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Nonlinear Instability and Chaos in Plasma Wave-Wave Interactions. I. Introduction

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Abstract

Conventional linear stability analyses may fail for fluid systems with an indefinite free energy functional. When such a system is linearly stable, it is said to possess negative energy modes. Instability may then occur either via dissipation of the negative energy modes, or nonlinearly via resonant wave-wave coupling, leading to explosive growth. In the dissipationless case, it is conjectured that intrinsic chaotic behavior may allow initially nonresonant systems to reach resonance by diffusion in phase space. In this and a companion paper [submitted to Physics of Plasmas], this phenomenon is demonstrated for a simple equilibrium involving cold counterstreaming ions. The system is described in the fluid approximation by a Hamiltonian functional and associated noncanonical Poisson bracket. By Fourier decomposition and appropriate coordinate transformations, the Hamiltonian for the perturbed energy is expressed in action-angle form. The normal modes correspond to Doppler-shifted ion-acoustic waves of positive and negative energy. Nonlinear coupling leads to decay instability via two-wave interactions, and to either decay or explosive instability via three-wave interactions. These instabilities are described for various (integrable) systems of waves interacting via single nonlinear terms. This discussion provides the foundation for the treatment of nonintegrable systems in the companion paper.

I. Introduction

This and a companion paper[1] address the effect of negative energy waves and intrinsic chaos on the nonlinear stability of plasma systems. Many plasma equilibria that appear stable by a linear analysis can contain negative energy modes.[2–5] If dissipation occurs, these negative energy modes become unstable as they give up energy; in the dissipationless case, nonlinear instability may occur via resonant coupling to positive energy modes. The growth of negative energy waves from dissipation or resonant wave-wave coupling is well known in plasma and beam physics.[6–14] Here we will consider wave-wave interactions in dissipationless systems, described using a Hamiltonian formulation.

Analytical treatments of coherent wave-wave interactions often proceed by considering a single wave triplet, whose component waves interact via a single nonlinear term in the Hamiltonian.[7,15] This formulation may be arrived at by averaging, where nonresonant nonlinear terms are assumed fast-varying compared to the single resonant or near-resonant term, and are therefore dropped. Such a Hamiltonian is integrable, so that the wave amplitudes and phases may be described exactly as an explicit function of time. When appropriate resonance conditions are satisfied, coupling between waves of positive and negative energy results in explosive growth (infinite amplitude in a finite time) for arbitrarily small perturbations, while coupling between waves of the same energy sign results in growth to a finite amplitude (decay instability), limited by energy conservation. When the resonance is "detuned," the equilibrium becomes stable to infinitesimal perturbations, with growth occurring for modes above some critical amplitude (outside a separatrix in phase space). Growth rates and critical amplitudes for growth may be calculated exactly in the integrable case.

It is often the case that more than one nonlinear term in the Hamiltonian is nearly resonant and should be retained. This will in general result in a nonintegrable system.[16–18]

In this case the motion cannot be described analytically, and some invariant surfaces in the phase space will be replaced by regions of chaotic motion. Chaos originates in the vicinity of separatrices between different types of phase space trajectories, so that in the case of a detuned resonance we expect destruction of the separatrix between stable and unstable motion, and therefore a change in the effective size of the stable region. If the system is nearly integrable, most of the invariant curves within the stable region will remain intact. For a two-degree-of-freedom Hamiltonian (i.e., some number of modes coupled by only two nonlinear terms), these invariant curves provide absolute barriers to transport in the phase space, and the system will be absolutely stable for small enough perturbations. If the system is very chaotic, most invariant curves within the stable region may be destroyed so that stability is effectively lost. For more than two degrees of freedom, the invariant surfaces are not of high enough dimensionality to partition the phase space, so that even in a nearly integrable system transport may occur across unlimited regions of phase space.[19] Thus waves whose amplitudes are initially well within a "stable" region may experience relatively slow growth until they reach sufficient amplitude for instability to occur. This process (Arnold diffusion) is generally quite slow; faster "thick layer" diffusion will occur if most of the invariant surfaces are destroyed.[20]

In general, therefore, for a system with negative energy modes, one might expect lack of long-term stability to be the rule. Our goal in this work is to see what type of transport occurs and to determine the relevant time scales for a simple example.

This paper provides a review of pertinent background material and describes the physical system to be studied and the mathematical modelling employed. The companion paper[1] describes numerical results for chaotic behavior of the system.

The paper is organized as follows. Section II contains a review of some relevant background material, which is included here for continuity. (See Refs. 21 and 22 for further discussion.) Subsection A discusses the noncanonical Hamiltonian formalism which will be used to describe our plasma model, and describe the free energy principle which yields a generalized definition of negative energy modes and a criterion for nonlinear stability.[23,3] In Sec. III we describe the physical system that was studied and the fluid model that was used to describe it. The system's energy is described by a Hamiltonian functional, and the equations of motion found from the corresponding noncanonical bracket. We describe the reduction to normal-mode variables. In Sec. IV we discuss the resonance properties arising from nonlinear interactions between the normal modes, and how chaotic motion arises.

II. Background

A. Noncanonical Hamiltonian Formalism and the Free Energy Principle

We will be investigating a Hamiltonian system with a noncanonical Hamiltonian structure. This is the natural framework for a system described in terms of Eulerian variables. Here we will review the noncanonical formalism for both finite-degree-of-freedom systems and for fields,[23] and we discuss the free energy principle and the related concept of negative energy modes[3] which will determine the stability of our system.

The Hamiltonian formulation allows us to exploit the wealth of techniques available for understanding such systems. Integrable systems to be described later are easily analyzed when in Hamiltonian form, and strict constraints on the phase space structure will lead to conclusions on what to expect in nonintegrable systems, as well as guiding the choice of numerical algorithms for studying these cases.

1. Finite-Degree-of-Freedom Systems

We consider first an M-degree-of-freedom Hamiltonian system where the dynamical variables are given by the vector $\mathbf{z} := (z^1, \ldots, z^M)$. Such a system is defined by a Hamiltonian function H and a Poisson bracket $\{\, ,\}$ with the time evolution of the dynamical variables given by Hamilton's equations:

$$\dot{z}^i = \{z^i, H\} = J^{ij}\frac{\partial H}{\partial z^j}, \qquad i = 1, \ldots, M \tag{1}$$

where the Poisson bracket is defined by

$$\{f, g\} := \frac{\partial f}{\partial z^i} J^{ij} \frac{\partial g}{\partial z^j} \tag{2}$$

for any functions f and g of the phase space variables. (Here we sum repeated indices from 1 to M.) The Poisson bracket (2) must satisfy the following algebraic properties:

$$\begin{gathered} \{f, g\} = -\{g, f\}, \\ \{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} = 0, \end{gathered} \tag{3}$$

where f, g, and h are functions of $\mathbf{z}$.

A special case is the canonical one[24] where the M dynamical variables split into N configuration variables and N momenta:

$$\mathbf{z} := (q_1, \ldots, q_N, p_1, \ldots, p_N) \tag{4}$$

and the Poisson bracket has the form

$$[f, g] := \frac{\partial f}{\partial z^i} J_c^{ij} \frac{\partial g}{\partial z^j}, \tag{5}$$

where the constant matrix

$$\mathbf{J}_c := \begin{bmatrix} 0_N & I_N \\ -I_N & 0_N \end{bmatrix} \tag{6}$$

is known as the cosymplectic form. (I_N and 0_N are the $N \times N$ unit and zero matrices.) Hamilton's equations (1) then take the familiar form

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}. \tag{7}$$

In the general (noncanonical) case, J^{ij} may be odd-dimensional and may be a function of the z^i. An important property of noncanonical brackets is the existence of Casimir invariants, which commute with any function of $\mathbf{z}$:

$$\{C, f(\mathbf{z})\} = 0. \tag{8}$$

Since f is arbitrary, we see from Eq. (2) that

$$J^{ij}\frac{\partial C}{\partial z^j} = 0, \qquad i = 1, \ldots, M. \tag{9}$$

This linear system clearly has nontrivial (nonconstant) Casimir solutions only if

$$\det(J^{ij}) = 0. \tag{10}$$

Since the cosymplectic form (6) has determinant 1, the canonical bracket has no nontrivial Casimir invariants. For noncanonical brackets, the Casimirs label "leaves" in phase space upon which trajectories are constrained to lie.

Consider perturbations about an equilibrium $\mathbf{z}_e$, which satisfies

$$0 = \dot{z}_e^i = \{z^i, H\} = \{z^i, F\} = J^{ij}\frac{\partial F(\mathbf{z}_e)}{\partial z^j}, \qquad i = 1, \ldots, M \tag{11}$$

where F, the "free energy," is defined by

$$F := H + \sum_i \lambda_i C_i; \tag{12}$$

evidently equilibria are given by

$$\frac{\partial F(\mathbf{z}_e)}{\partial z^i} = 0, \qquad i = 1, \ldots, M\,. \tag{13}$$

The Hamiltonian for the linear dynamics is given by expanding F about the equilibrium (see Refs. 21 and 22), i.e., upon setting $\mathbf{z} = \mathbf{z}_e + \delta\mathbf{z}$ and expanding in $\delta\mathbf{z}$ to second order. This yields

$$\delta^2 F := \frac{1}{2}\frac{\partial^2 F(\mathbf{z}_e)}{\partial z^i \partial z^j}\,\delta z^i\,\delta z^j. \tag{14}$$

The quantity $\delta^2 F$ is in fact the energy change under perturbations which conserve the Casimir constraints. Different values of λ_i label different equilibria.

If an equilibrium is such that $\delta^2 F$ is of definite sign for all perturbations, then nearby surfaces of constant F are topologically spheres and the equilibrium is stable. If $\delta^2 F$ is indefinite, then the system may be spectrally unstable, or if spectrally (linearly) stable, then it possesses modes of both "positive energy" and "negative energy." This may be taken as a general definition of a negative energy mode. Note that since energy is not a covariant quantity, we must require that $\delta^2 F$ must be indefinite in *any* reference frame.

In the theory of dielectric media the energy content of a linear wave is defined as the work done by an external agent in exciting the wave, and is given by[25]

$$\mathcal{E} = \frac{\partial}{\partial \omega}(\omega \varepsilon)|E|^2 = \omega \frac{\partial \varepsilon}{\partial \omega}|E_k|^2 \tag{15}$$

where $\varepsilon(k, \omega)$ is the dielectric function, E_k is the electric field strength for the mode of wavenumber k and $\omega(k)$ is the frequency as found from the zeroes of $\varepsilon(k, \omega)$. The energy signature is given by $\text{sgn}\left(\omega \frac{\partial \varepsilon}{\partial \omega}\right)$, so that waves of both negative and positive energy are possible. A negative energy wave is one such that the total energy of the system is lower in the presence of the excitation. Our previous definition of negative energy waves via $\delta^2 F$ agrees with the definition from dielectric theory, and provides a generalization which is valid for systems where the dielectric function is not defined or is difficult to calculate (see e.g., Ref. 26).

It is conjectured that systems with indefinite $\delta^2 F$ are generically unstable, via either dissipation or nonlinear resonance.[3] In following sections we will examine the role of resonant wave-wave interactions in a dissipationless plasma physics example.

2. Fields

The ideas just discussed are readily extended to the case of fields,[21–23] which will be necessary for the treatment of our fluid model. In this case the state of the system is specified by the field variables ψ^i, $i = 1, \ldots, M$. The time evolution is determined by a Hamiltonian functional H and a Poisson bracket

$$\{\mathcal{F}, \mathcal{G}\} = \int d\tau \frac{\delta \mathcal{F}}{\delta \psi^i} \mathfrak{J}^{ij} \frac{\delta \mathcal{G}}{\delta \psi^j}, \tag{16}$$

where $\mathfrak{J}$ is an operator (possibly a function of the ψ^i) making the bracket satisfy the properties (3), and $d\tau$ is the volume element. The equations of motion are

$$\dot{\psi}^i = \{\psi^i, H\} = \mathfrak{J}^{ij} \frac{\delta H}{\delta \psi^j}, \qquad i = 1, \ldots, M. \tag{17}$$

For a canonical field theory the dynamical variables split into N configuration variables η^i and N momenta π^i. In this case the operator $\mathfrak{J}$ is the $2N \times 2N$ constant matrix

$$\mathcal{J}_c = \begin{bmatrix} 0_N & I_N \\ -I_N & 0_N \end{bmatrix}, \tag{18}$$

giving for the Poisson bracket

$$\{\mathcal{F}, \mathcal{G}\} = \sum_{i=1}^{N} \int d^3x \left[\frac{\delta \mathcal{F}}{\delta \eta^i} \frac{\delta \mathcal{G}}{\delta \pi^i} - \frac{\delta \mathcal{G}}{\delta \eta^i} \frac{\delta \mathcal{F}}{\delta \pi^i} \right]. \tag{19}$$

Hamilton's equations take the form

$$\dot{\eta}^i = \{\eta^i, H\} = \frac{\delta H}{\delta \pi^i}, \qquad \dot{\pi}^i = \{\pi^i, H\} = -\frac{\delta H}{\delta \eta^i}. \tag{20}$$

For the noncanonical case we again have some number P (possibly infinite) of Casimir invariants satisfying

$$\{C_k, \mathcal{F}[\psi]\} = 0, \qquad k = 1, \ldots, P. \tag{21}$$

We can then find equilibria corresponding to extremals of the free energy functional

$$F = H + \sum_{k=1}^{P} \lambda_k C_k, \tag{22}$$

i.e., the Hamiltonian subject to the constraints of constant Casimirs. By definition, equilibria satisfy

$$\dot{\psi}^i = 0, \qquad i = 1, \ldots, M. \tag{23}$$

Since we have

$$\dot{\psi}^i = \{\psi^i, F\} = \{\psi^i, H\} = \mathfrak{J}^{ij}\frac{\delta F}{\delta \psi^j}, \tag{24}$$

we see that the equilibrium equations are

$$\frac{\delta F[\psi_e]}{\delta \psi^i} = 0, \qquad i = 1, \ldots, M, \tag{25}$$

where we have supposed that ψ_e solves (25). Then the first variation of F, relative to ψ_e in the direction $\phi = \left(\phi^1, \ldots, \phi^M\right)$, is denoted $\delta F[\psi; \phi]$ and given by

$$\delta F[\psi_e; \phi] := \frac{d}{d\epsilon}\, F[\psi_e + \epsilon\phi]|_{\epsilon=0} := \int d\tau \frac{\delta F}{\delta \psi^i}\phi^i. \tag{26}$$

Equation (25) implies that $\delta F = 0$ for all ϕ. A second variation at fixed ϕ yields

$$\delta^2 F[\psi_e; \phi_e] := \frac{d}{d\epsilon}\, \delta F[\psi_e + \epsilon\phi; \phi]|_{\epsilon=0} := \int d\tau\, \phi^i \frac{\delta^2 F}{\delta\psi^i \delta\psi^j}\phi^j. \tag{27}$$

Now analogous to the finite-dimensional case, definite $\delta^2 F$ implies stability, while indefinite $\delta^2 F$ implies either linear instability or the existence of negative energy modes.

III. Counterstreaming Ions: Basic Development

We will consider a simple one-dimensional plasma configuration consisting of two cold counterstreaming ion beams in a neutralizing isothermal electron background.[10]

We begin by describing the fluid equations governing the system, and the characteristic modes of oscillation that arise from the linearized equations. The Hamiltonian structure of the system is then described, and the normal modes are obtained from an energy functional by calculating the lowest-order expression for the perturbed energy about an equilibrium and transforming the corresponding system into action-angle variables. Higher-order terms in the Hamiltonian provide coupling between normal modes, which will be seen in following sections to lead to explosive instability as well as to strongly chaotic motion for a wide range of physical parameters.

The Hamiltonian nature of the system places strict constraints on its possible behavior. The Hamiltonian formulation employed here serves to illuminate these properties.

A. Linear Analysis

The dimensionless equations of motion and continuity for the two ion streams are

$$\frac{\partial v_\alpha}{\partial t} + v_\alpha \frac{\partial v_\alpha}{\partial x} + \frac{\partial \phi}{\partial x} = 0, \tag{28}$$

$$\frac{\partial n_\alpha}{\partial t} + \frac{\partial}{\partial x}\left(n_\alpha v_\alpha\right) = 0, \tag{29}$$

where $\alpha = \pm$ labels each ion stream with the sign of its velocity relative to the center-of-mass frame. Here the n_α are normalized to the total unperturbed ion density n_0, v_α is in units of the ion sound speed $c_s := \sqrt{\frac{T}{m_i}}$, the electric potential ϕ is in units of T_e/e where T is the electron temperature in energy units, x is in units of the electron Debye length $\lambda_d := \sqrt{\frac{T_e}{4\pi n_0 e^2}}$ and time is in units of the inverse ion plasma frequency $\omega_p^{-1} := \sqrt{\frac{m_i}{4\pi n_0 e^2}}$. These equations are supplemented with the isothermal ($\frac{m_e}{m_i} \to 0$) approximation for the electron motion and Poisson's equation:

$$n_e = e^\phi, \tag{30}$$

$$-\frac{\partial^2 \phi}{\partial x^2} = n_+ + n_- - n_e, \tag{31}$$

where n_e is normalized to n_0. Through the latter two equations we can in principle solve for $\phi\left(n_+, n_-\right)$ so that the entire system is described in terms of the dynamical variables $n_\pm$ and $v_\pm$.

For simplicity we consider an equilibrium of ion streams of equal density and speed:

$$n_+ = n_- = \frac{1}{2}, \qquad v_+ = -v_- = v, \qquad E = \phi = 0, \tag{32}$$

where the ion drift speed v is normalized to c_s. If we assume that perturbations have the dependence $e^{i(kx-\omega t)}$ and linearize Eqs. (28)–(31), the condition for the resulting system to have nontrivial solutions is the vanishing of the dielectric function, given by

$$\varepsilon(k,\omega) := 1 + \frac{1}{k^2} - \frac{1}{2}\left[\frac{1}{(\omega - kv)^2} + \frac{1}{(\omega + kv)^2}\right] = 0. \tag{33}$$

(Here k is normalized to λ_d^{-1} and ω to ω_{pi}.) Solving for ω yields

$$\omega^2 = k^2\left[\frac{1}{2(1+k^2)} + v^2 \pm \sqrt{\frac{1}{4(1+k^2)^2} + \frac{2v^2}{(1+k^2)}}\right]. \tag{34}$$

A plot of the four branches of $\omega(k)$ is shown in Fig. 1 for the cases $v > 1$ and $v < 1$. Taking the "+"-sign in Eq. (34) yields "fast modes," with phase speed ω/k greater than the drift speed v. These modes are always linearly stable (ω real). The "−"-sign yields "slow modes" with $\omega/k < v$, which are linearly stable when $v > \frac{1}{1+k^2}$. (Recall that v is the ratio of the equilibrium ion drift speed to the ion sound speed, or equivalently of the equilibrium ion kinetic energy to the electron temperature). For $v < 1$, the slow modes become linearly unstable, via the well-known ion-ion two-stream instability, for small k. (It should be noted that the linear stability criterion $v > 1$ would be replaced by the more stringent criterion $|\mathbf{v}|\cos\theta > 1$ in a two-dimensional model, where θ is the angle between $\mathbf{k}$ and $\mathbf{v}$.)

As discussed earlier, the energy signature of a wave is given by $\omega\frac{\partial\varepsilon}{\partial\omega}$. It is easily shown from Eqs. (33) and (34) that the fast modes have positive energy, while the slow modes are negative energy waves.

B. Hamiltonian Development

1. Hamiltonian Structure

The above normal modes may also be derived via a Hamiltonian formulation. The Hamiltonian is given by the energy, written as the functional

$$H = \Lambda \int_0^L dx \left[\frac{1}{2} n_+ v_+^2 + \frac{1}{2} n_- v_-^2 + \int_0^\phi d\phi \phi e^\phi + \frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 \right], \tag{35}$$

where $\Lambda = n_0 \lambda_d \gg 1$ is the (one-dimensional) plasma parameter, L is the size of the system in Debye lengths and H is normalized to the electron temperature. The first two terms are the ion kinetic energy densities, the third is the internal energy function for the electrons[27] and the fourth is the electric field energy. The electrons can be envisioned as supplying an effective ion pressure.

The appropriate Poisson bracket is given by[27,28]

$$\{\mathcal{F}, \mathcal{G}\} = \Lambda^{-1} \sum_{\alpha=\pm} \int_0^L dx \left[\frac{\delta \mathcal{G}}{\delta v_\alpha} \frac{\partial}{\partial x} \frac{\delta \mathcal{F}}{\delta n_\alpha} - \frac{\delta \mathcal{F}}{\delta v_\alpha} \frac{\partial}{\partial x} \frac{\delta \mathcal{G}}{\delta n_\alpha} \right] \tag{36}$$

where $\mathcal{F}$ and $\mathcal{G}$ are functions of the n_α and v_α, and $\frac{\delta f}{\delta u}$ denotes the variational derivative of f with respect to u. The Poisson bracket has units of $(\text{energy} \times \text{time})^{-1} = \text{action}^{-1}$; consistent with our earlier normalizations we have written it in units of $\omega_{pi} T_e^{-1}$. The equations of motion (28) and continuity equations (29) are obtained from the usual formulae for time derivatives in terms of Poisson brackets:

$$\dot{n}_\alpha = \{n_\alpha(x,t), H\} = \Lambda^{-1} \sum_{\alpha'=\pm} \int_0^L dx' \frac{\delta H}{\delta v_{\alpha'}} \frac{\partial}{\partial x'} \frac{\delta n_\alpha(x)}{\delta n_{\alpha'}(x')} = -\int_0^L dx' \frac{\partial}{\partial x'}(n_\alpha v_\alpha) \frac{\delta n_\alpha(x)}{\delta n_\alpha(x')}$$

$$= -\int_0^L dx' \frac{\partial}{\partial x'}(n_\alpha v_\alpha)\, \delta(x - x') = -\frac{\partial}{\partial x}(n_\alpha v_\alpha), \tag{37}$$

$$\dot{v}_\alpha = \{v_\alpha, H\} = -\Lambda^{-1} \int_0^L dx' \frac{\delta v_\alpha(x)}{\delta v_\alpha(x')} \frac{\partial}{\partial x'} \frac{\delta H}{\delta n_\alpha} = -\int_0^L dx'\, \delta(x - x') \frac{\partial}{\partial x'} \left(\frac{1}{2} v_\alpha^2 + \phi \right)$$

$$= -\frac{\partial}{\partial x} \left(\frac{1}{2} v_\alpha^2 + \phi \right), \tag{38}$$

where we have used Poisson's equation (31) and the Boltzmann relation (30) in evaluating $\frac{\delta H}{\delta n_\alpha}$ for the last equation. As described in Ref. 27 the Poisson bracket has the Casimir invariants

$$C_1^{\pm} = \int_0^L dx\; n_{\pm}, \qquad C_2^{\pm} = \int_0^L dx\; v_{\pm}. \tag{39}$$

The physical significance of these Casimirs may be traced back to the kinetic theory describing this system. The basic equations are the (dimensionless) Vlasov-Poisson equations:

$$\frac{\partial f}{\partial t} = -v\frac{\partial f}{\partial x} + \frac{\partial \phi(f;x)}{\partial x}\frac{\partial f}{\partial v}, \qquad \frac{\partial^2 \phi}{\partial x^2} = -\int f\, dv, \tag{40}$$

where $f(x, v, t)$ is the phase space density for a single species. This system has the invariant

$$C = \int \mathcal{C}(f)\; dx\, dv \tag{41}$$

where $\mathcal{C}$ is an arbitrary function; this invariant corresponds to conservation of phase space volume for a given value of f.[29] The correspondence between the fluid model and the Vlasov-Poisson system is made via the water-bag model.[12] This assumes the following form for f:

$$f(x, v) = \begin{cases} A & \text{for } \; v_- < v < v_+, \\ 0 & \text{otherwise} \end{cases} \tag{42}$$

where $A =$ constant, and v_+ and v_- are single-valued functions of x and t. We then obtain

$$\begin{aligned} n &= \int_{-\infty}^{\infty} f\; dv = A(v_+ - v_-), \\ v &= \frac{\displaystyle\int_{-\infty}^{\infty} vf\; dv}{\displaystyle\int_{-\infty}^{\infty} f\; dv} = \frac{1}{2}(v_+ + v_-) \end{aligned} \tag{43}$$

and

$$C = \int \mathcal{C}(A)[v_+ - v_-]\; dx. \tag{44}$$

Equation (39) is implied by Eqs. (43) and (44). Thus the Casimir invariants correspond to the conservation of phase space volume required by Liouville's theorem in the Vlasov theory.[30]

Equilibria are given by extrema of the energy subject to the Casimir constraints; we therefore consider variations of the free energy functional

$$F = H + \lambda_1^+ C_1^+ + \lambda_1^- C_1^- + \lambda_2^+ C_2^+ + \lambda_2^- C_2^- . \tag{45}$$

The equilibrium (32) is obtained from the first variation of F:

$$\begin{aligned} \delta F = 0 \quad &\Longrightarrow \quad \frac{\delta F}{\delta n_\pm} = 0 \qquad \text{and} \quad \frac{\delta F}{\delta v_\pm} = 0 \\ &\Longrightarrow \quad \lambda_1^\pm = -\frac{1}{2} v^2 \quad \text{and} \quad \lambda_2^\pm = \mp \frac{1}{2} v . \end{aligned} \tag{46}$$

The Lagrange multipliers may be recognized as the kinetic energy and momentum of the two streams.

Stability is determined from the second variation, evaluated on the equilibrium:

$$\begin{aligned} \delta^2 F = \Lambda \frac{1}{2} \int_0^L dx \Bigg[&\frac{1}{2} (\delta v^+)^2 + \frac{1}{2} (\delta v^-)^2 + 2 v \, \delta n^+ \delta v^+ \\ &- 2 v \, \delta n^- \delta v^- + (\delta \phi_x)^2 + (\delta \phi)^2 \Bigg] \end{aligned} \tag{47}$$

where $\delta\phi_x$ denotes $\frac{\partial}{\partial x}\delta\phi$. Here we have expanded the exponential in the internal energy function, keeping second-order terms. The sign of $\delta^2 F$ may be either positive or negative, depending on the perturbation; thus we may have either positive or negative energy waves in the system. It is important to note that, for this system, $\delta^2 F$ is indefinite regardless of reference frame.

2. Reduction to Normal-Mode Variables

We now Fourier decompose the perturbations:

$$\delta n^\pm = \sum_{m=-\infty}^{\infty} n_m^\pm e^{ik_m x}, \quad \delta v^\pm = \sum_{m=-\infty}^{\infty} v_m^\pm e^{ik_m x}, \quad \delta\phi = \sum_{m=-\infty}^{\infty} \phi_m e^{ik_m x}, \tag{48}$$

where $k_m := mk_1$, and $k_1 := \frac{2\pi}{L}$. If we linearize Eqs. (30) and (31) about the equilibrium (expanding the exponential) and insert the Fourier expansions for $\delta n^{\pm}$ and $\delta\phi$, we can solve for $\phi_m(n_m^+, n_m^-)$:

$$\phi_m = \frac{N_m}{1+k_m^2} - \frac{1}{2}\sum_{l=1}^{\infty}\left[\frac{N_l N_{m-l}}{(1+k_l^2)(1+k_m^2)(1+k_{m-l}^2)} + \frac{N_{-l}N_{m+l}}{(1+k_{-l}^2)(1+k_m^2)(1+k_{m+l}^2)}\right] + \mathcal{O}\left(N^3\right), \tag{49}$$

where $N_m := n_m^+ + n_m^-$. In terms of Fourier coefficients we then have

$$\delta^2 F = \frac{1}{2}\Lambda L \sum_{m=1}^{\infty}\left[\left(\left|v_m^+\right|^2 + \left|v_m^-\right|^2\right) + 2v\left(n_m^+ v_{-m}^+ - n_m^- v_{-m}^- + \text{C.C.}\right) + 2\frac{|N_m|^2}{1+k_m^2}\right] + \mathcal{O}\left(|N_m|^3\right), \tag{50}$$

where C.C. denotes complex conjugate. Here we have dropped terms of order $|N_m|^3$ which came from evaluating $|\phi_m|^2$; we shall reclaim them when we evaluate the cubic terms in the Hamiltonian.

The Poisson bracket (36) now takes the form (see Appendix A)

$$\{\mathcal{F}, \mathcal{G}\} = \sum_{\alpha=\pm}\sum_{m=1}^{\infty}\frac{\imath k_m}{\Lambda L}\left[\frac{\delta\mathcal{G}}{\delta n_m^\alpha}\frac{\delta\mathcal{F}}{\delta v_{-m}^\alpha} - \frac{\delta\mathcal{F}}{\delta n_m^\alpha}\frac{\delta\mathcal{G}}{\delta v_{-m}^\alpha} + \frac{\delta\mathcal{G}}{\delta n_{-m}^\alpha}\frac{\delta\mathcal{F}}{\delta v_m^\alpha} - \frac{\delta\mathcal{F}}{\delta n_{-m}^\alpha}\frac{\delta\mathcal{G}}{\delta v_m^\alpha}\right]. \tag{51}$$

This can be put into the familiar form via the transformation

$$\begin{aligned}
n_m^+ &= \frac{\sqrt{\pi v}k_m}{\sqrt{\Lambda L}}\left(p_2 - \imath q_1\right), & n_{-m}^+ &= (n_m^+)^*,\\
v_m^+ &= \frac{1}{2\sqrt{\Lambda}\sqrt{\pi v}}\left(p_1 - \imath q_2\right), & v_{-m}^+ &= (v_m^+)^*,\\
n_m^- &= \frac{\sqrt{\pi v}k_m}{\sqrt{\Lambda L}}\left(p_4 - \imath q_3\right), & n_{-m}^- &= (n_m^-)^*,\\
v_m^- &= \frac{1}{2\sqrt{\Lambda}\sqrt{\pi v}}\left(p_3 - \imath q_4\right), & v_{-m}^- &= (v_m^-)^*,
\end{aligned} \tag{52}$$

where we have suppressed the mode number labels on the p_i and q_i. (Here p_i and q_i have dimensions of action$^{\frac{1}{2}}$ = (energy $\times$ time)$^{\frac{1}{2}}$ and have been normalized to $(T_e/\omega_{pi})^{\frac{1}{2}}$.) The bracket then becomes

$$\{\mathcal{F},\mathcal{G}\} = \sum_{m=1}^{\infty}\sum_{i=1}^{4}\left(\frac{\partial\mathcal{F}}{\partial q_i}\frac{\partial\mathcal{G}}{\partial p_i} - \frac{\partial\mathcal{G}}{\partial q_i}\frac{\partial\mathcal{F}}{\partial p_i}\right). \tag{53}$$

Using the transformation (52) our expression (50) for the perturbed energy becomes

$$\delta^2 F = \frac{1}{2}\sum_{m=1}^{\infty}\sum_{i,j=1}^{4}\left(q_i A_{ij} q_j + p_i B_{ij} p_j\right)_m = \frac{1}{2}\sum_{m=1}^{\infty}\left(\tilde{\mathbf{q}}\mathbf{A}\mathbf{q} + \tilde{\mathbf{p}}\mathbf{B}\mathbf{p}\right)_m \tag{54}$$

where

$$\mathbf{A} = \begin{bmatrix} \dfrac{k_m^2 k_1 v}{1+k_m^2} & k_m v & \dfrac{k_m^2 k_1 v}{1+k_m^2} & 0 \\[2ex] k_m v & \dfrac{1}{2k_1 v} & 0 & 0 \\[2ex] \dfrac{k_m^2 k_1 v}{1+k_m^2} & 0 & \dfrac{k_m^2 k_1 v}{1+k_m^2} & -k_m v \\[2ex] 0 & 0 & -k_m v & \dfrac{1}{2k_1 v} \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} \dfrac{1}{2k_1 v} & k_m v & 0 & 0 \\[2ex] k_m v & \dfrac{k_m^2 k_1 v}{1+k_m^2} & 0 & \dfrac{k_m^2 k_1 v}{1+k_m^2} \\[2ex] 0 & 0 & \dfrac{1}{2k_1 v} & -k_m v \\[2ex] 0 & \dfrac{k_m^2 k_1 v}{1+k_m^2} & -k_m v & \dfrac{k_m^2 k_1 v}{1+k_m^2} \end{bmatrix}. \tag{55}$$

(The tilde denotes transpose.) The subscript "m" in Eq. (54) is a shorthand indicating that each quantity inside the parentheses is labelled with this mode number.

The wave momentum is given by

$$P = \Lambda \int_0^L dx \left(n^+ v^+ + n^- v^-\right). \tag{56}$$

The total momentum for our equilibrium is zero; the perturbed momentum is

$$\delta^2 P = \Lambda L \sum_{m=1}^{\infty} \left(n_m^+ v_{-m}^+ + n_m^- v_{-m}^- + \text{C.C.} \right)$$

$$= \sum_{m=1}^{\infty} k_m \left(p_1 p_2 + q_1 q_2 + p_3 p_4 + q_3 q_4 \right)_m = \frac{1}{2} \sum_{m=1}^{\infty} \left(\tilde{\mathbf{q}} \mathbf{C} \mathbf{q} + \tilde{\mathbf{p}} \mathbf{C} \mathbf{p} \right)_m \tag{57}$$

where

$$\mathbf{C} = \begin{bmatrix} 0 & k_m & 0 & 0 \\ k_m & 0 & 0 & 0 \\ 0 & 0 & 0 & k_m \\ 0 & 0 & k_m & 0 \end{bmatrix}. \tag{58}$$

We now seek a canonical transformation to diagonalize both $\delta^2 F$ and $\delta^2 P$. The appropriate transformation may be written in the form

$$\mathbf{q} = \mathbf{S} \mathbf{Q}, \qquad \mathbf{p} = \mathbf{T} \mathbf{P}, \tag{59}$$

where $\mathbf{Q} := (\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_4)$, $\mathbf{P} := (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4)$, and the 4×4 matrices $\mathbf{S}$ and $\mathbf{T}$ are given in Appendix B. In terms of the new variables we find

$$\delta^2 F = \sum_{m=1}^{\infty} \left[\omega_+ \frac{\mathcal{P}_1^2 + \mathcal{Q}_1^2}{2} - \omega_- \frac{\mathcal{P}_2^2 + \mathcal{Q}_2^2}{2} + \omega_+ \frac{\mathcal{P}_3^2 + \mathcal{Q}_3^2}{2} - \omega_- \frac{\mathcal{P}_4^2 + \mathcal{Q}_4^2}{2} \right]_m \tag{60}$$

and

$$\delta^2 P = \sum_{m=1}^{\infty} k_m \left[\frac{\mathcal{P}_1^2 + \mathcal{Q}_1^2}{2} - \frac{\mathcal{P}_2^2 + \mathcal{Q}_2^2}{2} - \frac{\mathcal{P}_3^2 + \mathcal{Q}_3^2}{2} + \frac{\mathcal{P}_4^2 + \mathcal{Q}_4^2}{2} \right]_m \tag{61}$$

where

$$\omega_\pm := k_m \left[\frac{1}{2\left(1 + k_m^2\right)} + v^2 \pm \sqrt{\frac{1}{4\left(1 + k_m^2\right)^2} + \frac{2v^2}{\left(1 + k_m^2\right)}} \right]^{\frac{1}{2}} > 0. \tag{62}$$

Making one further canonical transformation to action-angle variables

$$\mathcal{Q}_i = \sqrt{2\mathcal{J}_i} \sin \theta_i \qquad \mathcal{P}_i = \sqrt{2\mathcal{J}_i} \cos \theta_i, \tag{63}$$

we arrive at

$$\delta^2 F = \sum_{m=1}^{\infty} \left[\omega_+ \mathcal{J}_1 - \omega_- \mathcal{J}_2 + \omega_+ \mathcal{J}_3 - \omega_- \mathcal{J}_4\right]_m \tag{64}$$

$$\delta^2 P = \sum_{m=1}^{\infty} k_m \left[\mathcal{J}_1 - \mathcal{J}_2 - \mathcal{J}_3 + \mathcal{J}_4\right]_m . \tag{65}$$

The $\mathcal{J}_{i(m)}$ will be recognized as corresponding to the four eigenmodes of Fig. 1, and the negative energy character of branches 2 and 4 is now explicit in the Hamiltonian. To further illustrate the correspondence, let us write the perturbed energy (50) and momentum (57) in terms of the Fourier coefficients for the electric field $E_m := -\imath k_m \phi_m$. We obtain

$$\delta^2 F = \sum_{m=1}^{\infty} \omega(k_m) \left(L \frac{\partial \varepsilon(k_m, \omega)}{\partial \omega} \left|E_m\right|^2 \right), \tag{66}$$

$$\delta^2 P = \sum_{m=1}^{\infty} k_m \left(L \frac{\partial \varepsilon(k_m, \omega)}{\partial \omega} \left|E_m\right|^2 \right), \tag{67}$$

where $\omega(k_m)$ is one of the four roots (positive or negative) of the dispersion relation $\varepsilon(k_m, \omega) = 0$. We immediately identify the wave action, which is now clearly seen to be the Hamiltonian action, as

$$\mathcal{J} = N \frac{\partial \varepsilon}{\partial \omega} \left|E_m\right|^2 . \tag{68}$$

Again we see that the sign of the wave energy is given by $\omega \frac{\partial \varepsilon}{\partial \omega}$.

3. Nonlinear Coupling

We now consider cubic terms of the perturbed energy. Taking the third variation of F, inserting the Fourier expansions and carrying out the integration we obtain

$$\delta^3 F = \sum_{l,m=1}^{\infty} \left\{ \frac{\Lambda L}{2} \sum_{\alpha=\pm} \left[n_m^\alpha v_l^\alpha v_{-(m+l)}^\alpha + n_m^\alpha v_{-l}^\alpha v_{-(m-l)}^\alpha + \text{C.C.} \right] \right.$$
$$\left. - \frac{\Lambda L}{6} \left[\frac{N_m N_l N_{-(m+l)}}{(1+k_m^2)(1+k_l^2)(1+k_{m+l}^2)} + \frac{N_m N_{-l} N_{-(m-l)}}{(1+k_m^2)(1+k_l^2)(1+k_{m-l}^2)} + \text{C.C.} \right] \right\}$$

$$= \sum_{l,m=1}^{\infty} \Big\{ \alpha_m \Big[(p_2 - \imath q_1)_m (p_1 - \imath q_2)_l (p_1 + \imath q_2)_{m+l} + (p_2 - \imath q_1)_m (p_1 - \imath q_2)_{-l} (p_1 + \imath q_2)_{m-l}$$

$$+ \; (p_4 - \imath q_3)_m (p_3 - \imath q_4)_l (p_3 + \imath q_4)_{m+l} + (p_4 - \imath q_3)_m (p_3 - \imath q_4)_{-l} (p_3 + \imath q_4)_{m-l} \Big]$$

$$+ \beta_{l,m} \left[(p_2 \imath q_1 + p_4 - \imath q_3)_m (p_2 - \imath q_1 + p_4 - \imath q_3)_l (p_2 + \imath q_1 + p_4 + \imath q_3)_{m+l} \right]$$

$$+ \beta_{-l,m} \left[(p_2 - \imath q_1 + p_4 - \imath q_3)_m (p_2 - \imath q_1 + p_- \imath q_3)_{-l} (p_2 + \imath q_1 + p_4 + \imath q_3)_{m-l} \right]$$

$$+ \mathrm{C.C.} \Big\} \tag{69}$$

where

$$\begin{aligned} \alpha_m &= \frac{k_m}{8\pi^{1/2}\Lambda^{1/2}v^{1/2}} \\ \beta_{\pm l,m} &= \frac{k_1^2 v^{3/2}}{24\pi^{3/2}\Lambda^{3/2}} \, \frac{k_m k_{\pm l} k_{m\pm l}}{(1+k_m^2)(1+k_{\pm l}^2)(1+k_{m\pm l}^2)}. \end{aligned} \tag{70}$$

The only nonlinear terms surviving the integration are those corresponding to wave triplets that satisfy the wavenumber-matching conditions $n_1 k^{(1)} + n_2 k^{(2)} + n_3 k^{(3)} = 0$, where the n_i are integers satisfying $|n_1| + |n_2| + |n_3| = 3$. This condition corresponds to the conservation of wave momentum.

We can now substitute the transformations used earlier in Eqs. (59) and (63):

$$\begin{aligned} q_i &= \sum_{j=1}^{4} S_{ij} \mathcal{Q}_j = \sum_{j=1}^{4} S_{ij} \sqrt{2\mathcal{J}_j} \sin\theta_j \\ p_i &= \sum_{j=1}^{4} T_{ij} \mathcal{P}_j = \sum_{j=1}^{4} T_{ij} \sqrt{2\mathcal{J}_j} \cos\theta_j. \end{aligned} \tag{71}$$

Rather than attempt to write the general expression, we simply note for now that the general cubic term will have the form (up to a constant phase factor)

$$\alpha \, \mathcal{J}_a^{|m_a|/2} \, \mathcal{J}_b^{|m_b|/2} \, \mathcal{J}_c^{|m_c|/2} \sin(m_a\theta_a + m_b\theta_b + m_c\theta_c) \tag{72}$$

where $|m_a| + |m_a| + |m_a| = 3$ and where α here represents some combination of the α_m and $\beta_{\pm l,m}$ and the matrix elements S_{ij} and T_{ij}.

For later reference we note how the coefficients of the cubic terms scale with the equilibrium quantities. Using the S_{ij} and T_{ij} defined in Appendix B, we find that for wavenumbers of order mk_1,

$$\delta^3 F \sim \frac{m^{1/2} k_1^2}{\Lambda}. \tag{73}$$

IV. Resonances and Nonlinear Instability

Our interest is in resonant interactions between linearly stable modes; we therefore consider only sets of modes that interact resonantly (or near-resonantly) via the cubic terms of Eq. (69). Other modes yield nonlinear terms with rapidly varying phase and may be removed by averaging. The third-order resonance conditions to be satisfied are

$$n_1 k^{(1)} + n_2 k^{(2)} + n_3 k^{(3)} = 0 \tag{74}$$

$$n_1 \omega_1 \pm n_2 \omega_2 \pm n_3 \omega_3 = 0 \tag{75}$$

where ω_i denotes $\omega(k^{(i)})$, and the n_i are integers satisfying $|n_1|+|n_2|+|n_3| = 3$. Higher-order interactions will not be considered here.

We first consider the integrable cases of either two modes from the same branch of the dispersion relation, or three modes from different branches, interacting via a single nonlinear term, and then look at nonintegrable systems involving several nonlinear terms.

A. Phase Space Structure: Single Resonance

1. Two-Wave Interactions

A very important interaction is that of two nearly resonant modes from the same branch of the dispersion relation. This situation is generic for long wavelengths (small k), where we have $\omega_\pm \approx k_m \left[\frac{1}{2} + v^2 \pm \sqrt{\frac{1}{4} + 2v^2}\right]^{\frac{1}{2}}$, so that any pair of waves satisfying $2k_1 = k_2$ will also satisfy $2\omega_1 \approx \omega_2$. The frequency matching condition cannot be satisfied exactly, but is approached in the limit $k \to 0$. For a large system ($k_1 \ll 1$) there may be many such nearly resonant pairs of modes. We will see later that this can be a strong source of chaos.

Three modes for which this occurs are shown in Fig. 2. Here modes J_1, J_2 and J_3 have wavenumbers k_1, $2k_1$ and $4k_1$ respectively. For $k_1 \ll 1$ we will simultaneously have near-resonance between modes J_1 and J_2 and between modes J_2 and J_3. For now consider only modes J_1 and J_2. The Hamiltonian is

$$H = \omega_1 J_1 + \omega_2 J_2 + \alpha J_1 \sqrt{J_2} \sin(2\theta_1 - \theta_2), \tag{76}$$

where

$$\alpha = \frac{k_1^2}{\sqrt{8\pi\Lambda}} \Bigg[\widehat{a}_{3(k_1)} \left(2\widehat{a}_{3(2k_1)}\, \widehat{b}_{1(k_1)} + \widehat{a}_{3(k_1)}\, \widehat{b}_{1(2k_1)} \right) - \widehat{a}_{1(k_1)} \left(2\widehat{a}_{1(2k_1)}\, \widehat{b}_{3(k_1)} + \widehat{a}_{1(k_1)}\, \widehat{b}_{3(2k_1)} \right) + \frac{\left(\widehat{b}_{1(k_1)} - \widehat{b}_{3(k_1)}\right)^2 \left(\widehat{b}_{1(2k_1)} - \widehat{b}_{3(2k_1)}\right)}{(1+k_1^2)^2(1+4k_1^2)} \Bigg]. \tag{77}$$

(The subscripts in parentheses are the values of k at which $\widehat{a}_i$ and $\widehat{b}_i$ are evaluated.) Since the angles θ_1, θ_2 occur only in one combination, this can be reduced to a one-degree-of-freedom system. We do this by employing a canonical transformation to a new set of variables, which we call "resonance variables." One such transformation is provided by the mixed-variable generating function

$$F(I_1, I_3, \theta_1, \theta_2) = I_1(2\theta_1 - \theta_2) + I_3\theta_2, \tag{78}$$

which yields

$$\begin{aligned} J_1 &= \frac{\partial F}{\partial \theta_1} = 2\overline{I}_1, & \overline{\psi}_1 &= \frac{\partial F}{\partial \overline{I}_1} = 2\theta_1 - \theta_2, \\ J_2 &= \frac{\partial F}{\partial \theta_2} = \overline{I}_3 - \overline{I}_1, & \overline{\psi}_3 &= \frac{\partial F}{\partial \overline{I}_3} = \theta_2. \end{aligned} \tag{79}$$

The Hamiltonian becomes

$$\overline{H} = H - \omega_2 I_3 = \Omega_1 I_1 + 2\alpha I_1 \sqrt{I_3 - I_1} \sin\psi_1, \tag{80}$$

where

$$\Omega_1 := 2\omega_1 - \omega_2 \tag{81}$$

is the "detuning" of the resonance and

$$I_3 := J_2 + \frac{1}{2}J_1 = \text{constant.} \tag{82}$$

(The choice of notation "I_1, I_3" rather than "I_1, I_2" was made for later convenience.) Note that the constant of motion I_3 is just proportional to the wave momentum $k_1(J_1 + 2J_2)$.

For arbitrary pairs of waves, the phase space topology is determined by the "effective detuning" $\left|\frac{\Omega_\pm}{\alpha}\right|$, where "$\pm$" denotes waves of positive or negative energy. Expanding $\Omega_\pm$ for small k, we find

$$|\Omega_\pm| := |\omega_\pm(2k) - 2\omega_\pm(k)| = k^3 \frac{3}{\sqrt{2}} \frac{1 + 4v^2 \pm \sqrt{1+8v^2}}{\sqrt{1+8v^2}\sqrt{1+2v^2 \pm \sqrt{1+8v^2}}} + \mathcal{O}(k^5). \tag{83}$$

The detunings Ω_+ and Ω_- are nearly equal in magnitude for most v, differing in magnitude only for v near 1 (since Ω_- diverges as $v \to 1$).

Recalling from Eq. (73) that $\alpha \sim \frac{m^{3/2}k_1^2}{\Lambda}$ for small mk_1, we find for the effective detuning

$$\left|\frac{\Omega_\pm}{\alpha}\right| \sim \frac{m^3 k_1^3 \Lambda}{m^{3/2}k_1^2} \sim m^{3/2} k_1 \Lambda \sim m^{3/2}\frac{\Lambda}{L} \tag{84}$$

with a weaker dependence on v. We will therefore have the strongest resonance for the smallest Λ (dense plasma). We will also obtain stronger resonance by considering smaller k_1 (larger L); however, since k_1 enters into both Ω and α, this also increases the time scales required for the system to evolve.

The motion for the Hamiltonian (80) may be solved exactly in terms of elliptic integrals.[7,10] Our discussion of this and other integrable systems will focus primarily on graphical representations of the phase space to understand the motion. For nonintegrable systems to be considered later, there is of course no exact solution, and numerical methods will be used.

We now examine the phase space structure for the Hamiltonian (80). For graphical representations it is convenient to use the Cartesian form of the resonance variables:

$$P_i = \sqrt{2I_i}\cos\psi_i\,, \qquad Q_i = \sqrt{2I_i}\sin\psi_i\,. \tag{85}$$

In Fig. 3 we display constant-energy contours for two values of L and fixed Λ and v. I_3 was chosen to be 0.01. The motion is confined to the region $P_1^2 + Q_1^2 \le 2I_3$. When $\left|\frac{\Omega_1}{\alpha}\right| < \sqrt{I_3}$ there is an unstable fixed point at the origin and two stable fixed points on the positive and negative Q_1-axis. As $\left|\frac{\Omega_1}{\alpha}\right|$ passes through $\sqrt{I_3}$, one of the stable fixed points merges with the origin, which then becomes stable for $\left|\frac{\Omega_1}{\alpha}\right| > \sqrt{I_3}$.

Recall that in the Hamiltonian (80), $I_1 = \frac{1}{2}J_1$, so that in Fig. 3 we are examining the behavior of J_1. We could equally well have eliminated J_1 in order to examine the behavior of J_2. Equivalent to this, and more useful for later reference, is looking at the behavior of J_2 when coupled to the mode J_3 of Fig. 2. (In either case we will be considering the behavior of the high-frequency mode of a pair.) The Hamiltonian for this system is given by

$$H = \omega_2 J_2 + \omega_3 J_3 + \beta J_2\sqrt{J_3}\sin(2\theta_2 - \theta_3), \tag{86}$$

where

$$\beta = \frac{k_1^2}{\sqrt{\pi\Lambda}}\Bigg[\widehat{a}_{3(2k_1)}\left(2\widehat{a}_{3(4k_1)}\,\widehat{b}_{1(2k_1)} + \widehat{a}_{3(2k_1)}\,\widehat{b}_{1(4k_1)}\right) - \widehat{a}_{1(2k_1)}\left(2\widehat{a}_{1(4k_1)}\,\widehat{b}_{3(2k_1)} + \widehat{a}_{1(2k_1)}\,\widehat{b}_{3(4k_1)}\right)$$
$$+\frac{\left(\widehat{b}_{1(2k_1)} - \widehat{b}_{3(2k_1)}\right)^2\left(\widehat{b}_{1(4k_1)} - \widehat{b}_{3(4k_1)}\right)}{(1+4k_1^2)^2(1+16k_1^2)}\Bigg]. \tag{87}$$

Applying the canonical transformation

$$\begin{aligned} J_2 &= \widetilde{I}_3 - 2\widetilde{I}_2, & \widetilde{\psi}_2 &= \theta_3 - 2\theta_2, \\ J_3 &= \widetilde{I}_2, & \widetilde{\psi}_3 &= \theta_2, \end{aligned} \tag{88}$$

The Hamiltonian becomes

$$\widetilde{H} = H - \omega_2\widetilde{I}_3 = \widetilde{\Omega}_2\widetilde{I}_2 - \beta(\widetilde{I}_3 - 2\widetilde{I}_2)\sqrt{\widetilde{I}_2}\sin\widetilde{\psi}_2 \tag{89}$$

where

$$\widetilde{\Omega}_2 := \omega_3 - 2\omega_2, \tag{90}$$

and

$$\widetilde{I}_3 := J_2 + 2J_3 = \text{constant} \tag{91}$$

is again just proportional to the momentum. (Like the transformation (79) and other coordinate transformations to follow, the transformation (88) can be obtained from a mixed-variable generating function analogous to (78).)

The Hamiltonian (89) describes the behavior of the high-frequency wave of a near-resonant doublet, whereas Hamiltonian (80) describes the behavior of the low-frequency wave. Figure 4 shows the Cartesian phase space variables $\widetilde{P}_2, \widetilde{Q}_2$ for two values of L. We find that the origin is no longer a fixed point, no unstable fixed point ever occurs, and the two stable fixed points on the positive and negative $\widetilde{P}_2$-axis always exist, changing only in position as $\widetilde{\Omega}_2$ is varied.

2. Three-Wave Interactions

A large number of three-wave resonances is possible in our system. This may be seen graphically from Fig. 1. If the point $(k = 0, \omega = 0)$ on one of the negative-ω branches 3 or 4 is translated along branch 1, then its intersection with branch 2 defines a three-wave resonance. In the continuum limit such resonances occur regardless of the equilibrium parameters; for a finite system only discrete values of the parameters will yield exact resonances.

For our symmetric equilibrium there are only two physically distinct possibilities: a resonance involving two slow modes and one fast mode (where branch 4 connects branches 1 and 2), or two fast modes and one slow one (where branch 3 connects branches 1 and 2). Note that if the equilibrium parameters are such as to allow a resonance of either type, then there is another resonance given by the reflection of these modes about the k-axis (i.e., a resonance involving branches 1, 2 and 3 will be accompanied by a resonance involving branches 3, 4 and 1.) This degeneracy is a result of the symmetry which was included for analytical simplicity; general equilibria will yield isolated resonances.

Resonances involving modes of different energy signature result in explosive instability. If we go to a reference frame moving with speed u, the signed frequencies ω go to $\omega - ku$. (This does not affect the resonance condition (75).) The quantity $\frac{\partial \varepsilon(k,\omega-ku)}{\partial \omega}$ is invariant under frame shift, but the energy signature $(\omega - ku)\frac{\partial \varepsilon(k,\omega-ku)}{\partial \omega}$ is not. Thus if $\omega - ku$ has opposite sign from ω (i.e., if the branch crosses the k-axis) then the energy signature of the mode changes. If there exists a reference frame where all modes of a triplet have the same energy signature, then conservation of momentum and energy imply nonlinear stability (i.e., stability under the linear dynamics and where only limited growth is possible for sufficiently small perturbations).[12] It can be shown for general three-wave interactions that if and only if the wave of highest frequency has energy signature opposite in sign to that of the other two waves, then no such reference frame exists.[13] It is easily seen from the dispersion diagram for our system that this frequency relation holds if and only if the coupling is between one positive energy wave and two negative energy waves. These resonances therefore lead to explosive instability, while those involving two positive energy modes and one of negative energy will exhibit only decay instability and limited growth.

We now discuss three-wave resonances involving modes from different branches of the dispersion relation. We first look at "negative energy resonances" (involving two negative energy modes and one of positive energy) and then at "positive energy resonances" (two positive energy modes and one of negative energy).

Negative Energy Resonance Consider a three-wave negative energy resonance, involving the modes shown in Fig. 5. Modes J_2, J_4 and J_5 have wavenumbers $2k_1$, $2k_1$ and $4k_1$, respectively. (The labelling convention is chosen so that we may later couple these modes to those of Fig. 2 without changing mode labels.) The Hamiltonian is given by

$$H = \omega_2 J_2 - \omega_4 J_4 - \omega_5 J_5 + \gamma\sqrt{J_2 J_4 J_5}\sin(\theta_2 + \theta_4 + \theta_5), \tag{92}$$

where

$$\begin{aligned}\gamma = \frac{2k_1^2}{\sqrt{\pi}\Lambda}\Big[&-\widehat{a}_{2(2k_1)}\,\widehat{a}_{4(4k_1)}\,\widehat{b}_{1(2k_1)} + \widehat{a}_{1(2k_1)}\,\widehat{a}_{2(4k_1)}\,\widehat{b}_{2(2k_1)}\\ &- \widehat{a}_{2(2k_1)}\,\widehat{a}_{3(2k_1)}\,\widehat{b}_{2(4k_1)} + \widehat{a}_{2(4k_1)}\,\widehat{a}_{4(2k_1)}\,\widehat{b}_{3(2k_1)}\\ &- \widehat{a}_{3(2k_1)}\,\widehat{a}_{4(4k_1)}\,\widehat{b}_{4(2k_1)} + \widehat{a}_{1(2k_1)}\,\widehat{a}_{4(2k_1)}\,\widehat{b}_{4(4k_1)}\\ &+ \frac{\left(\widehat{b}_{1(2k_1)} - \widehat{b}_{3(2k_1)}\right)\left(\widehat{b}_{2(2k_1)} - \widehat{b}_{4(2k_1)}\right)\left(\widehat{b}_{2(4k_1)} - \widehat{b}_{4(4k_1)}\right)}{(1+4k_1^2)^2(1+16k_1^2)}\Big].\end{aligned} \tag{93}$$

The canonical transformation from the variables $(J_2, J_4, J_5, \theta_2, \theta_4, \theta_5)$ to the resonance variables $(I_3, I_4, I_5, \psi_3, \psi_4, \psi_5)$ is given by

$$\begin{aligned} J_2 &= I_3, & \psi_3 &= \theta_2 + \theta_4 + \theta_5,\\ J_4 &= I_4 + I_3, & \psi_4 &= \theta_4,\\ J_5 &= I_5 + I_3, & \psi_5 &= \theta_5, \end{aligned} \tag{94}$$

which yields the new Hamiltonian

$$\overline{H} = H + \omega_4 I_4 + \omega_5 I_5 = \Omega_3 I_3 + \gamma\sqrt{I_3(I_4+I_3)(I_5+I_3)}\sin\psi_3, \tag{95}$$

where

$$\Omega_3 := \omega_2 - \omega_4 - \omega_5, \quad I_4 := J_4 - J_2 = \text{constant}, \qquad I_5 := J_5 - J_2 = \text{constant}. \tag{96}$$

(The wave momentum is just $-2k_1(I_4 + I_5)$.)

The negative energy resonance condition $\omega_2 - \omega_4 - \omega_5 = 0$ is relatively insensitive to the value of L; the main control parameter is now v. For a given L, we may choose v to yield either exact resonance or a stable region of a desired size.

In Fig. 6 we plot Q_3 vs. P_3 for $L = 5000$ and two values of v, for the special case $I_4 = I_5 = 0$ (corresponding to equal mode amplitudes $J_2 = J_4 = J_5$). Figure 6(a) illustrates the case of exact resonance. All orbits (except for the equilibrium point $P_3 = Q_3 = 0$) go to infinity. Figure 6(b) shows the detuned case $\Omega_3 \neq 0$, with a small stable region around the equilibrium point and an accompanying unstable fixed point on the Q_3-axis. Orbits outside the separatrix still experience explosive growth, while those inside execute stable oscillations. The size of the stable region increases with $\left|\frac{\Omega_3}{\gamma}\right|$.

When one of the constants I_4 or I_5 is nonzero (i.e., either J_4 or J_5 has amplitude different from J_2), the origin $I_3 = 0$ will still be a fixed point, but will only become stable for $|\Omega_3| > \sqrt{I_4}\text{or}\sqrt{I_5}$. Thus there is a range of nonzero Ω_3 that yields instability at the origin. If both I_4 and I_5 are nonzero, (i.e., neither J_4 nor J_5 is equal to J_2), then no fixed points exist for small Ω_3. When $|\Omega_3|$ is large enough, a pair of fixed points (stable and unstable) appear, but the stable point is shifted off of the origin $I_3 = 0$.

Positive Energy Resonance We now consider a three-wave resonance involving two positive energy waves and one negative energy wave. This resonance will lead to decay instability just as in the case of two-wave interactions. An important difference is that a given wave triplet can be in *exact* resonance for some v; however, the resonance condition is strongly dependent on v.

We consider the modes shown in Fig. 7. The wavenumbers of J_2, J_6 and J_7 are $2k_1$, k_1 and $3k_1$, respectively. Mode J_2 is the same one featured in Figs. 2 and 5; the numbering of the other two modes was chosen to avoid confusion with modes already discussed. The Hamiltonian is given by

$$H = \omega_2 J_2 + \omega_6 J_6 - \omega_7 J_7 + \delta\sqrt{J_2 J_6 J_7}\sin(\theta_2 - \theta_6 + \theta_7), \tag{97}$$

where

$$\delta = \frac{\sqrt{6}k_1^2}{2\sqrt{\pi}\lambda}\Big(\widehat{a}_{3(2k_1)}\,\widehat{a}_{4(3k_1)}\,\widehat{b}_{3(k_1)} - \widehat{a}_{1(2k_1)}\,\widehat{a}_{2(3k_1)}\,\widehat{b}_{1(k_1)}$$

$$+\,\widehat{a}_{1(k_1)}\,\widehat{a}_{4(3k_1)}\,\widehat{b}_{1(2k_1)} - \widehat{a}_{2(k_3)}\,\widehat{a}_{3(k_1)}\,\widehat{b}_{3(2k_1)}$$

$$+\,\widehat{a}_{1(k_1)}\,\widehat{a}_{3(2k_1)}\,\widehat{b}_{2(3k_1)} - \widehat{a}_{1(2k_1)}\,\widehat{a}_{3(k_1)}\,\widehat{b}_{4(3k_1)}$$

$$+\frac{\left(\widehat{b}_{3(k_1)} - \widehat{b}_{1(k_1)}\right)\left(\widehat{b}_{3(2k_1)} - \widehat{b}_{1(2k_1)}\right)\left(\widehat{b}_{4(3k_1)} - \widehat{b}_{2(3k_1)}\right)}{(1+k_1^2)(1+4k_1^2)(1+9k_1^2)}\Bigg). \tag{98}$$

Analysis of this Hamiltonian proceeds exactly as in the three-wave negative energy case. The transformation

$$\begin{aligned} J_2 &= I_2, & \psi_2 &= \theta_2 - \theta_6 + \theta_7, \\ J_6 &= I_6 - I_2, & \psi_6 &= \theta_6, \\ J_7 &= I_7 + I_2, & \psi_7 &= \theta_7, \end{aligned} \tag{99}$$

yields

$$\overline{H} = H + \omega_6 I_6 - \omega_7 I_7 = \Omega_4 I_2 + \delta\sqrt{I_2(I_6 - I_2)(I_7 + I_2)}\sin\psi_2 \tag{100}$$

where

$$\Omega_4 := \omega_2 - \omega_6 - \omega_7 \tag{101}$$

and

$$I_6 = J_6 + J_2 = \text{constant}, \quad I_7 = J_7 - J_2 = \text{constant}. \tag{102}$$

The motion will be bounded by the restriction $I_2 < I_6$. If $I_7 = 0$ ($J_2 = J_7$), then the Hamiltonian has exactly the form of Hamiltonian (80), and the same analysis applies, except that we may now have exactly $\Omega_4 = 0$. When $I_7 > 0$, we find the same behavior as that of Hamiltonian (89). The phase space is shown in Fig. 8 for $\Omega_4 = 0$. The topologies of Figs. 8(a) and 8(b) look similar, but in the latter case there is no fixed point at the origin. We could also have $I_7 < 0$, corresponding to $J_2 > J_7$; this would imply a forbidden region around the origin of the P_2-Q_2 plane.

B. Multiple Resonances and Chaos

For the integrable systems just considered, small-amplitude motion about the stable fixed points will be stable for all time. Thus when a negative energy resonance is detuned, only motions with large enough amplitude (outside the separatrix of Fig. 6) will exhibit explosive instability. However, if we consider coupling to other modes via other nonlinear terms in the Hamiltonian, then chaotic motion will generally arise. In its mildest manifestation, this chaotic behavior will occur in thin layers near separatrices. Thus the separatrix between the stable and unstable orbits will be "smeared out." When chaotic motion is more widespread, many of the invariant curves within the stable region my be destroyed, effectively decreasing the size of the region. Moreover, in any chaotic system of more than two degrees of freedom, trajectories can make their way across the "stable" islets via Arnold diffusion,[19] so that a small perturbation may grow until it reaches the separatrix. The speed of this transport may vary widely depending on the how nearly resonant the various waves are. We now examine this phenomenon in detail.

1. General Discussion

Hamiltonians of the form

$$H = \omega_1 J_1 + \omega_2 J_2 + \omega_3 J_3 + \epsilon \Big[\alpha F(J_1, J_2, J_3) \cos(m_1\theta_1 + m_2\theta_2 + m_3\theta_3) + \beta G(J_1, J_2, J_3) \cos(n_1\theta_1 + n_2\theta_2 + n_3\theta_3)\Big], \qquad (103)$$

where F and G are higher-than-linear functions of the J_i, have been discussed by Contopoulos.[31] A particular case involving exact resonance in both nonlinear terms (i.e., $m_1\omega_1 + m_2\omega_2 + m_3\omega_3 = n_1\omega_1 + n_2\omega_2 + n_3\omega_3 = 0$) was studied numerically by Ford and Lunsford[32] and analytically by Kummer.[33] It was shown that, in general, globally chaotic motion occurs for arbitrarily small but nonzero α and β. This involves the breaking of most invariant surfaces and the consequent wandering of orbits over most of the energy surface. Contopoulos examined the transition to this situation from the nonresonant case. One exact constant of the motion besides H clearly exists since there are only two combinations of angles in the Hamiltonian. If only one resonance condition is satisfied (e.g., $\Omega_1 := m_1\omega_1 + m_2\omega_2 + m_3\omega_3 = 0$), then another approximate constant can be constructed, indicating the existence of invariant curves over most of the phase space. In this case chaotic motion will be confined to thin layers around broken separatrices. Very slow diffusion may occur along these thin layers; general estimates of the diffusion rate by Nekhoroshev[34] have been specialized by others[35] to the simple case of coupled linear oscillators that we are considering. As the other resonance condition is approached ($\Omega_2 := n_1\omega_1 + n_2\omega_2 + n_3\omega_3 \to 0$) and the "effective perturbation" ϵ/Ω_2 becomes large enough, then the approximate constant cannot be constructed, and the widespread dissolution of invariant surfaces occurs. This allows very fast transport in phase space.

2. Multiple Resonances in the Counterstreaming Plasma Problem

We now consider what sort of multiple resonances occur in our counterstreaming ion model. We have seen that exact three-wave resonances are possible, involving either decay instability ("positive energy resonance") or explosive instability ("negative energy resonance"). In addition, long-wavelength modes from the same branch of the dispersion relation will interact strongly due to the near-resonance that always exists between them. Whenever we include more than one nonlinear term in the Hamiltonian, we expect to find chaotic behavior originating in thin layers near separatrices. One result is that the well-defined stability boundary associated with a single negative energy resonance will be destroyed. We then expect that perturbations that were within the unperturbed separatrix may now become explosively unstable by crossing the chaotic layer. Another important consequence involves the breaking of separatrices associated with the positive energy resonances. Since these separatrices involve confined motion, the chaotic orbits that replace them can provide a long-time driver for chaotic transport through the phase space. When the system includes a negative energy triplet coupled to one or more such positive energy resonances, this chaotic motion may allow orbits that are initially deep within the "stable" negative energy islet to grow in amplitude until explosive instability occurs.

The extent of these chaotic regions may range from very thin layers to the entire phase space, depending on how many positive energy resonances or near-resonances occur. The two-wave near-resonance conditions may be arbitrarily well-satisfied depending only on the wavelengths, so that we would expect the possibility of approaching global chaos. This effect may be even stronger if a three-wave positive energy resonance condition is satisfied, which can occur for modes of any wavelength. With most or all invariant surfaces destroyed, orbits may reach any point in phase space via "thick-layer" diffusion. If positive energy resonance conditions are not nearly enough satisfied, then stochasticity will be confined to thin layers. Transport across large distances in the phase space may then be expected via Arnold diffusion (although on slow time scales) if the Hamiltonian has three or more degrees of freedom. We investigate these issues numerically in Part II of this work.

V. Conclusions

We have described a simple plasma system that, even when linearly stable, supports negative-energy waves which lead to explosive instability via resonant wave-wave interactions. Using a Hamiltonian formulation, these interactions were illustrated for cases where only a single resonance was considered. In Part II of this work[1] we turn to numerical modelling to study the behavior of the system when several resonances or near-resonances are considered. The Hamiltonian approach used here to describe the interactions will also guide us in the development of simple, fast numerical algorithms. This system will be shown to exhibit strongly chaotic behavior under very general circumstances, which will alter the thresholds for explosive instability that were described for the single-resonance case.

Other physical systems can also be studied using the methods and concepts described here. The plasma equilibrium considered was chosen partly because it allowed analytical reduction to the simple action-angle form (64); this is not possible for all systems. Nevertheless, the $\delta^2 F$ arguments addressing nonlinear stability are widely applicable, and the physical processes considered are quite common, so it is hoped that the results obtained will spur further investigations in these areas.

Appendix A. Transformation of the Bracket

The type of transformation described here was carried out by Gardner[36] in describing the Hamiltonian structure for the Korteweg-de Vries Equation.

We begin with Eq. (36):

$$\{\mathcal{F},\mathcal{G}\} = \Lambda^{-1}\sum_{\alpha=\pm}\int_0^L dx\left[\frac{\delta\mathcal{G}}{\delta v_\alpha}\frac{\partial}{\partial x}\frac{\delta\mathcal{F}}{\delta n_\alpha} - \frac{\delta\mathcal{F}}{\delta v_\alpha}\frac{\partial}{\partial x}\frac{\delta\mathcal{G}}{\delta n_\alpha}\right].$$

In terms of the perturbed quantities δn_α and δv_α, this becomes

$$\{\mathcal{F},\mathcal{G}\} = \Lambda^{-1}\sum_{\alpha=\pm}\int_0^L dx\left[\frac{\delta\mathcal{G}}{\delta(\delta v_\alpha)}\frac{\partial}{\partial x}\frac{\delta\mathcal{F}}{\delta(\delta n_\alpha)} - \frac{\delta\mathcal{F}}{\delta(\delta v_\alpha)}\frac{\partial}{\partial x}\frac{\delta\mathcal{G}}{\delta(\delta n_\alpha)}\right]. \tag{A1}$$

Now if $u = \sum_{m=-\infty}^{\infty} u_m e^{\imath k_m x}$ is one of our variables δn_α or δv_α, then

$$\frac{\partial f}{\partial u_m} = \int_0^L \frac{\delta\mathcal{F}}{\delta u}\frac{\partial u}{\partial u_m}\,dx = \int_0^L \frac{\delta\mathcal{F}}{\delta u}e^{\imath k_m x}\,dx. \tag{A2}$$

We then have for the Fourier expansion of a functional derivative

$$\frac{\delta\mathcal{F}}{\delta u} = \frac{1}{L}\sum_{m=-\infty}^{\infty}\frac{\partial\mathcal{F}}{\partial u_{-m}}e^{\imath k_m x}. \tag{A3}$$

Using this in expression (A1), we have

$$\begin{aligned}
\{\mathcal{F},\mathcal{G}\} &= \sum_{\alpha=\pm}\frac{1}{\Lambda L^2}\int_0^L dx\,\Bigg[\left(\sum_{m=-\infty}^{\infty}\frac{\partial\mathcal{G}}{\partial v^\alpha_{-m}}e^{\imath k_m x}\right)\frac{\partial}{\partial x}\left(\sum_{m=-\infty}^{\infty}\frac{\partial\mathcal{F}}{\partial n^\alpha_{-m}}e^{\imath k_m x}\right) \\
&\qquad\qquad - \left(\sum_{m=-\infty}^{\infty}\frac{\partial\mathcal{F}}{\partial v^\alpha_{-m}}e^{\imath k_m x}\right)\frac{\partial}{\partial x}\left(\sum_{m=-\infty}^{\infty}\frac{\partial\mathcal{G}}{\partial n^\alpha_{-m}}e^{\imath k_m x}\right)\Bigg] \\
&= \sum_{\alpha=\pm}\sum_{m=-\infty}^{\infty}\left(-\frac{\imath k_m}{\Lambda L}\right)\left[\frac{\partial\mathcal{G}}{\partial v^\alpha_{-m}}\frac{\partial\mathcal{F}}{\partial v^\alpha_{m}} - \frac{\partial\mathcal{F}}{\partial v^\alpha_{-m}}\frac{\partial\mathcal{G}}{\partial v^\alpha_{m}}\right] \\
&= \sum_{\alpha=\pm}\sum_{m=1}^{\infty}\left(\frac{\imath k_m}{\Lambda L}\right)\Bigg[\left(-\frac{\partial\mathcal{G}}{\partial v^\alpha_{-m}}\frac{\partial\mathcal{F}}{\partial v^\alpha_{m}} + \frac{\partial\mathcal{F}}{\partial v^\alpha_{-m}}\frac{\partial\mathcal{G}}{\partial v^\alpha_{m}}\right) \\
&\qquad\qquad + \left(\frac{\partial\mathcal{G}}{\partial v^\alpha_{-m}}\frac{\partial\mathcal{F}}{\partial v^\alpha_{m}} - \frac{\partial\mathcal{F}}{\partial v^\alpha_{-m}}\frac{\partial\mathcal{G}}{\partial v^\alpha_{m}}\right)\Bigg].
\end{aligned} \tag{A4}$$

Appendix B. Diagonalization of $\delta^2 F$

We have seen that the quadratic part of the perturbed energy and momentum may be written as

$$\delta^2 F = \frac{1}{2} \sum_{m=1}^{\infty} (\tilde{\mathbf{q}}\mathbf{A}\mathbf{q} + \tilde{\mathbf{p}}\mathbf{B}\mathbf{p})_m \tag{B1}$$

$$\delta^2 P = \frac{1}{2} \sum_{m=1}^{\infty} (\tilde{\mathbf{q}}\mathbf{C}\mathbf{q} + \tilde{\mathbf{p}}\mathbf{C}\mathbf{p})_m \tag{B2}$$

where A, B, and C are defined by Eqs. (55) and (58). For convenience we have define

$$\widehat{k}_m := \frac{k_m}{\sqrt{1+k_m^2}}, \qquad \widehat{v} := v\sqrt{1+k_m^2}. \tag{B3}$$

(Note that $\widehat{k}_1$ and $\widehat{v}$ both depend on the mode number m whereas k_1 and v do not; since the former occur only inside matrices that depend upon a single m, this will not lead to confusion.)

The equations of motion are then

$$\frac{d\mathbf{z}}{dt} = \mathbf{J}_c \nabla H = \mathbf{M}\mathbf{z} \tag{B4}$$

where

$$\mathbf{z} = (q_1, q_2, q_3, q_4, p_1, p_2, p_3, p_4) \tag{B5}$$

and

$$\mathbf{M} = \begin{bmatrix} \mathbf{0}\,\mathbf{B} \\ -\mathbf{A}\;\mathbf{0} \end{bmatrix}. \tag{B6}$$

The eigenvalues of $\mathbf{M}$ are given by

$$\lambda^2 = -\omega^2 \tag{B7}$$

where

$$\omega^2 = \widehat{k}_m^2 \left[\frac{1}{2} + \widehat{v}^2 \pm \sqrt{\frac{1}{4} + 2\widehat{v}^2} \right]. \tag{B8}$$

By taking appropriate linear combinations of the complex eigenvectors of $\mathbf{M}$, eight real eigenvectors are obtained; these provide a basis for a transformation of $\mathbf{z}$ that diagonalizes both $\delta^2 F$ and $\delta^2 P$. The eigenvectors contain eight unknown constants; the requirements (60) and (61) yield eight equations. The transformation decouples into separate transformations for $\mathbf{p}$ and $\mathbf{q}$, given by

$$\mathbf{q} = \mathbf{S}\mathbf{Q}, \qquad \mathbf{p} = \mathbf{T}\mathbf{P} \tag{B9}$$

where

$$\mathbf{S} = \begin{bmatrix} -b_3 & -b_4 & -b_1 & -b_2 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ a_3 & a_4 & a_1 & a_2 \end{bmatrix}, \qquad \mathbf{T} = \begin{bmatrix} a_1 & -a_2 & -a_3 & a_4 \\ -b_3 & b_4 & b_1 & -b_2 \\ a_3 & -a_4 & -a_1 & a_2 \\ b_1 & -b_2 & -b_3 & b_4 \end{bmatrix}. \tag{B10}$$

The matrix elements are given by

$$a_i := \widehat{a}_i \sqrt{\widehat{k}_m \widehat{k}_1 \widehat{v}}, \qquad b_i := \widehat{b}_i / \sqrt{\widehat{k}_m \widehat{k}_1 \widehat{v}}, \tag{B11}$$

where

$$\begin{aligned}
\widehat{a}_1 &:= \frac{\widehat{\omega}_+ + \widehat{v}}{\sqrt{4R\widehat{\omega}_+}}, & \widehat{b}_1 &:= \sqrt{\frac{(\widehat{\omega}_+ - \widehat{v})^2 - \frac{1}{2}}{8R\widehat{\omega}_+}}, \\
\widehat{a}_2 &:= \frac{\widehat{\omega}_- + \widehat{v}}{\sqrt{4R\widehat{\omega}_-}}, & \widehat{b}_2 &:= -\sqrt{\frac{(\widehat{\omega}_- - \widehat{v})^2 - \frac{1}{2}}{8R\widehat{\omega}_-}}, \\
\widehat{a}_3 &:= \frac{\widehat{\omega}_+ - \widehat{v}}{\sqrt{4R\widehat{\omega}_+}}, & \widehat{b}_3 &:= -\sqrt{\frac{(\widehat{\omega}_+ + \widehat{v})^2 - \frac{1}{2}}{8R\widehat{\omega}_+}}, \\
\widehat{a}_4 &:= \frac{\widehat{\omega}_- - \widehat{v}}{\sqrt{4R\widehat{\omega}_-}}, & \widehat{b}_4 &:= \sqrt{\frac{(\widehat{\omega}_- + \widehat{v})^2 - \frac{1}{2}}{8R\widehat{\omega}_-}},
\end{aligned} \tag{B12}$$

and we have defined

$$R := \sqrt{\frac{1}{4} + 2\widehat{v}^2}, \qquad \widehat{\omega}_\pm := \omega_\pm / \widehat{k}_m. \tag{B13}$$

Here the diagonalization transformation is expressed simply in terms of the a_i and b_i; the utility of introducing the $\widehat{a}_i$ and $\widehat{b}_i$ is that the latter variables are of order unity and depend relatively weakly on k_1 and v. This transformation then provides the diagonalized forms (60) and (61).

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FIGURE CAPTIONS

Fig. 1. Dispersion relation $\omega(k)$ for the cases (a) $v < 1$, where negative energy modes (branches 3 and 4) are linearly unstable for long wavelengths, and (b) $v > 1$, where all modes are linearly stable.

Fig. 2. Three modes which form two near-resonant doublets.

Fig. 3. Phase space topology for two-wave decay instability described by Hamiltonian (80) with $\Lambda = 100$, $v = 1.4796$, $I_3 = 0.01$ and (a) $L = 1000$; (b) $L = 5000$.

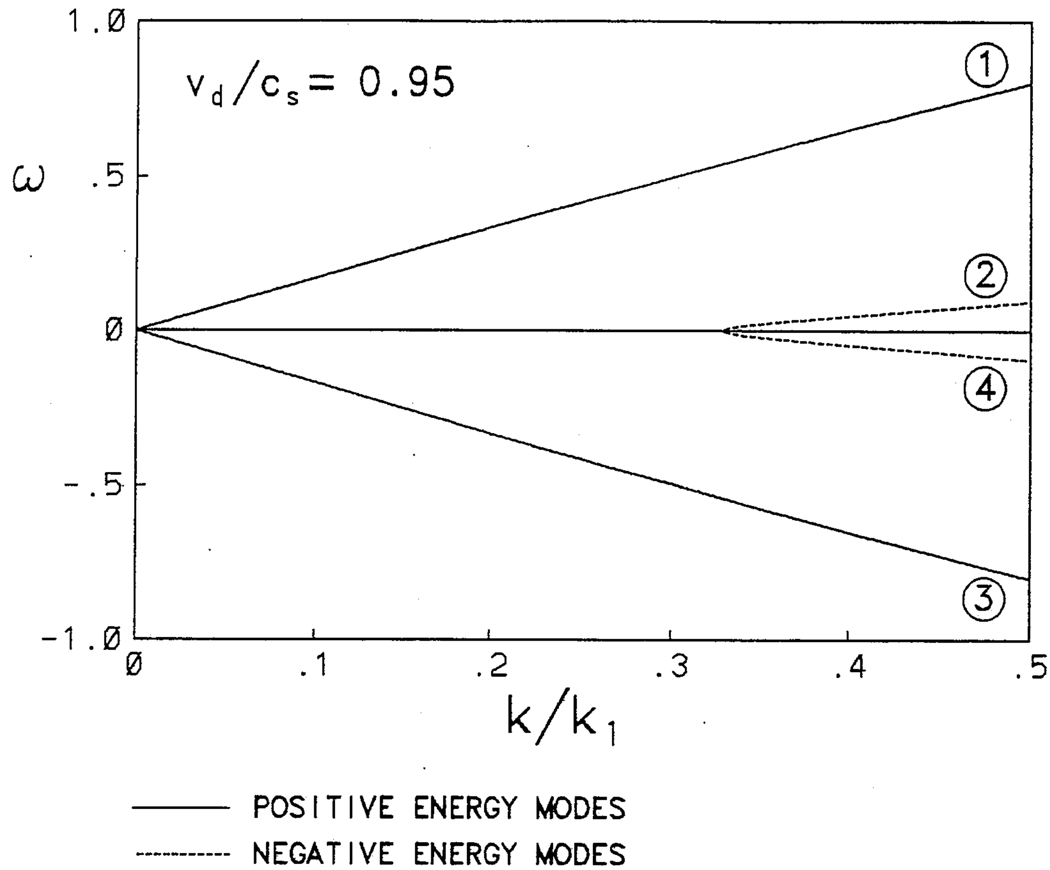
Fig. 4. Phase space topology for two-wave decay instability described by Hamiltonian (89) with $\Lambda = 100$, $v = 1.4796$, $\tilde{I}_3 = 0.02$ and (a) $L = 5000$; (b) $L = 10000$.

Fig. 5. Three modes which form a negative energy resonance.

Fig. 6. Phase space plots for Hamiltonian (95) with $I_4 = I_5 = 0$. $\Lambda = 100$, $L = 5000$, and (a) $v = 1.47959435$ ($\Omega_3 = 0$); (b) $v = 1.47959256$ ($\Omega_3 \neq 0$).

Fig. 7. Three modes which form a positive energy resonance.

Fig. 8. Phase space topology for three-wave decay instability described by Hamiltonian (100), with $\Omega_4 = 0$ and (a) $I_7 = 0$; (b) $I_7 \neq 0$.



1.Ø
.5
Ø
-.5
-1.Ø
ω
$v_d/c_s = 0.95$
1
2
3
4
Ø
.1
.2
.3
.4
.5
k/k_1
POSITIVE ENERGY MODES
NEGATIVE ENERGY MODES

Figure 1.a

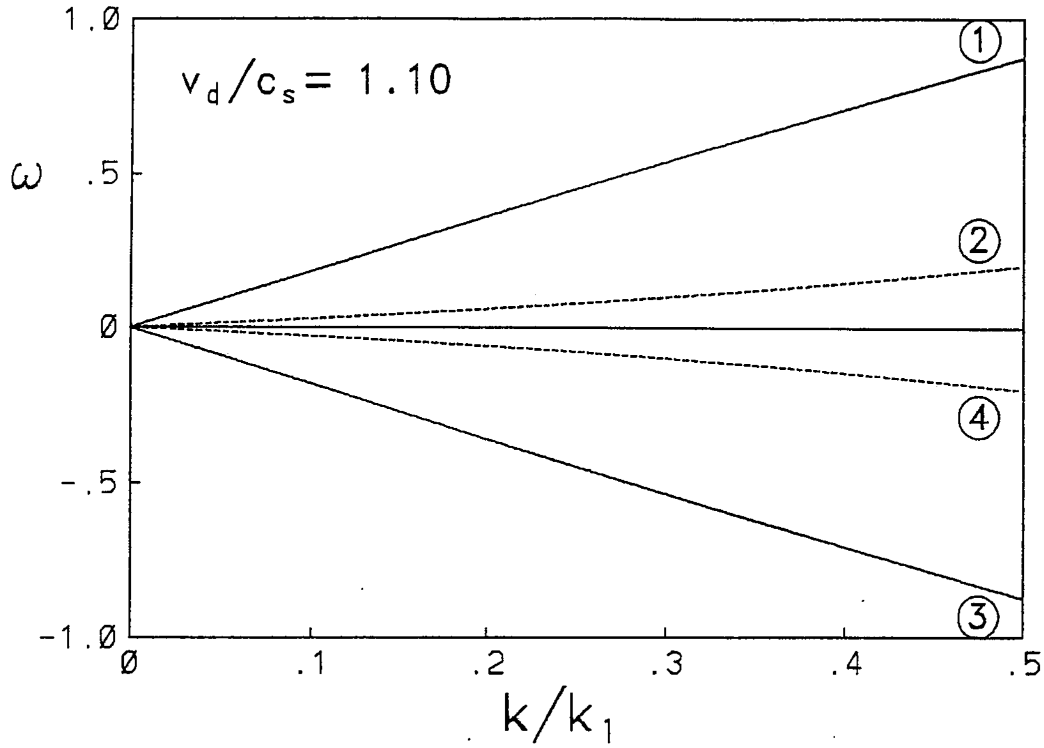


Figure 1.b

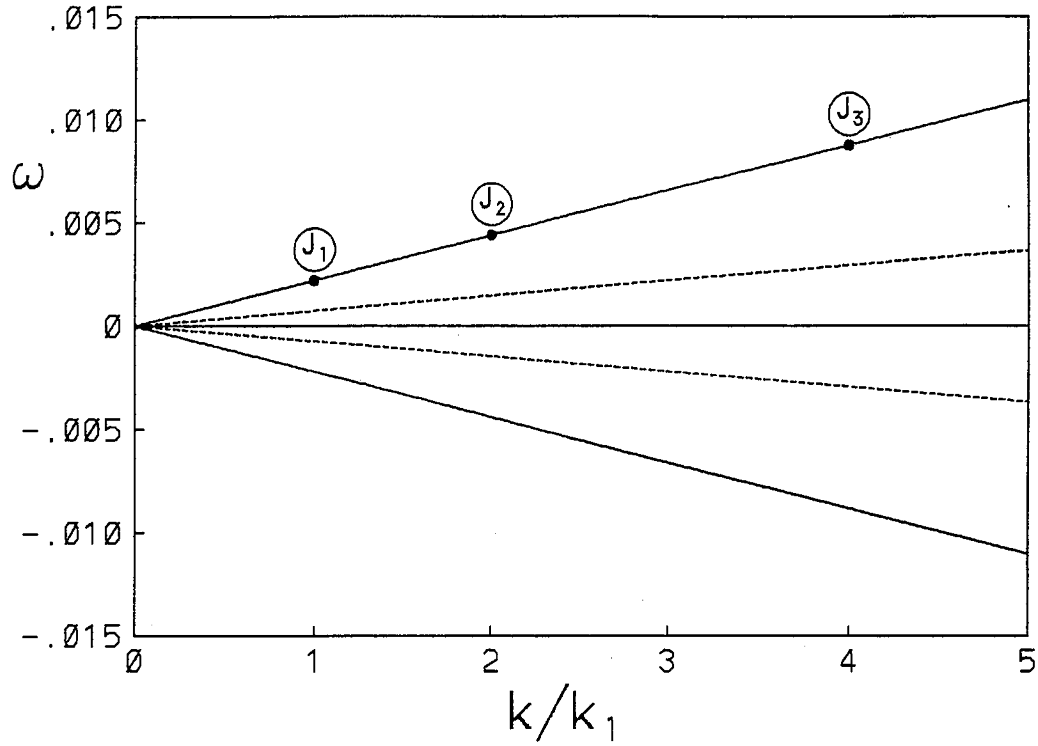
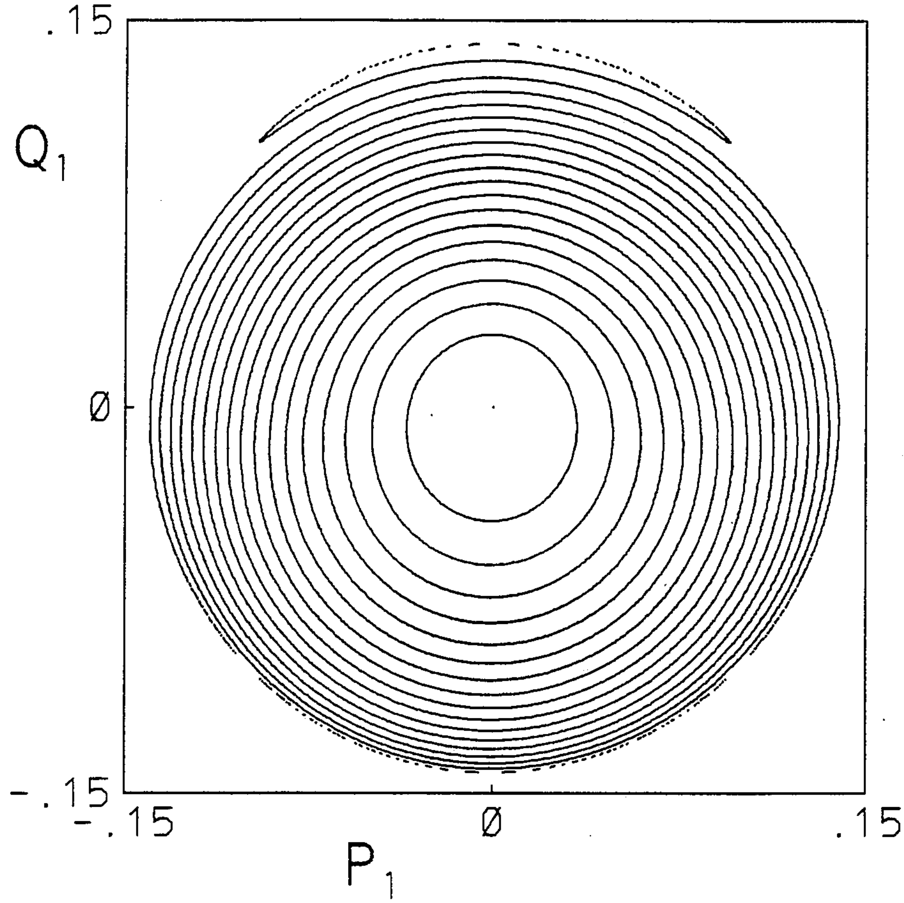


Figure 2



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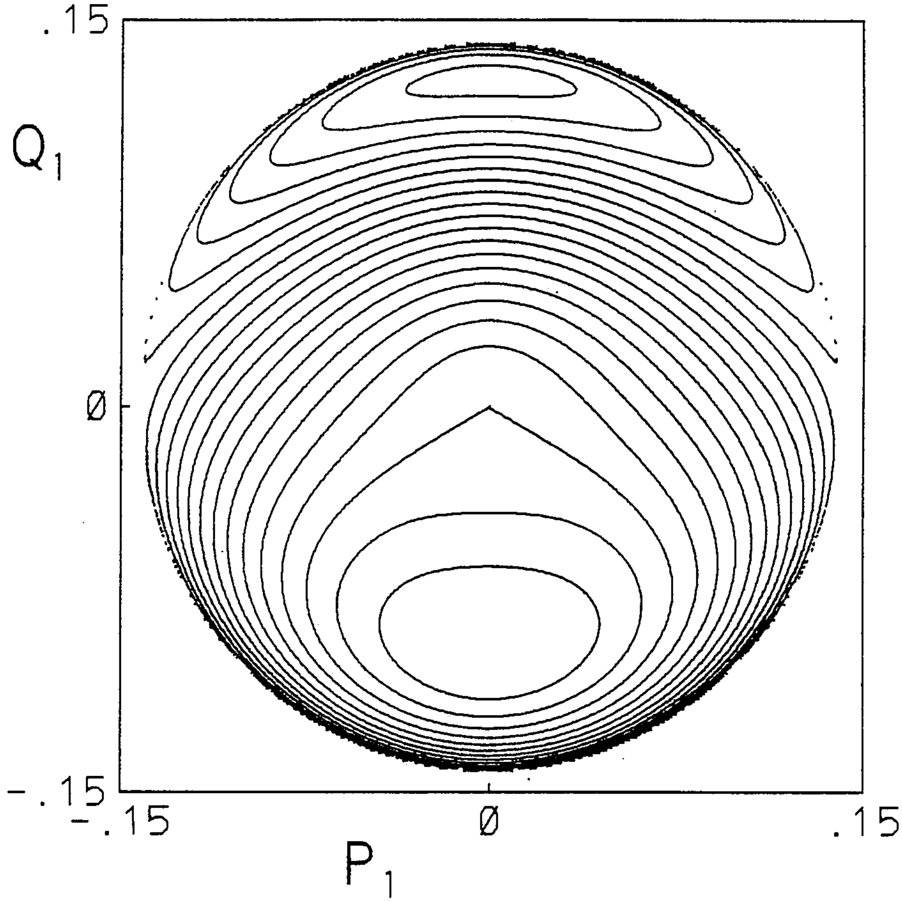


Figure 3.b

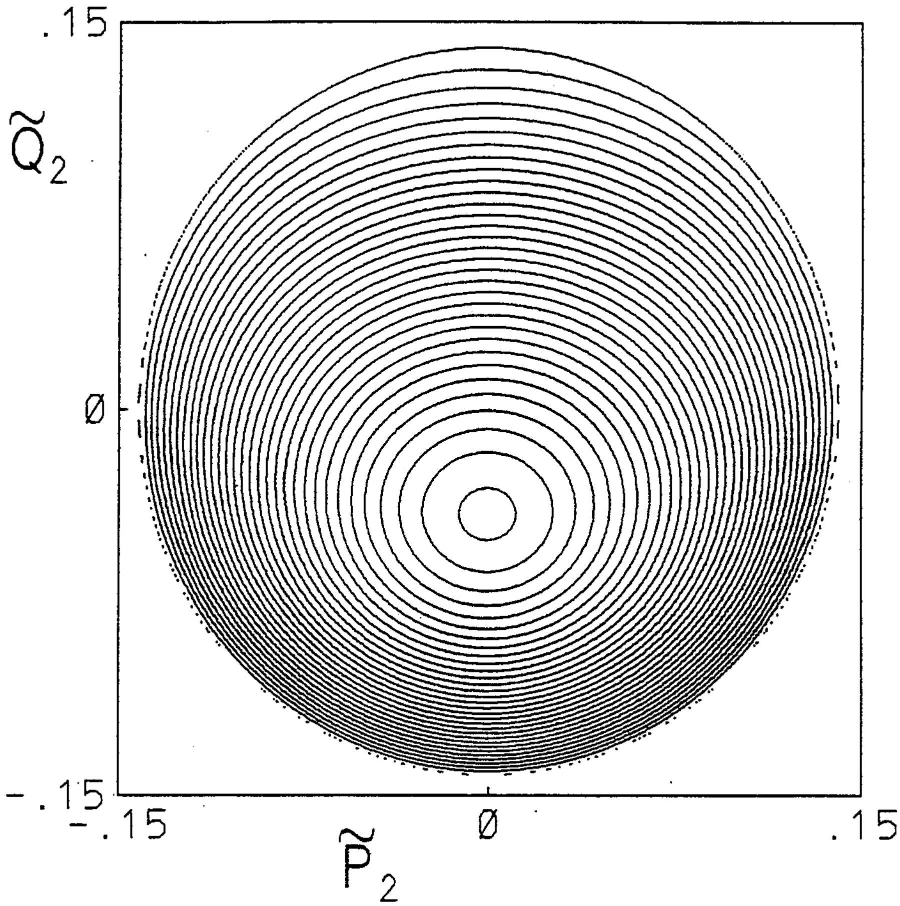


Figure 4.a

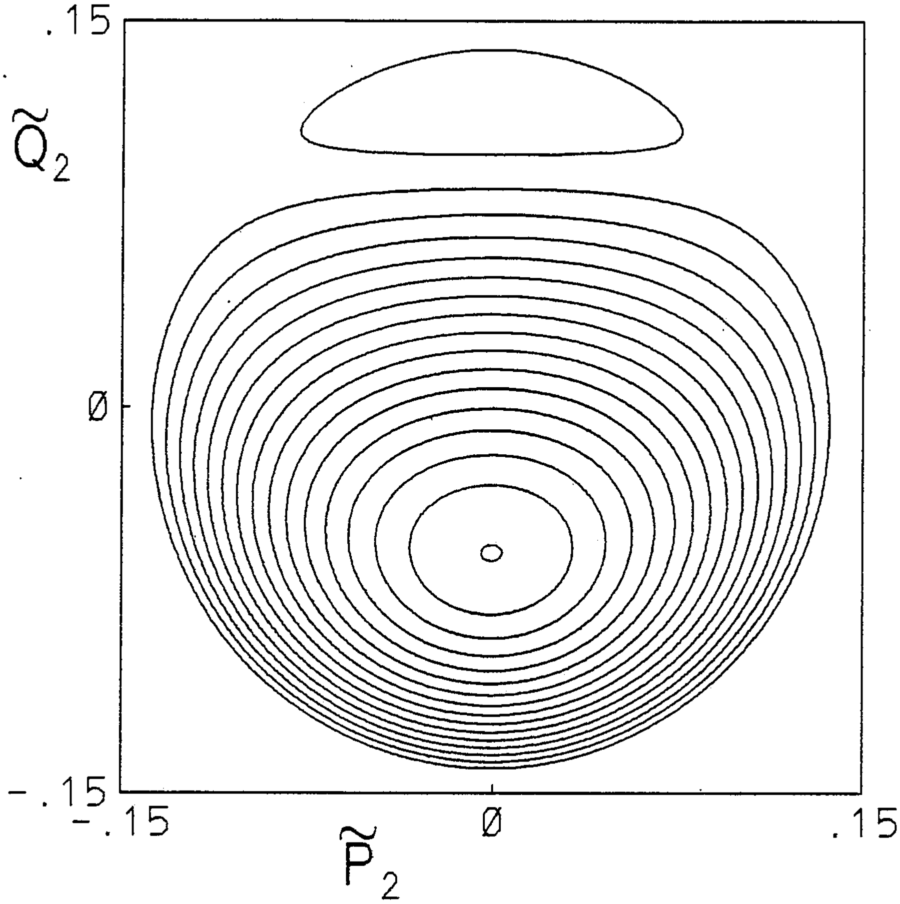


Figure 4.b

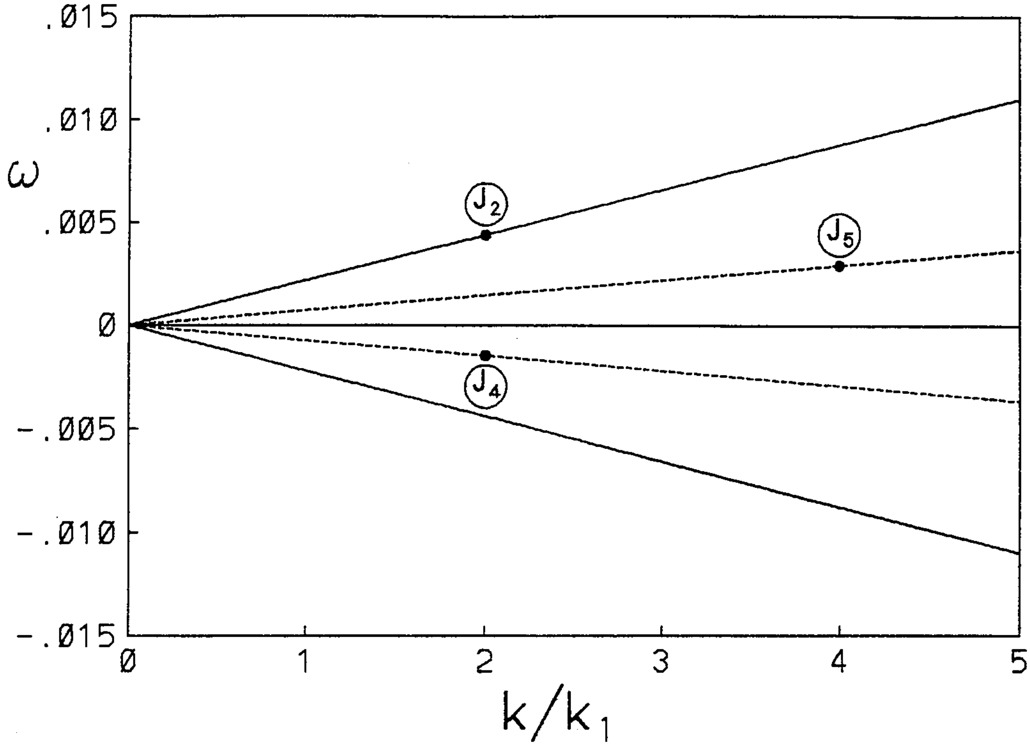


Figure 5

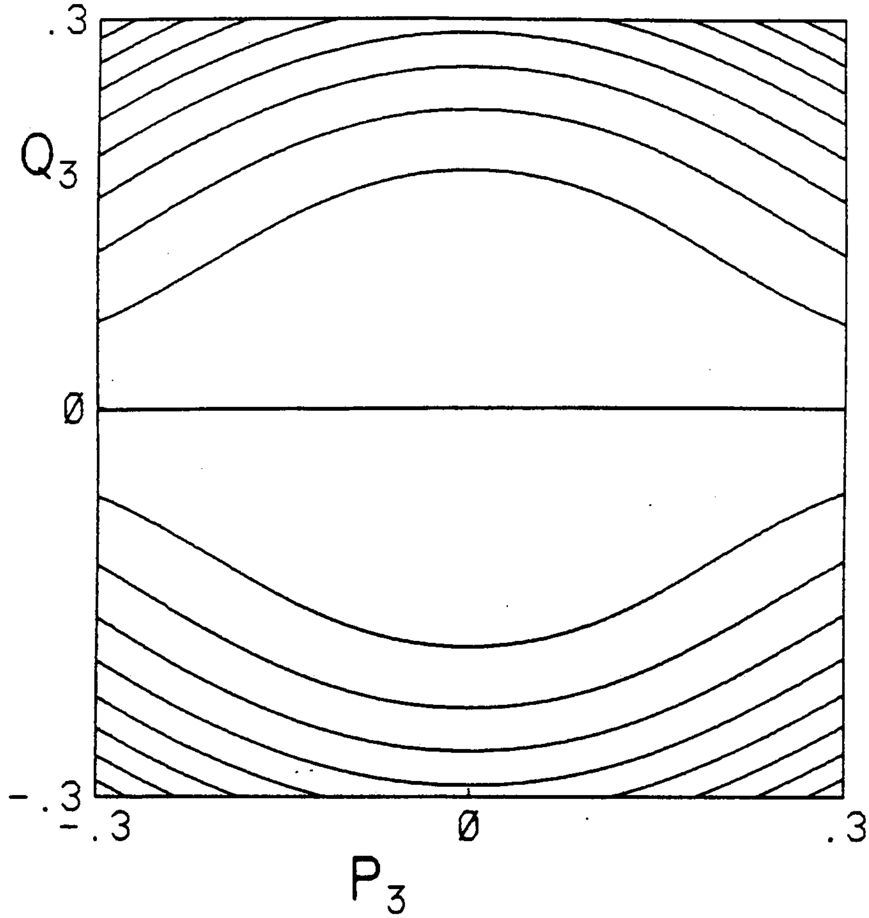
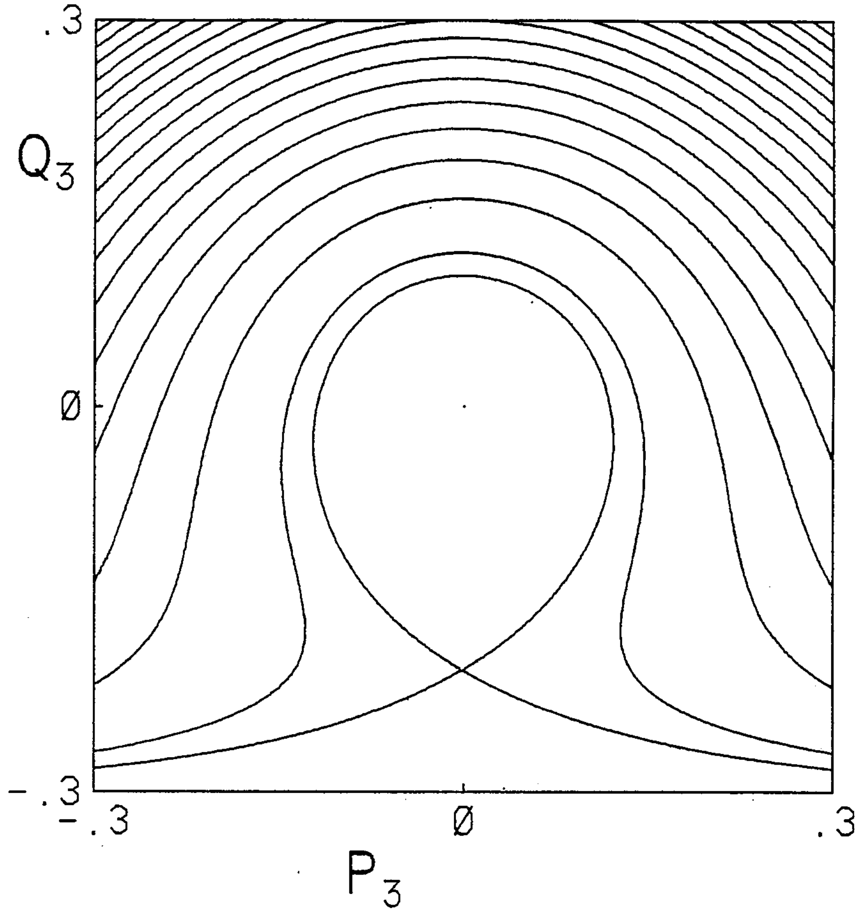


Figure 6.a



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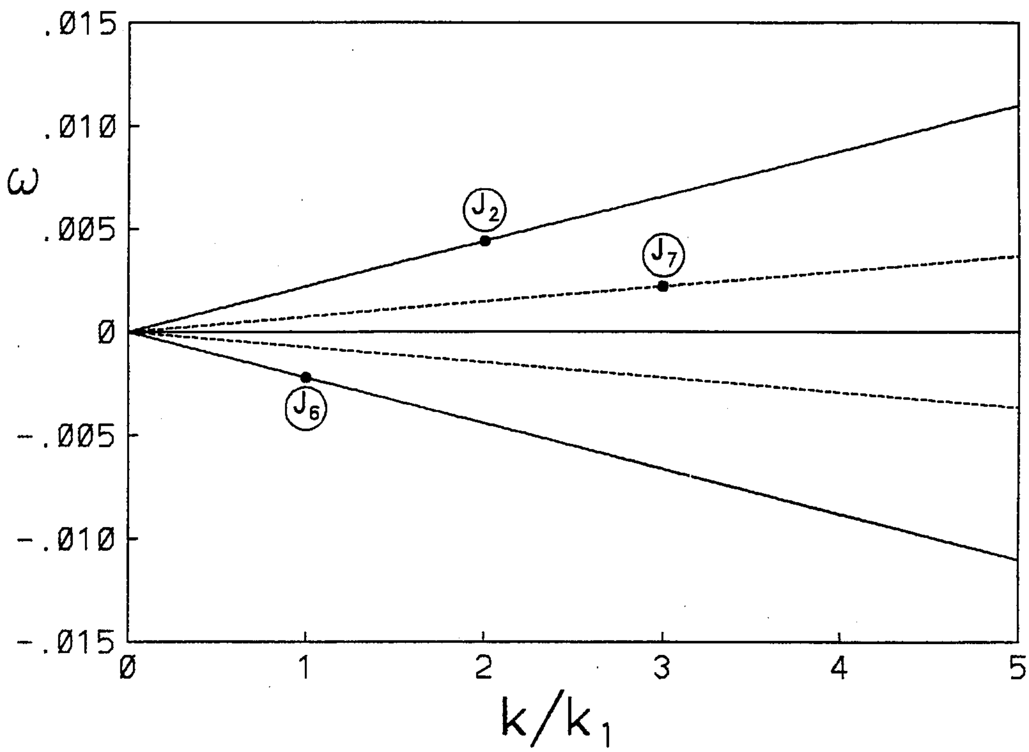

Figure 7

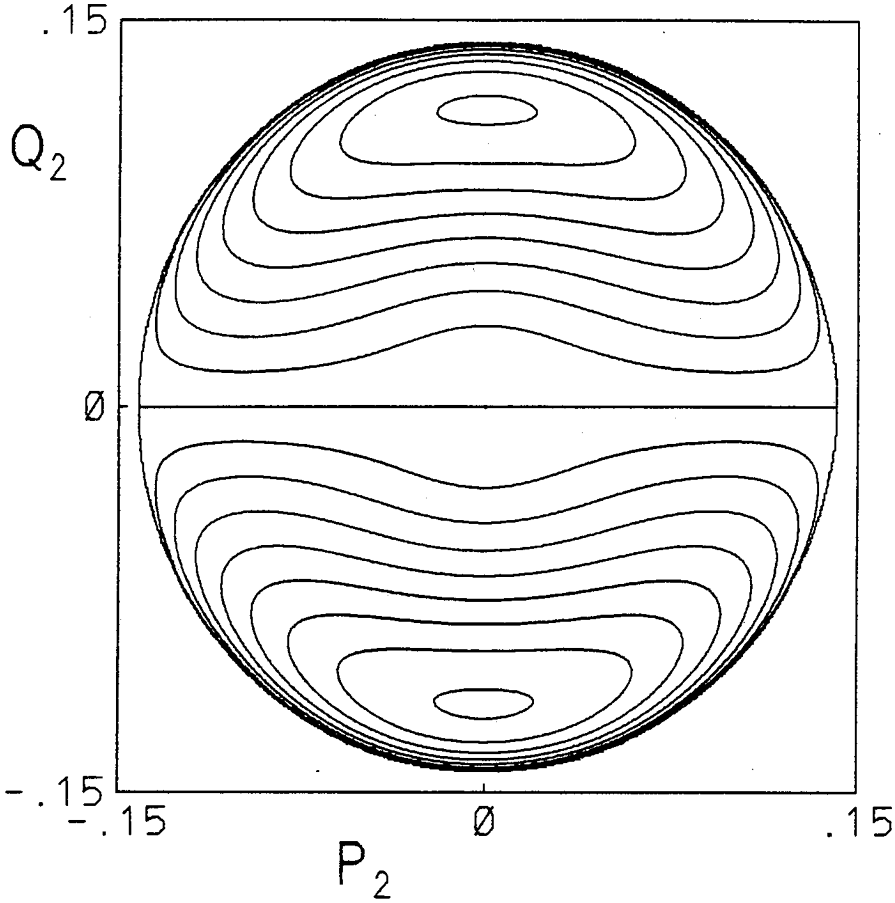

Figure 8.a

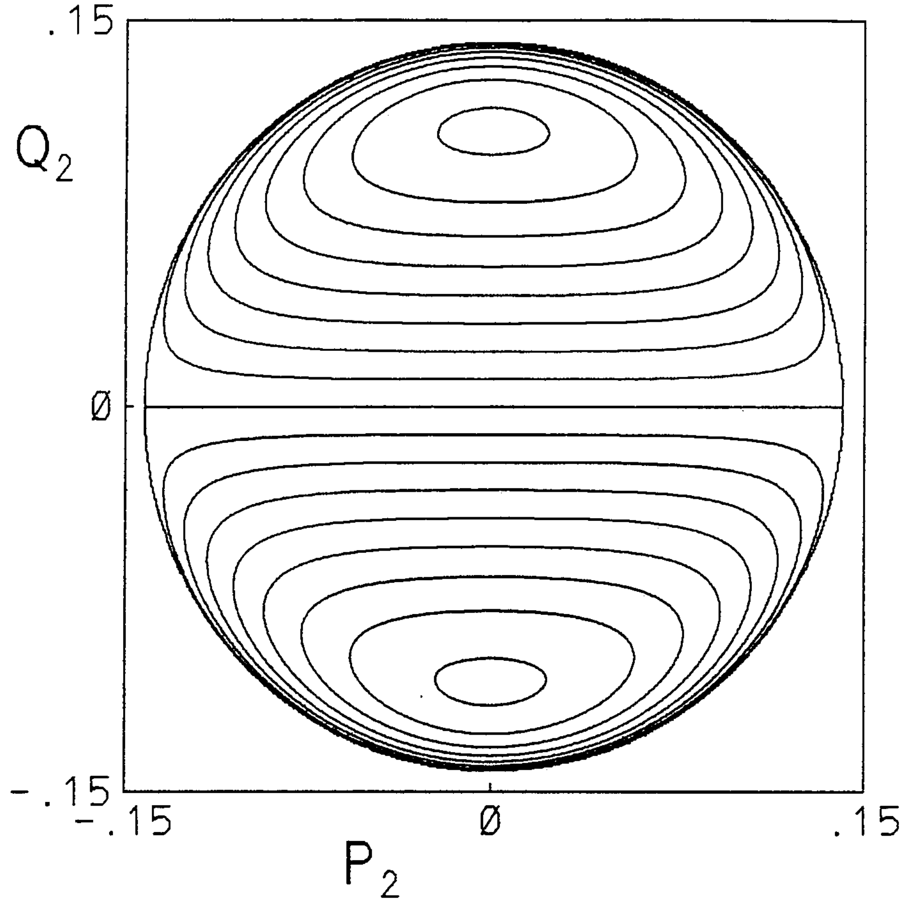

Figure 8.b