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**Nonlinear Symmetric Stability of Planetary Atmospheres**

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The energy–Casimir method is applied to the problem of symmetric stability in the context of a compressible, hydrostatic planetary atmosphere with a general equation of state. Linear stability criteria for symmetric disturbances to a zonally symmetric baroclinic flow are obtained. In the special case of a perfect gas the results of Stevens (1983) are recovered. Nonlinear stability conditions are also obtained that, in addition to implying linear stability, provide an upper bound on a certain positive-definite measure of disturbance amplitude.

## I. INTRODUCTION

A classical problem in geophysical fluid dynamics is that of symmetric stability. The adjective “symmetric” implies that both the equilibrium flow and allowed perturbations are independent of one coordinate. Symmetric instability is a kind of baroclinic instability that can occur even when the atmosphere is both statically and inertially stable (positively stratified with a monotonically increasing circulation). It has primarily been considered in the context of two physical applications: (i) mesoscale dynamics and (ii) planetary circulations. Sometimes referred to as slantwise convection, this instability produces symmetric convective rolls that have been identified with frontal rainbands in the mesoscale context and with axisymmetric cells in the planetary context.

Linear criteria for symmetric stability were first derived by Fjørtoft (1950). These criteria have recently been extended to finite amplitude in the case of the  $f$ -plane Boussinesq equations — relevant to the mesoscale context — by Cho, Shepherd & Vladimirov (1993), using the so-called energy–Casimir stability method. This method relies on the construction of a positive-definite invariant for disturbances to a given equilibrium, using the conservation of total energy together with the material conservation of entropy and angular momentum.

The present work extends the results of Cho *et al.* (1993) to planetary circulations, including the effects of both spherical geometry and compress-

ibility, in terms of a general equation of state. Beyond these additional physical effects, certain novel mathematical features are exhibited in the analysis: multi-valued inverse equilibrium profiles and a moving domain of integration. The latter leads to some technical difficulties with convexity.

The equations describing the compressible hydrostatic model, which assumes zonal (azimuthal) symmetry and neglects diabatic processes, are presented in §2. It is shown that the total energy and certain functionals of the angular momentum, entropy, and potential vorticity are conserved in time. In §3, we demonstrate that a general steady baroclinic zonal flow satisfying thermal-wind balance constitutes an equilibrium of the model system and corresponds to a conditional extremal of the total energy. In §4, we examine the linear stability of these baroclinic flows and obtain conditions equivalent to the classical criteria of inertial stability, static stability, and positivity of the product of the Coriolis parameter and the potential vorticity. In the special case of a perfect (ideal) gas, our linear criteria reduce to those of Stevens (1983). The nonlinear extension of this result, based on convexity arguments, is described in §5. Although we are not able to prove Liapunov stability, we do derive a rigorous upper bound on disturbance amplitude that is expressible solely in terms of initial amplitudes, with the bound going to zero as the initial disturbance tends to zero. We find that the nonlinear stability criteria imply linear stability and are expressible as generalizations of the linear symmetric stability conditions.

## II. GOVERNING EQUATIONS

We consider the model of hydrostatic compressible flow known in meteorology as the primitive equations. Denoting the latitude by  $\lambda$ , the longitude by  $\varphi$  and the pressure by  $p$ , we describe a longitudinally symmetric spherical shell at radius  $r = a$ , of thickness  $\Delta r \ll a$ , rotating at constant angular velocity  $\Omega$ . Using pressure coordinates  $(\lambda, \varphi, p)$  with corresponding velocities  $(u, v, \omega)$ , the model equations for axisymmetric flow are [e.g., see Lorenz (1967)]:

$$\frac{Du}{Dt} - \frac{\tan \varphi}{a} uv - 2\Omega v \sin \varphi = 0, \quad (1a)$$

$$\frac{Dv}{Dt} + \frac{\tan \varphi}{a} u^2 + 2\Omega u \sin \varphi = -\frac{1}{a} \Phi_\varphi, \quad (1b)$$

$$\frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (v \cos \varphi) + \frac{\partial \omega}{\partial p} = 0, \quad (1c)$$

$$\frac{Ds}{Dt} = 0, \quad (1d)$$

$$\Phi_p = -\frac{1}{\rho}. \quad (1e)$$

Here  $\rho$  is the fluid density,  $s$  is the specific entropy, and  $\Phi \doteq gr$  is the geopotential, where the gravitational acceleration  $g$  is treated as a constant. (We emphasize definitions with the notation “ $\doteq$ ”.) The upper surface is taken to be an isobaric surface on which, for the sake of simplicity, we suppose that  $p$  approximately vanishes. [This effectively requires that the density scale height be much less than the thickness of the shell:  $(\rho^{-1} \partial \rho / \partial r)^{-1} \ll \Delta r$ .] The appropriate boundary conditions are then  $\omega = 0$  on the upper

surface and  $D\Phi/Dt = 0$  on the lower surface (on which  $r = a$ ), where the material or Lagrangian derivative  $D/Dt$  is given by

$$\frac{D}{Dt} \doteq \frac{\partial}{\partial t} + \frac{v}{a} \frac{\partial}{\partial \varphi} + \omega \frac{\partial}{\partial p}. \quad (2)$$

Equation (1a) implies the material conservation of the axial component of the absolute angular momentum

$$m(\varphi, p) \doteq u(\varphi, p)a \cos \varphi + \Omega a^2 \cos^2 \varphi, \quad (3)$$

namely, that  $Dm/Dt = 0$ .

It is common in meteorology to formulate results in terms of the potential temperature

$$\theta(s) \doteq \exp \left( \int \frac{ds}{c_p(s, p_r)} \right), \quad (4)$$

instead of the entropy, where  $c_p > 0$  is the specific heat at constant pressure and  $p_r$  is a fixed but arbitrary reference pressure. However, the stability analysis in this work is most naturally formulated in terms of a generalized potential temperature  $\eta = \eta(s)$  defined by

$$\eta \doteq \int \theta ds = \int c_p d\theta. \quad (5)$$

For a fluid with constant  $c_p$ ,  $\eta$  is proportional to  $\theta$ . In the general situation,  $\eta$  appears to be the optimal variable in which to cast the stability analysis because the second derivative of the enthalpy

$$h(\eta, p) \doteq \mathcal{E}(\eta, p) + \frac{p}{\rho(\eta, p)} \quad (6)$$

with respect to this variable vanishes. Here  $\mathcal{E}(\eta, p)$  is the internal energy and  $\rho = \rho(\eta, p)$  is the equation of state. From the Maxwell relation

$$dh = T ds + \frac{1}{\rho} dp, \quad (7)$$

we see that

$$h_\eta = \frac{T}{\theta}, \quad (8a)$$

$$h_p = \frac{1}{\rho}, \quad (8b)$$

where  $T = T(\eta, p)$  is the temperature. The enthalpy is assumed to have continuous second derivatives so that  $T$  and  $\rho$  are related by

$$\frac{T_p}{\theta} = \frac{\partial}{\partial p} \bigg|_\eta \left( \frac{T}{\theta} \right) = \frac{\partial}{\partial \eta} \bigg|_p \left( \frac{1}{\rho} \right) = -\frac{\rho_\eta}{\rho^2}. \quad (9)$$

The thermodynamic relation

$$\frac{\partial T}{\partial s} \bigg|_p = \frac{T}{c_p} \quad (10)$$

and (5) imply that the second derivative of  $h$  with respect to  $\eta$  vanishes,

$$h_{\eta\eta} = \left( \frac{T}{\theta} \right)_\eta = \frac{T}{c_p \theta^2} - \frac{T \theta_\eta}{\theta^2} = 0. \quad (11)$$

It is also convenient to introduce the potential vorticity  $q \doteq \rho^{-1} \mathbf{Z} \cdot \nabla \theta$ , where in the shallow-layer approximation

$$\mathbf{Z} \doteq \nabla \times (u \hat{\lambda}) + f \hat{r} = \left( \frac{u}{a} \tan \varphi - \frac{1}{a} u_\varphi + 2\Omega \sin \varphi \right) \hat{r} + \frac{\partial u}{\partial r} \bigg|_\varphi \hat{\varphi} \quad (12)$$

is the absolute vorticity associated with an axisymmetric flow. In terms of the radial component of  $\mathbf{Z}$ ,

$$\mathbf{Z} \cdot \hat{\mathbf{r}} = \frac{u}{a} \tan \varphi - \frac{1}{a} u_\varphi + 2\Omega \sin \varphi = -\frac{m_\varphi}{a^2 \cos \varphi}, \quad (13)$$

one may express  $q$  as

$$q = -\frac{g}{c_p} \left( \mathbf{Z} \cdot \hat{\mathbf{r}} \eta_p + \frac{1}{a} u_p \eta_\varphi \right) = \frac{g}{c_p a^2 \cos \varphi} \frac{\partial(m, \eta)}{\partial(\varphi, p)} \equiv \frac{1}{\rho a \cos \varphi} \hat{\lambda} \cdot \nabla \theta \times \nabla m. \quad (14)$$

Using the Lagrangian invariance of  $m$  and  $\eta$  together with (1c), (14) may be used to show that  $Dq/Dt = 0$ .

The total energy for a spherical shell (of volume  $V$ ) of this fluid is given by

$$\bar{\mathcal{H}} \doteq \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_a^{a+\Delta r(\varphi, t)} \rho \left( \frac{u^2 + v^2}{2} + \mathcal{E} + \Phi \right) a^2 \cos \varphi \, dr \, d\varphi. \quad (15)$$

Upon using the hydrostatic relation and integrating by parts (taking  $p = 0$  on the upper surface), we may rewrite  $\bar{\mathcal{H}}$  in pressure coordinates:

$$\begin{aligned} \bar{\mathcal{H}}\{m, v, \eta, p_0\} &= \frac{1}{g} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{p_0(\varphi, t)} H(m, v, \eta; \varphi, p) a^2 \cos \varphi \, dp \, d\varphi \\ &\quad + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} p_0(\varphi, t) a^3 \cos \varphi \, d\varphi, \end{aligned} \quad (16)$$

where

$$H(m, v, \eta; \varphi, p) \doteq \frac{1}{2} \left( \frac{m}{a \cos \varphi} - \Omega a \cos \varphi \right)^2 + \frac{1}{2} v^2 + h(\eta, p) \quad (17)$$

and

$$p_0(\varphi, t) \doteq g \int_a^{a+\Delta r(\varphi, t)} \rho \, dr > 0 \quad (18)$$



is the (time-dependent) surface pressure, determined dynamically from the lower boundary condition  $D\Phi/Dt = 0$ , (1c) and (1e):

$$\frac{\partial p_0}{\partial t} + \frac{v}{a} \frac{\partial p_0}{\partial \varphi} = - \int_0^{p_0(\varphi, t)} \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (v \cos \varphi) dp.$$

For convenience, let us rescale  $\mathcal{H} \doteq g\bar{\mathcal{H}}/a^2$ .

The conservation properties of this system can be demonstrated with the following generalization of the transport theorem. For any quantity  $I$  satisfying

$$\frac{D}{Dt} I = S, \quad (19)$$

(*e.g.*, take  $I = u, v$ , or  $\eta$ ) and any arbitrary function  $f = f(I, p)$ , we see that

$$\begin{aligned} & \frac{d}{dt} \int_{-\pi/2}^{\pi/2} \int_0^{p_0} f(I, p) \cos \varphi dp d\varphi \\ &= \int_{-\pi/2}^{\pi/2} \int_0^{p_0} \left[ f_I (S - \mathbf{v} \cdot \nabla I) + f_p \left( \frac{Dp}{Dt} - \mathbf{v} \cdot \nabla p \right) \right] \cos \varphi dp d\varphi + \int_{-\pi/2}^{\pi/2} f \frac{\partial p_0}{\partial t} \cos \varphi d\varphi \\ &= \int_{-\pi/2}^{\pi/2} \int_0^{p_0} \left[ f_I S + f_p \frac{Dp}{Dt} - \nabla \cdot (f \mathbf{v}) \right] \cos \varphi dp d\varphi + \int_{-\pi/2}^{\pi/2} f \frac{\partial p_0}{\partial t} \cos \varphi d\varphi \\ &= \int_{-\pi/2}^{\pi/2} \int_0^{p_0} (f_I S + f_p \omega) \cos \varphi dp d\varphi. \end{aligned} \quad (20)$$

In the last line we used the divergence theorem and the fact that in pressure coordinates the normal component of the three-dimensional velocity  $\mathbf{v}$  on the lower surface is

$$\mathbf{v} \cdot \left( \hat{\mathbf{p}} - \frac{1}{a} \frac{\partial p_0}{\partial \varphi} \hat{\boldsymbol{\varphi}} \right) = \omega_0 - \frac{v}{a} \frac{\partial p_0}{\partial \varphi} = \frac{\partial p_0}{\partial t}, \quad (21)$$

where  $\omega_0 \doteq Dp_0/Dt$ , the value of  $\omega$  on the lower surface, is determined by vertically integrating (1c).

For example, one may readily verify that  $d\mathcal{H}/dt = 0$  by repeated application of (20) together with (8b) and the result

$$\begin{aligned} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{p_0} \frac{v}{a} \Phi_\varphi \cos \varphi \, dp \, d\varphi &= - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{p_0} \Phi \frac{\partial \omega}{\partial p} \cos \varphi \, dp \, d\varphi \\ &= - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a g \omega_0 \cos \varphi \, d\varphi - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{p_0} \frac{\omega}{\rho} \cos \varphi \, dp \, d\varphi, \end{aligned} \quad (22)$$

which follows from (1c) and (1e). The first term on the right-hand side of (22) is seen to vanish upon considering the volume integral of (1c). Note that the surface term of  $\mathcal{H}$  is just a constant proportional to the total mass, as can be seen by setting  $f = 1$  in (20).

In the special case where  $S = 0$  and  $f = f(I)$ , (20) implies that the density-weighted volume integral of an arbitrary function of any Lagrangian invariant  $I$  is constant in time:

$$\frac{d}{dt} \int \rho f(I) \, dV = \frac{1}{g} \frac{d}{dt} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{p_0} f(I) a^2 \cos \varphi \, dp \, d\varphi = 0. \quad (23)$$

In particular, the volume integral

$$C\{m, v, \eta, p_0\} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{p_0} C(m, \eta, q) \cos \varphi \, dp \, d\varphi \quad (24)$$

of any arbitrary function  $C$  of the Lagrangian invariants  $m$ ,  $\eta$ , and  $q$  will be conserved:  $dC/dt = 0$ .

### III. ZONAL EQUILIBRIUM

Consider an equilibrium state representing steady baroclinic zonal flow:

$$u = U(\varphi, p), \quad v = 0, \quad \eta = N(\varphi, p), \quad (25)$$

where  $U$  and  $N$  have continuous first derivatives. For this equilibrium the lower boundary condition  $D\Phi/Dt = 0$  is equivalent to the condition  $\omega = 0$  on the surface; (1c) then implies that  $\omega = 0$  everywhere. Equations (1a) and (1d) are therefore satisfied trivially, while (1b) becomes

$$U^2 \tan \varphi + 2\Omega U a \sin \varphi = -\Phi_\varphi. \quad (26)$$

Upon differentiating this result with respect to  $p$  and using the hydrostatic equation (1e), we obtain the thermal-wind relation

$$2(U \tan \varphi + \Omega a \sin \varphi)U_p = -\Phi_{\varphi p} = -\Phi_{p\varphi} = -\frac{1}{\rho^2} \left. \frac{\partial \rho}{\partial \varphi} \right|_p = -\frac{\rho_\eta}{\rho^2} N_\varphi, \quad (27)$$

where  $\rho$  and  $\rho_\eta$  are evaluated at  $(N, p)$ . In terms of the total equilibrium angular momentum

$$M(\varphi, p) \doteq U(\varphi, p)a \cos \varphi + \Omega a^2 \cos^2 \varphi, \quad (28)$$

the above relation may be written as

$$\frac{2 \tan \varphi}{a \cos \varphi} M U_p = -\frac{\rho_\eta}{\rho^2} N_\varphi. \quad (29)$$

Any flow of the form (25) that obeys (29) will consequently satisfy the governing equations (1).

It will prove convenient to introduce the equilibrium potential temperature  $\Theta(N) > 0$  and potential vorticity

$$Q \doteq -\frac{g}{c_p} \left( \zeta_{\text{abs}} N_p + \frac{1}{a} U_p N_\varphi \right) = \frac{g}{c_p a^2 \cos \varphi} \frac{\partial(M, N)}{\partial(\varphi, p)}, \quad (30)$$

where  $\zeta_{\text{abs}}$  is the radial component of the absolute vorticity:

$$\zeta_{\text{abs}} \doteq \frac{U}{a} \tan \varphi - \frac{1}{a} U_\varphi + 2\Omega \sin \varphi = -\frac{M_\varphi}{a^2 \cos \varphi}. \quad (31)$$

One sees from (30) that whenever  $Q$  is sign-definite, a single-valued transformation from  $(\varphi, p)$  to  $(M, N)$  will exist. However, we will soon see that our linear stability criterion requires that  $Q = 0$  at the equator. Let us restrict our attention to equilibrium flows where  $Q$  vanishes only on the equator and has different signs in each hemisphere. One can then construct inverse functions  $\varphi(M, N, Q)$  and  $p(M, N, Q)$  that depend only on  $M$ ,  $N$ , and the sign of  $Q$ . The latter dependence uniquely identifies the appropriate branch of the equilibrium profiles.

We wish to apply the so-called energy–Casimir method (Fjørtoft 1950; Kruskal & Oberman 1958; Arnol'd 1965, 1966; Holm *et al.* 1985; see also Shepherd 1990, §6) to obtain stability criteria for this equilibrium flow. In the absence of dissipation, it is well known that the fully compressible (nonhydrostatic) hydrodynamical equations may be written within a noncanonical Hamiltonian framework in which the cosymplectic operator  $\mathbf{J}$  is degenerate (Morrison & Greene 1980); *i.e.*, the phase space variable  $\mathbf{u}$  evolves in time according to

$$\frac{\partial}{\partial t} \mathbf{u} = \mathbf{J} \frac{\delta \mathcal{H}}{\delta \mathbf{u}}, \quad (32)$$

such that  $\det \mathbf{J} = 0$ . (The functional derivative  $\delta \mathcal{H} / \delta \mathbf{u}$  is defined in the usual way as the coefficient of the lowest-order term of the integrated power series

expansion in  $\delta\mathbf{u}$  of an induced perturbation  $\delta\mathcal{H}$ .) In addition to the invariants associated with symmetries of the Hamiltonian, such systems conserve additional quantities, known as Casimirs, that have gradients lying in the kernel of  $\mathbf{J}$ . While stationary solutions of the canonical Hamilton equations arise as extremals of the Hamiltonian, those of noncanonical systems correspond to extremizations constrained to surfaces defined by the invariance of the Casimirs. Although we will not prove that  $\mathcal{H}$  provides a Hamiltonian for our constrained dynamical system or that  $\mathcal{C}$  is indeed a Casimir, the energy–Casimir method is still applicable, provided we demonstrate that the resulting conditional extremals are steady solutions of the equations of motion. The stability of these equilibria can then be examined by considering second-order variations of the invariant  $\mathcal{H} + \mathcal{C}$ .

Let us therefore seek equilibria corresponding to conditional extremals of  $\mathcal{H}$  on surfaces of constant  $\mathcal{C}$ , appealing to the method of Lagrange multipliers. Because one of the dynamical variables,  $p_0$ , appears in the limits of integration, the first variation of  $\mathcal{H} + \mathcal{C}$  includes both volume and surface contributions (as is conventional, we absorb the multipliers into the function  $\mathcal{C}$ ):

$$\begin{aligned}
0 &= \delta(\mathcal{H} + \mathcal{C}) \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{p_0} \left[ (H_m + C_m)\delta m + (H_\eta + C_\eta)\delta\eta + H_v\delta v + C_q\delta q \right] \cos\varphi \, dp \, d\varphi \\
&\quad + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[ H + C + ag \right]_{p=p_0} \delta p_0 \cos\varphi \, d\varphi, \tag{33}
\end{aligned}$$

We use this condition to determine the as yet unspecified function  $C$ .

First, the condition  $C_q = 0$  can be satisfied by choosing  $C_q = 0$  in each region of the flow where  $Q$  is of definite sign and imposing the matching conditions

$$\lim_{Q \rightarrow 0^+} C(M, N, Q) = \lim_{Q \rightarrow 0^-} C(M, N, Q) = C(M, N, 0), \quad (34a)$$

$$\lim_{Q \rightarrow 0^+} C_q(M, N, Q) = \lim_{Q \rightarrow 0^-} C_q(M, N, Q) = C_q(M, N, 0) \quad (34b)$$

to guarantee that  $C_q = C_{qq} = 0$  over the entire flow.

In each region of the flow on which  $Q$  has a definite sign [and hence  $\partial(M, N)/\partial(\varphi, p) \neq 0$ ], the independent variables  $\varphi$  and  $p$  may then be expressed, *via* (25), in terms of  $M$  and  $N$ . The equilibrium conditions

$$H_m + C_m = 0, \quad (35a)$$

$$H_\eta + C_\eta = 0 \quad (35b)$$

can therefore be integrated to deduce that  $H + C$  in each region is independent of  $M$  and  $N$ , or equivalently, of  $\varphi$  and  $p$ . Equations (35) determine  $C$  only to within an arbitrary constant. [Constant values of  $C$  in (24) lead to invariants corresponding to the total mass.] In each region of sign-definite  $Q$ , let us choose this constant so that in equilibrium

$$H + C = \frac{1}{2}v^2 - ag = -ag; \quad (36)$$

this choice satisfies (34a) and ensures that the second integral in (33) vanishes. Equations (35) and (36) together with  $H_v = 0$  are sufficient to guarantee that  $\delta(\mathcal{H} + C) = 0$  in the equilibrium state.

Now

$$H_m = \frac{m}{a^2 \cos^2 \varphi} - \Omega, \quad H_v = v, \quad H_\eta = \frac{T}{\theta}, \quad (37)$$

upon using (8a). One finds for the equilibrium flow (25) that  $H_v = 0$  trivially, while evaluation of (35) leads to the prescriptions

$$C_m(M, N, Q) = -\frac{M}{a^2 \cos^2 \varphi(M, N, Q)} + \Omega \quad (38a)$$

$$C_\eta(M, N, Q) = -\frac{T(N, p(M, N, Q))}{\Theta(N)}. \quad (38b)$$

Note that the condition

$$\lim_{Q \rightarrow 0^+} \varphi(M, N, Q) = \lim_{Q \rightarrow 0^-} \varphi(M, N, Q) \quad (39)$$

guarantees that both  $C_m$  and  $C_\eta$  are continuous across  $Q = 0$ . Hence by (14), the matching condition (34b) is satisfied by equations (38).

These conditions determine the *form* of  $C$  as a function of  $M$ ,  $N$ , and  $Q$ ; all other variables, including the (normally) independent variables  $\varphi$  and  $p$  must be treated as functions of the conserved quantities  $M$ ,  $N$ , and  $Q$  (to which they are related through the equilibrium profiles). In contrast, when  $\mathcal{H}$  is functionally differentiated, the independent variables are held fixed.

It is important to establish the consistency of (38 a,b). To do this, we will need to determine certain partial derivative relations. Equation (30) allows us to write

$$\left. \frac{\partial \varphi}{\partial M} \right|_N = \frac{\partial(\varphi, N)}{\partial(M, N)} = \frac{\partial(\varphi, p)}{\partial(M, N)} \cdot \frac{\partial(\varphi, N)}{\partial(\varphi, p)} = \frac{g}{c_p a^2 \cos \varphi} Q^{-1} N_p \quad (40)$$

and

$$\left. \frac{\partial p}{\partial M} \right|_N = \frac{\partial(p, N)}{\partial(M, N)} = \frac{\partial(p, \varphi)}{\partial(M, N)} \cdot \frac{\partial(p, N)}{\partial(p, \varphi)} = -\frac{g}{c_p a^2 \cos \varphi} Q^{-1} N_\varphi. \quad (41)$$

In a similar manner one finds that

$$\left. \frac{\partial \varphi}{\partial N} \right|_M = -\frac{g}{c_p a^2 \cos \varphi} Q^{-1} M_p = -\frac{g}{c_p a} Q^{-1} U_p \quad (42)$$

and

$$\left. \frac{\partial p}{\partial N} \right|_M = \frac{g}{c_p a^2 \cos \varphi} Q^{-1} M_\varphi = -\frac{g}{c_p} Q^{-1} \zeta_{\text{abs}}, \quad (43)$$

upon using (31).

Now,  $C_\eta$  in (38b) is to be regarded as a function of  $M$ ,  $N$ , and  $Q$  only, with the  $M$  dependence entering through  $p$ . Hence

$$C_{\eta m} = -\left( \frac{T_p}{\Theta} \right) \left. \frac{\partial p}{\partial M} \right|_N. \quad (44)$$

Substitution of (9) and (41) into (44) yields

$$C_{\eta m} = \frac{-B}{a \cos \varphi}, \quad (45)$$

where

$$B \doteq \frac{g}{c_p a} \left( \frac{\rho_\eta}{\rho^2} \right) Q^{-1} N_\varphi. \quad (46)$$

Finally, upon using (38a), (42), (29), and (45) we verify that

$$\begin{aligned} C_{m\eta} &= -\frac{2 \sin \varphi}{a^2 \cos^3 \varphi} M \left. \frac{\partial \varphi}{\partial N} \right|_M = \frac{2g \tan \varphi}{c_p a^3 \cos^2 \varphi} M Q^{-1} U_p \\ &= \frac{g}{c_p a^2 \cos \varphi} \left( \frac{-\rho_\eta}{\rho^2} \right) Q^{-1} N_\varphi = C_{\eta m}. \end{aligned} \quad (47)$$

This establishes the existence of a twice-differentiable function  $C(m, \eta, q)$  satisfying (38).



## IV. LINEAR STABILITY THEOREM

Since  $\mathcal{H} + \mathcal{C}$  is conserved in time and  $\delta(\mathcal{H} + \mathcal{C}) = 0$  by construction, a sufficient condition for linear stability is that  $\delta^2(\mathcal{H} + \mathcal{C})$  be sign definite (for arbitrary perturbations). Upon taking the variation of (33), we find that

$$\begin{aligned} \delta^2(\mathcal{H} + \mathcal{C}) = & \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{p_0} \left[ \delta v^2 + G_{mm} \delta m^2 + 2G_{m\eta} \delta m \delta \eta + G_{\eta\eta} \delta \eta^2 \right] \cos \varphi dp d\varphi \\ & + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[ 2G_m \delta m \delta p_0 + 2G_\eta \delta \eta \delta p_0 + \frac{dG}{dp} \delta p_0^2 \right]_{p=p_0} \cos \varphi d\varphi, \end{aligned} \quad (48)$$

where

$$G(m, \eta, q; \varphi, p) \doteq H(m, v, \eta; \varphi, p) + C(m, \eta, q) - v^2/2 \quad (49)$$

and

$$\frac{dG}{dp} = G_m m_p + G_\eta \eta_p + G_p. \quad (50)$$

(Note that  $G_q = 0$ .) No  $\delta q$  contributions arise since the function  $C$  was chosen to satisfy the matching conditions (34) as well as  $C_q = 0$  and  $C_{qq} = 0$ .

For the equilibrium flow  $G_m = G_\eta = 0$  since  $C$  was constructed to satisfy (35). Moreover, (8b) implies that

$$G_p = \frac{1}{\rho} \geq 0. \quad (51)$$

We then see from (50) and (48) that  $\delta^2(\mathcal{H} + \mathcal{C})$  is sign definite (for arbitrary disturbances) if the symmetric matrix

$$\mathbf{A} \doteq \begin{pmatrix} H_{mm} + C_{mm} & H_{m\eta} + C_{m\eta} \\ H_{\eta m} + C_{\eta m} & H_{\eta\eta} + C_{\eta\eta} \end{pmatrix} \quad (52)$$

is positive definite almost everywhere. Note that the dependent variable  $v$  is decoupled from  $m$  and  $\eta$ .

In computing the Hessian (matrix of second derivatives) of  $H$ , we must hold the independent variables  $\varphi$  and  $p$  fixed. Since  $H_{\eta m} = H_{m\eta} = 0$ , we obtain a diagonal matrix with the elements

$$H_{mm} = \frac{1}{a^2 \cos^2 \varphi}, \quad (53a)$$

$$H_{\eta\eta} = 0, \quad (53b)$$

upon using (11).

On the other hand, in evaluating the Hessian of  $C$ , all variables are to be differentiated with respect to  $M$  or  $N$ . Using (38a), (40), (29), and (46) we calculate for the equilibrium flow,

$$\begin{aligned} C_{mm} &= -\frac{1}{a^2 \cos^2 \varphi} - \frac{2 \sin \varphi}{a^2 \cos^3 \varphi} M \left. \frac{\partial \varphi}{\partial M} \right|_N \\ &= \frac{1}{a^2 \cos^2 \varphi} \left[ -1 - 2 \tan \varphi M \left( \frac{g}{c_p a^2 \cos \varphi} \right) Q^{-1} N_p \right] \\ &= \frac{1}{a^2 \cos^2 \varphi} \left( -1 + \frac{g}{c_p a} \frac{\rho_\eta}{\rho^2} Q^{-1} \frac{N_\varphi}{U_p} N_p \right) \\ &= \frac{BN_p/U_p - 1}{a^2 \cos^2 \varphi}. \end{aligned} \quad (54a)$$

Next, we find from (38b), (9), (11), (43), and (46) that

$$C_{\eta\eta} = -\left( \frac{T}{\Theta} \right)_\eta - \frac{T_p}{\Theta} \left. \frac{\partial p}{\partial N} \right|_M = \left( \frac{-\rho_\eta}{\rho^2} \right) \frac{g}{c_p} Q^{-1} \zeta_{\text{abs}} = -\frac{Ba\zeta_{\text{abs}}}{N_\varphi}. \quad (54b)$$

Note that (30) can be used to eliminate  $\zeta_{\text{abs}}$  from (54b) to obtain

$$C_{\eta\eta} = \frac{BU_p}{N_p} \left( \frac{c_p a Q}{g U_p N_\varphi} + 1 \right). \quad (55)$$

The condition  $H_{mm} + C_{mm} > 0$  is seen to be equivalent to

$$\frac{BN_p}{U_p} > 0, \quad (56)$$

while the condition  $H_{\eta\eta} + C_{\eta\eta} > 0$  may be written, recalling (31), as

$$\rho_\eta Q M_\varphi > 0. \quad (57)$$

Upon using (53), (54a), (55), and (45), we can express the condition  $\det \mathbf{A} > 0$  as

$$\frac{B^2}{a^2 \cos^2 \varphi} \left( \frac{c_p a Q}{g U_p N_\varphi} \right) > 0, \quad (58)$$

or, after using (46), as  $\rho_\eta B U_p > 0$ . Together with (56), this result implies

$$\rho_\eta N_p > 0. \quad (59)$$

Note that the thermal-wind relation (29) allows us to rewrite (58) as

$$-\rho_\eta \left( \frac{f M Q}{\Omega} \right) > 0, \quad (60)$$

in terms of the Coriolis parameter  $f \doteq 2\Omega \sin \varphi$ . The three criteria given by (57), (59), and (60) are not independent of each other: the determinant condition (60) along with either of the other two inequalities implies the third one.

Given any equation of state for which  $\rho_\eta < 0$ <sup>1</sup> [e.g., for a perfect gas  $\rho_\eta = -\rho/(c_p \theta)$ ], these criteria reduce to

$$-Q M_\varphi > 0, \quad (61a)$$

---

<sup>1</sup>There are fluids (e.g., water between 0 and 4°C) with a positive  $\rho_\eta$ : since  $\partial\rho/\partial T|_p = \rho_\eta \eta_s \partial s/\partial T|_p = \rho_\eta \Theta c_p/T$ , one sees that  $\partial\rho/\partial T|_p$  and  $\rho_\eta$  have the same sign.

$$\frac{\partial\Theta}{\partial r} > 0, \tag{61b}$$

$$\frac{fMQ}{\Omega} > 0. \tag{61c}$$

It is instructive to compare the first criterion, which is equivalent to  $Q\zeta_{\text{abs}} > 0$ , with Rayleigh's condition for the stability of a cylindrical vortex to axisymmetric perturbations in an incompressible nonrotating fluid (Rayleigh 1916):

$$u\zeta \doteq \frac{u}{r} \frac{\partial}{\partial r}(ru) > 0. \tag{62}$$

We recognize the second criterion as the usual condition for static stability, while the third criterion is the result previously obtained by Stevens (1983). The three criteria are analogous to those found in the  $f$ -plane Boussinesq case (Cho *et al.* 1993).

As already noted, the criteria (61) need only hold almost everywhere in order to prove linear stability. [In fact, it is clear that the left-hand side of (61c) will necessarily vanish at the equator (where  $f = 0$ ) and at the poles (where  $M = 0$ ).] Recall that we restricted attention to equilibrium flows with  $Q$  sign-definite in each hemisphere and changing sign across the equator. It is easy to see that if  $Q$  changed sign anywhere except at the equator, the stability criteria could not be satisfied. For, in order to satisfy (61a) and (61c) simultaneously, both  $M_\varphi$  and  $M$  would have to change sign where  $Q$  changed sign; this would contradict the assumed continuity of  $M$  in  $\varphi$ . Within each hemisphere one may combine (61a) and (61c) to obtain the condition  $-fMM_\varphi/\Omega > 0$ , which precludes the existence of a local extremum

of  $M$  in  $\varphi$ . Given that  $M$  vanishes at the poles, it follows that  $M$  must be single-signed, having a single extremum at the equator.

We thus determine two possible configurations for stability: one with  $M/\Omega < 0$  and  $fQ < 0$  and the other with  $M/\Omega > 0$  and  $fQ > 0$  (almost everywhere). The latter represents the most likely situation for a planetary atmosphere, since  $M/\Omega > 0$  at the equator provided  $U(0, p)/\Omega > -a$ . (This condition is easily met for the Earth's atmosphere: since  $\Omega a = 465$  m/s, a violation would require supersonic wind speeds!)

Equation (60) may be interpreted in terms of the *Richardson number*  $R_i \doteq \mathcal{N}^2/U_r^2$ , where  $\mathcal{N}^2 \doteq g^2\rho_\eta N_p$  is the square of the Brunt–Väisälä frequency [which may be expressed in the more familiar form  $\mathcal{N}^2 = g\Theta_r/\Theta_0$ , where  $\Theta_0 \doteq -\rho(\partial\rho/\partial\theta)^{-1}$  is a characteristic potential temperature]. Note that (59) implies  $\mathcal{N}^2 > 0$ , verifying that (59) corresponds to static stability even when  $\rho_\eta < 0$ . With the help of (29), we rewrite (30) as

$$\begin{aligned}
Q &= -\frac{1}{c_p} \left[ g\zeta_{\text{abs}}N_p - \frac{1}{a\rho}U_rN_\varphi \right] \\
&= -\frac{1}{c_p} \left[ g\zeta_{\text{abs}}N_p - \frac{1}{ag} \left( \frac{1}{\rho_\eta} \right) \frac{2 \tan \varphi}{a \cos \varphi} MU_r^2 \right] \\
&= -\frac{1}{c_p g \rho_\eta} \left[ \zeta_{\text{abs}} \mathcal{N}^2 - f \left( \frac{M}{\Omega a^2 \cos^2 \varphi} \right) U_r^2 \right] \\
&= -\frac{f \mathcal{N}^2}{c_p g \rho_\eta} \left[ \frac{\zeta_{\text{abs}}}{f} - R_i^{-1} \left( 1 + \frac{U}{\Omega a \cos \varphi} \right) \right]. \tag{63}
\end{aligned}$$

Hence (60) can be rewritten as

$$\frac{M}{\Omega} \left[ \frac{\zeta_{\text{abs}}}{f} - R_i^{-1} \left( 1 + \frac{U}{\Omega a \cos \varphi} \right) \right] > 0, \tag{64}$$

or,

$$\frac{\zeta_{\text{abs}}}{f} \left( 1 + \frac{U}{\Omega a \cos \varphi} \right) > R_i^{-1} \left( 1 + \frac{U}{\Omega a \cos \varphi} \right)^2 > 0, \quad (65)$$

which leads to the stability criterion

$$R_i > \frac{f}{\zeta_{\text{abs}}} \left( 1 + \frac{U}{\Omega a \cos \varphi} \right) > 0. \quad (66)$$

For the special case  $U(\varphi, p) = U_0(p)/\cos \varphi$  one finds  $\zeta_{\text{abs}} = f$ ; (66) then reduces to

$$R_i > 1 + \frac{U}{\Omega a \cos \varphi} > 0. \quad (67)$$

## V. NONLINEAR STABILITY THEOREM

From the exact invariants  $\mathcal{H}$  and  $\mathcal{C}$ , let us construct the finite-amplitude disturbance invariant

$$\mathcal{A} \doteq [\mathcal{H} + \mathcal{C}](m, v, \eta, p_0) - [\mathcal{H} + \mathcal{C}](M, 0, N, P_0) \quad (68)$$

for the disturbed state defined by

$$m \doteq M + m', \quad v \doteq v', \quad \eta \doteq N + \eta', \quad p_0 \doteq P_0 + p'_0. \quad (69)$$

This invariant is called the disturbance *pseudoenergy* (e.g., Shepherd 1990).

If one defines

$$\Delta \doteq \int_0^{p_0} G(m, \eta, q; \varphi, p) dp - \int_0^{P_0} G(M, N, Q; \varphi, p) dp, \quad (70)$$

one may express  $\mathcal{A}$  in the compact form

$$\mathcal{A} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( \int_0^{p_0} \frac{1}{2} v^2 dp + \Delta + a g p'_0 \right) \cos \varphi d\varphi. \quad (71)$$

To analyze nonlinear stability, we must extend the domain of the equilibrium profiles  $M(\varphi, p)$  and  $N(\varphi, p)$  to values of  $p > P_0$  subject to the thermal-wind balance condition (29). This extension may be made in an arbitrary fashion, provided that the final nonlinear stability criterion (86) holds over the extended domain of  $M$  and  $N$ . (However, the details of the extension will affect the upper bound (89); one might be able to use this arbitrariness to reduce the bound as much as possible.) Bounds on the pseudoenergy can then be obtained by decomposing  $\Delta$  into its volume and surface contributions:  $\Delta = \Delta_V + \Delta_S$ , where (M. Mu, private communication, 1993)

$$\Delta_V \doteq \int_0^{p_0} [G(m, \eta, q; \varphi, p) - G(M, N, Q; \varphi, p)] dp, \quad (72a)$$

$$\Delta_S \doteq \int_{P_0}^{p_0} G(M, N, Q; \varphi, p) dp. \quad (72b)$$

Taylor's Remainder Theorem implies that for each value of  $\varphi$  and  $p$ , there exists some  $\tilde{m} \in (M, m)$ ,  $\tilde{\eta} \in (N, \eta)$  such that

$$\begin{aligned} \Delta_V = & \int_0^{p_0} \left[ G_m(M, N, Q)m' + G_\eta(M, N, Q)\eta' \right. \\ & \left. + \frac{1}{2}G_{mm}(\tilde{m}, \tilde{\eta}, \tilde{q})m'^2 + G_{m\eta}(\tilde{m}, \tilde{\eta}, \tilde{q})m'\eta' + \frac{1}{2}G_{\eta\eta}(\tilde{m}, \tilde{\eta}, \tilde{q})\eta'^2 \right] dp, \end{aligned} \quad (73)$$

where  $G(m, \eta, q) \equiv G(m, \eta, q; \varphi, p)$ . Similarly, the application of Taylor's Remainder Theorem to  $\Delta_S$  as a function of  $p_0$  about  $P_0$  implies that

$$\Delta_S = G(M, N, Q; \varphi, p_0)p_0' + \frac{1}{2} \frac{d}{dp} G(M, N, Q; \varphi, p) \Big|_{p=\tilde{p}_0} p_0'^2, \quad (74)$$

for some  $\tilde{p}_0 \in (P_0, p_0)$ .

The linear terms of  $\Delta_V$  vanish since for the equilibrium state  $C$  obeys (35). The first term of (74) cancels the third term in (71), because of (36). These results reflect the fact that  $\delta(\mathcal{H} + C) = 0$  for the equilibrium flow. Again, no  $\delta q$  contributions arise since the function  $C$  was chosen to satisfy (34).

It will be convenient to decompose the pseudoenergy (71) into volume and surface contributions such that  $\mathcal{A} = \mathcal{A}_V + \mathcal{A}_S$ , where

$$\mathcal{A}_V \doteq \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{p_0(\varphi, t)} (v'^2 + I_V) \cos \varphi dp d\varphi, \quad (75a)$$

$$\mathcal{A}_S \doteq \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} I_S \Big|_{p=\tilde{p}_0} \cos \varphi d\varphi. \quad (75b)$$

From (73) and (74) we identify

$$I_V \doteq \tilde{G}_{mm} m'^2 + 2\tilde{G}_{m\eta} m' \eta' + \tilde{G}_{\eta\eta} \eta'^2, \quad (76a)$$

$$I_S \doteq \frac{d}{dp} G(M, N, Q; \varphi, p) p_0'^2. \quad (76b)$$

Here, a tilde is used to indicate that the corresponding function is evaluated at  $(\tilde{m}, \tilde{\eta}, \tilde{q}; \varphi, p)$ .

For the surface contribution, (50) and (51) imply that  $I_S = p_0'^2 / \rho(N, p)$  since  $G_m = G_\eta = 0$  when evaluated at the equilibrium flow. Thus,

$$\mathcal{A}_S = \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{p_0'^2}{\rho(N(\varphi, \tilde{p}_0), \tilde{p}_0)} \cos \varphi d\varphi \geq 0. \quad (77)$$

The following lemma can be used to bound the interior contribution  $I_V$ .

LEMMA 1. *If  $\alpha > 0$  and  $x$  and  $y$  are real, then*

$$-\alpha |G|x^2 - \alpha^{-1} |G|y^2 \leq 2Gxy \leq \alpha |G|x^2 + \alpha^{-1} |G|y^2. \quad (78)$$



*Proof.* The two inequalities

$$0 \leq (\alpha^{1/2}x \pm \alpha^{-1/2}y)^2 = \alpha x^2 \pm 2xy + \alpha^{-1}y^2 \quad (79)$$

provide the bounds

$$-\alpha|G|x^2 - \alpha^{-1}|G|y^2 \leq \pm 2|G|xy \leq \alpha|G|x^2 + \alpha^{-1}|G|y^2, \quad (80)$$

which in turn imply (78).  $\square$

The application of Lemma 1 to (76a) yields, for any positive function  $\alpha \equiv \alpha(\varphi, p)$

$$I_V \geq (\tilde{G}_{mm} - \alpha|\tilde{G}_{m\eta}|) m'^2 + (\tilde{G}_{\eta\eta} - \alpha^{-1}|\tilde{G}_{m\eta}|) \eta'^2. \quad (81)$$

In order to prove nonlinear stability, we would ideally like to bound  $I_V$  and  $I_S$  from above and below by positive-definite quadratic forms. However this is problematical, because  $G$  depends on  $p$  as well as on  $m$ ,  $\eta$ , and  $q$ . The latter dependencies are not difficult to handle since the variables are Lagrangian invariants, and thus their range of values is always the same as the range provided by the initial conditions. But the surface pressure  $p_0$  is a dynamical quantity that can, in principle, take on any positive value whatsoever. Thus the  $p$  dependence in  $G$  is difficult to bound *a priori*. In particular, the surface density  $\rho$  in (77) could conceivably approach either zero or infinity.

Thus we are unable to obtain the sort of strong convexity properties that were employed in the analysis of Cho *et al.* (1993). Nevertheless, we can still

prove nonlinear stability. Upon noting that  $H_{m\eta} = 0$ , we obtain from (81) and (53) the bound

$$I_V \geq \left( \frac{1}{\alpha^2 \cos^2 \varphi} + \tilde{C}_{mm} - \alpha |\tilde{C}_{m\eta}| \right) m'^2 + \left( \tilde{C}_{\eta\eta} - \alpha^{-1} |\tilde{C}_{m\eta}| \right) \eta'^2. \quad (82)$$

To bound  $I_V$  in terms of the equilibrium quantities, we need to relate the range of the perturbed quantities appearing in (82) to the range of the equilibrium profiles. For each function  $F \in \{C_{mm}, |C_{m\eta}|, C_{\eta\eta}\}$  we suppose that

$$\{F(\tilde{m}, \tilde{\eta}, \tilde{q}) : (\varphi, p) \in V\} \subseteq \{F(M, N, Q) : (\varphi, p) \in V\}. \quad (83)$$

This will be the case if the perturbation globally introduces no new values of the Lagrangian invariants  $m$  and  $\eta$  (the  $q$ -dependence serving only to identify, by its sign, the branch of the function). The bound (82) may then be used to obtain a lower estimate for  $\mathcal{A}_V$ . [In the most general situation, one needs to extend the domain of the functions  $F$  to the new range of  $\tilde{m}$  and  $\tilde{\eta}$  without disturbing (35) and (82). Although this construction is always possible in the one-dimensional case (*cf.* Arnol'd 1966), it is not yet clear whether two-dimensional functions can always be extended in an analogous manner.]

Suppose now that there exists some positive constant  $c$  that provides the global bound (for  $-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$  and all  $p \geq 0$ )

$$\min \left( C_{mm} - \alpha |C_{m\eta}|, C_{\eta\eta} - \alpha^{-1} |C_{m\eta}| \right) \geq c > 0. \quad (84)$$

Equations (53), (82), (83), and (84) then imply that the pseudoenergy con-

tribution  $\mathcal{A}_V$  is bounded by

$$\mathcal{A}_V \geq \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{p_0} \left[ v'^2 + \left( \frac{1}{a^2 \cos^2 \varphi} + c \right) m'^2 + c\eta'^2 \right] \cos \varphi \, dp \, d\varphi. \quad (85)$$

Upon substitution of (54) and (45), the nonlinear stability criterion (84) appears as

$$\min \left( \frac{BN_p/U_p - \alpha_0|B| - 1}{a^2 \cos^2 \varphi}, -B \frac{a\zeta_{\text{abs}}}{N_\varphi} - \alpha_0^{-1}|B| \right) \geq c > 0, \quad (86)$$

where  $\alpha_0 \doteq \alpha a \cos \varphi$  and  $B$  is given by (46).

One may now establish nonlinear stability for the flow in terms of the disturbance measure  $\|\mathbf{x}'\|_c$  of the vector  $\mathbf{x}' \doteq (m', v', \eta', p'_0)$ , where

$$\|\mathbf{x}'\|_c^2 \doteq \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{P_0+p'_0} \left[ v'^2 + \left( \frac{1}{a^2 \cos^2 \varphi} + c \right) m'^2 + c\eta'^2 \right] \cos \varphi \, dp \, d\varphi. \quad (87)$$

Here, the surface pressure disturbance  $p'_0$  is related to the generalized potential temperature disturbance  $\eta'$  by

$$p'_0 \doteq g \int_a^{a+\Delta r} \rho_\eta \eta' \, dr. \quad (88)$$

Equations (85) and (77) imply that the evolution of the positive-definite quantity  $\|\mathbf{x}'\|_c^2$  is bounded according to

$$\|\mathbf{x}'(t)\|_c^2 \leq \mathcal{A}_V(t) \leq \mathcal{A}(t) = \mathcal{A}(0). \quad (89)$$

Hence finite-amplitude disturbances are bounded for all time by a quantity depending only on the initial state of the flow; the bound decreases to zero as the initial disturbance tends to zero.

The disturbance measure defined by (87) is not actually a norm since it does not satisfy the property of homogeneity: in general,  $\|\lambda \mathbf{x}\|_c \neq |\lambda| \|\mathbf{x}\|_c$  for a scalar  $\lambda$ . However, (87) still provides a meaningful measure of disturbance size since it satisfies the important property that  $\|\mathbf{x}\|_c^2 \geq 0$ , with equality occurring only when  $\mathbf{x} = 0$  almost everywhere (observe that  $\eta' = 0$  implies  $p'_0 = 0$ ).

Note that (84) requires that

$$C_{mm} > \alpha |C_{m\eta}| \geq 0, \quad C_{\eta\eta} > \alpha^{-1} |C_{m\eta}| \geq 0; \quad (90)$$

these inequalities in turn imply

$$C_{mm} > \frac{|C_{m\eta}|}{C_{\eta\eta}} |C_{m\eta}| = \frac{C_{m\eta}^2}{C_{\eta\eta}}. \quad (91)$$

Since for this case  $H_{mm} > 0$  and  $H_{\eta\eta} = 0$ , it follows that the nonlinear stability criterion (84) implies the linear stability conditions (61) obtained in Section IV.

Let us examine the nonlinear stability criterion (86) in further detail. Upon using (30) to eliminate  $\zeta_{\text{abs}}$  we find that

$$\frac{BN_p}{U_p} - 1 - ca^2 \cos^2 \varphi \geq \alpha_0 |B| \geq BN_p \left( \frac{c_p a Q}{g N_\varphi} + U_p \right)^{-1}. \quad (92)$$

Dividing through by the positive expression on the right-hand side of the second inequality, we obtain

$$\frac{c_p a Q}{g U_p N_\varphi} \geq (1 + ca^2 \cos^2 \varphi) \frac{1}{BN_p} \left( \frac{c_p a Q}{g N_\varphi} + U_p \right) > 0. \quad (93)$$

We now use (29) to eliminate  $U_p$  on the left-hand side. Upon recalling (46) we obtain a nonlinear generalization of (60):

$$-\rho_\eta \left( \frac{fMQ}{\Omega} \right) \geq a \cos^2 \varphi (1 + ca^2 \cos^2 \varphi) \frac{\rho_\eta Q}{N_p} \left( \frac{c_p a Q}{g} + U_p N_\varphi \right) > 0. \quad (94)$$

Analogues of the other linear stability conditions can also be obtained from (86), which requires that

$$-\rho_\eta Q \zeta_{\text{abs}} \geq \frac{c_p \rho^2 Q^2}{g} (c + \alpha_0^{-1} |B|) > 0, \quad (95a)$$

$$\frac{BN_p}{U_p} \geq 1 + ca^2 \cos^2 \varphi + \alpha_0 |B| > 0 \quad (95b)$$

for some positive constant  $c$ . Equations (95a) and (95b) are nonlinear generalizations of (57) and (56), respectively. Substitution of (46) and (29) into (95b), together with (94), leads to the criterion

$$\rho_\eta N_p \geq -\frac{c_p \rho_\eta}{g} \left( \frac{fMQ}{\Omega} \right)^{-1} Q^2 a^2 \cos^2 \varphi (1 + ca^2 \cos^2 \varphi + \alpha_0 |B|) > 0, \quad (96)$$

which is a nonlinear generalization of (59). Evidently these nonlinear conditions are all somewhat more restrictive than the corresponding linear stability conditions.

Let us emphasize that, although we have provided a bound on the nonlinear evolution of the disturbance measure  $\|\mathbf{x}\|_c$ , we have not proven Liapunov stability. Because of the possibility that  $\rho$  could approach arbitrarily close to zero as the flow evolves (*e.g.*, in the presence of cavitation), there appears to be no way of providing an upper bound on the surface contribution to the pseudoenergy. This treatment differs from the approach taken by Holm

*et al.* (1983) upon encountering a similar difficulty: since their calculation required a nonzero lower bound on the density, these authors simply made this a hypothesis. In contrast, the treatment here is dynamically self-consistent in that it requires no externally imposed assumptions on  $\rho$ .

## VI. DISCUSSION

We have applied the energy–Casimir method to the problem of symmetric stability of a compressible, hydrostatic planetary atmosphere with a general equation of state, including a proper treatment of the lower boundary condition. It has been shown that steady baroclinic zonal flows of the form (25) satisfy the variational principle (33), and are thus conditional extremals of the energy subject to constraints provided by a suitably constructed Casimir-like invariant. By considering the second variation of the corresponding energy–Casimir (or pseudoenergy) invariant functional about these equilibria, linear symmetric stability criteria have been derived: namely (57), (59), and (60). In the special case of a perfect gas these criteria reduce to (61), as previously derived by Stevens (1983).

In comparison with the  $f$ -plane, Boussinesq analysis of symmetric stability of Cho *et al.* (1993), the present analysis exhibits two interesting mathematical features: multi-valued inverse equilibrium profiles and a moving lower boundary in  $p$ -coordinates. The first of these is dealt with by allowing the Casimir-like functions to have an additional dependence on the sign

of the potential vorticity, which identifies the branch of the function. The second feature leads to an explicit contribution from the lower boundary to the pseudoenergy. However this contribution is positive definite, and thus presents no problem for the linear stability analysis.

For finite-amplitude disturbances, on the other hand, the moving domain of integration presents significant difficulties because the range of values of  $p$  cannot be bounded *a priori*. This means that strong convexity properties on the pseudoenergy, such as were employed by Cho *et al.* (1993), are not available in this case. In particular, Liapunov stability does not appear to be provable. Nevertheless, we show that the pseudoenergy is positive definite and that nonlinear stability conditions can be derived. By nonlinear stability we mean that a positive-definite measure of disturbance amplitude is bounded uniformly in time, with the bound decreasing to zero as the initial disturbance tends to zero [*cf.* (MU MU *et al.* 1994)]; this result is expressed by (89). The nonlinear stability conditions imply linear stability and are expressible as generalizations of the linear symmetric stability conditions.

From the stability bound (89) we also obtain a rigorous upper bound on the kinetic energy of the motion in the meridional plane:

$$\frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{p_0} v'^2(t) \cos \varphi dp d\varphi \leq \mathcal{A}(0). \quad (97)$$

Since (97) holds for any equilibrium for which  $I_V \geq 0$  everywhere, it may be used to define a generalized available potential energy for a given baroclinic zonal flow [see Shepherd (1993)]: simply choose a stable equilibrium, compute

$\mathcal{A}(0)$  for the resulting disturbance field (the difference between the initial and the equilibrium values), and then minimize  $\mathcal{A}(0)$  over all possible stable equilibria. This construction of a generalized available potential energy was used by Cho *et al.* (1993) to provide rigorous upper bounds on the nonlinear saturation of symmetric instabilities. A similar calculation would certainly be possible in the present case.

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# Nonlinear symmetric stability of planetary atmospheres

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## Abstract

The energy–Casimir method is applied to the problem of symmetric stability in the context of a compressible, hydrostatic planetary atmosphere with a general equation of state. Linear stability criteria for symmetric disturbances to a zonally symmetric baroclinic flow are obtained. In the special case of a perfect gas the results of Stevens (1983) are recovered. Nonlinear stability conditions are also obtained that, in addition to implying linear stability, provide an upper bound on a certain positive-definite measure of disturbance amplitude.

## I. INTRODUCTION

A classical problem in geophysical fluid dynamics is that of symmetric stability. In a zonally symmetric flow, this kind of baroclinic instability can occur even when the atmosphere is both statically and inertially stable (positively stratified with a radially increasing circulation). It has primarily been considered in the context of two physical applications: (i) mesoscale dynamics and (ii) planetary circulations. Symmetric instability, sometimes referred to as slantwise convection, produces symmetric convective rolls that have been identified with frontal rainbands in the mesoscale context and with axisymmetric cells in the planetary context.

Linear criteria for symmetric stability were first derived by Fjørtoft (1950). These criteria have recently been extended to finite amplitude in the case of the  $f$ -plane Boussinesq equations — relevant to the mesoscale context — by Cho, Shepherd & Vladimirov (1993), using the so-called energy-Casimir stability method. This method relies on the construction of a positive-definite invariant for disturbances to a given equilibrium, using the conservation of total energy together with the material conservation of entropy and angular momentum.

The present work extends the results of Cho *et al.* (1993) to planetary circulations, including the effects of both spherical geometry and compressibility, in terms of a general equation of state. Beyond these additional physical effects, certain novel mathematical features are exhibited in the analysis:

a multi-valued equilibrium profile and a moving domain of integration. The latter leads to some technical difficulties with convexity.

The equations describing the compressible hydrostatic model, which assumes zonal (azimuthal) symmetry and neglects diabatic processes, are presented in §2. It is shown that the total energy and certain functionals of the angular momentum, entropy, and potential vorticity are conserved in time. In §3, we demonstrate that a general steady baroclinic zonal flow satisfying thermal wind balance constitutes an equilibrium of the model system and corresponds to a conditional extremum of the total energy. In §4, we examine the linear stability of these baroclinic flows and obtain conditions equivalent to the classical criteria of inertial stability, static stability, and positivity of the product of the Coriolis parameter and the potential vorticity. In the special case of a perfect (ideal) gas, our linear criteria reduce to those of Stevens (1983). The nonlinear extension of this result, based on convexity arguments, is described in §5. Although we are not able to prove Liapunov stability, we do derive a rigorous upper bound on disturbance amplitude that is expressible solely in terms of initial amplitudes, with the bound going to zero as the initial disturbance tends to zero. We find that the nonlinear stability criteria imply linear stability and are expressible as generalizations of the linear symmetric stability conditions. The nonlinear stability arguments are based on an extension of Taylor's Remainder Theorem to a class of piecewise-differentiable functions with jump discontinuities, as described

in Appendix A.

## II. GOVERNING EQUATIONS

We consider the model of hydrostatic compressible flow known in meteorology as the primitive equations. Denoting the latitude by  $\lambda$ , the longitude by  $\varphi$  and the pressure by  $p$ , we describe a longitudinally symmetric spherical shell at radius  $r = a$ , of thickness  $\Delta r \ll a$ , rotating at constant angular velocity  $\Omega$ . Using pressure coordinates  $(\lambda, \varphi, p)$  with corresponding velocities  $(u, v, \omega)$ , the model equations for axisymmetric flow are [*e.g.*, see Lorenz (1967)]:

$$\frac{Du}{Dt} - \frac{\tan \varphi}{a} uv - 2\Omega v \sin \varphi = 0, \quad (1a)$$

$$\frac{Dv}{Dt} + \frac{\tan \varphi}{a} u^2 + 2\Omega u \sin \varphi = -\frac{1}{a} \Phi_{\varphi}, \quad (1b)$$

$$\frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (v \cos \varphi) + \frac{\partial \omega}{\partial p} = 0, \quad (1c)$$

$$\frac{D\theta}{Dt} = 0, \quad (1d)$$

$$\Phi_p = -\frac{1}{\rho}. \quad (1e)$$

Here  $\rho$  is the fluid density and  $\Phi \doteq gr$  is the geopotential, where the gravitational acceleration  $g$  is treated as a constant. (We emphasize definitions with the notation “ $\doteq$ ”.) The upper surface is taken to be an isobaric surface on which, for the sake of simplicity, we suppose that  $p$  approximately vanishes. [This effectively requires that the density scale height be much less than the thickness of the shell:  $(\rho^{-1} \partial \rho / \partial r)^{-1} \ll \Delta r$ .] The appropriate

boundary conditions are then  $\omega = 0$  on the upper surface and  $D\Phi/Dt = 0$  on the lower surface (on which  $r = a$ ). The potential temperature  $\theta \doteq \exp(\int d\eta/c_p(\eta, p_r)) > 0$  is defined in terms of the specific entropy  $\eta$ , specific heat at constant pressure  $c_p > 0$ , and a fixed but arbitrary reference pressure  $p_r$ . Finally, the material or Lagrangian derivative  $D/Dt$  is given by

$$\frac{D}{Dt} \doteq \frac{\partial}{\partial t} + \frac{v}{a} \frac{\partial}{\partial \varphi} + \omega \frac{\partial}{\partial p}. \quad (2)$$

Equation (1a) implies the material conservation of the axial component of the absolute angular momentum

$$m(\varphi, p) \doteq u(\varphi, p)a \cos \varphi + \Omega a^2 \cos^2 \varphi, \quad (3)$$

namely, that  $Dm/Dt = 0$ .

It will be convenient to introduce the potential vorticity  $q \doteq \rho^{-1} \mathbf{Z} \cdot \nabla \theta$ , where in the shallow-layer approximation

$$\mathbf{Z} \doteq \nabla \times (u \hat{\lambda}) + f \hat{\mathbf{r}} = \left( \frac{u}{a} \tan \varphi - \frac{1}{a} u_\varphi + 2\Omega \sin \varphi \right) \hat{\mathbf{r}} + \frac{\partial u}{\partial r} \Big|_\varphi \hat{\boldsymbol{\varphi}} \quad (4)$$

is the absolute vorticity associated with an axisymmetric flow. In terms of the radial component of  $\mathbf{Z}$ ,

$$\mathbf{Z} \cdot \hat{\mathbf{r}} = \frac{u}{a} \tan \varphi - \frac{1}{a} u_\varphi + 2\Omega \sin \varphi = -\frac{m_\varphi}{a^2 \cos \varphi}, \quad (5)$$

one may express  $q$  as

$$q = -g \left( \mathbf{Z} \cdot \hat{\mathbf{r}} \theta_p + \frac{1}{a} u_p \theta_\varphi \right) = \frac{g}{a^2 \cos \varphi} \frac{\partial(m, \theta)}{\partial(\varphi, p)} \equiv \frac{1}{\rho a \cos \varphi} \hat{\lambda} \cdot \nabla \theta \times \nabla m. \quad (6)$$

Using the Lagrangian invariance of  $m$  and  $\theta$  together with (1c), (6) may be used to show that  $Dq/Dt = 0$ .

The total energy for a spherical shell (of volume  $V$ ) of this fluid with internal energy per unit mass  $\mathcal{E}(\theta, p)$  and equation of state  $\rho = \rho(\theta, p)$  is given by

$$\bar{\mathcal{H}} \doteq \int_{-\pi/2}^{\pi/2} \int_a^{a+\Delta r(\varphi, t)} \rho \left( \frac{u^2 + v^2}{2} + \mathcal{E} + \Phi \right) a^2 \cos \varphi \, dr \, d\varphi. \quad (7)$$

Upon using the hydrostatic relation and integrating by parts (taking  $p = 0$  on the upper surface), we may rewrite  $\bar{\mathcal{H}}$  in pressure coordinates:

$$\begin{aligned} \bar{\mathcal{H}}\{m, v, \theta, p_0\} &= \frac{1}{g} \int_{-\pi/2}^{\pi/2} \int_0^{p_0(\varphi)} H(m, v, \theta; \varphi, p) a^2 \cos \varphi \, dp \, d\varphi \\ &\quad + \int_{-\pi/2}^{\pi/2} p_0(\varphi) a^3 \cos \varphi \, d\varphi, \end{aligned} \quad (8)$$

where

$$H(m, v, \theta; \varphi, p) \doteq \frac{1}{2} \left( \frac{m}{a \cos \varphi} - \Omega a \cos \varphi \right)^2 + \frac{1}{2} v^2 + \mathcal{E}(\theta, p) + \frac{p}{\rho(\theta, p)} \quad (9)$$

and

$$p_0(\varphi) \doteq g \int_a^{a+\Delta r} \rho \, dr > 0, \quad (10)$$

is the (time-dependent) surface pressure, determined dynamically from the lower boundary condition  $D\Phi/Dt = 0$ , (1c) and (1e):

$$\frac{\partial p_0}{\partial t} + \frac{v}{a} \frac{\partial p_0}{\partial \varphi} = - \int_0^{p_0(\varphi, t)} \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (v \cos \varphi) \, dp.$$

The functions  $\mathcal{E}$  and  $\rho$ , assumed to have continuous second derivatives, are related by Maxwell's thermodynamic relation

$$dE = T d\eta + \frac{p}{\rho^2} d\rho, \quad (11)$$

where  $E(\eta(\theta), \rho(\theta, p)) \doteq \mathcal{E}(\theta, p)$  and  $T$  is the temperature. Hence

$$\mathcal{E}_\theta = c_p \frac{T}{\theta} + \frac{p}{\rho^2} \rho_\theta, \quad (12)$$

$$\mathcal{E}_p = \frac{p}{\rho^2} \rho_p. \quad (13)$$

For convenience, let us rescale  $\mathcal{H} \doteq g\bar{\mathcal{H}}/a^2$ .

The conservation properties of this system can be demonstrated with the following generalization of the transport theorem. For any quantity  $I$  satisfying

$$\frac{D}{Dt} I = S, \quad (14)$$

(e.g., take  $I = u, v$ , or  $\theta$ ) and any arbitrary function  $f = f(I, p)$ , we see that

$$\begin{aligned} & \frac{d}{dt} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{p_0} f(I, p) \cos \varphi dp d\varphi \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{p_0} \left[ f_I (S - \mathbf{v} \cdot \nabla I) + f_p \left( \frac{Dp}{Dt} - \mathbf{v} \cdot \nabla p \right) \right] \cos \varphi dp d\varphi + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f \frac{\partial p_0}{\partial t} \cos \varphi d\varphi \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{p_0} \left[ f_I S + f_p \frac{Dp}{Dt} - \nabla \cdot (f \mathbf{v}) \right] \cos \varphi dp d\varphi + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f \frac{\partial p_0}{\partial t} \cos \varphi d\varphi \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{p_0} (f_I S + f_p \omega) \cos \varphi dp d\varphi. \end{aligned} \quad (15)$$

In the last line we used the divergence theorem and the fact that in pressure coordinates the normal component of the three-dimensional velocity  $\mathbf{v}$  on the



lower surface is

$$\mathbf{v} \cdot \left( \hat{\mathbf{p}} - \frac{1}{a} \frac{\partial p_0}{\partial \varphi} \hat{\varphi} \right) = \omega_0 - \frac{v}{a} \frac{\partial p_0}{\partial \varphi} = \frac{\partial p_0}{\partial t}, \quad (16)$$

where  $\omega_0 \doteq Dp_0/Dt$ , the value of  $\omega$  on the lower surface, is determined by vertically integrating (1c).

For example, one may readily verify that  $d\mathcal{H}/dt = 0$  by repeated application of (15) together with the results  $(\mathcal{E} + p/\rho)_p = 1/\rho$  and, from (1c) and (1e),

$$\begin{aligned} - \int_{-\pi/2}^{\pi/2} \int_0^{p_0} \frac{v}{a} \Phi_\varphi \cos \varphi dp d\varphi &= - \int_{-\pi/2}^{\pi/2} \int_0^{p_0} \Phi \frac{\partial \omega}{\partial p} \cos \varphi dp d\varphi \\ &= - \int_{-\pi/2}^{\pi/2} ag\omega_0 \cos \varphi d\varphi - \int_{-\pi/2}^{\pi/2} \int_0^{p_0} \frac{\omega}{\rho} \cos \varphi dp d\varphi. \end{aligned} \quad (17)$$

The first term on the right-hand side of (17) is seen to vanish upon considering the volume integral of (1c). Note that the surface term of  $\mathcal{H}$  is just a constant proportional to the total mass, as can be seen by setting  $f = 1$  in (15).

In the special case where  $S = 0$  and  $f = f(I)$ , (15) implies that the density-weighted volume integral of an arbitrary function of any Lagrangian invariant  $I$  is constant in time:

$$\frac{d}{dt} \int \rho f(I) dV = \frac{1}{g} \frac{d}{dt} \int_{-\pi/2}^{\pi/2} \int_0^{p_0} f(I) a^2 \cos \varphi dp d\varphi = 0. \quad (18)$$

In particular, the volume integral

$$\mathcal{C}\{m, v, \theta, p_0\} = \int_{-\pi/2}^{\pi/2} \int_0^{p_0(\varphi)} C(m, \theta, q) \cos \varphi dp d\varphi \quad (19)$$

of any arbitrary function  $C$  of the Lagrangian invariants  $m$ ,  $\theta$ , and  $q$  will be conserved:  $d\mathcal{C}/dt = 0$ .

### III. ZONAL EQUILIBRIUM

Consider an equilibrium state representing steady baroclinic zonal flow:

$$u = U(\varphi, p), \quad v = 0, \quad \theta = \Theta(\varphi, p), \quad (20)$$

where  $U$  and  $\Theta$  have continuous first derivatives. For this equilibrium the lower boundary condition  $D\Phi/Dt = 0$  is equivalent to the condition  $\omega = 0$  on the surface; (1c) then implies that  $\omega = 0$  everywhere. Equations (1a) and (1d) are therefore satisfied trivially, while (1b) becomes

$$U^2 \tan \varphi + 2\Omega U a \sin \varphi = -\Phi_\varphi. \quad (21)$$

Upon differentiating this result with respect to  $p$  and using the hydrostatic equation (1e), we obtain the thermal-wind relation

$$2(U \tan \varphi + \Omega a \sin \varphi)U_p = -\Phi_{\varphi p} = -\Phi_{p\varphi} = -\frac{1}{\rho^2} \frac{\partial \rho}{\partial \varphi} \Big|_p = -\frac{\rho_\theta}{\rho^2} \Theta_\varphi, \quad (22)$$

where  $\rho$  and  $\rho_\theta$  are evaluated at  $(\Theta, p)$ . In terms of the total equilibrium angular momentum

$$M(\varphi, p) \doteq U(\varphi, p)a \cos \varphi + \Omega a^2 \cos^2 \varphi, \quad (23)$$

the above relation may be written as

$$\frac{2 \tan \varphi}{a \cos \varphi} M U_p = -\frac{\rho_\theta}{\rho^2} \Theta_\varphi. \quad (24)$$

Any flow of the form (20) that obeys (24) will consequently satisfy the governing equations (1).

It will prove convenient to introduce the equilibrium potential vorticity

$$Q \doteq -g \left( \zeta_{\text{abs}} \Theta_p + \frac{1}{a} U_p \Theta_\varphi \right) = \frac{g}{a^2 \cos \varphi} \frac{\partial(M, \Theta)}{\partial(\varphi, p)}, \quad (25)$$

where  $\zeta_{\text{abs}}$  is the radial component of the absolute vorticity:

$$\zeta_{\text{abs}} \doteq \frac{U}{a} \tan \varphi - \frac{1}{a} U_\varphi + 2\Omega \sin \varphi = -\frac{M_\varphi}{a^2 \cos \varphi}. \quad (26)$$

One sees from (25) that whenever  $Q$  is of definite sign, a single-valued transformation from  $(\varphi, p)$  to  $(M, \Theta)$  will exist. For the stable equilibrium flows of interest, with  $M > 0$ , it will turn out that the condition  $Q \neq 0$  will be separately satisfied in each hemisphere (excluding the equator itself) [cf. (59)]. Anticipating this result, we will only consider flows for which  $Q(\varphi, p)$  vanishes on a set of measure zero (in two dimensions).

We wish to apply the so-called energy–Casimir method (Fjørtoft 1950; Kruskal & Oberman 1958; Arnol’d 1965, 1966; Holm *et al.* 1985; see also Shepherd 1990, §6) to obtain stability criteria for this equilibrium flow. In the absence of dissipation, it is well known that the fully compressible (non-hydrostatic) hydrodynamical equations may be written within a noncanonical Hamiltonian framework in which the cosymplectic tensor  $\mathbf{J}$  is degenerate (Morrison & Greene 1980); *i.e.*, the phase space variable  $\mathbf{u}$  evolves in time according to

$$\frac{\partial}{\partial t} \mathbf{u} = \mathbf{J} \frac{\delta \mathcal{H}}{\delta \mathbf{u}}, \quad (27)$$

such that  $\det \mathbf{J} = 0$ . In addition to the invariants associated with symmetries of the Hamiltonian, such systems conserve additional quantities, known as

Casimirs, that have gradients lying in the kernel of  $\mathbf{J}$ . (For canonical systems, the Casimirs are mere constants.) While stationary solutions of the canonical Hamilton equations correspond to extremals of the Hamiltonian, the existence of Casimirs modifies the expected steady solutions for non-canonical systems: the extremization must now be constrained to surfaces defined by the invariance of the Casimirs. Although we will not prove that  $\mathcal{H}$  provides a Hamiltonian for our constrained dynamical system or that  $\mathcal{C}$  is indeed a Casimir, the energy–Casimir method is still applicable, provided we demonstrate that the resulting conditional extremals are steady solutions of the equations of motion. The stability of these equilibria can then be examined by considering second-order variations of the energy-Casimir invariant.

Let us therefore seek equilibria corresponding to conditional extremals of  $\mathcal{H}$  on surfaces of constant  $\mathcal{C}$ :

$$\begin{aligned}
0 &= \delta(\mathcal{H} + \mathcal{C}) \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{p_0} \left[ (H_m + C_m) \delta m + (H_\theta + C_\theta) \delta \theta + H_v \delta v + C_q \delta q \right] \cos \varphi \, dp \, d\varphi \\
&\quad + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[ H + C + ag \right]_{p=p_0} \delta p_0(\varphi) \cos \varphi \, d\varphi, \tag{28}
\end{aligned}$$

upon appealing to the method of Lagrange multipliers. (As is conventional, we absorb the multipliers into the function  $\mathcal{C}$ .) We use this condition to determine the as yet unspecified function  $\mathcal{C}$ .

First, the condition  $C_q = 0$  can be satisfied by choosing  $C_q = 0$  in each

region of the flow where  $Q$  is of definite sign and imposing the matching conditions

$$\lim_{Q \rightarrow 0^+} C(M, \Theta, Q) = \lim_{Q \rightarrow 0^-} C(M, \Theta, Q) = C(M, \Theta, 0), \quad (29a)$$

$$\lim_{Q \rightarrow 0^+} C_q(M, \Theta, Q) = \lim_{Q \rightarrow 0^-} C_q(M, \Theta, Q) = C_q(M, \Theta, 0) \quad (29b)$$

to guarantee that  $C_q = C_{qq} = 0$  over the entire flow.

In each region of the flow on which  $Q$  has a definite sign (and hence  $\partial(M, \Theta)/\partial(\varphi, p) \neq 0$ ), the independent variables  $\varphi$  and  $p$  may then be expressed, *via* (20), in terms of  $M$  and  $\Theta$ . The equilibrium conditions

$$H_m + C_m = 0, \quad (30a)$$

$$H_\theta + C_\theta = 0 \quad (30b)$$

can therefore be integrated to deduce that  $H+C$  in each region is independent of  $M$  and  $\Theta$ , or equivalently, of  $\varphi$  and  $p$ . Equations (30) determine  $C$  only to within an arbitrary constant. [Constant values of  $C$  in (19) lead to invariants corresponding to the total mass.] In each region of sign-definite  $Q$ , let us choose this constant so that in equilibrium

$$H + C = \frac{1}{2}v^2 - ag = -ag; \quad (31)$$

this choice satisfies (29a) and ensures that the second integral in (28) vanishes. Equations (30) and (31) together with  $H_v = 0$  are sufficient to guarantee that  $\delta(\mathcal{H} + C) = 0$  in the equilibrium state.

Since

$$H_m = \frac{m}{a^2 \cos^2 \varphi} - \Omega, \quad H_v = v, \quad H_\theta = \mathcal{E}_\theta - \frac{p}{\rho^2} \rho_\theta, \quad (32)$$

one finds for the equilibrium flow (20) that  $H_v = 0$  trivially, while evaluation of (30) leads to the prescriptions

$$C_m(M, \Theta, Q) = \left[ -\frac{m}{a^2 \cos^2 \varphi} + \Omega \right]_{\substack{m=M \\ \varphi=\varphi(M, \Theta, Q)}}, \quad (33a)$$

$$C_\theta(M, \Theta, Q) = \left[ -\mathcal{E}_\theta + \frac{p}{\rho^2} \rho_\theta \right]_{\substack{\theta=\Theta \\ p=p(M, \Theta, Q)}}. \quad (33b)$$

Note that the condition

$$\lim_{Q \rightarrow 0^+} \varphi(M, \Theta, Q) = \lim_{Q \rightarrow 0^-} \varphi(M, \Theta, Q) \quad (34)$$

guarantees that both  $C_m$  and  $C_\theta$  are continuous across  $Q = 0$ . Hence by (6), the matching condition (29b) is satisfied by equations (33).

These conditions determine the *form* of  $C$  as a function of  $M$ ,  $\Theta$ , and  $Q$ ; all other variables, including the (normally) independent variables  $\varphi$  and  $p$  must be treated as functions of the conserved quantities  $M$ ,  $\Theta$ , and  $Q$  (to which they are related through the equilibrium profiles). In contrast, when  $\mathcal{H}$  is functionally differentiated, the independent variables are held fixed.

It is important to establish the consistency of (33 *a,b*). To do this, we will need to determine certain partial derivative relations. Equation (25) allows us to write

$$\left. \frac{\partial \varphi}{\partial M} \right|_{\Theta} = \frac{\partial(\varphi, \Theta)}{\partial(M, \Theta)} = \frac{\partial(\varphi, p)}{\partial(M, \Theta)} \cdot \frac{\partial(\varphi, \Theta)}{\partial(\varphi, p)} = \frac{g}{a^2 \cos \varphi} Q^{-1} \Theta_p \quad (35)$$

and

$$\left. \frac{\partial p}{\partial M} \right|_{\Theta} = \frac{\partial(p, \Theta)}{\partial(M, \Theta)} = \frac{\partial(p, \varphi)}{\partial(M, \Theta)} \cdot \frac{\partial(p, \Theta)}{\partial(p, \varphi)} = -\frac{g}{a^2 \cos \varphi} Q^{-1} \Theta_{\varphi}. \quad (36)$$

In a similar manner one finds that

$$\left. \frac{\partial \varphi}{\partial \Theta} \right|_M = -\frac{g}{a^2 \cos \varphi} Q^{-1} M_p = -\frac{g}{a} Q^{-1} U_p \quad (37)$$

and

$$\left. \frac{\partial p}{\partial \Theta} \right|_M = \frac{g}{a^2 \cos \varphi} Q^{-1} M_{\varphi} = -g Q^{-1} \zeta_{\text{abs}}, \quad (38)$$

upon using (26).

Now,  $C_{\theta}$  in (33b) is to be regarded as a function of  $M$ ,  $\Theta$ , and  $Q$  only, with the  $M$  dependence entering through  $p$ . That is,

$$C_{\theta} = -H_{\theta} = -F(\Theta, p(M, \Theta)), \quad (39)$$

where

$$F(\theta, p) \doteq \mathcal{E}_{\theta} - p\rho_{\theta}/\rho^2. \quad (40)$$

Hence

$$C_{\theta m} = -F_p \left. \frac{\partial p}{\partial M} \right|_{\Theta}. \quad (41)$$

Substitution of (12) into (40) yields  $F \equiv c_p T/\Theta$ . Equation (13) leads to

$$\mathcal{E}_{p\theta} = -\frac{2p}{\rho^3} \rho_{\theta} \rho_p + \frac{p}{\rho^2} \rho_{p\theta}, \quad (42)$$

which allows us to simplify

$$F_p = \mathcal{E}_{\theta p} - \frac{\rho_{\theta}}{\rho^2} + \frac{2p}{\rho^3} \rho_p \rho_{\theta} - \frac{p}{\rho^2} \rho_{\theta p} = -\frac{\rho_{\theta}}{\rho^2}. \quad (43)$$

Substitution of (43) and (36) into (41) yields

$$C_{\theta m} = \frac{-B}{a \cos \varphi}, \quad (44)$$

where

$$B \doteq \frac{g}{a} \left( \frac{\rho_\theta}{\rho^2} \right) Q^{-1} \Theta_\varphi. \quad (45)$$

Finally, upon using (33a), (37), (24), and (44) we verify that

$$\begin{aligned} C_{m\theta} &= -\frac{2 \sin \varphi}{a^2 \cos^3 \varphi} M \left. \frac{\partial \varphi}{\partial \Theta} \right|_M = \frac{2g \tan \varphi}{a^3 \cos^2 \varphi} M Q^{-1} U_p \\ &= \frac{g}{a^2 \cos \varphi} \left( \frac{-\rho_\theta}{\rho^2} \right) Q^{-1} \Theta_\varphi = C_{\theta m}. \end{aligned}$$

This establishes the existence of a twice-differentiable function  $C(m, \theta, q)$  satisfying (33).

#### IV. LINEAR STABILITY THEOREM

Since  $\mathcal{H} + \mathcal{C}$  is conserved in time and  $\delta(\mathcal{H} + \mathcal{C}) = 0$  by construction, a sufficient condition for linear stability is that  $\delta^2(\mathcal{H} + \mathcal{C})$  be sign definite (for arbitrary perturbations). In terms of

$$G(m, \theta, q; \varphi, p) \doteq H(m, v, \theta; \varphi, p) + C(m, \theta, q) - v^2/2,$$

$$K(m, \theta, q, p_0; \varphi, p) \doteq G(m, \theta, q; \varphi, p) H(p_0 - p), \quad (46)$$

where  $H$  is the Heaviside unit step function, one may express

$$\mathcal{H} + \mathcal{C} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[ \int_0^{p_0} \frac{1}{2} v^2 dp + \int_0^\infty K(m, \theta, q, p_0; \varphi, p) dp + a g p_0 \right] \cos \varphi d\varphi. \quad (47)$$



We will exploit the fact that

$$\int_0^\infty K_{p_0 p_0} dp = - \int_0^\infty G \frac{d}{dp} \hat{\delta}(p_0 - p) dp = \left. \frac{dG}{dp} \right|_{p=p_0}, \quad (48)$$

upon integrating by parts, where  $\hat{\delta}$  is the Dirac delta function and where

$$\frac{dG}{dp} = G_m m_p + G_\theta \theta_p + G_p. \quad (49)$$

(Note that  $G_q = 0$ .) Although partial derivatives of  $K$  with respect to  $p_0$  are to be taken at fixed  $m$  and  $\theta$ , the integration over  $p$  (at fixed  $\varphi$ ) introduces the implicit dependencies of  $G$  on  $p$  through  $m$  and  $\theta$ .

Since  $K_{mp_0} = G_m \hat{\delta}(p_0 - p)$  and  $K_{\theta p_0} = G_\theta \hat{\delta}(p_0 - p)$ , one finds that

$$\begin{aligned} \delta^2(\mathcal{H} + C) &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{p_0} \left[ \delta v^2 + G_{mm} \delta m^2 + 2G_{m\theta} \delta m \delta \theta + G_{\theta\theta} \delta \theta^2 \right] \cos \varphi dp d\varphi \\ &+ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[ 2G_m \delta m \delta p_0 + 2G_\theta \delta \theta \delta p_0 + \left. \frac{dG}{dp} \delta p_0^2 \right]_{p=p_0} \cos \varphi d\varphi. \end{aligned} \quad (50)$$

No  $\delta q$  contributions appear here since the function  $C$  was chosen to satisfy the matching conditions (29) as well as  $C_q = 0$  and  $C_{qq} = 0$ .

For the equilibrium flow  $G_m = G_\theta = 0$  since  $C$  was constructed to satisfy (30). Moreover, (13) allows us to simplify  $G_p = \mathcal{E}_p - p\rho_p/\rho^2 + 1/\rho$  to deduce that

$$G_p = \frac{1}{\rho} \geq 0. \quad (51)$$

We then see from (49) and (50) that  $\delta^2(\mathcal{H} + C)$  is sign definite (for arbitrary disturbances) if the symmetric matrix

$$\mathbf{A} \doteq \begin{pmatrix} H_{mm} + C_{mm} & H_{m\theta} + C_{m\theta} \\ H_{\theta m} + C_{\theta m} & H_{\theta\theta} + C_{\theta\theta} \end{pmatrix} \quad (52)$$

is positive-definite almost everywhere. Note that the dependent variable  $v$  is decoupled from  $m$  and  $\theta$ .

In computing the Hessian (matrix of second derivatives) of  $H$ , we must hold the independent variables  $\varphi$  and  $p$  fixed. Since  $H_{\theta m} = H_{m\theta} = 0$ , we obtain a diagonal matrix with the elements

$$H_{mm} = \frac{1}{a^2 \cos^2 \varphi}, \quad (53a)$$

$$H_{\theta\theta} = F_\theta, \quad (53b)$$

where  $F$  is given by (40).

On the other hand, in evaluating the Hessian of  $C$ , all variables are to be differentiated with respect to  $M$  or  $\Theta$ . Using (33a), (35), (24), and (45) we calculate for the equilibrium flow,

$$\begin{aligned} C_{mm} &= -\frac{1}{a^2 \cos^2 \varphi} - \frac{2 \sin \varphi}{a^2 \cos^3 \varphi} M \left. \frac{\partial \varphi}{\partial M} \right|_{\Theta} \\ &= \frac{1}{a^2 \cos^2 \varphi} \left[ -1 - 2 \tan \varphi M \left( \frac{g}{a^2 \cos \varphi} \right) Q^{-1} \Theta_p \right] \\ &= \frac{1}{a^2 \cos^2 \varphi} \left( -1 + \frac{g \rho_\theta}{a \rho^2} Q^{-1} \frac{\Theta_\varphi}{U_p} \Theta_p \right) \\ &= \frac{B \Theta_p / U_p - 1}{a^2 \cos^2 \varphi}. \end{aligned} \quad (54a)$$

Next, we find from (39), (38), (43), and (45) that

$$C_{\theta\theta} = -F_\theta - F_p \left. \frac{\partial p}{\partial \Theta} \right|_M = -F_\theta - \frac{\rho_\theta}{\rho^2} g Q^{-1} \zeta_{\text{abs}} = -F_\theta - \frac{B a \zeta_{\text{abs}}}{\Theta_\varphi}. \quad (54b)$$

Note that (25) can be used to eliminate  $\zeta_{\text{abs}}$  from (54b) to obtain

$$C_{\theta\theta} = -F_{\theta} + \frac{BU_p}{\Theta_p} \left( \frac{aQ}{gU_p\Theta_{\varphi}} + 1 \right). \quad (54c)$$

The condition  $H_{mm} + C_{mm} > 0$  is then seen to be equivalent to

$$\frac{B\Theta_p}{U_p} > 0, \quad (55)$$

while the condition  $H_{\theta\theta} + C_{\theta\theta} > 0$  reduces to

$$-\rho_{\theta}Q\zeta_{\text{abs}} > 0. \quad (56)$$

Upon using (53), (54a), (54c), and (44), we can express the condition  $\det \mathbf{A} > 0$  as

$$\frac{B^2}{a^2 \cos^2 \varphi} \left( \frac{aQ}{gU_p\Theta_{\varphi}} \right) > 0, \quad (57)$$

or, after using (45), as  $\rho_{\theta}BU_p > 0$ . Together with (55), this result implies

$$\rho_{\theta}\Theta_p > 0. \quad (58)$$

Note that the thermal-wind relation (24) allows us to rewrite (57) as

$$-\rho_{\theta} \left( \frac{fMQ}{\Omega} \right) > 0. \quad (59)$$

The three criteria given by (56), (58), and (59) are not independent of each other: the determinant condition (59) along with either of the other two inequalities implies the third one.

Given any equation of state for which  $\rho_\theta < 0$ <sup>1</sup> (e.g., for a perfect gas  $\rho_\theta = -\rho/\theta$ ), these criteria reduce to

$$Q\zeta_{\text{abs}} > 0, \quad (60a)$$

$$\frac{\partial\Theta}{\partial r} > 0, \quad (60b)$$

$$\frac{fMQ}{\Omega} > 0. \quad (60c)$$

It is instructive to compare the first criterion with Rayleigh's condition for the stability of a cylindrical vortex to axisymmetric perturbations in an incompressible nonrotating fluid (Rayleigh 1916):

$$u\zeta \doteq \frac{u}{r} \frac{\partial}{\partial r}(ru) > 0. \quad (61)$$

We recognize the second criterion as the usual condition for static stability, while the third criterion is the result previously obtained by Stevens (1983). The three criteria are analogous to those found in the  $f$ -plane Boussinesq case (Cho *et al.* 1993).

For  $u/\Omega > -a$ , the ratio  $M/\Omega$  is positive at the equator. (This condition is easily met for the Earth's atmosphere: since  $\Omega a = 465$  m/s, a violation would require supersonic wind speeds!) Since  $f$  changes sign across the equator, (59) implies that  $Q$  must also change sign. Although this means that  $Q = 0$  [and consequently that (59) is violated] at the equator, the behavior of the integrand in (50) at the point  $\varphi = 0$  does not contribute to

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<sup>1</sup>There are also fluids (e.g., water between 0 and 4°C) with a positive  $\rho_\theta$ : since  $\partial\rho/\partial T|_p = \rho_\theta \partial\theta/\partial T|_p$  and  $\partial\theta/\partial T|_p = c_p^{-1}\theta \partial\eta/\partial T|_p = \theta/T$  is positive,  $\partial\rho/\partial T|_p$  and  $\rho_\theta$  must have the same sign.

the integration over  $\varphi$ . Linear stability requires only that (59) hold almost everywhere.

Equation (59) may be interpreted in terms of the *Richardson number*  $R_i \doteq N^2/U_r^2$ , where  $N^2 \doteq g^2\rho_\theta\Theta_p$  is the square of the Brunt–Väisälä frequency (which may be expressed in the more familiar form  $N^2 = g\Theta_r/\Theta_0$ , where  $\Theta_0 \doteq -\rho/\rho_\theta$  is a characteristic potential temperature). Note that (58) implies that  $N^2 > 0$ , verifying that (58) corresponds to static stability even when  $\rho_\theta < 0$ . With the help of (24), we rewrite (25) as

$$\begin{aligned}
Q &= -g\zeta_{\text{abs}}\Theta_p + \frac{1}{a\rho}U_r\Theta_\varphi \\
&= -g\zeta_{\text{abs}}\Theta_p + \frac{1}{ag}\left(\frac{1}{\rho_\theta}\right)\frac{2\tan\varphi}{a\cos\varphi}MU_r^2 \\
&= -\frac{1}{g\rho_\theta}\left[\zeta_{\text{abs}}N^2 - f\left(\frac{M}{\Omega a^2\cos^2\varphi}\right)U_r^2\right] \\
&= -\frac{fN^2}{g\rho_\theta}\left[\frac{\zeta_{\text{abs}}}{f} - R_i^{-1}\left(1 + \frac{U}{\Omega a\cos\varphi}\right)\right]. \tag{62}
\end{aligned}$$

Hence (59) can be rewritten as

$$\frac{M}{\Omega}\left[\frac{\zeta_{\text{abs}}}{f} - R_i^{-1}\left(1 + \frac{U}{\Omega a\cos\varphi}\right)\right] > 0, \tag{63}$$

or,

$$\frac{\zeta_{\text{abs}}}{f}\left(1 + \frac{U}{\Omega a\cos\varphi}\right) > R_i^{-1}\left(1 + \frac{U}{\Omega a\cos\varphi}\right)^2 > 0, \tag{64}$$

which leads to the stability criterion

$$R_i > \frac{f}{\zeta_{\text{abs}}}\left(1 + \frac{U}{\Omega a\cos\varphi}\right) > 0. \tag{65}$$

For the special case  $U(\varphi, p) = U_0(p)/\cos\varphi$  one finds  $\zeta_{\text{abs}} = f$ ; (65) then reduces to

$$R_i > 1 + \frac{U}{\Omega a \cos\varphi} > 0. \quad (66)$$

## V. NONLINEAR STABILITY THEOREM

From the exact invariants  $\mathcal{H}$  and  $\mathcal{C}$ , let us construct the finite-amplitude disturbance invariant

$$\mathcal{A} \doteq [\mathcal{H} + \mathcal{C}](m, v, \theta, p_0) - [\mathcal{H} + \mathcal{C}](M, 0, \Theta, P_0) \quad (67)$$

for the disturbed state defined by

$$m \doteq M + m', \quad v \doteq v', \quad \theta \doteq \Theta + \theta', \quad p_0 \doteq P_0 + p'_0. \quad (68)$$

This invariant is referred to as the disturbance *pseudoenergy* (e.g., Shepherd 1990). If one defines

$$\Delta \doteq K(m, \theta, q, p_0; \varphi, p) - K(M, \Theta, Q, P_0; \varphi, p), \quad (69)$$

one may express  $\mathcal{A}$  in the compact form

$$\mathcal{A} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( \int_0^{p_0} \frac{1}{2} v^2 dp + \int_0^\infty \Delta dp + agp'_0 \right) \cos\varphi d\varphi. \quad (70)$$

To analyze nonlinear stability, we must extend the domain of the equilibrium profiles  $M(\varphi, p)$  and  $\Theta(\varphi, p)$  to values of  $p > P_0$  subject to the thermal-wind balance condition (24). Bounds on the pseudoenergy can

then be obtained by decomposing  $\Delta$  into its volume and surface contributions:  $\Delta = \Delta_V + \Delta_S$ , where (M. Mu, private communication, 1993)

$$\begin{aligned}\Delta_V &\doteq K(m, \theta, q, p_0; \varphi, p) - K(M, \Theta, Q, p_0; \varphi, p), \\ \Delta_S &\doteq K(M, \Theta, Q, p_0; \varphi, p) - K(M, \Theta, Q, P_0; \varphi, p).\end{aligned}\quad (71)$$

Taylor's Remainder Theorem implies that for each value of  $\varphi$  and  $p$ , there exists some  $\tilde{m} \in (M, m)$ ,  $\tilde{\theta} \in (\Theta, \theta)$  such that

$$\begin{aligned}\Delta_V &= K_m(M, \Theta, Q, p_0)m' + K_\theta(M, \Theta, Q, p_0)\theta' \\ &\quad + \frac{1}{2}K_{mm}(\tilde{m}, \tilde{\theta}, \tilde{q}, p_0)m'^2 + K_{m\theta}(\tilde{m}, \tilde{\theta}, \tilde{q}, p_0)m'\theta' + \frac{1}{2}K_{\theta\theta}(\tilde{m}, \tilde{\theta}, \tilde{q}, p_0)\theta'^2,\end{aligned}\quad (72)$$

where  $K(m, \theta, q, p_0) \equiv K(m, \theta, q, p_0; \varphi, p)$ . Furthermore, in Appendix A we show that Taylor's Remainder Theorem can be extended to the class of piecewise-differentiable functions (with jump discontinuities) having the form of  $\Delta_S$ ; hence, there exists a  $\tilde{p}_0 \in (P_0, p_0)$  such that

$$\Delta_S = K_{p_0}(M, \Theta, Q, P_0)p_0' + \frac{1}{2}K_{p_0p_0}(M, \Theta, Q, \tilde{p}_0)p_0'^2.\quad (73)$$

The linear terms of  $\Delta_V$  vanish since for the equilibrium state  $C$  obeys (30). Upon integration over  $p$ , the first term of (73) cancels the second integral in (70), because of (31). These results reflect the fact that  $\delta(\mathcal{H} + C) = 0$  for the equilibrium flow. Again, no  $\delta q$  contributions arise since the function  $C$  was chosen to satisfy (29).

It will be convenient to decompose the pseudoenergy into volume and surface contributions such that  $\mathcal{A} = \mathcal{A}_V + \mathcal{A}_S$ , where

$$\begin{aligned}\mathcal{A}_V &\doteq \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{p_0} (v'^2 + I_V) dp \cos \varphi d\varphi, \\ \mathcal{A}_S &\doteq \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} I_S \Big|_{p=\tilde{p}_0} \cos \varphi d\varphi.\end{aligned}\tag{74}$$

From (72) and (73), together with (48), we identify

$$I_V \doteq \tilde{G}_{mm} m'^2 + 2\tilde{G}_{m\theta} m' \theta' + \tilde{G}_{\theta\theta} \theta'^2,\tag{75}$$

$$I_S \doteq \frac{d}{dp} G(M, \Theta, Q; \varphi, p) p_0'^2.\tag{76}$$

Here, a tilde is used to indicate that the corresponding function is evaluated at  $(\tilde{m}, \tilde{\theta}; \varphi, p)$ .

For the surface contribution, (49) and (51) imply  $I_S = p_0'^2 / \rho(\Theta, p)$  since  $G_m = G_\theta = 0$  when evaluated for the equilibrium flow. Thus,

$$\mathcal{A}_S = \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{p_0'^2}{\rho(\Theta(\varphi, \tilde{p}_0), \tilde{p}_0)} \cos \varphi d\varphi \geq 0.\tag{77}$$

The following lemma can be used to bound the interior contribution  $I_V$ .

LEMMA 1. *If  $\alpha > 0$  and  $x$  and  $y$  are real, then*

$$-\alpha |G|x^2 - \alpha^{-1} |G|y^2 \leq 2Gxy \leq \alpha |G|x^2 + \alpha^{-1} |G|y^2.\tag{78}$$

*Proof.* The two inequalities

$$0 \leq (\alpha^{1/2} x \pm \alpha^{-1/2} y)^2 = \alpha x^2 \pm 2xy + \alpha^{-1} y^2\tag{79}$$



provide the bounds

$$-\alpha|G|x^2 - \alpha^{-1}|G|y^2 \leq \pm 2|G|xy \leq \alpha|G|x^2 + \alpha^{-1}|G|y^2, \quad (80)$$

which in turn imply (78).  $\square$

The application of Lemma 1 to (75) yields, for any positive function  $\alpha \equiv \alpha(\varphi, p)$

$$I_V \geq \left( \tilde{G}_{mm} - \alpha |\tilde{G}_{m\theta}| \right) m'^2 + \left( \tilde{G}_{\theta\theta} - \alpha^{-1} |\tilde{G}_{m\theta}| \right) \theta'^2. \quad (81)$$

In order to prove nonlinear stability, we would ideally like to bound  $I_V$  and  $I_S$  from above and below by positive-definite quadratic forms. However this is problematical, because  $G$  depends on  $p$  as well as on  $m$ ,  $\theta$ , and  $q$ . The latter dependencies are not difficult to handle since the variables are Lagrangian invariants, and thus their range of values is always the same as the range provided by the initial conditions. But the surface pressure  $p_0$  is a dynamical quantity that can, in principle, take on any positive value whatsoever. Thus the  $p$  dependence in  $G$  is difficult to bound *a priori*. In particular, the surface density  $\rho$  in (77) could conceivably approach either zero or infinity.

Thus we are unable to obtain the sort of strong convexity properties that were employed in the analysis of Cho *et al.* (1993). However, we can still prove nonlinear stability for fluids having the property  $F_\theta \geq 0$ ; it may be verified that this is equivalent to the condition

$$\left. \frac{\partial c_p}{\partial \theta} \right|_p \geq 0. \quad (82)$$

Although (82) does not appear to be a universal property, it includes the important case of a perfect gas with constant specific heats, for which  $F_\theta = 0$ . Given (82), then (81) and (53) imply that

$$I_V \geq \left( \frac{1}{a^2 \cos^2 \varphi} + \tilde{C}_{mm} - \alpha |\tilde{C}_{m\theta}| \right) m'^2 + \left( \tilde{C}_{\theta\theta} - \alpha^{-1} |\tilde{C}_{m\theta}| \right) \theta'^2, \quad (83)$$

noting that  $H_{m\theta} = 0$ .

Now suppose for each function  $h \in \{C_{mm}, |C_{m\theta}|, C_{\theta\theta}\}$  that

$$\{h(\tilde{m}, \tilde{\theta}, \tilde{q}) : (\varphi, p) \in V\} \subseteq \{h(M, \Theta, Q) : (\varphi, p) \in V\}. \quad (84)$$

This will be the case if the perturbation globally introduces no new values of the Lagrangian invariants  $m$ ,  $\theta$ , and  $q$ . The bound (83) may then be used to obtain a lower estimate for  $\mathcal{A}_V$ . [In the most general situation, one needs to extend the domain of the functions  $h$  to the new range of  $\tilde{m}$  and  $\tilde{\theta}$  so as to satisfy these bounds, without affecting (30) and (83). Although this construction is always possible in the one-dimensional case [*cf.* Arnol'd (1966)], it is not yet clear whether two-dimensional functions can be always be extended in an analogous manner.]

Suppose now that there exists some positive constant  $c$  that provides the global bound (for  $-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$  and all  $p \geq 0$ )

$$\min \left( C_{mm} - \alpha |C_{m\theta}|, C_{\theta\theta} - \alpha^{-1} |C_{m\theta}| \right) \geq c > 0. \quad (85)$$

Equations (53), (83), (84), and (85) then imply that the pseudoenergy con-

tribution  $\mathcal{A}_V$  is bounded by

$$\mathcal{A}_V \geq \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{p_0(\varphi)} \left[ v'^2 + \left( \frac{1}{a^2 \cos^2 \varphi} + c \right) m'^2 + c\theta'^2 \right] \cos \varphi \, dp \, d\varphi. \quad (86)$$

Upon substitution of (54) and (44), the nonlinear stability criterion (85) appears as

$$\min \left( \frac{B\Theta_p/U_p - \alpha_0|B| - 1}{a^2 \cos^2 \varphi}, -B \frac{a\zeta_{\text{abs}}}{\Theta_\varphi} - \alpha_0^{-1}|B| \right) \geq c > 0, \quad (87)$$

where  $\alpha_0 \doteq \alpha a \cos \varphi$  and  $B$  is given by (45).

One may now establish nonlinear stability for the flow in terms of the disturbance measure  $\|\mathbf{x}'\|_c$  of the vector  $\mathbf{x}' \doteq (m', v', \theta', p'_0)$ , where

$$\|\mathbf{x}'\|_c^2 \doteq \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{P_0+p'_0} \left[ v'^2 + \left( \frac{1}{a^2 \cos^2 \varphi} + c \right) m'^2 + c\theta'^2 \right] \cos \varphi \, dp \, d\varphi. \quad (88)$$

Here, the surface pressure disturbance  $p'_0$  is related to the potential temperature disturbance  $\theta'$  by

$$p'_0(\varphi) \doteq g \int_a^{a+\Delta r} \rho_\theta \theta' \, dr. \quad (89)$$

Equations (86) and (77) imply that the evolution of the positive-definite quantity  $\|\mathbf{x}'\|_c^2$  is bounded according to

$$\|\mathbf{x}'(t)\|_c^2 \leq \mathcal{A}_V(t) \leq \mathcal{A}(t) = \mathcal{A}(0). \quad (90)$$

Hence finite-amplitude disturbances are bounded for all time by a quantity depending only on the initial state of the flow; the bound decreases to zero as the initial disturbance tends to zero.

The disturbance measure defined by (88) is not actually a norm since it does not satisfy the property of homogeneity: in general,  $\|\lambda \mathbf{x}\|_c \neq |\lambda| \|\mathbf{x}\|_c$  for a scalar  $\lambda$ . However, (88) still provides a meaningful measure of disturbance size since it satisfies the important property that  $\|\mathbf{x}\|_c^2 \geq 0$ , with equality occurring only when  $\mathbf{x} = 0$  almost everywhere (observe that  $\theta' = 0$  implies  $p'_0 = 0$ ).

Note that (85) requires that

$$C_{mm} > \alpha |C_{m\theta}| \geq 0, \quad C_{\theta\theta} > \alpha^{-1} |C_{m\theta}| \geq 0; \quad (91)$$

these inequalities in turn imply

$$C_{mm} > \frac{|C_{m\theta}|}{C_{\theta\theta}} |C_{m\theta}| = \frac{C_{m\theta}^2}{C_{\theta\theta}}. \quad (92)$$

Since for this case  $H_{mm} > 0$  and  $H_{\theta\theta} \geq 0$ , it follows that the nonlinear stability criterion (85) implies the linear stability conditions (60) obtained in Section IV.

Let us examine the nonlinear stability criterion (87) in further detail.

Upon using (25) to eliminate  $\zeta_{\text{abs}}$  we find that

$$\frac{B\Theta_p}{U_p} - 1 - ca^2 \cos^2 \varphi \geq \alpha_0 |B| \geq B\Theta_p \left( \frac{aQ}{g\Theta_\varphi} + U_p \right)^{-1}. \quad (93)$$

Dividing through by the positive expression on the right-hand side of the second inequality, we obtain

$$\frac{aQ}{gU_p\Theta_\varphi} \geq (1 + ca^2 \cos^2 \varphi) \frac{1}{B\Theta_p} \left( \frac{aQ}{g\Theta_\varphi} + U_p \right) > 0. \quad (94)$$

We now use (24) to eliminate  $U_p$  on the left-hand side. Upon recalling (45) we obtain a nonlinear generalization of (59):

$$-\rho_\theta \left( \frac{fMQ}{\Omega} \right) \geq a \cos^2 \varphi (1 + ca^2 \cos^2 \varphi) \frac{\rho_\theta Q}{\Theta_p} \left( \frac{a}{g} Q + U_p \Theta_\varphi \right) > 0. \quad (95)$$

Analogues of the other linear stability conditions can also be obtained from (87), which requires that

$$-\rho_\theta Q \zeta_{\text{abs}} \geq \frac{\rho^2 Q^2}{g} (c + \alpha_0^{-1} |B|) > 0, \quad (96a)$$

$$\frac{B \Theta_p}{U_p} \geq 1 + ca^2 \cos^2 \varphi + \alpha_0 |B| > 0 \quad (96b)$$

for some positive constant  $c$ . Equations (96a) and (96b) are nonlinear generalizations of (56) and (55), respectively. Substitution of (45) and (24) into (96b), together with (95), leads to the criterion

$$\rho_\theta \Theta_p \geq -\frac{\rho_\theta}{g} \left( \frac{fMQ}{\Omega} \right)^{-1} Q^2 a^2 \cos^2 \varphi (1 + ca^2 \cos^2 \varphi + \alpha_0 |B|) > 0, \quad (97)$$

which is a nonlinear generalization of (58). Evidently these nonlinear conditions are all somewhat more restrictive than the corresponding linear stability conditions.

Let us emphasize that, although we have provided a bound on the nonlinear evolution of the disturbance measure  $\|\mathbf{x}\|_c$ , we have not proven Liapunov stability. Because of the possibility that  $\rho$  could approach arbitrarily close to zero as the flow evolves (*e.g.*, in the presence of cavitation), there appears to be no way of providing an upper bound on the surface contribution to

the pseudoenergy. This treatment differs from the approach taken by Holm *et al.* (1983) upon encountering a similar difficulty: since their calculation required a nonzero lower bound on the density, these authors simply made this a hypothesis. In contrast, the treatment here is dynamically self-consistent in that it requires no externally imposed assumptions on  $\rho$ .

## VI. DISCUSSION

We have applied the energy–Casimir method to the problem of symmetric stability of a compressible, hydrostatic planetary atmosphere with a general equation of state, including a proper treatment of the lower boundary condition. It has been shown that steady baroclinic zonal flows of the form (20) satisfy the variational principle (28), and are thus conditional extremals of the energy subject to constraints provided by a suitably constructed Casimir-like invariant. By considering the second variation of the corresponding energy–Casimir (or pseudoenergy) invariant functional about these equilibria, linear symmetric stability criteria have been derived: namely (56), (58), and (59). In the special case of a perfect gas these criteria reduce to (60), as previously derived by Stevens (1983).

In comparison with the  $f$ -plane, Boussinesq analysis of symmetric stability of Cho *et al.* (1993), the present analysis exhibits two interesting mathematical features: a multi-valued equilibrium profile and a moving lower boundary in  $p$ -coordinates. The first of these is dealt with by allowing the

Casimir-like functions to have an additional dependence on the potential vorticity, which identifies the branch of the function. The second feature leads to an explicit contribution from the lower boundary to the pseudoenergy. However this contribution is positive definite, and thus presents no problem for the linear stability analysis.

For finite-amplitude disturbances, on the other hand, the moving domain of integration presents significant difficulties because the range of values of  $p$  cannot be bounded *a priori*. This means that strong convexity properties on the pseudoenergy, such as were employed by Cho *et al.* (1993), are not available in this case. In particular, Liapunov stability does not appear to be provable. However, for fluids satisfying (82) — this includes the important case of a perfect gas with constant specific heats — we show that the pseudoenergy is positive definite and that nonlinear stability conditions can be derived. By nonlinear stability we mean that a positive definite measure of disturbance amplitude is bounded uniformly in time, with the bound decreasing to zero as the initial disturbance tends to zero; this result is expressed by (90). The nonlinear stability conditions imply linear stability, and are expressible as generalizations of the linear symmetric stability conditions.

From the stability bound (90) we also obtain a rigorous upper bound on the kinetic energy of the motion in the meridional plane:

$$\frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{p_0} v'^2(t) \cos \varphi dp d\varphi \leq \mathcal{A}(0). \quad (98)$$

Since (98) holds for *any* equilibrium for which  $I_V \geq 0$  everywhere, it may be

used to define a generalized available potential energy for a given baroclinic zonal flow [see Shepherd (1993)]: simply choose a stable equilibrium, compute  $\mathcal{A}(0)$  for the resulting disturbance field (the difference between the initial and the equilibrium values), and then minimize  $\mathcal{A}(0)$  over all possible stable equilibria. This construction of a generalized available potential energy was used by Cho *et al.* (1993) to provide rigorous upper bounds on the nonlinear saturation of symmetric instabilities. A similar calculation would certainly be possible in the present case.

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## Appendix A. GENERALIZED TAYLOR'S REMAINDER THEOREM

We show that Taylor's Remainder Theorem may be extended to the class of piecewise-differentiable functions (with jump discontinuities) represented by  $\Delta_S$ . Appealing to the theory of distribution functions, we will say that two distributions  $f$  and  $h$  are *equal* iff

$$\int f(p)g(p) dp = \int h(p)g(p) dp \quad (\text{A1})$$

for all non-negative differentiable functions  $g$ . Note that this definition of equality does not require  $f$  and  $h$  to agree pointwise.

**THEOREM A.1.** *Let  $G$  be a differentiable classical function on the real interval  $[a, b]$  and  $H$  be the Heaviside unit step function. Then there exists a real number  $c \in (a, b)$  such that, considered as distributions in  $p$ ,*

$$\begin{aligned} & G(p)H(b-p) - G(p)H(a-p) \\ &= (b-a)G(a)\hat{\delta}(a-p) + \frac{1}{2}(b-a)^2 \left[ G'(c)\hat{\delta}(c-p) + G(c)\hat{\delta}'(c-p) \right], \end{aligned} \quad (\text{A2})$$

where  $\hat{\delta}$  is the Dirac delta function.

*Proof.* Let  $g$  be any non-negative differentiable function. Consider the twice-differentiable function  $F(x) = \int_a^x G(p)g(p) dp$  on  $[a, b]$ . By Taylor's

Remainder Theorem, there exists a  $c \in (a, b)$  such that

$$F(b) - F(a) = (b - a)F'(a) + \frac{1}{2}(b - a)^2F''(c). \quad (\text{A3})$$

Upon substituting the expressions

$$\begin{aligned} F'(x) &= G(x)g(x), \\ F''(x) &= G'(x)g(x) + G(x)g'(x) \end{aligned} \quad (\text{A4})$$

and using the fact that

$$g'(x) = \int g'(p)\hat{\delta}(x-p) dp = - \int g(p)\frac{d}{dp}\hat{\delta}(x-p) dp = \int g(p)\hat{\delta}'(x-p) dp, \quad (\text{A5})$$

one concludes that

$$\begin{aligned} &\int [G(p)\text{H}(b-p) - G(p)\text{H}(a-p)] g(p) dp \\ &= (b-a) \int G(a)\hat{\delta}(a-p) g(p) dp \\ &\quad + \frac{1}{2}(b-a)^2 \int [G'(c)\hat{\delta}(c-p) + G(c)\hat{\delta}'(c-p)] g(p) dp. \end{aligned} \quad (\text{A6})$$

□

The distribution  $G(p)\text{H}(b-p) - G(p)\text{H}(a-p)$  corresponds to  $\Delta_S$ , with the identification  $a = P_0$  and  $b = p_0$ . Thus Theorem A.1 leads immediately to (73).

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