1\mathbf{v}_1 + \mathbf{v}_6)$$
$$\mathcal{H}_{29}(\mathbf{v}_3, \alpha_2\mathbf{v}_2 + \mathbf{v}_6) \qquad \mathcal{H}_{30}(\mathbf{v}_3, \alpha_1\mathbf{v}_1 + \mathbf{v}_4)$$
$$\mathcal{H}_{31}(\mathbf{v}_3, \alpha_4\mathbf{v}_4 + \mathbf{v}_5) \qquad \mathcal{H}_{32}(\mathbf{v}_3, \alpha_2\mathbf{v}_2 + \mathbf{v}_4)$$
$$\mathcal{H}_{33}(\mathbf{v}_3, \mathbf{v}_1) \qquad \mathcal{H}_{34}(\mathbf{v}_3, \mathbf{v}_2)$$
$$\mathcal{H}_{35}(\mathbf{v}_2, \alpha_1\mathbf{v}_1 + \mathbf{v}_6) \qquad \mathcal{H}_{36}(\mathbf{v}_2, \alpha_1\mathbf{v}_1 + \alpha_3\mathbf{v}_3 + \mathbf{v}_4)$$
$$\mathcal{H}_{37}(\mathbf{v}_2, \alpha_1\mathbf{v}_1 + \mathbf{v}_3) \qquad \mathcal{H}_{38}(\mathbf{v}_2, \mathbf{v}_1)$$
$$\mathcal{H}_{39}(\mathbf{v}_1, \alpha_2\mathbf{v}_2 + \mathbf{v}_6) \qquad \mathcal{H}_{40}(\mathbf{v}_1, \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 + \mathbf{v}_4)$$
$$\mathcal{H}_{41}(\mathbf{v}_1, \alpha_2\mathbf{v}_2 + \mathbf{v}_3) \qquad \mathcal{H}_{42}(\mathbf{v}_1, \mathbf{v}_2)$$
$$\mathcal{H}_{43}(\mathbf{v}_4, \alpha_3\mathbf{v}_3 + \mathbf{v}_5) \qquad \mathcal{H}_{44}(\mathbf{v}_4, \alpha_1\mathbf{v}_1 + \mathbf{v}_3)$$
$$\mathcal{H}_{45}(\mathbf{v}_4, \alpha_2\mathbf{v}_2 + \mathbf{v}_3) \qquad \mathcal{H}_{46}(\mathbf{v}_4, \mathbf{v}_2)$$

$$\mathcal{H}_{47}(\mathbf{v}_4, \mathbf{v}_1)\,.$$

We obtained 47 elements for the two-dimensional optimal system Θ_2, of which only 31 will generate the appropriate similarity variables to reduce the CHM equation to an ODE, and therefore yield group-invariant solutions. The actual criterion for the existence of group-invariant solutions consists of examining the rank of the matrix of the coordinate functions defined by each element $\mathcal{H}_i$.

More specifically, let $r_*(\xi, \eta)$ be the general rank of the tangent mapping of the algebra element $\mathcal{H}_i$. This $r_*(\xi, \eta)$ corresponds to the dimension of the orbits induced by $\mathcal{H}_i$ in the base space (x, y, t, ϕ). In addition, let $r_*(\xi)$ be the rank of the mapping of $\mathcal{H}_i$ in the space of independent variables (x, y, t). Then, the condition for having a non-singular $\mathcal{H}$-invariant solution is given by the relations

$$r_*(\xi,\eta) \leq n\,, \qquad r_*(\xi) = r_*(\xi,\eta)\,,$$

for each $\mathcal{H}_i$. Here n corresponds to the dimension of the space of independent variables (for details on this condition see [Ovsiannikov 82]). The algebras $\mathcal{H}_9$, $\mathcal{H}_{10}$, $\mathcal{H}_{11}$, $\mathcal{H}_{14}$, $\mathcal{H}_{20}$, $\mathcal{H}_{21}$, $\mathcal{H}_{22}$, $\mathcal{H}_{24}$, $\mathcal{H}_{25}$, $\mathcal{H}_{33}$, $\mathcal{H}_{34}$, and the last five elements of Θ_2, $\mathcal{H}_{43}$ thru $\mathcal{H}_{47}$, fail to satisfy the condition described above for the determination of group-invariant solutions. They have been included for completeness only.

To perform the reduction implied by all the remaining algebras, the ones that satisfy the invariant solution condition, we need to construct the invariants of the two subalgebras involved in each element of Θ_2. We start by calculating the invariants of the first of the two subalgebras, by solving the corresponding characteristic equations. Next, the second subalgebra is written in terms of these invariants (which must be possible since the reduced equation is invariant under this second algebra). The integration constants from this second set of characteristic equations are then invariants of both subalgebras, and define the so-called similarity variables.

As an example, consider the case of $\mathcal{H}_{15}(a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \mathbf{v}_4, \alpha_1\mathbf{v}_1 + \mathbf{v}_3)$. The characteristic equations for the first element of the algebra, $a_2\mathbf{v}_2+a_3\mathbf{v}_3+\mathbf{v}_4$, are

$$\frac{dy}{a_2} = \frac{dt}{a_3} = d\varphi\,. \tag{4.76}$$

From the integration of these equations we obtain the group invariants, which are equal to the integration constants, given by

$$C_1 = x\,, \qquad C_2 = y - \frac{a_2}{a_3}t\,, \qquad C_3 = \varphi - \frac{1}{a_3}t\,. \tag{4.77}$$

The second generator is written in terms of these invariants,

$$\alpha_1 \mathbf{v}_1 + \mathbf{v}_3 = \alpha_1 \frac{\partial}{\partial C_1} - \frac{a_2}{a_3}\frac{\partial}{\partial C_2} - \frac{1}{a_3}\frac{\partial}{\partial C_3}\,. \tag{4.78}$$

Therefore the second set of characteristic equations is

$$\frac{dC_1}{\alpha_1} = -\frac{dC_2}{a_2/a_3} = -\frac{dC_3}{1/a_3}\,, \tag{4.79}$$

whose integration constants are the new similarity variables given by

$$\eta = x + \frac{\alpha_1 a_3}{a_2}\left(y - \frac{a_2}{a_3}\right), \qquad \zeta = \varphi - \frac{1}{a_2}y\,. \tag{4.80}$$

These new variables η and ζ define the independent and dependent variables respectively, which will reduce the CHM equation to an ODE. They represent a combined boost in the x and y directions and the respective shift of the ϕ field. This transformation of coordinates linearizes the bracket nonlinearity and generates simple linear standing waves as the group-invariant solutions. The reduced form of the CHM equation is:

$$\frac{d^3\zeta}{d\eta^3} + k\frac{d\zeta}{d\eta} = 0\,, \tag{4.81}$$

where k is a constant given in terms of α_1, a_2 and a_3. Integrating (4.81) yields the solution

$$\zeta = A\sin(k^{1/2}\eta + b) + D \tag{4.82}$$

where A, b and D are integration constants, and η, ζ are given by (4.80) in terms of the original variables.

Using other members of Θ_2 we can obtain all posible reductions to ODE's, induced by point transformations, of the CHM equation. Among these reductions we distinguish families of solutions that characterize some special behavior. The first such family is the one that corresponds to monopole-type solutions. It is obtained from the algebras $\mathcal{H}_1$, $\mathcal{H}_4$, $\mathcal{H}_5$, and $\mathcal{H}_{12}$. The typical characteristic of this reduction is the choice of an isotropic independent coordinate, $\eta = r = (x^2 + y^2)^{1/2}$, which makes the nonlinear bracket automatically vanish, as was pointed out in the previous section for single reductions. Together with the transformation $\zeta = \varphi t$ we get the typical modified Bessel equation as the reduced form of the CHM equation

$$\eta^2 \frac{d^2\zeta}{d\eta^2} + \eta \frac{d\zeta}{d\eta} - \eta^2 \zeta = 0\,. \tag{4.83}$$

The general solutions of this equation are the functions I_0 and K_0, the modified Bessel functions of the first and second kind, which diverge as $r \longrightarrow \infty$ and $r \longrightarrow 0$, respectively. In order to get a continuous localized solution we can take K_0 as the outside solution and match it with an interior solution that remains finite at $r = 0$, in the same fashion as with the dipole solution presented earlier in this chapter.

Another interesting family of reductions can be obtained by using the elements $\mathcal{H}_{27}$, $\mathcal{H}_{28}$ and $\mathcal{H}_{29}$, of Θ_2. These three cases provide the only genuine nonlinear reductions of the CHM equation by means of the elements of Θ_2. They all correspond to stationary solutions, whose only explicit dependence on time t is through the shifted coordinate $y' = y - v_d t$, (recall (4.2)). Ultimately, they correspond to a "bracket equal to zero" type of equation.

For instance, take $\mathcal{H}_{28}(\mathbf{v}_3, \alpha_1 \mathbf{v}_1 + \mathbf{v}_6)$. The corresponding similarity variables are

$$\eta = y\,, \qquad\qquad \zeta = e^{x/\alpha_1}\varphi\,, \tag{4.84}$$

which immediately reduce the bracket to the form

$$\zeta\frac{d^3\zeta}{d\eta^3} - \frac{d\zeta}{d\eta}\frac{d^2\zeta}{d\eta^2} = 0\,. \tag{4.85}$$

This nonlinear equation is a disguised form of a simple linear ODE of second order. This can be easily seen dividing (4.85) by ζ^2 and noticing that the resulting form can be written as the total derivative in η of a second order linear ODE, as follows:

$$\frac{d}{d\eta}\left(\frac{1}{\zeta}\frac{d^2\zeta}{d\eta^2}\right) = 0\,, \tag{4.86}$$

which after an integration yields

$$\frac{d^2\zeta}{d\eta^2} + C\zeta = 0\,. \tag{4.87}$$

The solution to this simple harmonic oscillator equation (4.87) is given by

$$\zeta = Ae^{(-C)^{1/2}\eta} + Be^{-(-C)^{1/2}\eta}\,, \tag{4.88}$$

which, depending on the sign of the integration constant C, will represent oscillatory or exponentially decaying behavior in the variable y. If we consider $C \geq 0$, then the form of the solution in original variables is

$$\varphi = Ae^{-x/\alpha_1}\sin(k[y - v_d t])\,, \tag{4.89}$$

which constitutes a localized travelling-wave solution propagating in the y direction and decaying exponentially in the x direction.

The solution generated by $\mathcal{H}_{27}$ belongs in the same family of stationary solutions as the previous example, but with a slight generalization. In the present case the similarity variables turn out to be:

$$\eta = \left(x^2 + y^2\right)^{1/2}\,, \qquad \zeta = \varphi\, e^{\frac{1}{\alpha_5}\arctan(x/y)}\,, \tag{4.90}$$

and the reduction of the CHM equation is

$$\zeta\frac{d^3\zeta}{d\eta^3}-\frac{d\zeta}{d\eta}\left(\frac{d^2\zeta}{d\eta^2}\right)+\frac{\zeta}{\eta}\frac{d^2\zeta}{d\eta^2}-\frac{1}{\eta}\left(\frac{d\zeta}{d\eta}\right)^2-\frac{\zeta}{\eta^2}\frac{d\zeta}{d\eta}-\frac{2}{\alpha_5^2}\frac{\zeta^2}{\eta^3}=0\,. \tag{4.91}$$

Following the procedure of the previous example, we rewrite (4.91) as a linear equation in the form

$$\frac{d}{d\eta}\left(\frac{1}{\zeta}\frac{d^2\zeta}{d\eta^2}+\frac{1}{\eta\zeta}\frac{d\zeta}{d\eta}+\frac{1}{\alpha_5^2\eta^2}\right)=0\,. \tag{4.92}$$

Integrating once yields

$$\eta^2\frac{d^2\zeta}{d\eta^2}+\eta\frac{d\zeta}{d\eta}+\left(\frac{1}{\alpha_5^2}-C\eta^2\right)\zeta=0\,. \tag{4.93}$$

This equation can take different forms, and therefore solutions, depending on the value of the integration constant C. For $C=0$ we obtain an Euler homogeneous equation with solution $\zeta\sim\eta^n$. For $C>0$ we obtain a modified Bessel's equation with the monopole-type of solution already discussed in terms of the function K_0, and for $C<0$ we will get a Bessel equation, which generates a global solution. All these three cases require α_5 to be a pure imaginary number, $\alpha_5=i|\alpha_5|$. This represents a mild extension of the theory by allowing the constants to be complex numbers. If we assume that this extension holds, then we will get from the Bessel equation a new form of localized solution, one proportional to $J_{1/|\alpha_5|}$, which is finite for the whole domain $\mathcal{R}^2$. This solution, in terms of plane polar coordinates defined from the original variables: $r=(x^2+y^2)^2$, $\theta=\arctan(x/y)$, is

$$\varphi=A\cos\left(\frac{\theta}{|\alpha_5|}\right)J_{1/|\alpha_5|}\left(|C|^{1/2}r\right)\,. \tag{4.94}$$

It resemblances the form of the interior solution of the Larichev-Reznik dipole, recall (4.32), but here we have a globally valid solution, with physically proper limits at $r \to 0$ and $r \to \infty$. Thus the matching problem is avoided and our solution is C^∞ everywhere. It describes a dipole-like structure that decays slowly with the distance from the center $r = 0$.

Finally, the remaining reductions induced by elements of Θ_2 can be classified in three different families: 1) Homogeneous wave equations, which reduce to either the single harmonic oscillator or exponential decay depending on the value of the integration constants, 2) Euler homogeneous equations, which have solutions composed of powers of the independent variable, and 3) Inhomogeneous wave equations, leading to mixed type of solutions encompassing the former two cases. As a typical example, I present below the reduction implied by $\mathcal{H}_{19}(a_2\mathbf{v}_2 + \mathbf{v}_4, \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \mathbf{v}_3)$, which falls in the third category above. The similarity variables induced by this subalgebra are

$$\eta = x - \alpha_1 t \,, \qquad \zeta = \varphi - \frac{1}{a_2}y + \frac{\alpha_2}{a_2}t \,, \tag{4.95}$$

which are simple translations of the coordinates and fields, and which are well-known to induce travelling wave solutions. The form of the reduced CHM equation is

$$\frac{d^3\zeta}{d\eta^3} - A\frac{d\zeta}{d\eta} = B \,, \tag{4.96}$$

where A and B are given in terms of the group constants as $A = \alpha_1/(\alpha_1 + 1/a_2)$, $B = \alpha_2/a_2(\alpha_1 + 1/a_2)$. The general solution of this equation is given by

$$\zeta = e^{\pm\sqrt{A}\eta} - \frac{B}{A}\eta + C \,, \tag{4.97}$$

with C an integration constant. As can be seen, many of the group-invariant solutions have a singular behavior and therefore may not satisfy physical boundary or initial conditions.

Now we turn our attention in the next section to a more sophisticated plasma fluid model, which in turn will represent a richer example from the point of view of its Lie point symmetries.

4.2 Hazeltine's Three-Field Model: Symmetries and Reductions

In this section I present the results of the systematic application of the Lie-Group techniques developed in the last few sections for the CHM equation, to the three-field model of Hazeltine [Hazeltine 83], HTFM. The presentation here is more concise, with emphasis on results rather than the method used to obtain them. Further analysis will be given for some representative and physically interesting cases in the following chapter.

HTFM consists of a system of five coupled nonlinear PDE's that describe the evolution of the five fields U, φ, J, ψ and χ, in three-dimensional space and time. These equations are given by (2.105), (2.109), (2.110), and the relations between vorticity U and the stream function φ and the parallel current density J and the poloidal flux ψ. Written out explicitly, these equations are

$$\frac{\partial U}{\partial t} + [\varphi, U] + \frac{\partial J}{\partial z} - [\psi, J] = 0, \tag{4.98}$$

$$\frac{\partial \psi}{\partial t} + \frac{\partial \varphi}{\partial z} - [\psi, \varphi] - \alpha\left(\frac{\partial \chi}{\partial z} - [\psi, \chi]\right) = 0, \tag{4.99}$$

$$\frac{\partial \chi}{\partial t} + [\varphi, \chi] + \frac{\partial J}{\partial z} - [\psi, J] = 0, \tag{4.100}$$

$$U - \frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial y^2} = 0, \tag{4.101}$$

$$J - \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} = 0\,, \tag{4.102}$$

where $[f, g]$ stands for the usual Poisson bracket with respect to x and y (c.f. Chapter 2).

Evidently, this system of PDE's is substantially more complicated than the CHM equation, both from the physics and mathematics viewpoints. The base space is composed of the four independent variables t, x, y and z, and the five dependent variables: U, φ, J, ψ and χ. This means that the Lie point symmetry calculations will involve a very large number of linear coupled PDE's, the so-called determining equations. Again, we take advantage of the symbolic manipulation package SYMMGRP.MAX in MACSYMA to calculate these determining equations and solve them explicitly, obtaining as a final result the infinitesimals ξ^i and ϕ^ν, which correspond to the generators of the Lie point symmetries as was seen before. The general solution of the determining equations for the infinitesimals is

$$\xi^t(t,x,y,z,U,\varphi,J,\psi,\chi) = c_1 - c_3 t \tag{4.103}$$

$$\xi^x(t,x,y,z,U,\varphi,J,\psi,\chi) = f(t,z) + y h(t,z) \tag{4.104}$$

$$\xi^y(t,x,y,z,U,\varphi,J,\psi,\chi) = g(t,z) - x h(t,z) \tag{4.105}$$

$$\xi^z(t,x,y,z,U,\varphi,J,\psi,\chi) = c_2 - c_3 z \tag{4.106}$$

$$\phi^U(t,x,y,z,U,\varphi,J,\psi,\chi) = c_3 U - 2\frac{\partial h}{\partial t} \tag{4.107}$$

$$\begin{aligned}\phi^\varphi(t,x,y,z,U,\varphi,J,\psi,\chi) &= -y\frac{\partial f}{\partial t} + x\frac{\partial g}{\partial t} - 2\alpha\frac{\partial h}{\partial t} \\ &\quad -\frac{1}{2}(x^2+y^2)\frac{\partial h}{\partial t} + k(t) \\ &\quad -\int \frac{\partial l}{\partial t} dz + \alpha s(z) + c_3\varphi \end{aligned} \tag{4.108}$$

$$\begin{aligned}
\phi^{J}(t,x,y,z,U,\varphi,J,\psi,\chi) &= c_3 J + 2\frac{\partial h}{\partial z} && (4.109)\\
\phi^{\psi}(t,x,y,z,U,\varphi,J,\psi,\chi) &= c_3\psi - x\frac{\partial g}{\partial z} + y\frac{\partial f}{\partial z}\\
&\quad +\frac{1}{2}(x^2+y^2)\frac{\partial h}{\partial z} + l(t,z) && (4.110)\\
\phi^{\chi}(t,x,y,z,U,\varphi,J,\psi,\chi) &= c_3\chi + s(z) - 2\frac{\partial h}{\partial t}\,, && (4.111)
\end{aligned}$$

where f, g, h and l are arbitrary functions of time and the coordinate z, as indicated, and the functions $k(t)$, $s(z)$ are arbitrary functions of their arguments. The only constraint on any of these functions is that $h(t,z)$ has to satisfy the equation

$$\frac{\partial^2 h}{\partial t^2} - \frac{\partial^2 h}{\partial z^2} = 0\,, \tag{4.112}$$

a one-dimensional wave equation for the function $h(t,z)$, whose general solution is $h(t,z) = h^{+}(z+t) + h^{-}(z-t)$, a superposition of waves of arbitrary form traveling in opposite directions along the z-axis.

Therefore, the HTFM system of PDE's is invariant under an *infinite dimensional* Lie group of point transformations characterized by the set of arbitrary functions mentioned before. This property distinguishes the present analysis from the finite dimensional example of the last section, and opens a wider range of possibilities for symmetry reductions and solutions. The infinitesimal generators of the algebra can be obtained by setting each of the arbitrary constants equal to one while the others are set equal to zero and by setting the arbitrary functions to zero as well. This procedure yields a three dimensional subalgebra, which is the finite part of the full symmetry algebra for HTFM. By considering each of the functions mentioned above different from zero, while setting the others to zero (including the constants) yields the remaining part of the symmetry algebra, the infinite dimensional component. The final result is the following set of nine infinitesimal generators:

$$
\begin{aligned}
\mathbf{v}_1 &= \frac{\partial}{\partial t}, \qquad\qquad \mathbf{v}_2 = \frac{\partial}{\partial z}, \\
\mathbf{v}_3 &= -t\frac{\partial}{\partial t} - z\frac{\partial}{\partial z} + U\frac{\partial}{\partial U} + \varphi\frac{\partial}{\partial \varphi} \\
&\quad + J\frac{\partial}{\partial J} + \psi\frac{\partial}{\partial \psi} + \chi\frac{\partial}{\partial \chi}, \\
\mathbf{v}_4 &= f(t,z)\frac{\partial}{\partial x} - y\frac{\partial f(t,z)}{\partial t}\frac{\partial}{\partial \varphi} + y\frac{\partial f(t,z)}{\partial z}\frac{\partial}{\partial \psi}, \\
\mathbf{v}_5 &= g(t,z)\frac{\partial}{\partial y} + x\frac{\partial g(t,z)}{\partial t}\frac{\partial}{\partial \varphi} - x\frac{\partial g(t,z)}{\partial z}\frac{\partial}{\partial \psi}, \\
\mathbf{v}_6 &= h(t,z)\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) - 2\frac{\partial h(t,z)}{\partial t}\frac{\partial}{\partial U} \\
&\quad - \left\{\frac{1}{2}(x^2+y^2) + 2\alpha\right\}\frac{\partial h(t,z)}{\partial t}\frac{\partial}{\partial \varphi} \\
&\quad + \frac{\partial h(t,z)}{\partial z}\left(2\frac{\partial}{\partial J} + \frac{1}{2}(x^2+y^2)\frac{\partial}{\partial \psi}\right) \\
&\quad - 2\frac{\partial h(t,z)}{\partial t}\frac{\partial}{\partial \chi}, \\
\mathbf{v}_7 &= l(t,z)\frac{\partial}{\partial \psi} - \left(\int \frac{\partial l(t,z)}{\partial t}dz\right)\frac{\partial}{\partial \varphi}, \\
\mathbf{v}_8 &= k(t)\frac{\partial}{\partial \varphi}, \\
\mathbf{v}_9 &= \alpha s(z)\frac{\partial}{\partial \varphi} + s(z)\frac{\partial}{\partial \chi}.
\end{aligned}
\tag{4.113}
$$

From the explicit form of the generators given above, we can derive the finite transformations that leave equations (4.98)-(4.102) invariant under the group action. This is easily done again by solving the initial value problem of Lie's First Fundamental Theorem. Recall (3.6) and (3.7) yield, upon writing only the variables that are explicitly changed, the following transformations:

$$\mathbf{v}_1 = \frac{\partial}{\partial t} \implies \tilde{t} = t + \epsilon$$

$$\mathbf{v}_2 = \frac{\partial}{\partial z} \implies \tilde{z} = z + \epsilon$$

$$\mathbf{v}_3 = -t\frac{\partial}{\partial t} - z\frac{\partial}{\partial z} + U\frac{\partial}{\partial U} + \varphi\frac{\partial}{\partial \varphi} + J\frac{\partial}{\partial J} + \psi\frac{\partial}{\partial \psi} + \chi\frac{\partial}{\partial \chi}$$

$$\implies \begin{cases} \tilde{t} = te^{-\epsilon} \\ \tilde{z} = ze^{-\epsilon} \\ \tilde{U} = Ue^{\epsilon} \\ \tilde{\varphi} = \varphi e^{\epsilon} \\ \tilde{J} = Je^{\epsilon} \\ \tilde{\psi} = \psi e^{\epsilon} \\ \tilde{\chi} = \chi e^{\epsilon} \end{cases}$$

$$\mathbf{v}_4 = f\frac{\partial}{\partial x} - y\frac{\partial f}{\partial t}\frac{\partial}{\partial \varphi} + y\frac{\partial f}{\partial z}\frac{\partial}{\partial \psi} \implies \begin{cases} \tilde{x} = x + \epsilon f(t,z) \\ \tilde{\varphi} = \varphi - \epsilon y\frac{\partial f}{\partial t} \\ \tilde{\psi} = \psi + \epsilon y\frac{\partial f}{\partial z} \end{cases}$$

$$\mathbf{v}_5 = g\frac{\partial}{\partial y} + x\frac{\partial g}{\partial t}\frac{\partial}{\partial \varphi} - x\frac{\partial g}{\partial z}\frac{\partial}{\partial \psi} \implies \begin{cases} \tilde{y} = y + \epsilon g(t,z) \\ \tilde{\varphi} = \varphi + \epsilon x\frac{\partial g}{\partial t} \\ \tilde{\psi} = \psi - \epsilon x\frac{\partial g}{\partial z} \end{cases}$$

$$\begin{aligned} \mathbf{v}_6 &= hy\frac{\partial}{\partial x} - hx\frac{\partial}{\partial y} - 2\frac{\partial h}{\partial t}\frac{\partial}{\partial U} + \frac{\partial h}{\partial z}\left(\frac{\partial}{\partial J} + \frac{1}{2}(x^2+y^2)\frac{\partial}{\partial \psi}\right) \\ &\quad - \left\{\frac{1}{2}(x^2+y^2) + 2\alpha\right\}\frac{\partial h}{\partial t}\frac{\partial}{\partial \varphi} - 2\frac{\partial h}{\partial t}\frac{\partial}{\partial \chi} \end{aligned}$$

$$\implies \begin{cases} \tilde{x} = x cos(h\epsilon) + y sin(h\epsilon) \\ \tilde{y} = -x sin(h\epsilon) + y cos(h\epsilon) \\ \tilde{U} = U - 2\epsilon\frac{\partial h}{\partial t} \\ \tilde{J} = J + 2\epsilon\frac{\partial h}{\partial z} \\ \tilde{\varphi} = \varphi - \left\{\frac{1}{2}(x^2+y^2) + 2\alpha\right\}\frac{\partial h}{\partial t}\epsilon \\ \tilde{\psi} = \psi + \frac{1}{2}(x^2+y^2)\frac{\partial h}{\partial z}\epsilon \\ \tilde{\chi} = \chi - 2\frac{\partial h}{\partial t}\epsilon \end{cases}$$

$$\mathbf{v}_7 = l(t,z)\frac{\partial}{\partial\psi} - \left(\int \frac{\partial l}{\partial t} dz\right)\frac{\partial}{\partial\varphi} \implies \begin{cases} \tilde{\varphi} = \varphi - \left(\int \frac{\partial l}{\partial t} dz\right)\epsilon \\ \tilde{\psi} = \psi + l(t,z)\epsilon \end{cases}$$

$$\mathbf{v}_8 = k(t)\frac{\partial}{\partial\varphi} \implies \tilde{\varphi} = \varphi + k(t)\epsilon$$

$$\mathbf{v}_9 = \alpha s(z)\frac{\partial}{\partial\varphi} + s(z)\frac{\partial}{\partial\chi} \implies \begin{cases} \tilde{\varphi} = \varphi + \alpha s(z)\epsilon \\ \tilde{\chi} = \chi + s(z)\epsilon \end{cases}$$

The first three transformations constitute the finite subalgebra of the system (although we can consider three more finite symmetries contained in the generators $\mathbf{v}_4$, $\mathbf{v}_5$ and $\mathbf{v}_6$, by rewriting them appropriately using a simple integration, as we will see later) and correspond to symmetry invariance under translation in time, translation along the z-axis and scaling of t, z and the five dependent fields. The next six transformations are given in terms of six arbitrary functions of their arguments and constitute the infinite dimensional part of the algebra.

From $\mathbf{v}_4$ we obtain a generalized translational invariance along the x-axis, with time and z dependence, together with some interesting generalizations of the usual Galilean and space translations that naturally couple with gauge conditions of the potentials φ and ψ. As expected, the gauge conditions of the potential fields do not affect the value of the physical fields U and J. Notice that this transformation includes the usual x-translational invariance in the particular case of f=constant. For f equal to a linear function of time we get a Galilean boost along the x-axis. These limiting cases are well known symmetries of simpler models like MHD (see [Fuchs 91]). The next generator $\mathbf{v}_5$ corresponds to an equivalent invariance but along the y-axis.

From $\mathbf{v}_6$ we obtained invariance under time and z-dependent rotations, with the rotation angle being proportional to h, a solution of the one dimensional wave equation (4.112). This symmetry involves all five dependent variables: rotational transformations of the φ and ψ potentials, the corresponding transformations impling a shift in the vorticity U and the parallel current J, and a time dependent gauge of the perturbed plasma density χ. It is interesting to observe that the shift of vorticity and the gauge of the density field arise exclusively from the time dependence of the h function. On the other hand, the magnetic effect of the shift of the current density and the consistent change in ψ, appear only as functions of the z-dependence in h. This means that the magnetic effects under rotation are an explicit consequence of the model being three dimensional, with z being the toroidal direction. This describes a helical symmetric state where vorticity and toroidal current are generating "swirling" around the parallel direction to the toroidal field. The limiting case h=constant yields the usual rotational invariance in the perpendicular plane ($x - y$ plane), similar to the one previously studied for the CHM equation.

The last three symmetries $\mathbf{v}_7$, $\mathbf{v}_8$ and $\mathbf{v}_9$, imply invariance under different gauges of the potentials φ, ψ and the density field χ. From $\mathbf{v}_7$ we have invariance under a time and z-dependent gauge change for φ and ψ. The case of $\mathbf{v}_8$ implies a time dependent gauge change of the φ field alone. The last generator $\mathbf{v}_9$, implies a z-dependent gauge change of the φ and χ fields. All these gauge symmetries are independent of each other, and will be used for the symmetry reduction of the system under consideration. As was pointed out before, because of the computer aided calculation of the symmetries, we can rest assured that we have the complete group of Lie point symmetries allowed by HTFM.

The essence of the Lie algebra, composed of the infinitesimal generators, is given in terms of the commutator operation. In Tables 4.3 and 4.4, I present the commutation relations for the nine generators of HTFM. The algebraic information contained in the commutation relations constitutes the basis for a classification scheme of group-invariant solutions for the system in question. In the commutation Table 4.3, and in the remaining of this analysis, we have used uppercase calligraphic letters to denote arbitrary functions of the same argument as their lowercase counterparts. Recall, the functions f, g, h, and l, are arbitrary functions of (t, z), with h constrained to be a solution to the one dimensional wave equation (4.112). The function k depends on t alone and the function s depends on z only.

With this information we can study the adjoint action of the algebra on itself, which will generate the optimal system Θ_i of order i. Recall, this was shown to give all the possible symmetry reductions of the system of differential equations. It is important to notice that for the present case the space of independent coordinates has dimensionality equal to four, and therefore we would need to calculate Θ_3 in order to reduce the system to ODE's, as was done with the CHM equation. This procedure is rather involved and here I have chosen to get as much information with the least amount of effort, therefore studying only the consequences of using elements of Θ_1 and Θ_2, which will amount to single and double symmetry reductions of the number of independent variables, respectively. For the case of three-dimensional MHD, Fuchs [Fuchs 91] calculated Θ_3 and Θ_4 and presented some reductions to ODE's and to algebraic equations for the ideal MHD system.

Following the construction of the adjoint representation given by the

	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4(f)$	$\mathbf{v}_5(g)$
$\mathbf{v}_1$	0	0	$-\mathbf{v}_1$	$\mathbf{v}_4\left(\frac{\partial f}{\partial t}\right)$	$\mathbf{v}_5\left(\frac{\partial g}{\partial t}\right)$
$\mathbf{v}_2$	0	0	$-\mathbf{v}_2$	$\mathbf{v}_4\left(\frac{\partial f}{\partial z}\right)$	$\mathbf{v}_5\left(\frac{\partial g}{\partial z}\right)$
$\mathbf{v}_3$	$\mathbf{v}_1$	$\mathbf{v}_2$	0	$-\mathbf{v}_4\left(t\frac{\partial f}{\partial t} + z\frac{\partial f}{\partial z}\right)$	$-\mathbf{v}_5\left(t\frac{\partial g}{\partial t} + z\frac{\partial g}{\partial z}\right)$
$\mathbf{v}_4(\mathcal{F})$	$-\mathbf{v}_4\left(\frac{\partial \mathcal{F}}{\partial t}\right)$	$-\mathbf{v}_4\left(\frac{\partial \mathcal{F}}{\partial z}\right)$	$\mathbf{v}_4\left(t\frac{\partial \mathcal{F}}{\partial t} + z\frac{\partial \mathcal{F}}{\partial z}\right)$	0	$-\mathbf{v}_7\left(\frac{\partial \mathcal{F} g}{\partial z}\right)$
$\mathbf{v}_5(\mathcal{G})$	$-\mathbf{v}_5\left(\frac{\partial \mathcal{G}}{\partial t}\right)$	$-\mathbf{v}_5\left(\frac{\partial \mathcal{G}}{\partial z}\right)$	$\mathbf{v}_5\left(t\frac{\partial \mathcal{G}}{\partial t} + z\frac{\partial \mathcal{G}}{\partial z}\right)$	$\mathbf{v}_7\left(\frac{\partial f \mathcal{G}}{\partial z}\right)$	0
$\mathbf{v}_6(\mathcal{H})$	$-\mathbf{v}_6\left(\frac{\partial \mathcal{H}}{\partial t}\right)$	$-\mathbf{v}_6\left(\frac{\partial \mathcal{H}}{\partial z}\right)$	$\mathbf{v}_6\left(t\frac{\partial \mathcal{H}}{\partial t} + z\frac{\partial \mathcal{H}}{\partial z}\right)$	$-\mathbf{v}_5\left(f\mathcal{H}\right)$	$\mathbf{v}_4\left(g\mathcal{H}\right)$
$\mathbf{v}_7(\mathcal{L})$	$-\mathbf{v}_7\left(\frac{\partial \mathcal{L}}{\partial t}\right)$	$-\mathbf{v}_7\left(\frac{\partial \mathcal{L}}{\partial z}\right)$	$\mathbf{v}_7\left(t\frac{\partial \mathcal{L}}{\partial t} + z\frac{\partial \mathcal{L}}{\partial z} + \mathcal{L}\right)$	0	0
$\mathbf{v}_8(\mathcal{K})$	$-\mathbf{v}_8\left(\frac{d\mathcal{K}}{dt}\right)$	0	$\mathbf{v}_8\left(\frac{d(t\mathcal{K})}{dt}\right)$	0	0
$\mathbf{v}_9(\mathcal{S})$	0	$-\mathbf{v}_9\left(\frac{d\mathcal{S}}{dz}\right)$	$\mathbf{v}_9\left(\frac{d(z\mathcal{S})}{dz}\right)$	0	0

Table 4.3: Commutation relations for the infinite dimensional Lie algebra $\mathcal{G}^9$ of HTFM. First part. The entry in row i and column j represents $[\mathbf{v}_i, \mathbf{v}_j]$.

	$\mathbf{v}_6(h)$	$\mathbf{v}_7(l)$	$\mathbf{v}_8(k)$	$\mathbf{v}_9(s)$
$\mathbf{v}_1$	$\mathbf{v}_6\left(\frac{\partial h}{\partial t}\right)$	$\mathbf{v}_7\left(\frac{\partial l}{\partial t}\right)$	$\mathbf{v}_8\left(\frac{dk}{dt}\right)$	0
$\mathbf{v}_2$	$\mathbf{v}_6\left(\frac{\partial h}{\partial z}\right)$	$\mathbf{v}_7\left(\frac{\partial l}{\partial z}\right)$	0	$\mathbf{v}_9\left(\frac{ds}{dz}\right)$
$\mathbf{v}_3$	$-\mathbf{v}_6\left(t\frac{\partial h}{\partial t} + z\frac{\partial h}{\partial z}\right)$	$-\mathbf{v}_7\left(t\frac{\partial l}{\partial t} + z\frac{\partial l}{\partial z} + l\right)$	$-\mathbf{v}_8\left(\frac{d(tk)}{dt}\right)$	$-\mathbf{v}_9\left(\frac{d(zs)}{dz}\right)$
$\mathbf{v}_4(\mathcal{F})$	$\mathbf{v}_5\left(\mathcal{F}h\right)$	0	0	0
$\mathbf{v}_5(\mathcal{G})$	$-\mathbf{v}_4\left(\mathcal{G}h\right)$	0	0	0
$\mathbf{v}_6(\mathcal{H})$	0	0	0	0
$\mathbf{v}_7(\mathcal{L})$	0	0	0	0
$\mathbf{v}_8(\mathcal{K})$	0	0	0	0
$\mathbf{v}_9(\mathcal{S})$	0	0	0	0

Table 4.4: Commutation relations for the infinite dimensional Lie algebra $\mathcal{G}^9$ of HTFM. Second part

sum of the Lie series (4.39), we obtain the adjoint table 4.5 and 4.6 for the symmetry algebra $\mathcal{G}^9$ of HTFM. We can use this information to construct the optimal system of first order Θ_1, which constitutes the basis of further analysis.

As was done before for the CHM equation, we calculate the first order optimal system Θ_1 by judicious applications of adjoint transformations to a general element of the algebra $\mathbf{v}$, composed of a linear combination of all the generators of the algebra. What we obtain is a much simpler irreducible form that can be used to reduce the number of variables in the system. By considering all possible cases, we get a set of independent vector fields that span the space of possible reductions. These are the elements of Θ_1. Below is a list of all the elements of the first order optimal system for HTFM:

$$
\begin{array}{cc}
(a) & \mathbf{v}_3 \\
(b) & a_1\mathbf{v}_1 + \mathbf{v}_2 \\
(c) & \mathbf{v}_1 + a_9\mathbf{v}_9 \\
(d) & \mathbf{v}_2 + a_8\mathbf{v}_8 \\
(e) & \mathbf{v}_6 + a_7\mathbf{v}_7 + a_8\mathbf{v}_8 + a_9\mathbf{v}_9 \\
(f) & a_4\mathbf{v}_4 + \mathbf{v}_6 + a_8\mathbf{v}_8 + a_9\mathbf{v}_9 \\
(g) & \mathbf{v}_4 + a_8\mathbf{v}_8 + a_9\mathbf{v}_9 \\
(h) & \mathbf{v}_5 + a_8\mathbf{v}_8 + a_9\mathbf{v}_9 \\
(i) & a_7\mathbf{v}_7 + a_8\mathbf{v}_8 + \mathbf{v}_9 \\
(j) & a_7\mathbf{v}_7 + \mathbf{v}_8 \\
(k) & \mathbf{v}_7
\end{array}
$$

Ad	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4(f)$
$\mathbf{v}_1$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3 + \epsilon\mathbf{v}_1$	$e^{-\epsilon\frac{\partial}{\partial t}}\mathbf{v}_4(f)$
$\mathbf{v}_2$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3 + \epsilon\mathbf{v}_2$	$e^{-\epsilon\frac{\partial}{\partial z}}\mathbf{v}_4(f)$
$\mathbf{v}_3$	$e^{-\epsilon}\mathbf{v}_1$	$e^{-\epsilon}\mathbf{v}_2$	$\mathbf{v}_3$	$e^{\epsilon\left(t\frac{\partial}{\partial t}+z\frac{\partial}{\partial z}\right)}\mathbf{v}_4(f)$
$\mathbf{v}_4(\mathcal{F})$	$\mathbf{v}_1 + \epsilon\mathbf{v}_4\left(\frac{\partial\mathcal{F}}{\partial t}\right)$	$\mathbf{v}_2 + \epsilon\mathbf{v}_4\left(\frac{\partial\mathcal{F}}{\partial z}\right)$	$\mathbf{v}_3 - \epsilon\mathbf{v}_4\left(t\frac{\partial\mathcal{F}}{\partial t} + z\frac{\partial\mathcal{F}}{\partial z}\right)$	$\mathbf{v}_4(f)$
$\mathbf{v}_5(\mathcal{G})$	$\mathbf{v}_1 + \epsilon\mathbf{v}_5\left(\frac{\partial\mathcal{G}}{\partial t}\right)$	$\mathbf{v}_2 + \mathbf{v}_5\left(\frac{\partial\mathcal{G}}{\partial z}\right)$	$\mathbf{v}_3 - \epsilon\mathbf{v}_5\left(t\frac{\partial\mathcal{G}}{\partial t} + z\frac{\partial\mathcal{G}}{\partial z}\right)$	$\mathbf{v}_4(f) - \epsilon\mathbf{v}_7\left(\frac{\partial\mathcal{G}f}{\partial z}\right)$
$\mathbf{v}_6(\mathcal{H})$	$\mathbf{v}_1 + \epsilon\mathbf{v}_6\left(\frac{\partial\mathcal{H}}{\partial t}\right)$	$\mathbf{v}_2 + \epsilon\mathbf{v}_6\left(\frac{\partial\mathcal{H}}{\partial z}\right)$	$\mathbf{v}_3 - \epsilon\mathbf{v}_6\left(t\frac{\partial\mathcal{H}}{\partial t} + z\frac{\partial\mathcal{H}}{\partial z}\right)$	$\mathbf{v}_4(f)cos() + \mathbf{v}_5(g)sin()$
$\mathbf{v}_7(\mathcal{L})$	$\mathbf{v}_1 + \epsilon\mathbf{v}_7\left(\frac{\partial\mathcal{L}}{\partial t}\right)$	$\mathbf{v}_2 + \epsilon\mathbf{v}_7\left(\frac{\partial\mathcal{L}}{\partial z}\right)$	$\mathbf{v}_3 - \epsilon\mathbf{v}_7\left(t\frac{\partial\mathcal{L}}{\partial t} + z\frac{\partial\mathcal{L}}{\partial z} + \mathcal{L}\right)$	$\mathbf{v}_4(f)$
$\mathbf{v}_8(\mathcal{K})$	$\mathbf{v}_1 + \epsilon\mathbf{v}_8\left(\frac{d\mathcal{K}}{dt}\right)$	$\mathbf{v}_2$	$\mathbf{v}_3 - \epsilon\mathbf{v}_8\left(\frac{d(t\mathcal{K})}{dt}\right)$	$\mathbf{v}_4(f)$
$\mathbf{v}_9(\mathcal{S})$	$\mathbf{v}_1$	$\mathbf{v}_2 + \epsilon\mathbf{v}_9\left(\frac{d\mathcal{S}}{dz}\right)$	$\mathbf{v}_3 - \epsilon\mathbf{v}_9\left(\frac{d(z\mathcal{S})}{dz}\right)$	$\mathbf{v}_4(f)$

Table 4.5: Adjoint table for the Lie algebra $\mathcal{G}^9$ of HTFM. The (i, j) entry represents $\mathrm{Ad}(\exp(\epsilon\mathbf{v}_i))\mathbf{v}_j$. First part.

Ad	$\mathbf{v}_5(g)$	$\mathbf{v}_6(h)$	$\mathbf{v}_7(l)$	$\mathbf{v}_8(k)$	$\mathbf{v}_9(s)$
$\mathbf{v}_1$	$e^{-\epsilon\frac{\partial}{\partial t}}\mathbf{v}_5(g)$	$e^{-\epsilon\frac{\partial}{\partial t}}\mathbf{v}_6(h)$	$e^{-\epsilon\frac{\partial}{\partial t}}\mathbf{v}_7(l)$	$e^{-\epsilon\frac{d}{dt}}\mathbf{v}_8(k)$	$\mathbf{v}_9(s)$
$\mathbf{v}_2$	$e^{-\epsilon\frac{\partial}{\partial z}}\mathbf{v}_5(g)$	$e^{-\epsilon\frac{\partial}{\partial z}}\mathbf{v}_6(h)$	$e^{-\epsilon\frac{\partial}{\partial z}}\mathbf{v}_7(l)$	$\mathbf{v}_8(k)$	$e^{-\epsilon\frac{d}{dz}}\mathbf{v}_9(s)$
$\mathbf{v}_3$	$e^{\epsilon\left(t\frac{\partial}{\partial t}+z\frac{\partial}{\partial z}\right)}\mathbf{v}_5$	$e^{\epsilon\left(t\frac{\partial}{\partial t}+z\frac{\partial}{\partial z}\right)}\mathbf{v}_6$	$e^{\epsilon\left(t\frac{\partial}{\partial t}+z\frac{\partial}{\partial z}\right)}\mathbf{v}_7$	$e^{\epsilon\left(t\frac{\partial}{\partial t}+1\right)}\mathbf{v}_8$	$e^{\epsilon\left(z\frac{\partial}{\partial z}+1\right)}\mathbf{v}_9$
$\mathbf{v}_4(\mathcal{F})$	$\mathbf{v}_5(g)+\epsilon\mathbf{v}_7\left(\frac{\partial\mathcal{F}g}{\partial z}\right)$	$\mathbf{v}_6(h)-\epsilon\mathbf{v}_5(\mathcal{F}h)$ $-\frac{\epsilon^2}{2}\mathbf{v}_7\left(\frac{\partial\mathcal{F}^2h}{\partial z}\right)$	$\mathbf{v}_7(l)$	$\mathbf{v}_8(k)$	$\mathbf{v}_9(s)$
$\mathbf{v}_5(\mathcal{G})$	$\mathbf{v}_5(g)$	$\mathbf{v}_6(h)+\epsilon\mathbf{v}_4(\mathcal{G}h)$ $-\frac{\epsilon^2}{2}\mathbf{v}_7\left(\frac{\partial\mathcal{G}^2h}{\partial z}\right)$	$\mathbf{v}_7(l)$	$\mathbf{v}_8(k)$	$\mathbf{v}_9(s)$
$\mathbf{v}_6(\mathcal{H})$	$\mathbf{v}_5(g)cos()-\mathbf{v}_4sin()$	$\mathbf{v}_6(h)$	$\mathbf{v}_7(l)$	$\mathbf{v}_8(k)$	$\mathbf{v}_9(s)$
$\mathbf{v}_7(\mathcal{L})$	$\mathbf{v}_5(g)$	$\mathbf{v}_6(h)$	$\mathbf{v}_7(l)$	$\mathbf{v}_8(k)$	$\mathbf{v}_9(s)$
$\mathbf{v}_8(\mathcal{K})$	$\mathbf{v}_5(g)$	$\mathbf{v}_6(h)$	$\mathbf{v}_7(l)$	$\mathbf{v}_8(k)$	$\mathbf{v}_9(s)$
$\mathbf{v}_9(\mathcal{S})$	$\mathbf{v}_5(g)$	$\mathbf{v}_6(h)$	$\mathbf{v}_7(l)$	$\mathbf{v}_8(k)$	$\mathbf{v}_9(s)$

Table 4.6: Adjoint table for the Lie algebra $\mathcal{G}^9$ of HTFM. Second part.

Evidently, the last three cases, (i), (j) and (k), which deal with gauge symmetries of the fields, do not induce reductions of the number of independent variables. However, all the remaining cases, (a) thru (h), do generate single reductions on the number of independent variables and, although we are dealing here with a four dimensional space of independent variables, in some instances these single reductions can simplify substantially the form of the nonlinear system of PDE's. As an example, consider element (g) of the optimal system above:

$$\mathbf{v}_4 + a_8\mathbf{v}_8 + a_9\mathbf{v}_9 = f\frac{\partial}{\partial x} - \left(y\frac{\partial f}{\partial t} + a_8\alpha k - a_9 s\right)\frac{\partial}{\partial \varphi} + y\frac{\partial f}{\partial z}\frac{\partial}{\partial \psi} + \frac{a_9}{\alpha}s\frac{\partial}{\partial \chi} \quad . \tag{4.114}$$

Integrating the characteristics for this element yields the invariants that represent the new similarity variables for the symmetry reduction. The independent variables are given by

$$\eta_1 = t\,, \quad \eta_2 = y\,, \quad \eta_3 = z\,,$$

while the dependent variables are

$$\zeta_1 = U\,, \quad \zeta_2 = f\varphi + x\left(y\frac{\partial f}{\partial t} + a_8\alpha k - a_9 s\right)\,, \quad \zeta_3 = J\,,$$

$$\zeta_4 = f\psi - xy\frac{\partial f}{\partial z}\,, \quad \zeta_5 = f\chi - \frac{a_9}{\alpha}xs\,.$$

The new independent variables are simply equal to three of the original independent variables: t, y, z, while x is added to the potential fields φ and ψ, and to the density field χ as part of a gauge transformation that leaves U and J invariant. The net effect of this symmetry transformation in the three field model equations (4.98)-(4.102), is to linearize the Poisson brackets appearing in them. This is done by breaking the symmetry between the coordinates x and y. Therefore, we are left with a *linear* system of five coupled PDE's, which is much simpler to analyze than the original nonlinear set. The form of the reduced system is

$$\frac{\partial \zeta_1}{\partial \eta_1} - \frac{1}{f}\left(\eta_2 \frac{\partial f}{\partial \eta_1} + a_8 \alpha k(\eta_1) - a_9 s(\eta_3)\right)\frac{\partial \zeta_1}{\partial \eta_2} + \frac{\partial \zeta_3}{\partial \eta_3} - \frac{\eta_2}{f}\frac{\partial f}{\partial \eta_3}\frac{\partial \zeta_3}{\partial \eta_2} = 0\,, \tag{4.115}$$

$$\frac{1}{f}\frac{\partial \zeta_4}{\partial \eta_1} - a_8 \alpha \frac{k}{f^2}\frac{\partial \zeta_4}{\partial \eta_2} - \frac{1}{f^2}\frac{\partial f}{\partial \eta_1}\frac{\partial}{\partial \eta_2}(\eta_2 \zeta_4) + \frac{1}{f^2}\frac{\partial f}{\partial \eta_3}\frac{\partial}{\partial \eta_2}(\eta_2(\alpha\zeta_5 - \zeta_2)) - \frac{1}{f}\frac{\partial}{\partial \eta_3}(\alpha\zeta_5 - \zeta_2) = 0\,, \tag{4.116}$$

$$\frac{\partial}{\partial \eta_1}\left(\frac{1}{f}\zeta_5\right) - \frac{1}{f^2}\frac{\partial \zeta_5}{\partial \eta_2}\left(\eta_2 \frac{\partial f}{\partial \eta_1} + a_8 \alpha k - a_9 s\right) + \frac{a_9}{\alpha}\frac{s}{f^2}\frac{\partial \zeta_2}{\partial \eta_2} + \frac{\partial \zeta_3}{\partial \eta_3} - \frac{\eta_2}{f}\frac{\partial f}{\partial \eta_3}\frac{\partial \zeta_3}{\partial \eta_2} = 0\,, \tag{4.117}$$

$$\zeta_1 - \frac{1}{f}\frac{\partial^2 \zeta_2}{\partial \eta_2^2} = 0\,, \tag{4.118}$$

$$\zeta_3 - \frac{1}{f}\frac{\partial^2 \zeta_4}{\partial \eta_2^2} = 0\,, \tag{4.119}$$

$$\tag{4.120}$$

where $f(\eta_1, \eta_3)$, $k(\eta_1)$ and $s(\eta_3)$, are arbitrary functions of their arguments and a_8, a_9 and α are arbitrary constants. Besides the freedom afforded by the presence of these arbitrary functions, which can be chosen at will to further reduce the problem, we have reduced the number of independent variables by one and also linearized a complicated system of PDE's without making any approximations. The trade-off that has to be made to achieve all these advantages is that solutions corresponding to the linearized system shown above won't necessarily be of physical interest. In many cases, group invariant solutions describe asymptotic behavior of dynamical systems or constrained solutions, which are valid only in a restricted region and describe physical situations near a singularity. This is only known after the solution has been calculated.

The next chapter will be devoted to an in-depth exploration of symmetry reductions and solutions of HTFM, with emphasis on physical interpretation and connection with the underlying ideas of plasma physics.

Chapter 5

Physical Interpretation of Symmetry Solutions

In the last chapter, symmetry group techniques were implemented for two extensively studied nonlinear plasma fluid models, HTFM and its electrostatic limit the CHM equation. By using the CHM equation as a working example, we were able to see the benefits afforded by a thorough and systematic analysis of the symmetry reductions based on the use of the Lie algebra associated with the generators of the symmetries. Following this program, we obtained some analytical solutions and learned the fundamental principle of succesive application of symmetries for multidimensional reduction of the number of independent variables.

In this chapter we concentrate on the more general HTFM. The Lie point symmetries of this model were calculated in the last section of the preceding chapter and were shown to form an infinite dimensional Lie algebra. This property of the underlying group makes the analysis of the present problem far more complicated, but at the same time with an enormously richer structure than the finite dimensional case exemplified by the CHM system.

Besides the actual calculation of different symmetry reductions, we place special emphasis on possible physical scenarios for the calculated solutions. Particular consideration is given to the two-dimensional limit of the model and to the RMHD reduction of it, mentioned before. We will start out from the most general, three-dimensional case and later revert to particular limits.

5.1 Hazeltine's Model: Three-Dimensional Results

5.1.1 Extension of the Algebra

As noted above, a salient feature of the symmetry group calculated for HTFM is the fact that the infinitesimals depend on arbitrary functions of time and the space coordinate z (see eqs. (4.103)-(4.111)). What this means is that the corresponding Lie algebra is infinite dimensional, allowing for a wealth of particular cases describing the three dimensional nature of the model and of the corresponding symmetry solutions.

In order to compare with the finite dimensional case of CHM and the simplified two-dimensional and RMHD limits, it is convenient to rewrite the symmetries given by (4.113) in a more explicit form, extracting in a unique way, the structure of simple translations and rotations included in the general form of the generators $\mathbf{v}_4$, $\mathbf{v}_5$ and $\mathbf{v}_6$. This can be done by formally integrating out the known symmetries and expressing the result as a set of equivalent "renormalized" symmetries, which still preserve the infinite dimensional nature of the algebra.

More explicitly, consider the form of the generalized Galilean translation in the x-direction, given by the generator $\mathbf{v}_4$ in (4.113)

$$\mathbf{v}_4 = f(z,t)\frac{\partial}{\partial x} - y\frac{\partial f(z,t)}{\partial t}\frac{\partial}{\partial \varphi} + y\frac{\partial f(z,t)}{\partial z}\frac{\partial}{\partial \psi}\,. \tag{5.1}$$

Evidently, this generator contains as particular cases the following two very fundamental symmetries:

1) Invariance under translation along the x-axis. This symmetry is obtained for the special choice $f(z,t)$ = constant and has the infinitesimal generator of the simple form

$$\mathbf{v} = \frac{\partial}{\partial x}\,.$$

2) Invariance under a generalized Galilean boost in the x-direction. This symmetry is obtained from the general case $\mathbf{v}_4$ by letting $f(z,t) = n(t)$, yielding the following infinitesmal generator:

$$\mathbf{v} = n(t)\frac{\partial}{\partial x} - y\frac{dn}{dt}\frac{\partial}{\partial \varphi}\,,$$

which reduces to the well-known simple Galilean boost $t\frac{\partial}{\partial x} - y\frac{\partial}{\partial \varphi}$, when $n(t)$ is a linear function of time.

In order to extract the information contained in these symmetries we have to formally integrate the functional dependence of $f(z,t)$ with respect to its arguments. This can be done as follows: First differentiate f with respect to z and then define it as a new function of z and t, $f^*(z,t)$

$$\frac{\partial f(z,t)}{\partial z} =: f^*(z,t)\,. \tag{5.2}$$

Then integrate the z-dependence, obtaining

$$f(z,t) = \int f^*(z,t)dz + \int n(t)dt\,,$$

where the last term is just a convenient way to write the arbitrary function of t that comes from the integration in z. Next, we take the time derivative of the above expression for f, and integrate with respect to its time dependence, yielding

$$f(z,t) = \int\int \frac{\partial f^*}{\partial t} dzdt + \int n(t)dt + c_4. \tag{5.3}$$

This is just the function $f(z,t)$ rewritten in a more convenient way, where some of its features have been made explicit thru the integration constant c_4 and the arbitrary function of time $n(t)$. What this means is that each separate function and constant of integration in (5.3) will represent a separate symmetry of the system of PDE's, when substituted in the place of $f(z,t)$. The constant c_4 corresponds to simple translation along the x-axis, the function $n(t)$ corresponds to a generalized Galilean transformation in the x-direction, with arbitrary time dependent coefficients, and the function f^* corresponds to the general case, excluding the two previous symmetries. Therefore, the generator $\mathbf{v}_4$ can be replaced, without loss of generality, by the three symmetries:

$$\mathbf{v}_4^{'} = \frac{\partial}{\partial x}, \qquad \mathbf{v}_4^{''} = \left(\int n(t)dt\right)\frac{\partial}{\partial x} - yn(t)\frac{\partial}{\partial \varphi},$$
$$\mathbf{v}_4^{'''} = \left(\int\int \frac{\partial f}{\partial t} dzdt\right)\frac{\partial}{\partial x} - y\left(\int \frac{\partial f}{\partial t} dz\right)\frac{\partial}{\partial \varphi} + yf\frac{\partial}{\partial \psi},$$

where we have replaced f^* by $f(z,t)$ in the last generator.

If we follow the same procedure with $\mathbf{v}_5$ instead of $\mathbf{v}_4$ from (4.113), we will end up with three symmetries equivalent to the ones presented above, but describing invariance in the y-direction. Their explicit form is given by

$$\mathbf{v}_5^{'} = \frac{\partial}{\partial y}, \qquad \mathbf{v}_5^{''} = \left(\int m(t)dt\right)\frac{\partial}{\partial y} + xm(t)\frac{\partial}{\partial \varphi},$$
$$\mathbf{v}_5^{'''} = \left(\int\int \frac{\partial g}{\partial t} dzdt\right)\frac{\partial}{\partial y} + x\left(\int \frac{\partial g}{\partial t} dz\right)\frac{\partial}{\partial \varphi} - xg\frac{\partial}{\partial \psi},$$

where $m(t)$ is an arbitrary function of time and $g = g(z,t)$ is an arbitrary function of its arguments.

The last function that we would like to consider under the present scheme is the function $h(z,t)$, related to the generalized rotational symmetry $\mathbf{v}_6$ in (4.113). Recall that $h(z,t)$ is constrained to be a solution of a one dimensional wave equation (4.112). Our goal here is to single out the invariance under rotations in the $x-y$ plane, contained in the form of $\mathbf{v}_6$ when $h(z,t)$ =constant. This can be achieved by formally integrating the z and t dependencies of the function $h(z,t)$, in the same fashion as was done before for the generator $\mathbf{v}_4$. The end result is again a splitting of the general symmetry into three independent symmetries covering all possible cases of the original generator. These three symmetries have the form:

$$\mathbf{v}_6^{'} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y},$$

$$\mathbf{v}_6^{''} = \left(\int r(t)dt\right)\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) - r(t)\left(\frac{1}{2}(x^2+y^2)+2\alpha\right)\frac{\partial}{\partial \varphi}$$
$$-r(t)\left(2\frac{\partial}{\partial U} + 2\frac{\partial}{\partial \chi}\right),$$

$$\mathbf{v}_6^{'''} = \left(\int\int \frac{\partial h}{\partial t}dtdz\right)\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) - 2\left(\int \frac{\partial h}{\partial t}dz\right)\left(\frac{\partial}{\partial U} + \frac{\partial}{\partial \chi}\right)$$
$$-\left(\int \frac{\partial h}{\partial t}dz\right)\left(\frac{1}{2}(x^2+y^2)+2\alpha\right)\frac{\partial}{\partial \varphi} + h(z,t)\left(2\frac{\partial}{\partial J} + \frac{1}{2}(x^2+y^2)\frac{\partial}{\partial \psi}\right),$$

where the function $h(z,t)$ has to satisfy an integral version of the wave equation constraint (4.112), given by

$$\int \frac{\partial^2 h}{\partial t^2}dz + \frac{dr(t)}{dt} = \frac{\partial h}{\partial z}. \tag{5.4}$$

What we have accomplished with this procedure is to expand the number of basic generators, from nine to fifteen, for the representation of the infinite dimensional algebra admitted by the three field model, therefore uncovering a richer algebraic structure in the new representation, but most importantly, separating a finite dimensional subalgebra that can be directly compared with simpler models, like the CHM equation previously studied, and whose algebraic properties have been thoroughly analyzed. All this has been done without having to choose a particular case for the arbitrary functions involved, and therefore keeping the freedom afforded by these undetermined functions.

The extended finite subalgebra is given by the six generators:

$$\mathbf{v}_1 = \frac{\partial}{\partial t}, \qquad \mathbf{v}_2 = \frac{\partial}{\partial z}, \qquad \mathbf{v}_3 = \frac{\partial}{\partial x}, \qquad \mathbf{v}_4 = \frac{\partial}{\partial y},$$
$$\mathbf{v}_5 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y},$$
$$\mathbf{v}_6 = -t\frac{\partial}{\partial t} - z\frac{\partial}{\partial z} + U\frac{\partial}{\partial U} + \varphi\frac{\partial}{\partial \varphi} + J\frac{\partial}{\partial J} + \psi\frac{\partial}{\partial \psi} + \chi\frac{\partial}{\partial \chi}.$$

They correspond to invariance under time and space translations, rotation in the x-y plane and scaling of the five fields together with time and the z-coordinate, respectively.

The extended infinite dimensional subalgebra for the system is given by the generators $\mathbf{v}_4^{''}$, $\mathbf{v}_4^{'''}$, $\mathbf{v}_5^{''}$, $\mathbf{v}_5^{'''}$, $\mathbf{v}_6^{''}$, $\mathbf{v}_6^{'''}$, listed above, plus the three gauge symmetries: $\mathbf{v}_7$, $\mathbf{v}_8$ and $\mathbf{v}_9$ listed in (4.113). Notice that the six symmetries $\mathbf{v}_1$, $\mathbf{v}_3$, $\mathbf{v}_4$, $\mathbf{v}_5$, $\mathbf{v}_6$ and $\mathbf{v}_8$ correspond to a three dimensional generalization of the symmetry algebra $\mathcal{G}^6$ for the CHM equation, with one of the generators, $\mathbf{v}_8$, being an element of an infinite dimensional algebra.

In the two dimensional limit, the generalization of the algebra $\mathcal{G}^6$ stems not from the fact that we are dealing with a system described by additional fields, but by the occurrence of the free time dependent function $k(t)$ involved in the $\mathbf{v}_8$ generator

$$\mathbf{v}_8 = k(t)\frac{\partial}{\partial \varphi}. \tag{5.5}$$

If we consider the particular case $k(t) = \text{constant}$, we will have a six parameter subalgebra isomorphic to the CHM symmetry algebra $\mathcal{G}^6$, and therefore with known elements of the first and second order optimal systems, Θ_1 and Θ_2 derived in the last chapter.

However, since we are exploring the most general set of reductions given by the infinite dimensional algebra for HTFM, we will try to keep the functions of z and time t, which appear in the generators of the symmetries, arbitrary whenever possible. Particular reductions and detailed comparisons with lower dimensional systems will be left for later sections.

Now we proceed with the analysis of the extended representation of the Lie algebra for the three field model, $\mathcal{G}^{15}$, by following the procedure used with the CHM equation in the previous chapter. First, we calculate the basic commutation relations for the symmetry generators, which describe the properties of the infinite dimensional algebra and lead to the determination of the optimal system through the adjoint transformation. The commutation table, divided into three parts, Table 5.1, Table 5.2 and Table 5.3, is given below and contains all the information that is needed to generate the possible symmetry reductions of HTFM.

As was pointed out before, the extended representation of the algebra $\mathcal{G}^{15}$ used here enables us to study a variety of possible cases, some containing exclusively elements of the finite subalgebra $\mathcal{G}^6$ composed by the first six generators, $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6)$ (see Table 5.1). Some of the cases contain elements of the infinite dimensional subalgebra composed of the generators $\mathbf{v}_7$ through $\mathbf{v}_{15}$, where we have made the following identifications for these elements:

$$\mathbf{v}_7 = \mathbf{v}_4^{'''}, \quad \mathbf{v}_8 = \mathbf{v}_4^{''}, \quad \mathbf{v}_9 = \mathbf{v}_5^{'''},$$
$$\mathbf{v}_{10} = \mathbf{v}_5^{''}, \quad \mathbf{v}_{11} = \mathbf{v}_6^{'''}, \quad \mathbf{v}_{12} = \mathbf{v}_6^{''},$$

and the last three generators of (4.113) are

$$\begin{aligned}
\mathbf{v}_{13} &= l(t,z)\frac{\partial}{\partial\psi} - \left(\int \frac{\partial l(t,z)}{\partial t} dz\right)\frac{\partial}{\partial\varphi}, \\
\mathbf{v}_{14} &= k(t)\frac{\partial}{\partial\varphi}, \\
\mathbf{v}_{15} &= \alpha s(z)\frac{\partial}{\partial\varphi} + s(z)\frac{\partial}{\partial\chi}.
\end{aligned}$$

These latter three generators can describe quite general situations in terms of the free functions of space and time. Other mixed cases containing elements of both the finite and infinite subalgebras are considered. In any case, as we will see below, the set of possible reductions given by the elements of the optimal system for $\mathcal{G}^{15}$ is larger, in a nontrivial way, than the set of reductions calculated for the unextended algebra $\mathcal{G}^9$ of HTFM (see the last section of the last chapter). This could be understood as an indirect proof of the need for an extended representation, which would extract valuable information from the structure of the algebra that was hidden in the original compact form of $\mathcal{G}^9$.

Of course this situation arises in the context of an infinite dimensional Lie algebra, where very few problems have been solved (see for example

	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$	$\mathbf{v}_6$
$\mathbf{v}_1$	0	0	0	0	0	$-\mathbf{v}_1$
$\mathbf{v}_2$	0	0	0	0	0	$-\mathbf{v}_2$
$\mathbf{v}_3$	0	0	0	0	$-\mathbf{v}_4$	0
$\mathbf{v}_4$	0	0	0	0	$\mathbf{v}_3$	0
$\mathbf{v}_5$	0	0	$\mathbf{v}_4$	$-\mathbf{v}_3$	0	0
$\mathbf{v}_6$	$\mathbf{v}_1$	$\mathbf{v}_2$	0	0	0	0
$\mathbf{v}_7(\mathcal{F})$	$-\mathbf{v}_7\left(\frac{\partial\mathcal{F}}{\partial t}\right)$	$-\mathbf{v}_7\left(\frac{\partial\mathcal{F}}{\partial z}\right)$	0	$-\mathbf{v}_{13}(\mathcal{F})$	$-\mathbf{v}_9(\mathcal{F})$	$\mathbf{v}_7\left(t\frac{\partial\mathcal{F}}{\partial t}+z\frac{\partial\mathcal{F}}{\partial z}\right)$
$\mathbf{v}_8(\mathcal{N})$	$-\mathbf{v}_8\left(\frac{d\mathcal{N}}{dt}\right)$	0	0	$\mathbf{v}_{14}(\mathcal{N})$	$-\mathbf{v}_{10}(\mathcal{N})$	$\mathbf{v}_8\left(\frac{d(\mathcal{N}t)}{dt}\right)$
$\mathbf{v}_9(\mathcal{G})$	$-\mathbf{v}_9\left(\frac{\partial\mathcal{G}}{\partial t}\right)$	$-\mathbf{v}_9\left(\frac{\partial\mathcal{G}}{\partial z}\right)$	$\mathbf{v}_{13}(\mathcal{G})$	0	$\mathbf{v}_7(\mathcal{G})$	$\mathbf{v}_9\left(t\frac{\partial\mathcal{G}}{\partial t}+\frac{\partial(z\mathcal{G})}{\partial z}\right)$
$\mathbf{v}_{10}(\mathcal{M})$	$-\mathbf{v}_{10}\left(\frac{d\mathcal{M}}{dt}\right)$	0	$-\mathbf{v}_{14}(\mathcal{M})$	0	$\mathbf{v}_8(\mathcal{M})$	$\mathbf{v}_{10}\left(\frac{d(t\mathcal{M})}{dt}\right)$
$\mathbf{v}_{11}(\mathcal{H})$	$-\mathbf{v}_{11}\left(\frac{\partial\mathcal{H}}{\partial t}\right)$	$-\mathbf{v}_{11}\left(\frac{\partial\mathcal{H}}{\partial z}\right)$	$\mathbf{v}_9(\mathcal{H})$	$-\mathbf{v}_7(\mathcal{H})$	0	$\mathbf{v}_{11}\left(\frac{\partial t\mathcal{H}}{\partial t}+z\frac{\partial\mathcal{H}}{\partial z}\right)$
$\mathbf{v}_{12}(\mathcal{R})$	$-\mathbf{v}_{12}\left(\frac{\partial\mathcal{R}}{\partial t}\right)$	0	$\mathbf{v}_{10}(\mathcal{R})$	$-\mathbf{v}_8(\mathcal{R})$	0	$\mathbf{v}_{12}\left(\frac{d(t\mathcal{R})}{dt}\right)$
$\mathbf{v}_{13}(\mathcal{L})$	$-\mathbf{v}_{13}\left(\frac{\partial\mathcal{L}}{\partial t}\right)$	$-\mathbf{v}_{13}\left(\frac{\partial\mathcal{L}}{\partial z}\right)$	0	0	0	$\mathbf{v}_{13}\left(t\frac{\partial\mathcal{L}}{\partial t}+\frac{\partial(z\mathcal{L})}{\partial z}\right)$
$\mathbf{v}_{14}(\mathcal{K})$	$-\mathbf{v}_{14}\left(\frac{d\mathcal{K}}{dt}\right)$	0	0	0	0	$\mathbf{v}_{14}\left(\frac{d(t\mathcal{K})}{dt}\right)$
$\mathbf{v}_{15}(\mathcal{S})$	0	$-\mathbf{v}_{15}\left(\frac{d\mathcal{S}}{dz}\right)$	0	0	0	$\mathbf{v}_{15}\left(\frac{d(z\mathcal{S})}{dz}\right)$

Table 5.1: Commutation table for the symmetry algebra $\mathcal{G}^{15}$ of HTFM. First part.

	$\mathbf{v}_7(f)$	$\mathbf{v}_8(n)$	$\mathbf{v}_9(g)$	$\mathbf{v}_{10}(m)$
$\mathbf{v}_1$	$\mathbf{v}_7\left(\frac{\partial f}{\partial t}\right)$	$\mathbf{v}_8\left(\frac{dn}{dt}\right)$	$\mathbf{v}_9\left(\frac{\partial g}{\partial t}\right)$	$\mathbf{v}_{10}\left(\frac{dm}{dt}\right)$
$\mathbf{v}_2$	$\mathbf{v}_7\left(\frac{\partial f}{\partial z}\right)$	0	$\mathbf{v}_9\left(\frac{\partial g}{\partial z}\right)$	0
$\mathbf{v}_3$	0	0	$-\mathbf{v}_{13}(g)$	$\mathbf{v}_{14}(m)$
$\mathbf{v}_4$	$\mathbf{v}_{13}(f)$	$-\mathbf{v}_{14}(n)$	0	0
$\mathbf{v}_5$	$\mathbf{v}_9(f)$	$\mathbf{v}_{10}(n)$	$-\mathbf{v}_7(g)$	$-\mathbf{v}_8(m)$
$\mathbf{v}_6$	$-\mathbf{v}_7\left(t\frac{\partial f}{\partial t} + z\frac{\partial f}{\partial z} + f\right)$	$-\mathbf{v}_8\left(\frac{d(tn)}{dt}\right)$	$-\mathbf{v}_9\left(t\frac{\partial g}{\partial t} + \frac{\partial zg}{\partial z}\right)$	$-\mathbf{v}_{10}\left(\frac{d(tm)}{dt}\right)$
$\mathbf{v}_7(\mathcal{F})$	0	0	$-\mathbf{v}_{13}\left(g\int \mathcal{F}dz + \mathcal{F}\int gdz\right)$	$-\mathbf{v}_{13}\left(\mathcal{F}\int mdt\right)$
$\mathbf{v}_8(\mathcal{N})$	0	0	$-\mathbf{v}_{13}\left(g\int \mathcal{N}dt\right)$	$\mathbf{v}_{14}\left(m\int \mathcal{N}dt + \mathcal{N}\int mdt\right)$
$\mathbf{v}_9(\mathcal{G})$	$\mathbf{v}_{13}\left(\mathcal{G}\int fdz + f\int \mathcal{G}dz\right)$	$\mathbf{v}_{13}\left(\mathcal{G}\int ndt\right)$	0	0
$\mathbf{v}_{10}(\mathcal{M})$	$\mathbf{v}_{13}\left(f\int \mathcal{M}dt\right)$	$-\mathbf{v}_{14}\left(\mathcal{M}\int ndt + n\int \mathcal{M}dt\right)$	0	0
$\mathbf{v}_{11}(\mathcal{H})$	$\mathbf{v}_9\left(\mathcal{H}\int fdz + f\int \mathcal{H}dz\right)$	$\mathbf{v}_9\left(\mathcal{H}\int ndt\right)$	$-\mathbf{v}_7\left(g\int \mathcal{H}dz + \mathcal{H}\int gdz\right)$	$-\mathbf{v}_7\left(\mathcal{H}\int mdt\right)$
$\mathbf{v}_{12}(\mathcal{R})$	$\mathbf{v}_9\left(f\int \mathcal{R}dt\right)$	$\mathbf{v}_{10}\left(\mathcal{R}\int ndt + n\int \mathcal{R}dt\right)$	$-\mathbf{v}_7\left(g\int \mathcal{R}dt\right)$	$-\mathbf{v}_8\left(\mathcal{R}\int mdt + m\int \mathcal{R}dt\right)$
$\mathbf{v}_{13}(\mathcal{L})$	0	0	0	0
$\mathbf{v}_{14}(\mathcal{K})$	0	0	0	0
$\mathbf{v}_{15}(\mathcal{S})$	0	0	0	0

Table 5.2: Commutation table for the symmetry algebra $\mathcal{G}^{15}$. Second part.

	$\mathbf{v}_{11}(h)$	$\mathbf{v}_{12}(r)$	$\mathbf{v}_{13}(l)$	$\mathbf{v}_{14}(k)$	$\mathbf{v}_{15}(s)$
$\mathbf{v}_1$	$\mathbf{v}_{11}\left(\frac{\partial h}{\partial t}\right)$	$\mathbf{v}_{12}\left(\frac{dr}{dt}\right)$	$\mathbf{v}_{13}\left(\frac{\partial l}{\partial t}\right)$	$\mathbf{v}_{14}\left(\frac{dk}{dt}\right)$	0
$\mathbf{v}_2$	$\mathbf{v}_{11}\left(\frac{\partial h}{\partial z}\right)$	0	$\mathbf{v}_{13}\left(\frac{\partial l}{\partial z}\right)$	0	$\mathbf{v}_{15}\left(\frac{ds}{dz}\right)$
$\mathbf{v}_3$	$-\mathbf{v}_9(h)$	$-\mathbf{v}_{10}(r)$	0	0	0
$\mathbf{v}_4$	$\mathbf{v}_7(h)$	$\mathbf{v}_8(r)$	0	0	0
$\mathbf{v}_5$	0	0	0	0	0
$\mathbf{v}_6$	$-\mathbf{v}_{11}\left(\frac{\partial(th)}{\partial t} + z\frac{\partial h}{\partial z}\right)$	$-\mathbf{v}_{12}\left(\frac{d(tr)}{dt}\right)$	$-\mathbf{v}_{13}\left(t\frac{\partial l}{\partial t} + \frac{\partial(zl)}{\partial z}\right)$	$-\mathbf{v}_{14}\left(\frac{d(tk)}{dt}\right)$	$-\mathbf{v}_{15}\left(\frac{d(zs)}{dz}\right)$
$\mathbf{v}_7(\mathcal{F})$	$-\mathbf{v}_9\left(\mathcal{F}\int hdz + h\int \mathcal{F}dz\right)$	$-\mathbf{v}_9\left(\mathcal{F}\int rdt\right)$	0	0	0
$\mathbf{v}_8(\mathcal{N})$	$-\mathbf{v}_9\left(h\int \mathcal{N}dt\right)$	$-\mathbf{v}_{10}\left(r\int \mathcal{N}dt + \mathcal{N}\int rdt\right)$	0	0	0
$\mathbf{v}_9(\mathcal{G})$	$\mathbf{v}_7\left(h\int \mathcal{G}dz + \mathcal{G}\int hdz\right)$	$\mathbf{v}_7\left(\mathcal{G}\int rdt\right)$	0	0	0
$\mathbf{v}_{10}(\mathcal{M})$	$\mathbf{v}_7\left(h\int \mathcal{M}dt\right)$	$\mathbf{v}_8\left(r\int \mathcal{M}dt + \mathcal{M}\int rdt\right)$	0	0	0
$\mathbf{v}_{11}(\mathcal{H})$	0	0	0	0	0
$\mathbf{v}_{12}(\mathcal{R})$	0	0	0	0	0
$\mathbf{v}_{13}(\mathcal{L})$	0	0	0	0	0
$\mathbf{v}_{14}(\mathcal{K})$	0	0	0	0	0
$\mathbf{v}_{15}(\mathcal{S})$	0	0	0	0	0

Table 5.3: Commutation table for the symmetry algebra $\mathcal{G}^{15}$. Third part.

[Salmon-Hollerbach 91] and [Won 90]) and our understanding of the theory is still at a basic level. However, we know that for fluid models of plasma represented in terms of potentials (φ, ψ) we are going to inevitably face the infinite dimensional Lie algebra case, [Acevedo-Morrison 93] lending importance to the present study.

We now calculate the adjoint representation of the algebra $\mathcal{G}^{15}$, that will allow us to determine the linear combinations of generators spanning the space of similarity solutions. Making use of the commutation relations shown above we can calculate explicitly the adjoint table, Tables 5.4-5.7, which is divided in four sections to present in detail the elements that generate the optimal system. The operators $\boldsymbol{\mathcal{D}}(\epsilon)$ and $\boldsymbol{\mathcal{D}}'(\epsilon)$ represent integro-differential operators whose form is too complicated to display in the table and unnecessary for the calculation of symmetry reductions. From the tables we can see that the presence of the arbitrary functions of time and the space coordinate z, $(f, \mathcal{F}, g, \mathcal{G}, h, \mathcal{H}, l, \mathcal{L})$, arbitrary functions of time only, $(n, \mathcal{N}, m, \mathcal{M}, r, \mathcal{R}, k, \mathcal{K})$, and arbitrary functions of z alone, $(s, \mathcal{S})$, enrich the algebraic structure under consideration and will play an important role in determining most of the members of the optimal system.

Following the standard construction of the optimal system of first order, shown in the previous chapter, we obtain from the adjoint action associated to $\mathcal{G}^{15}$ the elements of Θ_1:

$$(a) \qquad a_5\mathbf{v}_5 + \mathbf{v}_6 + a_8\mathbf{v}_8 + a_9\mathbf{v}_9$$

$$(b) \qquad a_3\mathbf{v}_3 + \mathbf{v}_6$$

$$(c) \qquad a_1\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_5 + a_8\mathbf{v}_8 + a_9\mathbf{v}_9$$

Ad	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$
$\mathbf{v}_1$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$
$\mathbf{v}_2$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$
$\mathbf{v}_3$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$
$\mathbf{v}_4$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$
$\mathbf{v}_5$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3 \cos\epsilon - \mathbf{v}_4 \sin\epsilon$	$\mathbf{v}_4 \cos\epsilon + \mathbf{v}_3 \sin\epsilon$
$\mathbf{v}_6$	$e^{-\epsilon}\mathbf{v}_1$	$e^{-\epsilon}\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$
$\mathbf{v}_7(\mathcal{F})$	$\mathbf{v}_1 + \epsilon\mathbf{v}_7\left(\frac{\partial\mathcal{F}}{\partial t}\right)$	$\mathbf{v}_2 + \epsilon\mathbf{v}_7\left(\frac{\partial\mathcal{F}}{\partial z}\right)$	$\mathbf{v}_3$	$\mathbf{v}_4 + \epsilon\mathbf{v}_{13}(\mathcal{F})$
$\mathbf{v}_8(\mathcal{N})$	$\mathbf{v}_1 + \epsilon\mathbf{v}_8\left(\frac{d\mathcal{N}}{dt}\right)$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4 - \epsilon\mathbf{v}_{14}(\mathcal{N})$
$\mathbf{v}_9(\mathcal{G})$	$\mathbf{v}_1 + \epsilon\mathbf{v}_9\left(\frac{\partial\mathcal{G}}{\partial t}\right)$	$\mathbf{v}_2 + \epsilon\mathbf{v}_9\left(\frac{\partial\mathcal{G}}{\partial z}\right)$	$\mathbf{v}_3 - \epsilon\mathbf{v}_{13}(\mathcal{G})$	$\mathbf{v}_4$
$\mathbf{v}_{10}(\mathcal{M})$	$\mathbf{v}_1 + \epsilon\mathbf{v}_{10}\left(\frac{d\mathcal{M}}{dt}\right)$	$\mathbf{v}_2$	$\mathbf{v}_3 + \epsilon\mathbf{v}_{14}(\mathcal{M})$	$\mathbf{v}_4$
$\mathbf{v}_{11}(\mathcal{H})$	$\mathbf{v}_1 + \epsilon\mathbf{v}_{11}\left(\frac{\partial\mathcal{H}}{\partial t}\right)$	$\mathbf{v}_2 + \epsilon\mathbf{v}_{11}\left(\frac{\partial\mathcal{H}}{\partial z}\right)$	$\mathbf{v}_3 - \mathcal{D}(\epsilon)[\mathbf{v}_9]$ $-\mathcal{D}'(\epsilon)[\mathbf{v}_7]$	$\mathbf{v}_4 + \mathcal{D}(\epsilon)[\mathbf{v}_7]$ $-\mathcal{D}'(\epsilon)[\mathbf{v}_9]$
$\mathbf{v}_{12}(\mathcal{R})$	$\mathbf{v}_1 + \epsilon\mathbf{v}_{12}\left(\frac{d\mathcal{R}}{dt}\right)$	$\mathbf{v}_2$	$\mathbf{v}_3 - \mathcal{D}(\epsilon)[\mathbf{v}_{10}]$ $-\mathcal{D}'(\epsilon)[\mathbf{v}_8]$	$\mathbf{v}_4 + \mathcal{D}(\epsilon)[\mathbf{v}_8]$ $-\mathcal{D}'(\epsilon)[\mathbf{v}_{10}]$
$\mathbf{v}_{13}(\mathcal{L})$	$\mathbf{v}_1 + \epsilon\mathbf{v}_{13}\left(\frac{\partial\mathcal{L}}{\partial t}\right)$	$\mathbf{v}_2 + \epsilon\mathbf{v}_{13}\left(\frac{\partial\mathcal{L}}{\partial z}\right)$	$\mathbf{v}_3$	$\mathbf{v}_4$
$\mathbf{v}_{14}(\mathcal{K})$	$\mathbf{v}_1 + \epsilon\mathbf{v}_{14}\left(\frac{d\mathcal{K}}{dt}\right)$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$
$\mathbf{v}_{15}(\mathcal{S})$	$\mathbf{v}_1$	$\mathbf{v}_2 + \epsilon\mathbf{v}_{15}\left(\frac{d\mathcal{S}}{dz}\right)$	$\mathbf{v}_3$	$\mathbf{v}_4$

Table 5.4: Adjoint table for the generators of the Lie algebra $\mathcal{G}^{15}$ of HTFM. The (i,j) entry represents $\mathrm{Ad}(\exp(\epsilon\mathbf{v}_i))\mathbf{v}_j$. First part.

Ad	$\mathbf{v}_5$	$\mathbf{v}_6$	$\mathbf{v}_7(f)$	$\mathbf{v}_8(n)$
$\mathbf{v}_1$	$\mathbf{v}_5$	$\mathbf{v}_6+\epsilon\mathbf{v}_1$	$e^{-\frac{\partial}{\partial t}}\mathbf{v}_7(f)$	$e^{-\frac{\partial}{\partial t}}\mathbf{v}_8(n)$
$\mathbf{v}_2$	$\mathbf{v}_5$	$\mathbf{v}_6+\epsilon\mathbf{v}_2$	$e^{-\frac{\partial}{\partial z}}\mathbf{v}_7(f)$	$\mathbf{v}_8(n)$
$\mathbf{v}_3$	$\mathbf{v}_5+\epsilon\mathbf{v}_4$	$\mathbf{v}_6$	$\mathbf{v}_7(f)$	$\mathbf{v}_8(n)$
$\mathbf{v}_4$	$\mathbf{v}_5-\epsilon\mathbf{v}_3$	$\mathbf{v}_6$	$\mathbf{v}_7(f)-\epsilon\mathbf{v}_{13}(f)$	$\mathbf{v}_8+\epsilon\mathbf{v}_{14}(n)$
$\mathbf{v}_5$	$\mathbf{v}_5$	$\mathbf{v}_6$	$\mathbf{v}_7(f)\cos\epsilon-\mathbf{v}_9(f)\sin\epsilon$	$\mathbf{v}_8(n)\cos\epsilon-\mathbf{v}_{10}(n)\sin\epsilon$
$\mathbf{v}_6$	$\mathbf{v}_5$	$\mathbf{v}_6$	$e^{t\frac{\partial}{\partial t}+z\frac{\partial}{\partial z}+1}\mathbf{v}_7(f)$	$e^{t\frac{d}{dt}+1}\mathbf{v}_8(n)$
$\mathbf{v}_7(\mathcal{F})$	$\mathbf{v}_5+\epsilon\mathbf{v}_9(\mathcal{F})$ $+\frac{\epsilon^2}{2}\mathbf{v}_{13}\left(2\mathcal{F}\int\mathcal{F}dz\right)$	$\mathbf{v}_6-\epsilon\mathbf{v}_7\left(t\frac{\partial\mathcal{F}}{\partial t}+z\frac{\partial\mathcal{F}}{\partial z}\right)$	$\mathbf{v}_7(f)$	$\mathbf{v}_8(n)$
$\mathbf{v}_8(\mathcal{N})$	$\mathbf{v}_5+\epsilon\mathbf{v}_{10}(\mathcal{N})$ $+\frac{\epsilon^2}{2}\mathbf{v}_{14}\left(2\mathcal{N}\int\mathcal{N}dt\right)$	$\mathbf{v}_6-\epsilon\mathbf{v}_8\left(\frac{d\mathcal{N}t}{dt}\right)$	$\mathbf{v}_7(f)$	$\mathbf{v}_8(n)$
$\mathbf{v}_9(\mathcal{G})$	$\mathbf{v}_5-\epsilon\mathbf{v}_7(\mathcal{G})$ $+\frac{\epsilon^2}{2}\mathbf{v}_{13}\left(2\mathcal{G}\int\mathcal{G}dz\right)$	$\mathbf{v}_6-\epsilon\mathbf{v}_9\left(t\frac{\partial\mathcal{G}}{\partial t}+\frac{\partial(z\mathcal{G})}{\partial z}\right)$	$\mathbf{v}_7(f)-\epsilon\mathbf{v}_{13}\left(\mathcal{G}\int fdz\right.$ $\left.+f\int\mathcal{G}dz\right)$	$\mathbf{v}_8(n)-\epsilon\mathbf{v}_{13}\left(\mathcal{G}\int ndt\right)$
$\mathbf{v}_{10}(\mathcal{M})$	$\mathbf{v}_5-\epsilon\mathbf{v}_8(\mathcal{M})$ $-\frac{\epsilon^2}{2}\mathbf{v}_{14}\left(2\mathcal{M}\int\mathcal{M}dt\right)$	$\mathbf{v}_6-\epsilon\mathbf{v}_{10}\left(\frac{d(t\mathcal{M})}{dt}\right)$	$\mathbf{v}_7(f)-\epsilon\mathbf{v}_{13}\left(f\int\mathcal{M}dt\right)$	$\mathbf{v}_8(n)+\epsilon\mathbf{v}_{14}\left(\mathcal{M}\int ndt\right.$ $\left.+n\int\mathcal{M}dt\right)$
$\mathbf{v}_{11}(\mathcal{H})$	$\mathbf{v}_5$	$\mathbf{v}_6-\epsilon\mathbf{v}_{11}\left(\frac{\partial(t\mathcal{H})}{\partial t}+z\frac{\partial\mathcal{H}}{\partial z}\right)$	$\mathcal{D}(\epsilon)[\mathbf{v}_7]-\mathcal{D}'(\epsilon)[\mathbf{v}_9]$	$\mathbf{v}_8(n)-(\mathcal{D}(\epsilon)-\mathbf{1})[\mathbf{v}_7]$ $-\mathcal{D}'(\epsilon)[\mathbf{v}_9]$
$\mathbf{v}_{12}(\mathcal{R})$	$\mathbf{v}_5$	$\mathbf{v}_6-\epsilon\mathbf{v}_{12}\left(\frac{d(t\mathcal{R})}{dt}\right)$	$\mathcal{D}(\epsilon)[\mathbf{v}_7]-\mathcal{D}'(\epsilon)[\mathbf{v}_9]$	$\mathcal{D}(\epsilon)[\mathbf{v}_8]-\mathcal{D}'(\epsilon)[\mathbf{v}_{10}]$
$\mathbf{v}_{13}(\mathcal{L})$	$\mathbf{v}_5$	$\mathbf{v}_6-\epsilon\mathbf{v}_{13}\left(t\frac{\partial\mathcal{L}}{\partial t}+\frac{\partial(z\mathcal{L})}{\partial z}\right)$	$\mathbf{v}_7(f)$	$\mathbf{v}_8(n)$
$\mathbf{v}_{14}(\mathcal{K})$	$\mathbf{v}_5$	$\mathbf{v}_6-\epsilon\mathbf{v}_{14}\left(\frac{d(t\mathcal{K})}{dt}\right)$	$\mathbf{v}_7(f)$	$\mathbf{v}_8(n)$
$\mathbf{v}_{15}(\mathcal{S})$	$\mathbf{v}_5$	$\mathbf{v}_6-\epsilon\mathbf{v}_{15}\left(\frac{d(z\mathcal{S})}{dz}\right)$	$\mathbf{v}_7(f)$	$\mathbf{v}_8(n)$

Table 5.5: Adjoint table for the generators of the Lie algebra $\mathcal{G}^{15}$ of HTFM. Second part.

Ad	$\mathbf{v}_9(g)$	$\mathbf{v}_{10}(m)$	$\mathbf{v}_{11}(h)$
$\mathbf{v}_1$	$e^{-\frac{\partial}{\partial t}}\mathbf{v}_9(g)$	$e^{-\frac{\partial}{\partial t}}\mathbf{v}_{10}(m)$	$e^{-\frac{\partial}{\partial t}}\mathbf{v}_{11}(h)$
$\mathbf{v}_2$	$e^{-\frac{\partial}{\partial z}}\mathbf{v}_9(g)$	$\mathbf{v}_{10}(m)$	$e^{-\frac{\partial}{\partial z}}\mathbf{v}_{11}(h)$
$\mathbf{v}_3$	$\mathbf{v}_9(g)+\epsilon\mathbf{v}_{13}(g)$	$\mathbf{v}_{10}(m)-\epsilon\mathbf{v}_{14}(m)$	$\mathbf{v}_{11}(h)+\epsilon\mathbf{v}_9(h)$ $+\frac{\epsilon^2}{2}\mathbf{v}_{13}(h)$
$\mathbf{v}_4$	$\mathbf{v}_9(g)$	$\mathbf{v}_{10}(m)$	$\mathbf{v}_{11}(h)-\epsilon\mathbf{v}_7(h)$ $+\frac{\epsilon^2}{2}\mathbf{v}_{13}(h)$
$\mathbf{v}_5$	$\mathbf{v}_9(g)\cos\epsilon+\mathbf{v}_7(g)\sin\epsilon$	$\mathbf{v}_{10}(m)\cos\epsilon+\mathbf{v}_8(m)\sin\epsilon$	$\mathbf{v}_{11}(h)$
$\mathbf{v}_6$	$e^{t\frac{\partial}{\partial t}+z\frac{\partial}{\partial z}+1}\mathbf{v}_9(g)$	$e^{t\frac{d}{dt}+1}\mathbf{v}_{10}(m)$	$e^{t\frac{\partial}{\partial t}+z\frac{\partial}{\partial z}+1}\mathbf{v}_{11}(h)$
$\mathbf{v}_7(\mathcal{F})$	$\mathbf{v}_9(g)+\epsilon\mathbf{v}_{13}\left(g\int\mathcal{F}dz\right.$ $\left.+\mathcal{F}\int gdz\right)$	$\mathbf{v}_{10}(m)+\epsilon\mathbf{v}_{13}\left(\mathcal{F}\int mdt\right)$	$\mathbf{v}_{11}(h)+\epsilon\mathbf{v}_9(g^*)$ $+\frac{\epsilon^2}{2}\mathbf{v}_{13}\left(g^*\int\mathcal{F}dz+\mathcal{F}\int g^*dz\right)$
$\mathbf{v}_8(\mathcal{N})$	$\mathbf{v}_9(g)+\epsilon\mathbf{v}_{13}\left(g\int\mathcal{N}dt\right)$	$\mathbf{v}_{10}(m)-\epsilon\mathbf{v}_{14}\left(m\int\mathcal{N}dt\right.$ $\left.+\mathcal{N}\int mdt\right)$	$\mathbf{v}_{11}(h)+\epsilon\mathbf{v}_9(g^*)$ $+\frac{\epsilon^2}{2}\mathbf{v}_{13}\left(g^*\int\mathcal{N}dt\right)$
$\mathbf{v}_9(\mathcal{G})$	$\mathbf{v}_9(g)$	$\mathbf{v}_{10}(m)$	$\mathbf{v}_{11}(h)-\epsilon\mathbf{v}_7(f^*)$ $+\frac{\epsilon^2}{2}\mathbf{v}_{13}\left(\mathcal{G}\int f^*dz+f^*\int\mathcal{G}dz\right)$
$\mathbf{v}_{10}(\mathcal{M})$	$\mathbf{v}_9(g)$	$\mathbf{v}_{10}(m)$	$\mathbf{v}_{11}(h)-\epsilon\mathbf{v}_7\left(h\int\mathcal{M}dt\right)$ $+\frac{\epsilon^2}{2}\mathbf{v}_{13}\left(h\left(\int\mathcal{M}dt\right)^2\right)$
$\mathbf{v}_{11}(\mathcal{H})$	$\mathcal{D}(\epsilon)[\mathbf{v}_9]+\mathcal{D}'(\epsilon)[\mathbf{v}_7]$	$\mathbf{v}_{10}+\left(\mathcal{D}(\epsilon)-\mathbf{1}\right)[\mathbf{v}_9]$ $+\mathcal{D}'(\epsilon)[\mathbf{v}_7]$	$\mathbf{v}_{11}(h)$
$\mathbf{v}_{12}(\mathcal{R})$	$\mathcal{D}(\epsilon)[\mathbf{v}_9]+\mathcal{D}'(\epsilon)[\mathbf{v}_7]$	$\mathcal{D}(\epsilon)[\mathbf{v}_{10}]+\mathcal{D}'(\epsilon)[\mathbf{v}_8]$	$\mathbf{v}_{11}(h)$
$\mathbf{v}_{13}(\mathcal{L})$	$\mathbf{v}_9(g)$	$\mathbf{v}_{10}(m)$	$\mathbf{v}_{11}(h)$
$\mathbf{v}_{14}(\mathcal{K})$	$\mathbf{v}_9(g)$	$\mathbf{v}_{10}(m)$	$\mathbf{v}_{11}(h)$
$\mathbf{v}_{15}(\mathcal{S})$	$\mathbf{v}_9(g)$	$\mathbf{v}_{10}(m)$	$\mathbf{v}_{11}(h)$

Table 5.6: Adjoint table for the generators of the Lie algebra $\mathcal{G}^{15}$ of HTFM. Third part.

Ad	$\mathbf{v}_{12}(r)$	$\mathbf{v}_{13}(l)$	$\mathbf{v}_{14}(k)$	$\mathbf{v}_{15}(s)$
$\mathbf{v}_1$	$e^{-\frac{\partial}{\partial t}}\mathbf{v}_{12}(r)$	$e^{-\frac{\partial}{\partial t}}\mathbf{v}_{13}(l)$	$e^{-\frac{\partial}{\partial t}}\mathbf{v}_{14}(k)$	$\mathbf{v}_{15}(s)$
$\mathbf{v}_2$	$\mathbf{v}_{12}(r)$	$e^{-\frac{\partial}{\partial z}}\mathbf{v}_{13}(l)$	$\mathbf{v}_{14}(k)$	$e^{-\frac{\partial}{\partial z}}\mathbf{v}_{15}$
$\mathbf{v}_3$	$\mathbf{v}_{12}(r) + \epsilon\mathbf{v}_{10}(r) + \frac{\epsilon^2}{2}\mathbf{v}_{14}(r)$	$\mathbf{v}_{13}(l)$	$\mathbf{v}_{14}(k)$	$\mathbf{v}_{15}(s)$
$\mathbf{v}_4$	$\mathbf{v}_{12}(r) - \epsilon\mathbf{v}_8(r) - \frac{\epsilon^2}{2}\mathbf{v}_{14}(r)$	$\mathbf{v}_{13}(l)$	$\mathbf{v}_{14}(k)$	$\mathbf{v}_{15}(s)$
$\mathbf{v}_5$	$\mathbf{v}_{12}(r)$	$\mathbf{v}_{13}(l)$	$\mathbf{v}_{14}(k)$	$\mathbf{v}_{15}(s)$
$\mathbf{v}_6$	$e\mathbf{v}_{12}(r)$			
$\mathbf{v}_7(\mathcal{F})$	$\mathbf{v}_{12}(r) + \epsilon\mathbf{v}_9\left(\mathcal{F}\int r dt\right)$	$\mathbf{v}_{13}(l)$	$\mathbf{v}_{14}(k)$	$\mathbf{v}_{15}(s)$
$\mathbf{v}_8(\mathcal{N})$	$\mathbf{v}_{12}(r) + \epsilon\mathbf{v}_{10}(m^*)$	$\mathbf{v}_{13}(l)$	$\mathbf{v}_{14}(k)$	$\mathbf{v}_{15}(s)$
$\mathbf{v}_9(\mathcal{G})$	$\mathbf{v}_{12}(r) - \epsilon\mathbf{v}_7\left(\mathcal{G}\int r dt\right)$	$\mathbf{v}_{13}(l)$	$\mathbf{v}_{14}(k)$	$\mathbf{v}_{15}(s)$
$\mathbf{v}_{10}(\mathcal{M})$	$\mathbf{v}_{12}(r) - \epsilon\mathbf{v}_8\left(r\int \mathcal{M} dt\right)$	$\mathbf{v}_{13}(l)$	$\mathbf{v}_{14}(k)$	$\mathbf{v}_{15}(s)$
$\mathbf{v}_{11}(\mathcal{H})$	$\mathbf{v}_{12}(r)$	$\mathbf{v}_{13}(l)$	$\mathbf{v}_{14}(k)$	$\mathbf{v}_{15}(s)$
$\mathbf{v}_{12}(\mathcal{R})$	$\mathbf{v}_{12}(r)$	$\mathbf{v}_{13}(l)$	$\mathbf{v}_{14}(k)$	$\mathbf{v}_{15}(s)$
$\mathbf{v}_{13}(\mathcal{L})$	$\mathbf{v}_{12}(r)$	$\mathbf{v}_{13}(l)$	$\mathbf{v}_{14}(k)$	$\mathbf{v}_{15}(s)$
$\mathbf{v}_{14}(\mathcal{K})$	$\mathbf{v}_{12}(r)$	$\mathbf{v}_{13}(l)$	$\mathbf{v}_{14}(k)$	$\mathbf{v}_{15}(s)$
$\mathbf{v}_{15}(\mathcal{S})$	$\mathbf{v}_{12}(r)$	$\mathbf{v}_{13}(l)$	$\mathbf{v}_{14}(k)$	$\mathbf{v}_{15}(s)$

Table 5.7: Adjoint table for the generators of the Lie algebra $\mathcal{G}^{15}$ of HTFM. Fourth part.

$$(d) \quad a_1\mathbf{v}_1 + \mathbf{v}_2 + a_3\mathbf{v}_3$$

$$(e) \quad \mathbf{v}_2 + a_3\mathbf{v}_3 + a_9\mathbf{v}_9 + a_{12}\mathbf{v}_{12}$$

$$(f) \quad \mathbf{v}_1 + a_3\mathbf{v}_3 + a_{15}\mathbf{v}_{15}$$

$$(g) \quad a_3\mathbf{v}_3 + a_{11}\mathbf{v}_{11} + \mathbf{v}_{12} + a_{13}\mathbf{v}_{13} + a_{14}\mathbf{v}_{14} + a_{15}\mathbf{v}_{15}$$

$$(h) \quad a_3\mathbf{v}_3 + a_8\mathbf{v}_8 + a_{10}\mathbf{v}_{10} + \mathbf{v}_{11} + a_{13}\mathbf{v}_{13} + a_{15}\mathbf{v}_{15}$$

$$(i) \quad a_3\mathbf{v}_3 + a_4\mathbf{v}_4 + a_8\mathbf{v}_8 + \mathbf{v}_{11} + a_{13}\mathbf{v}_{13} + a_{15}\mathbf{v}_{15}$$

$$(j) \quad a_3\mathbf{v}_3 + a_4\mathbf{v}_4 + a_7\mathbf{v}_7 + a_8\mathbf{v}_8 + \mathbf{v}_{10} + a_{15}\mathbf{v}_{15}$$

$$(k) \quad a_3\mathbf{v}_3 + a_4\mathbf{v}_4 + a_7\mathbf{v}_7 + a_8\mathbf{v}_8 + \mathbf{v}_9 + a_{15}\mathbf{v}_{15}$$

$$(l) \quad a_3\mathbf{v}_3 + a_4\mathbf{v}_4 + a_7\mathbf{v}_7 + \mathbf{v}_8 + a_{15}\mathbf{v}_{15}$$

$$(m) \quad a_3\mathbf{v}_3 + a_4\mathbf{v}_4 + \mathbf{v}_7 + a_{15}\mathbf{v}_{15}$$

$$(n) \quad a_3\mathbf{v}_3 + \mathbf{v}_{15}$$

$$(o) \quad \mathbf{v}_3$$

$$(p) \quad a_{13}\mathbf{v}_{13} + \mathbf{v}_{14}$$

$$(q) \quad \mathbf{v}_{13}$$

which yield, by definition, all the possible single reductions of the number of independent variables of the system, leading to solutions not connected by a group transformation.

Comparing this list of seventeen elements of Θ_1 for $\mathcal{G}^{15}$ with the equivalent list for $\mathcal{G}^9$ (composed of ten elements) given at the end of the last chapter, we note that there are some additional elements of the optimal system arising from the explicit choice for the coefficients of the three generators: generalized Galilean transformations along x and y, and generalized rotational invariance in the x-y plane, that otherwise would be impossible to generate as particular cases of the reduced representation of the algebra, $\mathcal{G}^9$. Therefore, our extended algebra $\mathcal{G}^{15}$, has a richer algebraic structure that brings out some properties that were hidding in the compact form of the original representation, $\mathcal{G}^9$, helping us discern new potential reductions of the system.

This result supports the argument that our extension of the number of fundamental symmetries for HTFM, with the construction shown at the beginning of the present chapter, is a nontrivial new representation for the elements of the associated Lie algebra.

5.1.2 Single and Double Reductions

Now we turn our attention to some explicit reductions of HTFM. In order to accomplish this task, we make explicit use of some elements of the optimal system of first order, Θ_1, shown above, and elements of the optimal system of second order, Θ_2, whose construction is based on the previous knowledge of Θ_1 and the properties of the algebra contained in the commutation tables. Recall this was shown in detail for the CHM equation in the last chapter. The common denominator of the single and double reductions is the fact that they generate a three and two dimensional base space, respectively, from the original four dimensional space of independent variables (x, y, z, t). In general, the new similarity variables will involve the three space coordinates and time, making the symmetry solutions of the reduced system time dependent and fully three dimensional.

Consider first the simple reductions of HTFM, those of equations (4.98)-(4.102) that are induced by element (f) of the Optimal system Θ_1 shown above, and the corresponding element of Θ_2, $\mathcal{H}_6$ given by

$$\mathcal{H}_6\left(\mathbf{v}_1 + a_3\mathbf{v}_3 + a_{15}\mathbf{v}_{15},\ \mathbf{v}_4 + \alpha_{15}\mathbf{v}_{15}\right) . \tag{5.6}$$

The physical motivation for this example becomes clear if we recall the reduction of the CHM equation that lead to the Larichev-Reznik dipole solution, eq. (4.32). In that example, invariance under time translation combined with space translation along one of the axis, yielded a similarity variable corresponding to a moving frame along the invariant direction. In the present case, element (f) of Θ_1 also includes translations in time and space, $\mathbf{v}_1$ and $\mathbf{v}_3$, respectively, plus an additional z-dependent gauge transformation of the fields φ and χ. Therefore, we should expect some travelling wave solutions under this reduction.

Starting with the single reduction induced by element (f) of Θ_1, we obtain the new independent variables

$$\eta_1 = x - a_3 t\,, \qquad \eta_2 = y\,, \qquad \eta_3 = z\,, \tag{5.7}$$

and the new dependent variables

$$\zeta_1 = U\,, \quad \zeta_2 = \varphi - \alpha a_{15} s(z) t\,, \quad \zeta_3 = J\,, \quad \zeta_4 = \psi\,, \quad \zeta_5 = \chi - a_{15} s(z) t\,, \tag{5.8}$$

which yield, upon substitution into the HTFM equations, the following set of equations:

$$\frac{\partial \zeta_3}{\partial \eta_3} + [\zeta_2 + a_3\eta_2\,,\, \zeta_1] - [\zeta_4\,,\, \zeta_3] = 0\,, \tag{5.9}$$

$$\frac{\partial(\zeta_2 - \alpha\zeta_5)}{\partial \eta_3} - a_3\frac{\partial \zeta_4}{\partial \eta_1} + [\zeta_2 - \alpha\zeta_5\,,\, \zeta_4] = 0\,, \tag{5.10}$$

$$\frac{\partial \zeta_3}{\partial \eta_3} + [\zeta_2 + a_3\eta_2\,,\, \zeta_5] - [\zeta_4\,,\, \zeta_3] + [a_{15}\eta_1\,,\, s(\eta_3)\eta_2] = 0\,, \tag{5.11}$$

$$\frac{\partial^2 \zeta_2}{\partial \eta_1^2} + \frac{\partial^2 \zeta_2}{\partial \eta_2^2} - \zeta_1 = 0\,, \tag{5.12}$$

$$\frac{\partial^2 \zeta_4}{\partial \eta_1^2} + \frac{\partial^2 \zeta_4}{\partial \eta_2^2} - \zeta_3 = 0\,, \tag{5.13}$$

where the brackets are the usual Poisson bracket in terms of η_1 and η_2.

From eq. (5.11) we can see that the net effect of $\mathbf{v}_{15}$ is the inhomogeneous term written last in the equation. The easiest simplification, and the closest to the CHM reduction mentioned above, is given in the limit when a_{15} vanishes. In this case the dependent variables ζ_2 and ζ_5 are reduced to φ and χ respectively, as is seen from (5.8). Thus, the difference of eqs. (5.9) and (5.11) yields

$$[\zeta_2 + a_3\eta_2\,,\, \zeta_1 - \zeta_5] = 0\,, \tag{5.14}$$

which has a general solution given by

$$\zeta_1 - \zeta_5 = \mathcal{F}(\zeta_2 + a_3\eta_2)\,, \tag{5.15}$$

where $\mathcal{F}$ is an arbitrary function of its argument. This relation together with eq. (5.12), which gives ζ_1 in terms of ζ_2, imply that we can eliminate ζ_1 and ζ_5 in favor of ζ_2, obtaining

$$\zeta_5 = \nabla^2\zeta_2 - \mathcal{F}(\zeta_2 + a_3\eta_2)\,. \tag{5.16}$$

Also, notice that eq. (5.10) can be cast in the following form

$$\frac{\partial \mathcal{U}}{\partial \eta_3} + [\mathcal{U}\,,\, \zeta_4] = 0\,, \tag{5.17}$$

which has the generic form of the nonlinear parallel gradient $\mathbf{B}\cdot\nabla\mathcal{U} \sim \partial\mathcal{U}/\partial z - [\psi, \mathcal{U}]$ equal to zero. The new variable $\mathcal{U}$ is defined by

$$\mathcal{U} = \zeta_2 - \alpha\nabla^2\zeta_2 + a_3\eta_2 + \alpha\mathcal{F}(\zeta_2 + a_3\eta_2)\,. \tag{5.18}$$

Depending on the form of the free function $\mathcal{F}$ one can make $\mathcal{U}$ proportional to the vorticity field $\zeta_1 = \nabla^2\zeta_2$, or to a more complicated combination of both the fields U and $\varphi = \zeta_2$. The simple choice $\mathcal{F} = -1/\alpha(\zeta_2 + a_3\eta_2)$ implies $\mathcal{U} = -\alpha\zeta_1 = -\alpha\nabla_\perp^2\zeta_2$. Finally, eq. (5.17) couples with the equation obtained from adding eqs. (5.9) and (5.11), yielding

$$\frac{\partial \nabla^2 \zeta_4}{\partial \eta_3} - \left[\zeta_4\,,\, \nabla^2\zeta_4\right] + \left[\zeta_2 + a_3\eta_2\,,\, \nabla^2\zeta_2\right] = 0\,. \tag{5.19}$$

Equations (5.17) and (5.19), together with the definition of $\mathcal{U}$, eq. (5.18), represent a set of two nonlinear PDE's for the fields ζ_2 and ζ_4, and even though they are still quite difficult to solve, we have achieved a substantial reduction from the original set of equations by using some of its simplest symmetries. However, considering the fact that they still depend on three independent variables, and we have a substantially large subalgebra of the original system that we have not yet used, we propose taking a further symmetry reduction that will eliminate another of the independent variables and obtain a system of equations in two variables related to element (f) of Θ_1.

The most general element of Θ_2 that can be derived from element (f) of Θ_1 is given in eq. (5.6). In order to obtain the corresponding similarity variables for the ensuing reduction, we need to determine the invariants of the first element of the two dimensional algebra in $\mathcal{H}_6$, $\mathbf{v}_1 + a_3\mathbf{v}_3 + a_{15}\mathbf{v}_{15}$, which we just did while calculating the single reduction shown above. The invariants are given by eqs. (5.7) and (5.8). Then we write the second element of the algebra, $\mathbf{v}_4 + \alpha_{15}\mathbf{v}_{15}$, in terms of these first invariants. Finally, we determine the invariants of the transformed second element, which will yield the similarity variables corresponding to the double reduction. Following this program we obtain from $\mathcal{H}_6$

$$\eta_1 = x - a_3 t\,, \qquad \eta_2 = z\,, \tag{5.20}$$

as the new independent variables, and

$$\begin{gathered}\zeta_1 = U\,, \quad \zeta_2 = \varphi - \alpha s(z)\left(\alpha_{15} y + a_{15} t\right)\,, \\ \zeta_3 = J\,,\ \zeta_4 = \psi\,, \quad \zeta_5 = \chi - s(z)\left(\alpha_{15} y + a_{15} t\right)\,,\end{gathered} \tag{5.21}$$

as the new dependent variables of the double reduction. This particular case linearizes the HTFM by reducing the brackets for ζ_2 and ζ_5 to linear functions and making all other nonlinear terms zero. The final form of the reduced equations is

$$a_3 \frac{\partial \zeta_1}{\partial \eta_1} + \alpha s(\eta_2) \alpha_{15} \frac{\partial \zeta_1}{\partial \eta_1} - \frac{\partial \zeta_3}{\partial \eta_2} = 0\,, \tag{5.22}$$

$$a_3 \frac{\partial \zeta_4}{\partial \eta_1} - \frac{\partial \zeta_2}{\partial \eta_2} + \alpha \frac{\partial \zeta_5}{\partial \eta_2} = 0\,, \tag{5.23}$$

$$a_3 \frac{\partial \zeta_5}{\partial \eta_1} + \alpha_{15} s(\eta_2) \left(\alpha \frac{\partial \zeta_5}{\partial \eta_1} - \frac{\partial \zeta_2}{\partial \eta_1} \right) - \frac{\partial \zeta_3}{\partial \eta_2} = a_{15} s(\eta_2)\,, \tag{5.24}$$

$$\frac{\partial^2 \zeta_2}{\partial \eta_1^2} - \zeta_1 = 0\,, \tag{5.25}$$

$$\frac{\partial^2 \zeta_4}{\partial \eta_1^2} - \zeta_3 = 0\,. \tag{5.26}$$

If we eliminate the fields ζ_1 and ζ_3 in favor of ζ_2 and ζ_4 respectively, by means of eqs. (5.25) and (5.26), and substitute into (5.22), we notice that we can integrate twice with respect to η_1, yielding the following:

$$\left(a_3 + \alpha \alpha_{15} s(\eta_2)\right) \frac{\partial \zeta_2}{\partial \eta_1} - \frac{\partial \zeta_4}{\partial \eta_2} = F_1(\eta_2)\eta_1 + F_2(\eta_2)\,, \tag{5.27}$$

where F_1 and F_2 are arbitrary functions of η_2. Following the same line of thinking, if we substitute ζ_3 from (5.26) into (5.24) and integrate with respect to η_1, we obtain

$$(a_3 + \alpha\alpha_{15}s(\eta_2))\,\zeta_5 - \alpha_{15}s(\eta_2)\zeta_2 - \frac{\partial^2\zeta_4}{\partial\eta_1\partial\eta_2} = a_{15}s(\eta_2)\eta_1 + F_3(\eta_2)\,, \qquad (5.28)$$

where F_3 is an arbitrary function of η_2. This equation gives a simple relation between ζ_5 and the other two remaining fields ζ_2 and ζ_4. The remaining equation (5.23) coupled with eqs. (5.27) and (5.28) yields a linear system of PDE's that can be written as a single fourth order linear PDE for one of the fields by substituting subsequently the other two fields. The general form of this single PDE, when written in terms of ζ_4 is

$$\frac{\partial^2\zeta_4}{\partial\eta_1^2} + \frac{\alpha}{a_3}A(\eta_2)\frac{\partial^4\zeta_4}{\partial\eta_1^2\partial\eta_2^2} - \frac{\alpha}{a_3}A'(\eta_2)\frac{\partial^3\zeta_4}{\partial\eta_1^2\partial\eta_2}$$
$$- A^2(\eta_2)\frac{\partial^2\zeta_4}{\partial\eta_2^2} + AA'(\eta_2)\frac{\partial\zeta_4}{\partial\eta_2} = B(\eta_2) + C(\eta_2)\eta_1\,, \qquad (5.29)$$

where the functions A, B and C are given functions of $s(\eta_2)$, $F_1(\eta_2)$, and $F_2(\eta_2)$. Although eq. (5.29) is still a complicated linear PDE, we have made great progress from the original nonlinear system of PDE's. In particular, if we recall that α is a physical parameter such that in the RMHD limit $\alpha = 0$ and the second and third terms vanish, A becomes equal to a constant which cancels the fifth term, and B and C become proportional to F_1' and F_2', respectively. The end result is

$$a_3^2\frac{\partial^2\zeta_4}{\partial\eta_1^2} - \frac{\partial^2\zeta_4}{\partial\eta_2^2} = F_2'(\eta_2)\eta_1 + F_1'(\eta_2)\,, \qquad (5.30)$$

which represents a one dimensional inhomogeneous wave equation with the velocity of propagation equal to a_3, and $\eta_2 = z$ taking the place of time. The solution to the homogeneous equation is simply

$$\zeta_4 = A_1 f(\eta_1 - a_3\eta_2) + A_2 g(\eta_1 + a_3\eta_2)\,, \qquad (5.31)$$

which is the well-known general wave solution, where A_1 and A_2 are constants and f and g are arbitrary functions of their arguments. Depending on the form of the functions F_1 and F_2 we could obtain a general solution for the complete inhomogeneous equation. In any case, from the homogeneous problem we learn the type of behavior that we can expect under the present reduction. It describes a perturbation moving in the poloidal direction x, that at any fixed time changes its form as a function of the toroidal direction z. Of course (5.29) will correspond to a generalization of this wave-like behavior with the characteristic velocity depending explicitly on z. This reduction generalizes previously found particular solutions using other more intuitive methods [Prahović et al. 92].

5.2 Two-Dimensional Limit

A natural simplification for HTFM, given the complicated nature of three-dimensional systems presented above, is to take the two-dimensional limit of (4.98)-(4.102), where we drop any dependencies on the z coordinate. Such configurations still represent some interesting physics particularly when referring to a helical symmetric state, which is two-dimensional. The big advantage from the symmetry reduction point of view is the fact that we will be getting systems of ODE's by simply using elements of the optimal system of second order Θ_2, which generates double reductions in the number of independent variables from a base space of order three, (x, y, t).

Exclusion of the z dependence reduces the number of degrees of freedom in the problem and also reduces the number of free functions appearing in the symmetry generators. A computer aided calculation using MACSYMA shows that the two dimensional system admits an infinite dimensional Lie algebra with eleven basic generators in contrast to fifteen generators for the three dimensional case. The explicit form of the infinitesimal generators is

$$\begin{aligned}
\mathbf{v}_1 &= \frac{\partial}{\partial x}, & \mathbf{v}_2 &= \frac{\partial}{\partial y},\\
\mathbf{v}_3 &= \frac{\partial}{\partial t}, & \mathbf{v}_4 &= \frac{\partial}{\partial \psi},\\
\mathbf{v}_5 &= \frac{\partial}{\partial \chi}, & \mathbf{v}_6 &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y},\\
\mathbf{v}_7 &= -2ty\frac{\partial}{\partial x} + 2tx\frac{\partial}{\partial y} + 4\frac{\partial}{\partial U} + (x^2+y^2)\frac{\partial}{\partial \varphi}, \\
\mathbf{v}_8 &= t\frac{\partial}{\partial t} - U\frac{\partial}{\partial U} - \varphi\frac{\partial}{\partial \varphi} - J\frac{\partial}{\partial J} - \psi\frac{\partial}{\partial \psi} - \chi\frac{\partial}{\partial \chi},\\
\mathbf{v}_9 &= f(t)\frac{\partial}{\partial \varphi},\\
\mathbf{v}_{10} &= \left(\int g(t)dt\right)\frac{\partial}{\partial x} - yg(t)\frac{\partial}{\partial \varphi},\\
\mathbf{v}_{11} &= \left(\int h(t)dt\right)\frac{\partial}{\partial y} + xh(t)\frac{\partial}{\partial \varphi},
\end{aligned} \tag{5.32}$$

where f, g and h are arbitrary functions of time. We have again found invariance under translations in space and time ($\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$), a constant gauge invariance for the fields ψ and χ ($\mathbf{v}_4, \mathbf{v}_5$), invariance under rotation in the perpendicular plane ($\mathbf{v}_6$). The symmetry given by $\mathbf{v}_7$, representing invariance under a time dependent rotation, is a reminiscent of the generalized rotations encountered in the three dimensional case. The generator of dilations, $\mathbf{v}_8$, is the two dimensional limit of the one used before. The time dependent gauge for φ ($\mathbf{v}_9$) is the same as before, and the generalized Galilean invariance along x and y, given by ($\mathbf{v}_{10}$, $\mathbf{v}_{11}$), are just two dimensional limits of the ones used before.

Notice that the first five generators form an abelian subalgebra, and the first six generators together with $\mathbf{v}_8$ constitute the largest finite subalgebra (dimension 7). Upon making use of these properties and the adjoint action explained before, we can calculate the elements of the optimal system of first order, Θ_1, given by

$$(a) \quad a_6\mathbf{v}_6 + \mathbf{v}_8 + a_{11}\mathbf{v}_{11}\,.$$

$$(b) \quad a_3\mathbf{v}_3 + a_7\mathbf{v}_7 + \mathbf{v}_8 + a_9\mathbf{v}_9 + a_{10}\mathbf{v}_{10}\,.$$

$$(c) \quad a_3\mathbf{v}_3 + a_4\mathbf{v}_4 + a_5\mathbf{v}_5 + \mathbf{v}_7 + a_{11}\mathbf{v}_{11}\,.$$

$$(d) \quad \mathbf{v}_3 + a_4\mathbf{v}_4 + a_5\mathbf{v}_5\,.$$

$$(e) \quad a_4\mathbf{v}_4 + a_5\mathbf{v}_5 + \mathbf{v}_6 + a_9\mathbf{v}_9\,.$$

$$(f) \quad \mathbf{v}_1 + a_2\mathbf{v}_2 + a_4\mathbf{v}_4 + a_5\mathbf{v}_5 + a_{10}\mathbf{v}_{10}\,.$$

$$(g) \quad \mathbf{v}_1 + a_4\mathbf{v}_4 + a_5\mathbf{v}_5 + a_{10}\mathbf{v}_{10} + a_{11}\mathbf{v}_{11}\,.$$

$$(h) \quad \mathbf{v}_2 + a_4\mathbf{v}_4 + a_5\mathbf{v}_5 + a_{10}\mathbf{v}_{10} + a_{11}\mathbf{v}_{11}\,.$$

$$(i) \quad a_4\mathbf{v}_4 + a_5\mathbf{v}_5 + \mathbf{v}_{10} + a_{11}\mathbf{v}_{11}\,.$$

$$(j) \quad a_4\mathbf{v}_4 + a_5\mathbf{v}_5 + \mathbf{v}_{11}\,.$$

$$(k) \quad a_4\mathbf{v}_4 + a_5\mathbf{v}_5 + \mathbf{v}_9\,.$$

$$(l) \quad a_4\mathbf{v}_4 + \mathbf{v}_5\,.$$

$$(m) \quad \mathbf{v}_4\,.$$

The direct use of any of the above elements, except for the last three, will lead to a single reduction in the number of independent variables of the system. Since we are interested in double reductions that reduce the two dimensional HTFM equations to a set of ODE's; we will use an element of the optimal system of second order, Θ_2, to illustrate the kind of results that can be obtained for such a complex system.

Consider element (c) of the optimal system of first order given above, in the case $a_3 = a_4 = a_5 = a_{11} = 0$. The corresponding element of Θ_2 is

$$\mathcal{H}\left(\mathbf{v}_7,\ \alpha_6\mathbf{v}_6 + \mathbf{v}_8\right)\,, \tag{5.33}$$

where we have used the commutation rules of the algebra. For the sake of simplicity consider the particular case where $\alpha_6 = 0$. Then, we first calculate the differential invariants corresponding to $\mathbf{v}_7$

$$C_1 = (x^2+y^2)^{1/2}\,, \quad C_2 = t\,, \quad C_3 = U - \frac{2}{t}\arctan\left(\frac{y}{x}\right)\,, \tag{5.34}$$

$$C_4 = \varphi - \frac{x^2+y^2}{2t}\arctan\left(\frac{y}{x}\right)\,, \quad C_5 = J\,, \quad C_6 = \psi\,, \quad C_7 = \chi\,, \tag{5.35}$$

and using this information we write the second element of the algebra, $\mathbf{v}_8$, in terms of the invariants of the first one, and integrate the corresponding characteristics, yielding

$$\eta = C_1 = (x^2+y^2)^{1/2}\,, \tag{5.36}$$

as the single independent variable, and

$$\zeta_1 = C_2C_3 = tU - 2\arctan\left(\frac{y}{x}\right)\,, \qquad \zeta_3 = C_2C_5 = tJ\,, \tag{5.37}$$

$$\zeta_2 = C_2C_4 = t\varphi - \frac{x^2+y^2}{2}\arctan\left(\frac{y}{x}\right)\,, \quad \zeta_4 = C_2C_6 = t\psi\,, \tag{5.38}$$

$$\zeta_5 = C_2C_7 = t\chi\,, \tag{5.39}$$

as the new dependent variables. Under this transformation the HTFM system of equations becomes

$$\frac{1}{2}\eta\frac{d\zeta_1}{d\eta} + \zeta_1 - \frac{2}{\eta}\frac{d\zeta_2}{d\eta} = 0\,, \tag{5.40}$$

$$\eta \frac{d\zeta_4}{d\eta} + 2\zeta_4 = 0\,, \tag{5.41}$$

$$\eta \frac{d\zeta_5}{d\eta} + 2\zeta_5 = 0\,, \tag{5.42}$$

$$\frac{d^2\zeta_2}{d\eta^2} + \frac{1}{\eta}\frac{d\zeta_2}{d\eta} - \zeta_1 = 0\,, \tag{5.43}$$

$$\frac{d^2\zeta_4}{d\eta^2} + \frac{1}{\eta}\frac{d\zeta_4}{d\eta} - \zeta_3 = 0\,, \tag{5.44}$$

where the second and third equations are identical and readily solvable. The reason for the decoupling of the ζ_4 and ζ_5 fields is due to the fact that their mutual Poisson bracket vanishes which is equivalent to take the $\alpha = 0$ limit. Substituting ζ_1 from (5.43) into (5.40) yields a simple equation for ζ_2 given by

$$\eta^3 \frac{d^3\zeta_2}{d\eta^3} + 3\eta^2 \frac{d^2\zeta_2}{d\eta^2} - 3\eta \frac{d\zeta_2}{d\eta} = 0\,. \tag{5.45}$$

This is an Euler homogeneous equation with solutions

$$\zeta_2 = A + B\eta^2 + C\eta^{-2}\,, \tag{5.46}$$

where A, B, and C are constants of integration. The solution of (5.41) and (5.42) is also a power of η

$$\zeta_4 = D_1\eta^{-2}\,, \quad \zeta_5 = D_2\eta^{-2}\,, \tag{5.47}$$

where D_1 and D_2 are constants. The solution for the other two fields ζ_1 and ζ_3 is given by eqs. (5.43) and (5.44), respectively.

The solution presented above corresponds to a case where the density perturbation field decouples from the dynamics of the other fields, which is the case in the RMHD limit. This will be the subject of the next subsection.

5.2.1 The RMHD Limit

As was discussed in chapter 2, when the coupling constant α vanishes in the HTFM, we recover the RMHD equations for plasma evolution. In the three-dimensional case these equations provide physical insight into the solutions of the more complicated HTFM equations. However, as was shown in the last section, the two-dimensional limit provides the fastest route to obtain analytic solutions, by taking advantage of the elements of the optimal system of second order acting on a three-dimensional base space. Another interesting feature of two-dimensional RMHD is that it admits an additional symmetry than its HTFM generalization. This fact shows again that a system of differential equations derived from a parent system by symmetry reduction or any other method, might contain additional symmetries that were hidden in the original one. What this means is that at every level of the symmetry reduction we can start a new symmetry calculation from scratch, and not only work with the symmetries that were inherited by the reduced system. This problem has been studied and conditions have been found for some families of ODE's (see [Abraham-Shrauner-Guo 92]).

The eleven generators for the two-dimensional RMHD symmetry algebra are given by

$$\mathbf{v}_1 = \frac{\partial}{\partial x}, \qquad \mathbf{v}_2 = \frac{\partial}{\partial y},$$

$$\mathbf{v}_3 = \frac{\partial}{\partial t}, \qquad \mathbf{v}_4 = \frac{\partial}{\partial \psi},$$

$$\mathbf{v}_5 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y},$$

$$\mathbf{v}_6 = 2ty\frac{\partial}{\partial x} - 2tx\frac{\partial}{\partial y} - 4\frac{\partial}{\partial U} - (x^2+y^2)\frac{\partial}{\partial \varphi},$$

$$\mathbf{v}_7 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2\psi\frac{\partial}{\partial \psi} + 2\varphi\frac{\partial}{\partial \varphi}, \tag{5.48}$$
$$\mathbf{v}_8 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - 2U\frac{\partial}{\partial U} - 2J\frac{\partial}{\partial J},$$
$$\mathbf{v}_9 = f(t)\frac{\partial}{\partial \varphi},$$
$$\mathbf{v}_{10} = \left(\int g(t)dt\right)\frac{\partial}{\partial x} - yg(t)\frac{\partial}{\partial \varphi},$$
$$\mathbf{v}_{11} = \left(\int h(t)dt\right)\frac{\partial}{\partial y} + xh(t)\frac{\partial}{\partial \varphi},$$

where f, g, and h are arbitrary functions of time. The new generator $\mathbf{v}_7$ corresponds to a scaling symmetry involving the two space coordinates and the two potentials ψ and φ. Notice that $\mathbf{v}_8$, another scaling symmetry, is different in nature from the corresponding symmetry of the two-dimensional HTFM (see $\mathbf{v}_8$ in last section). In this case, we have a scaling that involves time, space and the two physical quantities U and J.

As an explicit example of a group invariant analytical solution, let us consider the following element of Θ_2:

$$\mathcal{H}(\mathbf{v}_{11},\ \mathbf{v}_7)\ . \tag{5.49}$$

The first element of the two dimensional algebra $\mathbf{v}_{11}$ corresponds to a generalized Galilean transformation in the y direction. The second is the new scaling symmetry discussed above.

In order to combine the action of both symmetries, we need to determine their differential invariants and obtain the resulting similarity variables. During the analysis, we will try to keep the arbitrary function of time, $h(t)$, as general as possible. The characteristic equations for the first generator $\mathbf{v}_{11}$ are

$$\frac{dy}{\int h(t)dt} = \frac{d\varphi}{xh(t)}, \tag{5.50}$$

which generate the six differential invariants

$$C_1 = t\,, \qquad C_2 = x\,, \qquad C_3 = \frac{h(t)}{\int h(t)dt} y - \frac{\varphi}{x}\,,$$
$$C_4 = \psi\,, \qquad C_5 = U\,, \qquad C_6 = J\,.$$

Now we write the second generator $\mathbf{v}_7$ in terms of these invariants, yielding

$$\mathbf{v}_7 = C_2 \frac{\partial}{\partial C_2} + C_3 \frac{\partial}{\partial C_3} + 2C_4 \frac{\partial}{\partial C_4}\,. \tag{5.51}$$

From the corresponding differential invariants, we derive the following similarity variables

$$C_1 = t = \eta\,, \tag{5.52}$$

$$\frac{C_3}{C_2} = \frac{h(t)}{\int h(t)dt} \frac{y}{x} - \frac{\varphi}{x^2} = \zeta_1\,, \tag{5.53}$$

$$\frac{C_4}{C_2^2} = \frac{\psi}{x^2} = \zeta_2\,, \tag{5.54}$$

$$C_5 = U = \zeta_3\,, \tag{5.55}$$

$$C_6 = J = \zeta_4\,, \tag{5.56}$$

which define the new independent variable for the reduced system η, and the new dependent variables $\zeta_1 - \zeta_4$. Given the simple form of ζ_3 and ζ_4, and the fact that the new independent variable is equal to time, the Poisson brackets involving U and J will vanish identically. Then the vorticity equation takes the simple form

$$\frac{d\zeta_3}{d\eta} = 0\,, \tag{5.57}$$

which implies the vorticity U is constant in time. The other evolution equation, the parallel component of Ohm's law, becomes

$$\frac{d\zeta_2}{d\eta} = 2 \frac{h(\eta)}{\int h(\eta)d\eta} \zeta_2\,, \tag{5.58}$$

whose exact solution is given by

$$\zeta_2 = c\left(\int h(\eta)d\eta\right)^2 . \tag{5.59}$$

This function ζ_2 is proportional to ψ, recall (5.54), and therefore we have obtained an exact solution to RMHD. The other variables will be found by analyzing the Laplacian relations between J and ψ, and U and φ, which reduce to the algebraic relations

$$\zeta_4 = 2\zeta_2 , \qquad \zeta_3 = -2\zeta_1 . \tag{5.60}$$

If we write all these results in the original variables, we get

$$U = \mathcal{C} , \tag{5.61}$$

$$\varphi = \frac{h(t)}{\int h(t)dt}xy + \frac{1}{2}\mathcal{C}x^2 , \tag{5.62}$$

$$\psi = c\left(\int h(t)dt\right)^2 x^2 , \tag{5.63}$$

$$J = 2c\left(\int h(t)dt\right)^2 , \tag{5.64}$$

where c and $\mathcal{C}$ are two constants of integration and $h(t)$ is an arbitrary function of time.

In order to obtain a physical picture of this analytical solution, it is necessary to recall the relation between the potential fields and the physically measurable velocity and poloidal magnetic fields. The electrostatic potential or stream function φ is related to the flow velocity through the relation $\mathbf{V} = v_A\hat{\mathbf{z}} \times \nabla_\perp\varphi$. The poloidal magnetic field is given by $\mathbf{B}_P = \nabla_\perp\psi \times \hat{\mathbf{z}}$. From these definitions we obtain

$$V_x = -\frac{h(t)}{\int h(t)dt}x , \tag{5.65}$$

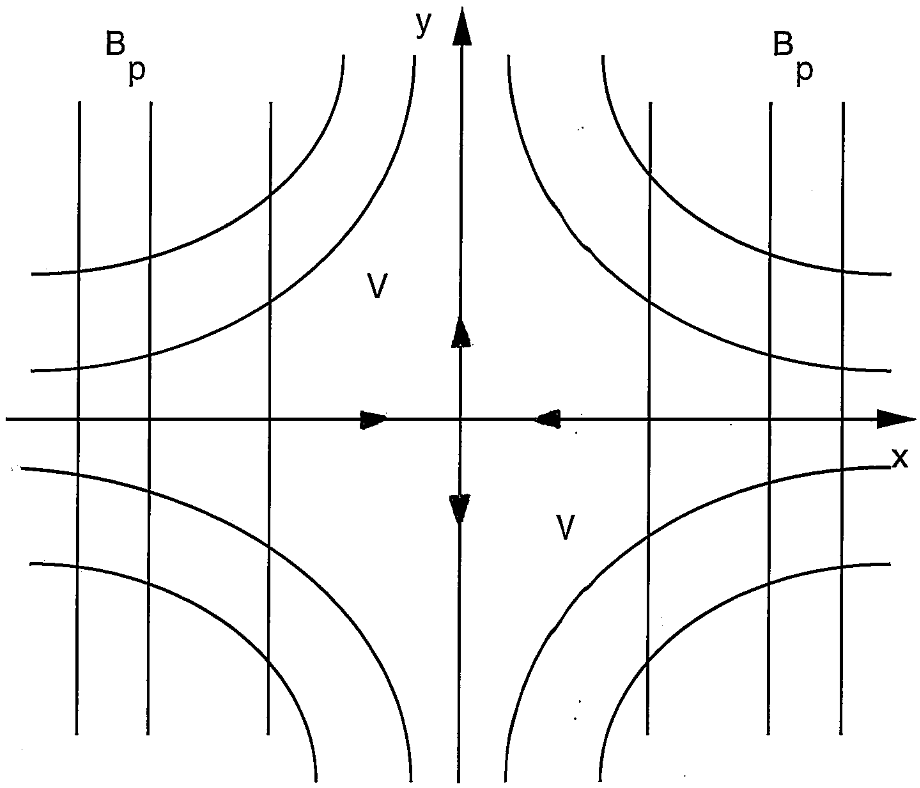

Figure 5.1: Velocity flow and magnetic field lines for the X-point solution of RMHD.

$$V_y = \frac{h(t)}{\int h(t)dt} y + Cx, \tag{5.66}$$

$$B_y = -2c \left(\int h(t)dt\right)^2 x. \tag{5.67}$$

This corresponds to a flow around an X-point with a time dependent magnetic field. The function h is arbitrary and can be made proportional to a harmonic function of time. Then as the field oscillates with strength varying in the position x, the plasma flows in along the x-direction and flows out along the y-direction (For more detail see figure 5.1).

This example shows how we can solve the system of nonlinear PDE's regardless of the boundary conditions and at the end try to understand the physical picture corresponding to the actual solution. Following this method a number of interesting new solutions have been obtained. It will the subject of future work to explore further reductions and solutions.

Chapter 6

Conclusions and Future Directions

Lie group methods for solving differential equations have been implemented to study the solution space of systems of nonlinear PDE's that describe the fluid evolution of plasmas. One of the main features of Lie's techniques is the applicability to a wide range of equations regardless of their order, type of nonlinearity, or integrability, making these techniques an important tool for analytical study and solution of complicated physical systems.

From the family of plasma-fluid models derived in Chapter 2 we have found that the typical nonlinear term of this description is of the Poisson bracket type. Therefore the solutions found in Chapters 4 and 5 represent a case study of this type of nonlinearity, an ubiquitous feature in fluid descriptions of continuous media. In many instances we have shown that the Poisson bracket nonlinearity can be linearized through a nontrivial symmetry reduction of the number of independent variables, thus providing a much simpler equation to be solved.

In general, Lie group invariant solutions often do not satisfy initial or boundary conditions pertaining to a particular physical situation. However, if one has an initial condition near one of these solutions, since the equations are local, due to the fact that we are dealing with PDE's, the solutions for the physical initial or boundary condition will behave like the symmetry solutions away from the singularities. This situation is often seen with soliton solutions when the solution is near a localized travelling wave.

The local nature of the symmetry solutions, like the one found at the end of Chapter 5 for RMHD, suggests that perhaps it is possible to "stitch" together several of these solutions in a manner similar to the procedure used to construct the modon solution for the CHM equation. This will be the subject of future work.

Of course, the great number of potential reductions for the three dimensional HTFM, implied by its infinite dimensional Lie algebra $\mathcal{G}^{15}$, has only been explored on the surface. The construction of Θ_3-optimal system of third order, would imply symmetry reductions to ODE's that promise to yield a number of interesting analytical solutions. This will also be pursued in the future.

So far, we have based our study exclusively on Lie point symmetries for the plasma-fluid systems, which are just the simplest geometrical symmetries consistent with Lie's original idea. However, as was mentioned in Chapter 3, there exist more general symmetries involving not only transformations of the dependent and independent variables but higher order derivatives of the dependent variables as well. This concept defines what are called "generalized symmetries", which constitute an extension of Lie's work and have been found in early work by Noether on variational symmetries and conservation laws. The plasma-fluid systems that we have considered seem especially suitable for study because of their Lagrangian and/or Hamiltonian nature, leading naturally to the concept of their generalized symmetries. In this direction, little progress has been made, particularly because of the difficulty in calculating the generalized symmetries. This requires more powerful computer programs than the ones used for Lie point symmetries. This is another area that needs to be explored, using the newest computer manipulation programs, which we believe will yield new conservation laws and integration schemes for fluid systems.

In summary, the present work has demonstrated the use of an important tool for analytical solution of systems of PDE's. Lie's methods are still being developed and we believe will be more widely used and shown to be of practical relevance. The solutions obtained represent some interesting local behavior of plasmas, and can be used for instance to test computer codes for numerical simulation.

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