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Hybrid Model in General Geometry

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Abstract

We propose hybrid model equations in toroidal (or more general) geometry for magnetically confined plasmas. This is suitable for low frequency toroidal modes, for example, the trapped electron and current diffusive ballooning instabilities. This model consists of fluid ions and drift kinetic electrons. We discuss the numerical algorithm of these model equations. The linear dispersion relation of this model equations that defines the requirements of the model for describing these modes is also discussed.

I. INTRODUCTION

In a study of nonlinear developments of low frequency instabilities in magnetically confined toroidal plasmas such as tokamaks, it is important to reduce noise, while retaining the relevant correct physics such as the low frequency hydrodynamic and/or drift motions and the toroidal geometry. It demands sophisticated algorithms to do so. In this article we explore a toroidal hybrid algorithm to accomplish this goal. The hybrid models have been pioneered by many authors for their advantages of reducing noises while retaingin necessary physics.[1–3] Global toroidal kinetic ion simulations have been carried out to show the radially extended streamer signatures reminiscent of TFTR beam emission spectroscopy measurements[4] and to suggest the exponentially radially increasing χ_i.[6] Such an effect is not observed in cylindrical plasma simulations, otherwise identical to this toroidal simulation. In these simulations the modes are above the marginal stability but close to it, to make the modes heavily affected by kinetic resonance effects, thus demonstrating the importance of geometric effects on long range interaction of the Coulombic force in magnetically confined plasmas. In these runs, however, electron kinetic effects, i.e. trapped particle effects have been neglected. There is some evidence that the electron transport is distinct from the ion transport in recent large tokamak experiments.[5]

Recently, Itoh *et al.*[7] proposed current diffusive ballooning mode to explain the so-called L-mode physics and they have shown this model produces the typical L-mode scaling. As they have shown, the electron is easily destabilized by electron nonlinearity and it give rise to a higher saturation level.[8]

To investigate electron dynamics more accurately, we use the drift kinetic electron response. We use, however, the fluid ion response for simplicity, although the kinetic ion response is also an important factor to explain the ion power loss channel as demonstrated in Lebrun *et al.*[6] and Tajima *et al.*[9] The target modes are the trapped electron modes and current diffusive ballooning modes.

II. MODEL EQUATIONS

In this section, we discuss our hybrid model with electrons being treated as particles and ions as fluid. For electrons, we solve the drift kinetic equations given by

$$\mathbf{v}_d = \mathbf{u} + \frac{\mathbf{b}}{\Omega_e} \times \left\{ \frac{\mu}{m} \nabla B + v_\parallel^2 (\mathbf{b} \cdot \nabla \mathbf{b}) \right\} \ , \tag{1}$$

and

$$\frac{dv_\parallel}{dt} = -\frac{e}{m} E_\parallel - \frac{\mu}{m} \mathbf{b} \cdot \nabla B \ , \tag{2}$$

and

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}_d + v_\parallel \mathbf{b} \ , \tag{3}$$

where $\mathbf{u} = \mathbf{E} \times \mathbf{b}/B$, $\mathbf{b} = \mathbf{B}/B$, $\mu = mv_\perp^2/(2B)$, $\Omega_e = eB/m$, and m is the electron mass. For ions, we solve the vorticity equation given by

$$\nabla \cdot \left(\overline{n} m_i \frac{\mathbf{B}}{B_s B^s} \times \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \mathbf{u} \right) = -\nabla \cdot \mathbf{j}_e \ , \tag{4}$$

where we take into account only the equilibrium part for the density and the repeated sub/super script $B_s B^s$ means to sum over for $s = 1, 2, 3$. The left-hand side of Eq. (4) represents the ion polarization current terms. This is equivalent to demand the quasineutrality condition $\nabla \cdot j_i + \nabla \cdot j_e = 0$. The electron current is calculated from

$$\mathbf{j}_{(e)} = -p_{e\perp} \frac{c\mathbf{b} \times \nabla B}{B^2} - p_{e\|} \frac{c\mathbf{b} \times (\mathbf{b} \cdot \nabla \mathbf{b})}{B} + j_{\|}\mathbf{b} \ . \tag{5}$$

We difine following quantities

$$p_{e\perp} = \int \frac{1}{2} m v_\perp^2 \delta f d^3 v \ , p_{e\|} = \int m v_\|^2 \delta f d^3 v \ , j_\| = -e \int v_\| \delta f d^3 v \ ,$$

where $f = f_M + \delta f$.

For convenience we introduce the following normalizations

$$\begin{aligned} &E_* = \frac{eE}{m\omega_p^2 \Delta} \ , \quad && B_* = \frac{eB}{m\omega_p c} \ , \quad && v_* = \frac{v}{\omega_p \Delta} \ , \\ &t_* = t\omega_p \ , && m_i^* = \frac{m_i}{m_e} \ , && \overline{n}_* = \frac{\overline{n}}{n_0} \ , \\ &\delta f_* = \frac{\delta f}{n_0} \ , \end{aligned} \tag{6}$$

where Δ is the grid spacing, ω_p, the plasma frequency. We then obtain the normalized dynamical equations for our system:

$$\frac{dv_{\|*}}{dt_*} = -E_* - \mu_* \mathbf{b} \cdot \nabla_* B_* \ ,$$

$$\frac{d\mathbf{x}_*}{dt} = v_{\|*}\mathbf{b} + \frac{\mathbf{b} \times \nabla_* \phi_*}{B_*} + \frac{v_{\|*}^2 \mathbf{b} \times (\mathbf{b} \cdot \nabla_* \mathbf{b})}{B_*} + \mu_* \frac{\mathbf{b} \times \nabla_* B_*}{B_*} \ ,$$

$$\nabla_* \cdot \left(\frac{\overline{n}_* m_{i*}}{B_*} \mathbf{b} \times \left(\frac{\partial}{\partial t_*} + \mathbf{u}_* \cdot \nabla_* \right) \mathbf{u}_* \right) = -\nabla_* \cdot \mathbf{j}_{e*} \ ,$$

$$\mathbf{j}_{e*} = -p_{\perp*} \frac{\mathbf{b} \times \nabla_* B_*}{B_*^2} - p_{\parallel*} \frac{\mathbf{b} \times (\mathbf{b} \cdot \nabla_* \mathbf{b})}{B_*} + j_{\parallel*} \mathbf{b} \ , \tag{7}$$

where $\mathbf{u}_* = \mathbf{b} \times \nabla_* \phi_* / B_*$, $\mu_* = v_{\perp*}^2/(2B_*)$ and

$$p_{\perp*} = \frac{1}{2} \int v_{\perp*}^2 \delta f_* d^3v \ , p_{\parallel*} = \int v_{\parallel*}^2 \delta f d^3v \ , j_{\parallel*} = -\int v_{\parallel*} \delta f_* d^3 v \ . \tag{8}$$

In the following, we will drop asterisks for simplicity.

We now write down the explicit form of Eq. (3) in general geometry. As well known, in general metric contravariant and covariant vectors are no longer the same as is the case in Cartesian geometry. First we define a contravariant vector $\{u^1, u^2, u^3\}$

$$\begin{aligned} u^1 &= \mathbf{u} \cdot \mathbf{e}^1 = \frac{1}{\sqrt{g}\, B_s B^s} \left(B_2 \frac{\partial \widetilde{\phi}}{\partial x^3} - B_3 \frac{\partial \widetilde{\phi}}{\partial x^2} \right) \ , \\ u^2 &= \mathbf{u} \cdot \mathbf{e}^2 = \frac{1}{\sqrt{g}\, B_s B^s} \left(B_3 \frac{\partial \widetilde{\phi}}{\partial x^1} - B_1 \frac{\partial \widetilde{\phi}}{\partial x^3} \right) \ , \\ u^3 &= \mathbf{u} \cdot \mathbf{e}^3 = \frac{1}{\sqrt{g}\, B_s B^s} \left(B_1 \frac{\partial \widetilde{\phi}}{\partial x^2} - B_2 \frac{\partial \widetilde{\phi}}{\partial x^1} \right) \ , \end{aligned} \tag{9}$$

where $B_1 = \mathbf{B} \cdot \mathbf{e}_1$, $B_2 = \mathbf{B} \cdot \mathbf{e}_2$, $B_3 = \mathbf{B} \cdot \mathbf{e}_3$ and $B^1 = \mathbf{B} \cdot \mathbf{e}^1$, $B^2 = \mathbf{B} \cdot \mathbf{e}^2$, $B^3 = \mathbf{B} \cdot \mathbf{e}^3$.

If we set

$$\mathbf{F} = \overline{n} m_i \frac{\mathbf{B}}{B_s B^s} \times \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \mathbf{u} \ , \tag{10}$$

then we write

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x}^{i} \sqrt{g}\, F^i = -\frac{1}{\sqrt{g}} \frac{\partial}{\partial x}^{i} \sqrt{g}\, j_e^i \ , \tag{11}$$

where

$$F^1 = \frac{\overline{n} m_i}{B_s B^s} B^2 \sqrt{g} g^{11} \left\{ \frac{du^3}{dt} + u^i u^j \left\{ \begin{matrix} 3 \\ i\,j \end{matrix} \right\} \right\}$$

$$- \frac{\overline{n} m_i}{B_s B^s} B^3 \sqrt{g}\, g^{11} \left\{ \frac{du^2}{dt} + u^i u^j \left\{ \begin{matrix} 2 \\ i\,j \end{matrix} \right\} \right\} , \qquad (12)$$

$$F^2 = -\frac{\overline{n} m_i}{B_s B^s} B^1 \sqrt{g} g^{22} \left\{ \frac{du^3}{dt} + u^i u^j \left\{ \begin{matrix} 3 \\ i\,j \end{matrix} \right\} \right\}$$

$$+ \frac{\overline{n} m_i}{B_s B^s} B^3 \sqrt{g} g^{22} \left\{ \frac{du^1}{dt} + u^i u^j \left\{ \begin{matrix} 1 \\ i\,j \end{matrix} \right\} \right\} , \qquad (13)$$

$$F^3 = \frac{\overline{n} m_i}{B_s B^s} B^1 \sqrt{g} g^{33} \left\{ \frac{du^2}{dt} + u^i u^j \left\{ \begin{matrix} 2 \\ i\,j \end{matrix} \right\} \right\}$$

$$- \frac{\overline{n} m_i}{B_s B^s} B^2 \sqrt{g} g^{33} \left\{ \frac{du^1}{dt} + u^i u^j \left\{ \begin{matrix} 1 \\ i\,j \end{matrix} \right\} \right\} , \qquad (14)$$

where

$$[i, jk] = \frac{1}{2} \left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i} \right) = [i, kj] , \qquad (15)$$

and

$$\left\{ \begin{matrix} i \\ jk \end{matrix} \right\} = \frac{1}{2} g^{il} \left(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \right) = \left\{ \begin{matrix} i \\ kj \end{matrix} \right\} , \qquad (16)$$

where Eqs. (15) and (16) express the Christoffel symbols of the first and second kinds in terms of the components of the metric tensor of the underlying curvilinear coordinate system.[10] The electron current components are

$$j_e^1 = -\frac{p_{e\perp}}{B} \frac{\mathbf{e}^1 \cdot \mathbf{b} \times \nabla B}{B} - \frac{p_{e\parallel}}{B} \mathbf{e}^1 \cdot \mathbf{b} \times (\mathbf{b} \cdot \nabla \mathbf{b}) + j_\parallel \frac{B^1}{B} , \qquad (17)$$

$$j_e^2 = -\frac{p_{e\perp}}{B}\frac{\mathbf{e}^2\cdot\mathbf{b}\times\nabla B}{B} - \frac{p_{e\parallel}}{B}\,\mathbf{e}^2\cdot\mathbf{b}\times(\mathbf{b}\cdot\nabla\mathbf{b}) + j_\parallel\frac{B^2}{B}\,, \tag{18}$$

$$j_e^3 = -\frac{p_{e\perp}}{B}\frac{\mathbf{e}^3\cdot\mathbf{b}\times\nabla B}{B} - \frac{p_{e\parallel}}{B}\,\mathbf{e}^3\cdot\mathbf{b}\times(\mathbf{b}\cdot\nabla\mathbf{b}) + j_\parallel\frac{B^3}{B}\,, \tag{19}$$

where

$$\frac{\mathbf{e}^1\cdot\mathbf{b}\times\nabla B}{B} = \frac{1}{B_sB^s}\frac{1}{\sqrt{g}}\left(B_2\frac{\partial B}{\partial x^3} - B_3\frac{\partial B}{\partial x^2}\right)\,, \tag{20}$$

$$\begin{aligned}\mathbf{e}^1\cdot\mathbf{b}\times(\mathbf{b}\cdot\nabla\mathbf{b}) = &-\frac{1}{B_sB^s}\frac{1}{\sqrt{g}}\left(\frac{\partial B_3}{\partial x^2} - \frac{\partial B_2}{\partial x^3}\right)\frac{B_2B^2 + B_3B^3}{\sqrt{B_sB^s}}\\ &+\frac{1}{B_sB^s}\frac{1}{\sqrt{g}}\left(\frac{\partial B_1}{\partial x^3} - \frac{\partial B_3}{\partial x^1}\right)\frac{B_2B^1}{\sqrt{B_sB^s}}\\ &+\frac{1}{B_sB^s}\frac{1}{\sqrt{g}}\left(\frac{\partial B_2}{\partial x^1} - \frac{\partial B_1}{\partial x^2}\right)\frac{B_3B^1}{\sqrt{B_sB^s}}\,,\end{aligned} \tag{21}$$

$$\frac{\mathbf{e}^2\cdot\mathbf{b}\times\nabla B}{B} = \frac{1}{B_sB^s}\frac{1}{\sqrt{g}}\left(B_3\frac{\partial B}{\partial x^1} - B_1\frac{\partial B}{\partial x^3}\right)\,, \tag{22}$$

$$\begin{aligned}\mathbf{e}^2\cdot\mathbf{b}\times(\mathbf{b}\cdot\nabla\mathbf{b}) = &\frac{1}{B_sB^s}\frac{1}{\sqrt{g}}\left(\frac{\partial B_3}{\partial x^2} - \frac{\partial B_2}{\partial x^3}\right)\frac{B_1B^2}{\sqrt{B_sB^s}}\\ &-\frac{1}{B_sB^s}\frac{1}{\sqrt{g}}\left(\frac{\partial B_1}{\partial x^3} - \frac{\partial B_3}{\partial x^1}\right)\frac{B_1B^1 + B_3B^3}{\sqrt{B_sB^s}}\\ &-\frac{1}{B_sB^s}\frac{1}{\sqrt{g}}\left(\frac{\partial B_2}{\partial x^1} - \frac{\partial B_1}{\partial x^2}\right)\frac{B_3B^2}{\sqrt{B_sB^s}}\,,\end{aligned} \tag{23}$$

$$\frac{\mathbf{e}^3\cdot\mathbf{b}\times\nabla B}{B} = \frac{1}{B_sB^s}\frac{1}{\sqrt{g}}\left(B_1\frac{\partial B}{\partial x^2} - B_2\frac{\partial B}{\partial x^1}\right)\,, \tag{24}$$

$$\mathbf{e}^3 \cdot \mathbf{b} \times (\mathbf{b} \cdot \nabla \mathbf{b}) = \frac{1}{B_s B^s} \frac{1}{\sqrt{g}} \left(\frac{\partial B_3}{\partial x^2} - \frac{\partial B_2}{\partial x^3} \right) \frac{B_1 B^3}{\sqrt{B_s B^s}}$$
$$- \frac{1}{B_s B^s} \frac{1}{\sqrt{g}} \left(\frac{\partial B_1}{\partial x^3} - \frac{\partial B_3}{\partial x^1} \right) \frac{B_2 B^3}{\sqrt{B_s B^s}}$$
$$+ \frac{1}{B_s B^s} \frac{1}{\sqrt{g}} \left(\frac{\partial B_2}{\partial x^1} - \frac{\partial B_1}{\partial x^2} \right) \frac{B_1 B^1 + B_2 B^2}{\sqrt{B_s B^s}} . \quad (25)$$

The contravariants of the equation of motion are written as

$$\frac{dx^1}{dt} = u^1 + \frac{v_\perp^2}{2B} \frac{\mathbf{e}^1 \cdot \mathbf{b} \times \nabla B}{B} + \frac{v_\parallel^2}{B} \mathbf{e}^1 \cdot \mathbf{b} \times (\mathbf{b} \cdot \nabla \mathbf{b}) + v_\parallel \frac{B^1}{B} , \quad (26)$$

$$\frac{dx^2}{dt} = u^2 + \frac{v_\perp^2}{2B} \frac{\mathbf{e}^2 \cdot \mathbf{b} \times \nabla B}{B} + \frac{v_\parallel^2}{B} \mathbf{e}^2 \cdot \mathbf{b} \times (\mathbf{b} \cdot \nabla \mathbf{b}) + v_\parallel \frac{B^2}{B} , \quad (27)$$

$$\frac{dx^3}{dt} = u^3 + \frac{v_\perp^2}{2B} \frac{\mathbf{e}^3 \cdot \mathbf{b} \times \nabla B}{B} + \frac{v_\parallel^2}{B} \mathbf{e}^3 \cdot \mathbf{b} \times (\mathbf{b} \cdot \nabla \mathbf{b}) + v_\parallel \frac{B^3}{B} , \quad (28)$$

and

$$\frac{dv_\parallel}{dt} = \frac{1}{B} \left(B^1 \frac{\partial \phi}{\partial x^1} + B^2 \frac{\partial \phi}{\partial x^2} + B^3 \frac{\partial \phi}{\partial x^3} \right)$$
$$- \frac{v_\perp^2}{2B} \left(B^1 \frac{\partial B}{\partial x^1} + B^2 \frac{\partial B}{\partial x^2} + B^3 \frac{\partial B}{\partial x^3} \right) . \quad (29)$$

III. ORTHOGONAL COORDINATE SYSTEM

We now consider orthogonal coordinate systems. In an orthogonal system we can write the metric as

$$\begin{aligned} &g_{11} = h_1^2 \;, \qquad g_{22} = h_2^2 \;, \qquad g_{33} = h_3^2 \;, \\ &g^{11} = \frac{1}{h_1^2} \;, \qquad g^{22} = \frac{1}{h_2^2} \;, \qquad g^{33} = \frac{1}{h_3^2} \;, \\ &g_{ij} = 0 \;, (i \neq j) \; g^{ij} = 0 \;, (i \neq i) \; \sqrt{g} = h_1 h_2 h_3 \;, \end{aligned} \tag{30}$$

and

$$\begin{aligned}
&\left\{ \begin{matrix} 1 \\ 1\,1 \end{matrix} \right\} = \frac{1}{h_1}\frac{\partial h_1}{\partial x^1} \,, \left\{ \begin{matrix} 1 \\ 2\,2 \end{matrix} \right\} = -\frac{h_2}{h_1^2}\frac{\partial h_2}{\partial x^1} \,, \left\{ \begin{matrix} 1 \\ 3\,3 \end{matrix} \right\} = -\frac{h_3}{h_1^2}\frac{\partial h_3}{\partial x^1} \,, \\
&\left\{ \begin{matrix} 1 \\ 1\,3 \end{matrix} \right\} = \frac{1}{h_1}\frac{\partial h_1}{\partial x^3} \,, \left\{ \begin{matrix} 1 \\ 1\,2 \end{matrix} \right\} = \frac{1}{h_1}\frac{\partial h_1}{\partial x^2} \,, \left\{ \begin{matrix} 1 \\ 2\,3 \end{matrix} \right\} = 0 \,, \\
&\left\{ \begin{matrix} 2 \\ 1\,1 \end{matrix} \right\} = -\frac{h_1}{h_2^2}\frac{\partial h_1}{\partial x^2} \,, \left\{ \begin{matrix} 2 \\ 2\,2 \end{matrix} \right\} = \frac{1}{h_2}\frac{\partial h_2}{\partial x^2} \,, \left\{ \begin{matrix} 2 \\ 3\,3 \end{matrix} \right\} = -\frac{h_3}{h_2^2}\frac{\partial h_3}{\partial x^2} \,, \\
&\left\{ \begin{matrix} 2 \\ 1\,2 \end{matrix} \right\} = \frac{1}{h_2}\frac{\partial h_2}{\partial x^1} \,, \left\{ \begin{matrix} 2 \\ 2\,3 \end{matrix} \right\} = \frac{1}{h_2}\frac{\partial h_2}{\partial x^3} \,, \left\{ \begin{matrix} 2 \\ 3\,1 \end{matrix} \right\} = 0 \,, \\
&\left\{ \begin{matrix} 3 \\ 1\,1 \end{matrix} \right\} = -\frac{h_1}{h_3^2}\frac{\partial h_1}{\partial x^3} \,, \left\{ \begin{matrix} 3 \\ 2\,2 \end{matrix} \right\} = -\frac{h_2}{h_3^2}\frac{\partial h_2}{\partial x^3} \,, \left\{ \begin{matrix} 3 \\ 3\,3 \end{matrix} \right\} = \frac{1}{h_3}\frac{\partial h_3}{\partial x^3} \,, \\
&\left\{ \begin{matrix} 3 \\ 1\,3 \end{matrix} \right\} = \frac{1}{h_3}\frac{\partial h_3}{\partial x^1} \,, \left\{ \begin{matrix} 3 \\ 2\,3 \end{matrix} \right\} = \frac{1}{h_3}\frac{\partial h_3}{\partial x^2} \,, \left\{ \begin{matrix} 3 \\ 1\,2 \end{matrix} \right\} = 0 \,.
\end{aligned} \tag{31}$$

More specifically we concentrate on the toroidal coordinate system (r, χ, ζ), which reads

$$
\begin{aligned}
u^1 &= \frac{B_\chi}{h_\zeta B^2}\frac{\partial \tilde{\phi}}{\partial \zeta} - \frac{B_\zeta}{h_\chi B^2}\frac{\partial \tilde{\phi}}{\partial \chi} , \\
u^2 &= \frac{B_\zeta}{h_\chi B^2}\frac{\partial \tilde{\phi}}{\partial r} , \\
u^3 &= -\frac{B_\chi}{h_\zeta B^2}\frac{\partial \tilde{\phi}}{\partial r} ,
\end{aligned} \tag{32}
$$

where $|B|^2 = B_\chi^2 + B_\zeta^2$, $h_r = 1$, $h_\chi = r/r_0$, and $h_\zeta = R/R_0$. Here we have

$$
F^1 = \frac{\overline{n}m_i}{B^2} B_\chi h_\zeta \frac{\partial u^3}{\partial t} - \frac{\overline{n}m_i}{B^2} B_\zeta h_\chi \frac{\partial u^2}{\partial t} + S^1 , \tag{33}
$$

$$
\begin{aligned}
S^1 = &+\frac{\overline{n}m_i}{B^2} B_\chi h_\zeta \left(u^i \frac{\partial u^3}{\partial x^i} + 2\frac{1}{h_\zeta}\frac{\partial h_\zeta}{\partial r} u^1 u^3 + 2\frac{1}{h_\zeta}\frac{\partial h_\zeta}{\partial \chi} u^2 u^3 \right) \\
&-\frac{\overline{n}m_i}{B^2} B_\zeta h_\chi \left(u^i \frac{\partial u^2}{\partial x^i} - \frac{h_\zeta}{h_\chi^2}\frac{\partial h_\zeta}{\partial \chi} u^3 u^3 + 2\frac{1}{h_\chi}\frac{\partial h_\chi}{\partial r} u^1 u^2 \right) ,
\end{aligned} \tag{34}
$$

where $x^1 = r$, $x^2 = \chi$, and $x^3 = \zeta$,

$$
F^2 = \frac{\overline{n}m_i}{B^2} B_\zeta \frac{1}{h_\chi}\frac{\partial u^1}{\partial t} + S^2 , \tag{35}
$$

$$
S^2 = +\frac{\overline{n}m_i}{B^2} B_\zeta \frac{1}{h_\chi} \left(u^i \frac{\partial u^1}{\partial x^i} - h_\chi \frac{\partial h_\chi}{\partial r} u^2 u^2 - h_\zeta \frac{\partial h_\zeta}{\partial r} u^3 u^3 \right) , \tag{36}
$$

and

$$
F^3 = -\frac{\overline{n}m_i}{B^2} B_\chi \frac{1}{h_\zeta}\frac{\partial u^1}{\partial t} + S^3 , \tag{37}
$$

$$
S^3 = -\frac{\overline{n}m_i}{B^2} B_\chi \frac{1}{h_\zeta} \left(u^i \frac{pu^1}{\partial x^i} - h_\chi \frac{\partial h_\chi}{\partial r} u^2 u^2 - h_\zeta \frac{\partial h_\zeta}{\partial r} u^3 u^3 \right) . \tag{38}
$$

We have also

$$\frac{\partial}{\partial r}\left(h_r h_\chi h_\zeta F^1\right)+\frac{\partial}{\partial \chi}\left(h_r h_\chi h_\zeta F^2\right)+\frac{\partial}{\partial \zeta}\left(h_r h_\chi h_\zeta F^3\right)$$

$$=-\left(\frac{\partial}{\partial r}\left(h_\chi h_\zeta j_1\right)+\frac{\partial}{\partial \chi}\left(h_\zeta h_r j_2\right)+\frac{\partial}{\partial \zeta}\left(h_r h_\chi j_3\right)\right) , \tag{39}$$

where

$$j_1=-\frac{p_{e\perp}}{B^2}\left(\mathbf{b}\times\nabla B\right)_1-\frac{p_{e\parallel}}{B}\left(\mathbf{b}\times(\mathbf{b}\cdot\nabla\mathbf{b})\right)_1 ,$$

$$j_2=-\frac{p_{e\perp}}{B^2}\left(\mathbf{b}\times\nabla B\right)_2-\frac{p_{e\parallel}}{B}\left(\mathbf{b}\times(\mathbf{b}\cdot\nabla\mathbf{b})\right)_2+j_\parallel\frac{B_\chi}{B} ,$$

$$j_3=-\frac{p_{e\perp}}{B^2}\left(\mathbf{b}\times\nabla B\right)_3-\frac{p_{e\parallel}}{B}\left(\mathbf{b}\times(\mathbf{b}\cdot\nabla\mathbf{b})\right)_3+j_\parallel\frac{B_\zeta}{B} , \tag{40}$$

and

$$(\mathbf{b}\times\nabla B)_1=-\frac{B_\zeta}{h_\chi B^2}\left(B_\chi\frac{\partial B_\chi}{\partial\chi}+B_\zeta\frac{\partial B_\zeta}{\partial\chi}\right) ,$$

$$(\mathbf{b}\times\nabla B)_2=\frac{B_\zeta}{B^2}\left(B_\chi\frac{\partial B_\chi}{\partial r}+B_\zeta\frac{\partial B_\zeta}{\partial r}\right) ,$$

$$(\mathbf{b}\times\nabla B)_3=-\frac{B_\chi}{B^2}\left(B_\chi\frac{\partial B_\chi}{\partial r}+B_\zeta\frac{\partial B_\zeta}{\partial r}\right) ,$$

$$(\mathbf{b}\times(\mathbf{b}\cdot\nabla\mathbf{b}))_1=-\frac{B_\zeta}{B^2}\frac{1}{h_\chi B}\left(B_\chi\frac{\partial B_\chi}{\partial\chi}+B_\zeta\frac{\partial B_\zeta}{\partial\chi}\right) ,$$

$$(\mathbf{b}\times(\mathbf{b}\cdot\nabla\mathbf{b}))_2=\frac{B_\zeta}{B^2}\left[\frac{1}{B}\left(B_\chi\frac{\partial B_\chi}{\partial r}+B_\zeta\frac{\partial B_\zeta}{\partial x}\right)-\frac{B_\chi}{Bh_\chi}\frac{\partial}{\partial r}\left(h_\chi B_\chi\right)\right] ,$$

$$(\mathbf{b}\times(\mathbf{b}\cdot\nabla\mathbf{b}))_3 = -\frac{B_\chi}{B^2}\left[\frac{1}{B}\left(B_\chi\frac{\partial B_\chi}{\partial r}+B_\zeta\frac{\partial B_\zeta}{\partial x}\right)-\frac{B_\chi}{Bh_\chi}\frac{\partial}{\partial r}(h_\chi B_\chi)\right] .$$

Then we obtain the vorticity equation from the Poisson equation using current j as

$$\frac{\partial}{\partial a}t\mathcal{L}\tilde{\phi} = \left(\frac{\partial}{\partial r}(h_\chi h_\zeta j_1)+\frac{\partial}{\partial\chi}(h_\zeta h_r j_2)+\frac{\partial}{\partial\zeta}(h_r h_\chi j_3)\right)$$

$$+\frac{\partial}{\partial r}(h_r h_\chi h_\zeta S^1)+\frac{\partial}{\partial\chi}h_r h_\chi h_\zeta S^2)+\frac{\partial}{\partial\zeta}(h_r h_\chi h_\zeta S^3) , \quad (41)$$

where the poisson operator

$$\mathcal{L} = a_{11}\frac{\partial^2}{\partial r^2}+a_{22}\frac{\partial^2}{\partial\chi^2}+a_{33}\frac{\partial^2}{\partial\zeta^2}$$

$$-a_{23}\frac{\partial^2}{\partial\chi\partial\zeta}-b_1\frac{\partial}{\partial r}-b_2\frac{\partial}{\partial\chi}-b_3\frac{\partial}{\partial\zeta} , \quad (42)$$

and

$$a_{11} = \frac{\overline{n}m_i}{B^2}h_\chi h_\zeta , a_{22} = \frac{\overline{n}m_i}{B^4}\frac{h_\zeta}{h_\chi}B_\zeta^2 , a_{33} = \frac{\overline{n}m_i}{B^4}\frac{h_\chi}{h_\zeta}B_\chi^2 ,$$

$$a_{23} = 2\frac{\overline{n}m_i}{B^4}B_\chi B_\zeta \, b_1 = -\frac{\partial}{\partial r}\left(\frac{\overline{n}m_i}{B^2}h_\chi h_\zeta\right) ,$$

$$b_2 = -\frac{\partial}{\partial\chi}\left(\frac{\overline{n}m_i}{B^4}\frac{h_\zeta}{h_\chi}B_\zeta^2\right)+\frac{\partial}{\partial\zeta}\left(\frac{\overline{n}m_i}{B^4}B_\chi B_\zeta\right) ,$$

$$b_3 = -\frac{\partial}{\partial\zeta}\left(\frac{\overline{n}m_i}{B^4}\frac{h_\chi}{h_\zeta}B_\chi^2\right)+\frac{\partial}{\partial\chi}\left(\frac{\overline{n}m_i}{B^4}B_\chi B_\zeta\right) . \quad (43)$$

From Eqs. (1) and (3), we write

$$\frac{dr}{dt} = \frac{B_\chi}{B^2}\frac{1}{h_\zeta}\frac{\partial\tilde{\phi}}{\partial\zeta}-\frac{B_\zeta}{B^2}\frac{1}{h_\chi}\frac{\partial\tilde{\phi}}{\partial\chi}-\frac{v_\perp^2/2+v_\parallel^2}{B}\frac{B_\zeta}{B^2}\frac{1}{h_\chi}\frac{\partial B}{\partial\chi} , \quad (44)$$

$$h_\chi \frac{d\chi}{dt} = \frac{B_\zeta}{B^2}\frac{\partial\tilde{\phi}}{\partial r} + \frac{v_\perp^2/2 + v_\parallel^2}{B}\frac{B_\zeta}{B^2}\frac{\partial B}{\partial r} + \frac{v_\parallel^2}{B^2}\frac{B_\chi B_\zeta}{B^2} j^* + \frac{B_\chi}{B} v_\parallel \,, \tag{45}$$

$$h_\zeta \frac{d\zeta}{dt} = -\frac{B_\chi}{B^2}\frac{\partial\tilde{\phi}}{\partial r} - \frac{v_\perp^2/2 + v_\parallel^2}{B}\frac{B_\chi}{B^2}\frac{\partial B}{\partial r} - \frac{v_\parallel^2}{B^2}\frac{B_\chi^2}{B^2} j^* + \frac{B_\zeta}{B} v_\parallel \,, \tag{46}$$

where

$$j^* = \frac{1}{h_\chi}\frac{\partial(h_\chi B_\chi)}{\partial r} \,. \tag{47}$$

Equation (2) is written as

$$\frac{dv_\parallel}{dt} = \frac{B_\chi}{B}\frac{1}{h_\chi}\frac{\partial\tilde{\phi}}{\partial\chi} + \frac{B_\zeta}{B}\frac{1}{h_\zeta}\frac{\partial\tilde{\phi}}{\partial\zeta} - \frac{v_\perp^2}{2B}\frac{B_\chi}{B}\frac{1}{h_\chi}\frac{\partial B}{\partial\chi} \,. \tag{48}$$

IV. FIELD SOLVER

To solve the vorticity equation Eq. (43), we use the over-relaxation method. To do this, we decouple the operator $\mathcal{L}$ into the poloidal angle dependent part and independent part $\overline{\mathcal{L}} + \tilde{\mathcal{L}}$ where

$$\overline{\mathcal{L}} = \bar{a}_{11}\frac{\partial^2}{\partial r^2} - \bar{a}_{22}k_\chi^2 - \bar{a}_{33}k_\zeta^2 + \bar{a}_{23}k_\chi k_\zeta - \bar{b}_1\frac{\partial}{\partial r} \,, \tag{49}$$

and

$$\tilde{\mathcal{L}} = (a_{11} - \bar{a}_{11})\frac{\partial^2}{\partial r^2} - (a_{22} - \bar{a}_{22})k_\chi^2 - (a_{33} - \bar{a}_{33})k_\zeta^2 + (a_{23} - \bar{a}_{23})k_\chi k_\zeta$$
$$-(b_1 - \bar{b}_1)\frac{\partial}{\partial r} - b_2 i k_\chi - b_3 i k_\zeta \,, \tag{50}$$

where

$$\overline{a}_{11} = \frac{\overline{n}m_i}{B_{\zeta 0}^2}\, h_\chi \;, \overline{a}_{22} = \frac{\overline{n}m_i}{B_{\zeta 0}^2}\,\frac{1}{h_\chi} \;, \overline{a}_{33} = \frac{\overline{n}m_i}{B_{\zeta 0}^4}\, h_\chi B_{\chi 0}^2 \;,$$

$$\overline{a}_{23} = \frac{2\overline{n}m_i}{B_{\zeta 0}^2}\, B_{\chi 0}B_{\zeta 0} \;, \overline{b}_1 = \frac{2\overline{n}m_i h_\chi}{B_{\zeta 0}^4}\, B_{\chi 0}\frac{\partial B_{\chi 0}}{\partial r} - \frac{m_i}{B_{\zeta 0}^2}\frac{\partial \overline{n}}{\partial r}\, h_\chi - \frac{\overline{n}m_i}{B_{\zeta 0}^2}\frac{\partial h_\chi}{\partial r} \;. \tag{51}$$

Then we can write

$$(\overline{\mathcal{L}}\phi^{n+1})_i = -(\widetilde{\mathcal{L}}\phi^{n+1})_i + (\mathcal{L}\phi^n)_i + \Delta t S_i^{n+1/2} \;, \tag{52}$$

where $(\overline{\mathcal{L}}\phi^{n+1})_i = \overline{\alpha}_i\phi_{i+1}^{n+1} + \overline{\beta}_i\phi_i^{n+1} + \overline{\gamma}_i\phi_{i-1}^{n+1}$ and

$$\overline{\alpha}_i = \left(\frac{\partial\xi}{\partial r}\right)^2 \overline{a}_{11} + \left(\frac{\partial^2\xi}{\partial r^2}\right)\frac{\overline{a}_{11}}{2} - \left(\frac{\partial\xi}{\partial r}\right)\frac{\overline{b}_1}{2} \;,$$

$$\overline{\beta}_i = -2\left(\frac{\partial\xi}{\partial r}\right)^2 \overline{a}_{11} - \overline{a}_{22}k_\chi^2 - \overline{a}_{33}k_\zeta^2 + \overline{a}_{23}k_\chi k_\zeta \;,$$

$$\overline{\gamma}_i = \left(\frac{\partial\xi}{\partial r}\right)^2 \overline{a}_{11} - \left(\frac{\partial^2\xi}{\partial r^2}\right)\frac{\overline{a}_{11}}{2} + \left(\frac{\partial\xi}{\partial r}\right)\frac{\overline{b}_1}{2} \;, \tag{53}$$

$(\widetilde{\mathcal{L}}\phi^{n+1})_i = \widetilde{\alpha}_i\phi_{i+1}^{n+1} + \widetilde{\beta}_i\phi_i^{n+1} + \widetilde{\gamma}_i\phi_{i-1}^{n+1}$,

$$\widetilde{\alpha}_i = \left(\frac{\partial\xi}{\partial r}\right)^2 \widetilde{a}_{11} + \left(\frac{\partial^2\xi}{\partial r^2}\right)\frac{\widetilde{a}_{11}}{2} - \left(\frac{\partial\xi}{\partial r}\right)\frac{\widetilde{b}_1}{2} \;,$$

$$\widetilde{\beta}_i = -2\left(\frac{\partial\xi}{\partial r}\right)^2 \widetilde{a}_{11} - \widetilde{a}_{22}k_\chi^2 - \widetilde{a}_{33}k_\zeta^2 + \widetilde{a}_{23}k_\chi k_\zeta - i\widetilde{b}_2 k_\chi - i\widetilde{b}_3 k_\zeta \;,$$

$$\widetilde{\gamma}_i = \left(\frac{\partial\xi}{\partial r}\right)^2 \widetilde{a}_{11} - \left(\frac{\partial^2\xi}{\partial r^2}\right)\frac{\widetilde{a}_{11}}{2} + \left(\frac{\partial\xi}{\partial r}\right)\frac{\widetilde{b}_1}{2} \;, \tag{54}$$

$$S_i^{n+1/2} = \left(\frac{\partial}{\partial r} h_\chi h_\zeta j_1\right)_i^{n+1/2} + \left(\frac{\partial}{\partial \chi} h_\zeta j_2\right)_i^{n+1/2} + \left(\frac{\partial}{\partial \zeta} h_\chi j_3\right)_i^{n+1/2}$$

$$+ \left(\frac{\partial}{\partial r} h_\chi h_\zeta S^1\right)_i^{n+1/2} + \left(\frac{\partial}{\partial \chi} h_\chi h_\zeta S^2\right)_i^{n+1/2} + \left(\frac{\partial}{\partial \zeta} h_\chi h_\zeta S^3\right)_i^{n+1/2} \quad (55)$$

Then we obtain

$$({}^{p+1}\phi^{n+1}) = (1-k)({}^{p}\phi^{n+1}) + k\overline{\mathcal{L}}^{-1}\left[\mathcal{L}\phi^n + \Delta t S_i^{n+1/2} - \widetilde{\mathcal{L}}({}^{p}\phi^{n+1})\right] , \quad (56)$$

where k is a relaxation parameter and p represents the iteration count.

V. PREDICTOR-CORRECTOR METHOD

In advancing the time steps, we use the predictor-corrector method. The parallel velocity component is pushed in the normal way. We have $(v_{\parallel j}^{(n-1/2)}, \mathbf{x}_j^{(n-1)}, \widetilde{\phi}^{(n-1)}, \mathbf{x}_j^{(n)}, \widetilde{\phi}^{(n)})$ for the previous time step.

The predictor step

$$v_{\parallel j}^{(n)} = v_{\parallel j}^{(n-1/2)} + F_\parallel(\widetilde{\phi}(\mathbf{x}_j^{(n)}))\,\frac{\delta t}{2} , \quad (57)$$

$$\mathbf{v}_{dj}^{(n)} = \frac{\mathbf{b}}{B}\times\nabla\widetilde{\phi}(\mathbf{x}_j^{(n)}) + \frac{\mathbf{b}}{\Omega_e}\times\left(\frac{\mu_j}{m}\nabla B(\mathbf{x}_j^{(n)}) + (v_{\parallel j}^{(n)})^2(\mathbf{b}\cdot\nabla\mathbf{b})\right) , \quad (58)$$

$$(\mathbf{x}_j^{(n+1)})^* = \mathbf{x}_j^{(n-1)} + 2\delta t(\mathbf{v}_{dj}^{(n)} + v_{\parallel j}^{(n)}\mathbf{b}) , \quad (59)$$

$$\widetilde{j}_{\parallel(e)}^{(n)} = -\sum_j q_j^{(n)} v_{\parallel j}^{(n)}\delta(\mathbf{x}-\mathbf{x}_j^{(n)}) , \quad (60)$$

$$p_{e\perp}^{(n)} = \frac{m_e}{2} \sum_j \mathbf{v}_{\perp j}^{(n)} \cdot \mathbf{v}_{\perp j}^{(n)} \delta(\mathbf{x} - \mathbf{x}_j^{(n)}) \ , \tag{61}$$

$$p_{e\parallel}^{(n)} = m_e \sum_j v_{\parallel j}^{(n)} v_{\parallel j}^{(n)} \delta(\mathbf{x} - \mathbf{x}_j^{(n)}) \ . \tag{62}$$

Then we subtract the equilibrium part from the total pressure $\widetilde{p}_{e\perp} = p_{e\perp} - p_{e\perp}^{(0)} - p_{e\perp}^{(m=0)}$, and $\widetilde{p}_{e\parallel} = p_{e\parallel} - p_{e\parallel}^{(0)} - p_{e\parallel}^{(m=0)}$ where $p_{e\perp}^{(0)} = (m/2) \int \mathbf{v}_\perp \cdot \mathbf{v}_\perp f_M d^3v$, and $p_{e\parallel}^{(0)} = m \int v_\parallel v_\parallel f_M d^3v$ and $p_{e\perp}^{(m=0)}$, $p_{e\parallel}^{(m=0)}$ means the quasilinear part (this is an option)

$$(\mathcal{L}\widetilde{\phi}^{(n+1)})^* = \mathcal{L}\widetilde{\phi}^{(n-1)} + 2\delta t S\, \widetilde{j}_{\parallel(e)}^{(n)}\Big) \ , \tag{63}$$

$$\widetilde{p_{e\perp}}^{(n)} \ , \widetilde{p_{e\parallel}}^{(n)} \ , \widetilde{\phi}^{(n)}\Big) \ . \tag{64}$$

The corrector step

$$v_{\parallel j}^{(n+1/2)} = v_{\parallel j}^{(n)} + F_\partial|(\widetilde{\phi}(\mathbf{x}_j^{(n)}))\, \frac{\delta t}{2} \ , \tag{65}$$

$$v_{\parallel j}^{(n+1)*} = v_{\parallel j}^{(n+1/2)} + F_\parallel(\widetilde{\phi}(\mathbf{x}_j^{(n+1)*}))\, \frac{\delta t}{2} \ , \tag{66}$$

$$\mathbf{v}_{dj}^{(n+1)*} = \frac{\mathbf{b}}{B} \times \nabla\widetilde{\phi}(\mathbf{x}_j^{(n+1)*}) + \frac{\mathbf{b}}{\Omega_e} \times \left(\frac{\mu_j}{m} \nabla B(\mathbf{x}_j^{(n+1)*}) + v_{\parallel j}^{(n+1)*2} (\mathbf{b} \cdot \nabla \mathbf{b})\right) \ , \tag{67}$$

$$\mathbf{x}_j^{(n+1)} = \mathbf{x}_j^{(n)} + \delta t \left(\frac{\mathbf{v}_{dj}^{(n)} + \mathbf{v}_{dj}^{(n+1)*}}{2} + v_{\parallel j}^{(n+1/2)}\, \mathbf{b} \right) , \tag{68}$$

$$\tilde{j}_{\|(e)}^{(n+1/2)*} = -\frac{\sum_j q_j^{(n+1)*} v_{\|j}^{(n+1)*} \delta(\mathbf{x} - \mathbf{x}_j^{(n+1)*}) + \sum_j q_j^{(n)} v_{\|j}^{(n)} \delta(\mathbf{x} - \mathbf{x}_j^{(n)})}{2} , \tag{69}$$

$$\widetilde{p_{e\perp}}^{(n+1/2)*} = \frac{\widetilde{p_{e\perp}}^{(n)} + \widetilde{p_{e\perp}}^{(n+1)*}}{2} , \tag{70}$$

$$\widetilde{p_{e\|}}^{(n+1/2)*} = \frac{\widetilde{p_{e\|}}^{(n)} + \widetilde{p_{e\|}}^{(n+1)*}}{2} , \tag{71}$$

$$p_{e\perp}^{(n+1)*} = \frac{m_e}{2} \sum_j \mathbf{v}_{\perp j}^{(n+1)*} \cdot \mathbf{v}_{\perp j}^{(n+1)*} \delta(\mathbf{x} - \mathbf{x}_j^{(n+1)*}) , \tag{72}$$

$$p_{e\|}^{(n+1)*} = m_e \sum_j v_{\|j}^{(n+1)*} v_{\|j}^{(n+1)*} \delta(\mathbf{x} - \mathbf{x}_j^{(n+1)*}) , \tag{73}$$

$$\tilde{\phi}^{(n+1/2)*} = \frac{\tilde{\phi}^{(n)} + \tilde{\phi}^{(n+1)*}}{2} , \tag{74}$$

$$\mathcal{L}\tilde{\phi}^{(n+1)} = \mathcal{L}\tilde{\phi}^{(n)} + \delta t S \left(\tilde{j}_{\|(e)}^{(n+1/2)*}, \widetilde{p_{e\perp}}^{(n+1/2)*}, \widetilde{p_{e\|}}^{(n+1/2)*}, \tilde{\phi}^{(n+1/2)*} \right) . \tag{75}$$

This closes the one complete time loop.

VI. APPLICATION

Using the hybrid model equations, we can investigate trapped electron/CDBM branch. We demonstrate here that the present model can describe the correct linear behavior both in the kinetic and fluid limits of electron dynamics. The fluctuating part of the distribution function can be expressed as

$$\delta f = \frac{e\phi}{T_e} F_M + h^p + h^T \ , \tag{76}$$

where

$$h^p = -\frac{\omega - \omega_{*T_e}}{\omega - k_\parallel v_\parallel - \omega_{D_e}} F_M \frac{e\phi}{T_e} \ , \tag{77}$$

$$h^T = -\frac{\omega - \omega_{*T_e}}{\omega - \langle \omega_{D_e} \rangle} F_M \frac{e \langle \phi \rangle}{T_e} \ . \tag{78}$$

Here $\omega_{D_e} = \epsilon_n \omega_* (\cos\theta + s\theta \sin\theta) m (v_\perp^2/2 + v_\parallel^2)/T_e$, $\omega_{*T_e} = \omega_*(1 + \eta_e(E/T_e - 3/2))$, $\omega_* = k_\theta T_e/(eBL_n)$, s is the shear parameter, θ represents the ballooning coordinate and $\langle \ \ \rangle$ means the bounce average, although in the model $\langle \ \ \rangle$ is not taken and full kinetic resonance is incorporated (this in fact is close to reality).

The parallel current is given by

$$\begin{aligned} j_\parallel &= -e \int v_\parallel h^p d^3 v \\ &= -\frac{\omega - \omega_*}{k_\parallel} \frac{e^2 \overline{n} \phi}{T_e} + \frac{1}{k_\parallel} \frac{e^2 \phi}{T_e} \int d^3 v F_M \frac{(\omega - \omega_{D_e})(\omega - \omega_{*T_e})}{\omega - k_\parallel v_\parallel - \omega_{D_e}} \ . \end{aligned} \tag{79}$$

The perpendicular current is given by

$$\mathbf{j}_\perp = -e \int \mathbf{v}_D f d^3 v \ . \tag{80}$$

For simplicity we assume $\theta = 0$, $\langle \omega_{D_e} \rangle = \omega_{D_e}$ and using the constant energy resonance approximation ($v_\perp^2/2 + v_\parallel^2 \to \alpha v^2/2$, $\alpha = 4/3$) for trapped particles, we obtain

$$\mathbf{k} \cdot \mathbf{j} = -2\epsilon_n \omega_* \frac{e^2 \overline{n}}{T_e} \phi - (\omega - \omega_*) \frac{e^2 \overline{n}}{T_e} \phi$$

$$+ \frac{e^2 \overline{n}}{T_e} \phi \sqrt{2\epsilon} \left(\omega_* - \omega + \frac{\eta_e \omega}{\epsilon_n} I_1 \right) + \frac{e^2 \overline{n}}{T_e} \phi I_2 \ , \tag{81}$$

where

$$I_1 = \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{y - 3/2 - (\omega - \omega_*)/(\eta_e \omega_*)}{y - \omega/(\alpha \omega_* \epsilon_n)} y^{1/2} \exp(-y) dy \ , \tag{82}$$

$$I_2 = \left(\frac{m}{2\pi T_e} \right)^{3/2} \int_0^\infty 2\pi v_\perp dv_\perp$$

$$\times \int_{-\infty}^{+\infty} dv_\parallel \exp\left(-\frac{mv^2}{2T_e} \right) \frac{\omega(\omega - \omega_{*T_e})}{\omega - k_\parallel v_\parallel - \omega_{D_e}} \ . \tag{83}$$

Then we obtain the dispersion relation using the vorticity equation $(\overline{n} m_i / B_0^2) \partial_t \nabla_\perp^2 \phi = i\mathbf{k} \cdot \mathbf{j}$. The dispersion relation is given by

$$\omega \rho_s^2 k_\perp^2 = -2\epsilon_n \omega_* + (\omega_* - \omega)(1 + \sqrt{2\epsilon}) + \sqrt{2\epsilon} \frac{\eta_e \omega}{\epsilon_n \alpha} I_1 + I_2 \ . \tag{84}$$

If we neglect I_2 (kinetic limit $\omega/\omega_{De} \to 0$), then we recover the dispersion for the collisionless trapped electron modes for $k_\perp \rho_s \ll 1$

$$\omega \sim \frac{\omega_*}{1 + k_\perp^2 \rho_s^2} \sim \omega_* \ ,$$

$$\frac{\gamma}{\omega_*} \sim 2\sqrt{\pi} \frac{\eta_e \sqrt{2\epsilon}}{\epsilon_n} \left(\frac{1}{\alpha \epsilon_n} - \frac{3}{2} \right) \frac{1}{\sqrt{\alpha \epsilon_n}} \exp\left(-\frac{1}{\alpha \epsilon_n} \right) \ . \tag{85}$$

Next we consider the effect of I_2. Taking the fluids limit $k_\parallel v_\parallel/\omega \to 0$, $\omega_{D_e}/\omega \to 0$, and expanding the denominator, we obtain

$$\omega\rho_s^2 k_\perp^2 = -2\epsilon_n\omega_*(1+\eta_e)\frac{\omega_*}{\omega} + \sqrt{2\epsilon}\left(\omega_* - \omega + \frac{\eta_e\omega}{\epsilon_n}I_1\right)$$

$$+\frac{k_\parallel^2 T_e}{m\omega^2}(\omega - (1+\eta_e)\omega_*) + 7\frac{\epsilon_n^2\omega_*^2}{\omega^2}(\omega - (1+2\eta_e)\omega_*) \ . \qquad (86)$$

In the limit $\omega_*/\omega \to 0$, we obtain the dispersion relation for the inertia driven interchange modes ($k_\parallel \to k_y\rho_s s x$, $S = s(2l+1)$, $v_{the} = \sqrt{T_e/m}$)

$$\frac{\gamma}{\omega_*} = -\frac{1}{2}\frac{k_y v_{the}}{\omega_*}\frac{S}{k_y^2\rho_s^2} + \frac{1}{2}\sqrt{\left(\frac{k_y v_{the}}{\omega_*}\frac{S}{k_y^2\rho_s^2}\right)^2 + 4\frac{2\epsilon_n(1+\eta_e)}{k_y^2\rho_s^2}} \ . \qquad (87)$$

Notice that this dispersion is also obtained based on the three-field model equations

$$\frac{cnm_i}{B_0}\frac{d}{dt}\nabla_\partial^2\phi = \frac{B_0}{c}\nabla_\parallel j_\parallel + \nabla p_e \times \nabla(2r\cos\theta/R_0)\cdot\hat{z} \ , \qquad (88)$$

$$\frac{m_e}{e^2 n}\frac{d}{dt}j_\parallel = -\nabla_\parallel\phi \ , \qquad (89)$$

$$\frac{d}{dt}p_e = 0 \ , \qquad (90)$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{c}{B_0}[\phi, \ ] \ . \qquad (91)$$

Therefore, we see that the passing electron drives the nonlinear instability.

$$-i(\omega - \omega_{D_e} - k_\parallel v_\parallel)h^p = i(\omega - \omega_{*T_e})F_M\frac{e\phi}{T_e}$$

$$\underbrace{-\hat{z}\cdot\nabla\frac{e\phi}{T_e}\times\nabla h^p}_{\text{CDBM driving term}} \ , \qquad (92)$$

$$
\begin{aligned}
-i(\omega - \langle\omega_{D_e}\rangle)h^T = {} & i(\omega - \omega_{*T_e})F_M \frac{e}{T_e}\langle\phi\rangle \\
& \underbrace{-\hat{z}\cdot\nabla\frac{e}{T_e}\langle\phi\rangle\times\nabla h^T}_{\text{stabilizing}} ,
\end{aligned}
\tag{93}
$$

where $-e\int \mathbf{v}_{D_e} h^T d^3v \equiv \mathbf{j}_\perp^T \to p^T$, Therefore if we renormalize $[\phi, p^T]$ nonlinearity, this term can be writen as $\to \chi\nabla_\perp^2 p^T$, which contributes to stabilizing the mode. We show that this model in fact reproduces two limits of fluid and kinetic electron dynamics correctly.

In this paper we presented an appropriate algorithm that retains relevant electron dynamics coupled with fluid ions in toroidal (or more general) metric. The algorithm should be useful in investigations of nonlinear electron dynamics in tokamak transport studies in low frequencies.

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Appendix A. General equation

Generaly, kinetic equation is given by

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m\,(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}})\ . \tag{A.1}$$

Then we can write the equation explicitly as

$$\frac{dv^i}{dt} + \left\{ \begin{matrix} i \\ rk \end{matrix} \right\} v^r v^k = \frac{q}{m}\left(E^i + g^{il}\epsilon_{ljk}v^j B^k\right) \tag{A.2}$$

where

$$\epsilon_{ijk} = \begin{cases} \sqrt{g} \text{ if i, j, k is a cyclic permutation of } 1,2,3 \\ -\sqrt{g} \text{ if i, j, k is a cyclic permutation of } 2,1,3 \\ 0\ . \end{cases} \tag{A.3}$$

If we take the toroidal coordinate system (r, χ, ζ), then the above equation is reduced as

$$h_i\frac{dv_i}{dt} + 2v_l v_i\frac{\partial h_i}{\partial x^l} - v_l v_l\frac{h_l}{h_i}\frac{\partial h_l}{\partial x^i} = \frac{q}{m}(E_i + \epsilon_{ijk}v_j B_k)\ ,(i = r, \chi, \zeta) \tag{A.4}$$

where $\mathbf{v}\cdot\mathbf{e}_i/|\mathbf{e}_i| = h_i v_i$, $\mathbf{E}\cdot\mathbf{e}_i/|\mathbf{e}_i| = E_i$, $\mathbf{B}\cdot\mathbf{e}_i/|\mathbf{e}_i| = B_i$, and

$$\epsilon_{ijk} = \begin{cases} 1 \text{ if i, j, k is a cyclic permutation of } 1,2,3 \\ -1 \text{ if i, j, k is a cyclic permutation of } 2,1,3 \\ 0 \end{cases} \tag{A.5}$$