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NONLINEAR ELECTRON LANDAU DAMPING OF ION-ACOUSTIC SOLITONS

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I. Introduction

Linear electron Landau damping of ion-acoustic solitons was first studied by Ott and Sudan.[1] They derived a Korteweg- de Vries (KdV) equation with a source term that models the lowest order effects of resonant electrons. Their equation contains the lowest order nonlinear terms in addition to terms that correspond to the linear dispersion relation for ion-acoustic waves. The derivation is justified by a formal procedure whereby these terms in the equation are of the same order.

Van Dam and Taniuti[2] pointed out that Ott and Sudan neglected trapped particle effects, which are of the same order as the linear Landau damping terms included in the treatment of Ref. 1. The situation is analogous to that of nonlinear Landau damping of a large amplitude plasma wave.[3] For times longer than the electron bounce time, $\omega_{be}^{-1} = (m_e/ek^2\Phi)^{1/2}$, the linear theory breaks down; thus if the Landau time, $\gamma_L^{-1} = [k^2\pi m_e T_e/(8m_i^2)]^{-1/2}$, is longer than ω_{be}^{-1} nonlinear effects are important.

For the ion acoustic soliton $\gamma_L << \omega_{be}$ provided the amplitude is mildly large: $e\Phi/T_e >> (m_e/m_i)^2$. It will be shown that for the soliton, as for the plasma wave, phase mixing of electron orbits effectively stops the damping after a few bounce periods.

This time dependent damping problem has not been previously treated. Schamel[4] assumes a stationary trapped electron distribution, showing that trapped particles can modify the relationship between soliton speed, amplitude and width. Karpman[5] and Lotko[6] note that for time $t << \omega_{be}^{-1}$ the theory of Ott and Sudan is valid and then treat the effects of ion Landau damping for $t >> \omega_{bi}^{-1}$. They are forced to assume an unperturbed KdV soliton as an initial condition, noting that within a time $t \sim \omega_{bi}^{-1}$ the electrons will have phase mixed.

Our calculation is valid before the ion orbit effects become important (i.e. for $t < \omega_{bi}^{-1}$) and thus yields the appropriate initial condition for studies of ion effects. These differ from electron effects because an ion-acoustic soliton is a localized pulse with $\Phi > 0$. The soliton reflects ions and thus continually exchanges momentum with ions arriving at the pulse from infinity.[6]

The derivation presented here begins with the coupled Vlasov-Poisson-ion fluid equations. Utilizing the standard ordering scheme of Gardner and Morikawa[7] we obtain in Sec. II a reduced system: the coupled Vlasov-KdV equations. Instead of artificially separating resonant and nonresonant contributions, we use a subtraction procedure[8] to isolate the nonadiabatic portion of the electron response.

In Sec. III the Vlasov equation is solved by integrating along the electron orbits in a soliton with frozen amplitude, following O'Neil.[3] This approximation requires the amplitude change to be small; yet $e\Phi/T_e >> (m_e/m_i)^2$. Several authors have extended the O'Neil analysis to larger γ_L/ω_b (in an attempt to approach self-consistency) by treating the adiabatic modification of the particle orbits in the damping wave.[9-11] This procedure is not applicable to the soliton since untrapped particles do not have periodic orbits. We leave the self-consistent treatment of the Vlasov-KdV system to future work.

The damping of the soliton is treated by the method of perturbed conservation laws in Sec. IV.[12,13] We obtain an equation for the soliton speed as a function time. As the soliton damps and oscillates at the bounce frequency, its speed, width and amplitude remain related as in the unperturbed case. More rigorous perturbation theories[14,15] show the approximate validity of this method. Our final result, Fig. 4, is the asymptotic speed of the soliton as a function of initial condition. Figure 4 shows that $e\Phi/T_e \sim (m_e/m_i)^2$ is an effective threshold for existence of the soliton: for amplitudes larger than this electron Landau damping is a small effect.

For an experimental measurement of this effect several criteria must be met. First, linear ion Landau damping must be weak compared to that due to electrons. This implies $T_e/T_i > 16$. As shown by Van Dam and Taniuti[2] collisions are relatively unimportant; however, the transverse dimension of the soliton must be large enough so that wall collisions can be neglected: $L > \lambda_{De} (T_e/e\Phi)^{1/2}$. We mention that trapped electrons have been experimentally observed in an ion acoustic soliton by Tran and Means.[16]

II. Kinetic Electron KdV Equation

In this section the ion-acoustic KdV equation with corrections due to kinetic electron effects is obtained. Preparing for this derivation we write the electron Vlasov, ion fluid and Poisson equations in terms of the following dimensionless variables, which are appropriate for ion-acoustic waves:

$$\omega_{pi}t' = t, \qquad k_{De}x' = x, \qquad v'/v_e = v$$

$$e\Phi'/T_e = \Phi, \qquad v_e f'/n_o = f, \qquad n'/n_o = n, \qquad u'/c_o = u \quad . \tag{1}$$

Here the primed quantities are the unscaled variables; $f(x,v)$ is the electron distribution function; n and u are respectively the ion fluid density and velocity; and Φ is the electrostatic potential. The scaling parameters are the ion-acoustic speed $c_o^2 = T_e/m_i$; the unperturbed density n_o; the electron thermal speed $v_e^2 = T_e/m_e$; the ion plasma frequency; and the electron Debye wave number k_{De}.

In terms of these variables the equations contain the mass ratio $\delta \equiv m_e/m_i$ as a small parameter. The scaled equations are

$$\delta^{1/2}\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{\partial \Phi}{\partial x}\frac{\partial f}{\partial v} = 0 \tag{2}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{\partial \Phi}{\partial x} \tag{3}$$

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nu) = 0 \tag{4}$$

$$\frac{\partial^2 \Phi}{\partial x^2} = \int f\, dv - n \quad . \tag{5}$$

Following the standard derivation of the KdV equation[7], we assume that the perturbed potential is small, $\Phi \sim \varepsilon$, and introduce stretched time and space coordinates. It is assumed that the characteristic scale length of the perturbation is $k_s \sim \varepsilon^{1/2}$. This implies for ion-acoustic oscillations that typical frequencies will be $\omega_o = k_s c_o \sim \varepsilon^{1/2}$. In addition to this time scale, a slow soliton time scale is introduced: $\omega_s \sim \varepsilon^{3/2}$. These time scales are sufficient when the electron response is adiabatic. However, in the present case several additional time scales must be considered. The first, the electron plasma frequency, we neglect - simply assuming plasma waves are not present. The second is the electron bounce frequency, ω_b (here and henceforth we drop the subscript e). Finally the nonadiabatic electron motion gives rise to Landau damping at the rate γ_L. The scaled values (in terms of ω_{pi}) of these frequencies are

$$\omega_{pe} \sim \delta^{-1/2}$$

$$\omega_b \sim \varepsilon\delta^{-1/2} \tag{6}$$

$$\gamma_L \sim (\varepsilon\delta)^{1/2} \quad .$$

The relative values of the five time scales are plotted as a function of ε in Fig. 1.

In the seminal work of Ott and Sudan it was assumed that $\varepsilon \sim \delta^{1/2}$. As seen in Fig. 1 this is the point where $\gamma_L \sim \omega_s$ and hence where Landau damping is the same order as the terms of the usual KdV equation. Ott and Sudan[1] assert that when $\gamma_L \gg \omega_s$ linear Landau damping causes the wave to damp away before nonlinear evolution occurs. In the opposite limit the soliton effectively does not damp at all.

However, at the point where $\varepsilon \sim \delta^{1/2}$ the electron bounce frequency is O(1) and thus the analysis of Ott and Sudan, which neglects this motion, is invalid. These authors justify their use of straight line orbits by the ad hoc introduction of a small amount of noise. In the absence of noise one can show that for $t \ll \omega_b^{-1}$ the theory of Ott and Sudan is valid; however, during this time negligible Landau damping occurs.[2,5,6] In Appendix A we show that our result, Eq. (29), reduces to that of Ref. 1 in the limit $\omega_b t \to 0$.

Thus we are led to assume that the bounce time scale enters. In Sec. III, as in Ref. 3, we solve the Vlasov equation with the bounce time dependence of Φ suppressed. Note from Fig. 1 that for all values of ε of interest ($\varepsilon < \delta^{-1}$) we have $\omega_b \gg \omega_s$; furthermore, when $\varepsilon > \delta^2$ we have $\omega_b > \gamma_L$. We will see, however, that it is consistent to neglect changes in the quantities n, Φ and u on the bounce time scale. The primary reason for this is that the coupling between the ion fluid quantities and the nonadiabatic portion of the electron motion occurs only at the final order in our expansion $[O(\varepsilon^2)]$.

Proceeding to the fluid equations we introduce the usual variables[7]

$$\xi = \varepsilon^{1/2}(x-t)$$

$$\tau = \varepsilon^{3/2}t \quad , \tag{7}$$

where ξ takes into account ω_o variations and τ varies on the scale ω_s. Expanding,

$$n(x,t) = 1 + \varepsilon n_1(\xi,\tau) + \ldots$$

$$u(x,t) = \varepsilon u_1(\xi,\tau) + \ldots \tag{8}$$

$$\Phi(x,t) = \varepsilon\Phi_1(\xi,\tau) + \ldots \quad ,$$

and substituting Eqs. (7) and (8) into Eqs. (3) - (5) we obtain the desired equation. For convenience we define the adiabatic electron density

$$n_a \equiv 1 + \varepsilon\Phi_1 + \varepsilon^2\left[\Phi_2 + \frac{1}{2}(\Phi_1)^2\right] + O(\varepsilon^3) \quad . \tag{9}$$

This density results from assuming $\int f dv = e^{\Phi}$ and expanding using Eq. (8). To obtain the correct result it is necesary to assume

$$\int f dv - n_a \lesssim \varepsilon^2 \quad . \tag{10}$$

This will be verified in Sec. IV.

Carrying out the expansion to second order yields

$$n_1 = \Phi_1 = u_1 \ ,$$

$$\frac{\partial\Phi_1}{\partial\tau} + \Phi_1\frac{\partial\Phi_1}{\partial\xi} + \frac{1}{2}\frac{\partial^3\Phi_1}{\partial\xi^3} = \frac{1}{2\varepsilon^2}\frac{\partial}{\partial\xi}\left[\int f dv - n_a\right]$$

$$\equiv \frac{\partial S}{\partial\xi} \quad . \tag{11}$$

The right hand side is the kinetic correction to the KdV equation. Equation (11) is equivalent to that derived by Karpman[5] and Van Dam and Taniuti[2] except that instead of splitting off the adiabatic portion of the electron response, they remove the non-resonant response. This necessitates defining the electron density by an integral over the non-resonant portion of phase space, which is not a well defined procedure when the orbits are nonlinear. Furthermore with their procedure the non-resonant electron density is assumed to be independent of the position of the phase space boundary separating the resonant and non-resonant regions. This is difficult to justify. In contrast our procedure is analogous to the subtraction procedure of Morales and O'Neil[8]: they split off the linear response in their study of large amplitude plasma waves.

III. Electron Vlasov Equation

In this section we solve the Vlasov equation with an ion-acoustic soliton potential. We allow f to depend on the bounce time scale, $t_b \equiv \varepsilon\delta^{-1/2}t$. This time scale will appear [as in Ref. (3)] in the generalized damping coefficient obtained in Sec. IV. The characteristic velocity width in phase space over which orbits differ significantly from free particle orbits (i.e., the trapping velocity) is $v_T \sim \sqrt{\Phi} \sim \sqrt{\varepsilon}$. If we rescale the velocity of Eq. (2) in terms of v_T, $v \equiv \sqrt{\varepsilon}\, w$, then to lowest order the Vlasov equation becomes

$$\frac{\partial f}{\partial t_b} + w\frac{\partial f}{\partial \xi} + \frac{\partial \Phi_1}{\partial \xi}\frac{\partial f}{\partial w} = 0 \quad . \tag{12}$$

Observe that all terms of Eq. (12) are $O(\varepsilon)$. We solve this equation by integrating along the electron orbits in a soliton with frozen amplitude. In the waveframe Φ_1 depends on τ but does not depend on t_b [$\varepsilon >> \delta^2$ c.f. Fig. (1)].

Thus, as potential we take the solution to the KdV equation in the absence of kinetic effects (S=0)

$$\Phi_S(\xi,\tau) = 3\ c\ \mathrm{sech}^2 \left[\sqrt{c/2}\ (\xi - c\tau)\right] \quad , \tag{13}$$

where c represents the speed of the soliton in excess of the sound speed.

Noting that the electron density must satisfy Poisson's equation (5) at $t=0$, using the fact that $\Phi_S(x,0)$ satisfies the unperturbed KdV equation (11), and finally using the expansions of Eq. (8) yields

$$\int f(\xi,v,0)dv = n_a(\xi,0) \quad , \tag{14}$$

where n_a is given by Eq. (9). We further assume that the initial velocity distribution is Maxwellian in the lab frame, and hence in the soliton frame is

$$f(\xi,v,0) = \frac{n_a(\xi,0)}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}(v+\delta^{1/2})^2\right] \quad , \tag{15}$$

where the $\delta^{1/2}$ term is the ion sound speed in units of v_e. For the purposes of our analysis we require only the lowest order source term for the KdV equation; hence, it is sufficient to take $n_a(\xi,0) = 1 + O(\varepsilon)$ in Eq. (15).

Thus far we have made assumptions equivalent to those of O'Neil. There are two small parameters

$$\varepsilon \sim \omega_b/\omega_{pe} \qquad\qquad \delta = m_e/m_i$$

and the ordering $1 \gg \varepsilon \gg \delta^2$ is equivalent to O'Neil's $\omega_{pe} \gg \omega_b \gg \gamma_L$.

Equation (12) is integrated along orbits to obtain

$$f(\xi,w,t_b) = f\big(\xi_0(\xi,w,t_b),\ w_0(\xi,w,t_b),\ 0\ \big)\ , \tag{16}$$

where (ξ_0,w_0) is the initial phase point, which evolves to (ξ,w) at time t_b. The characteristics of Eq. (12) are

$$\frac{\partial\xi}{\partial t_b} = w \tag{17}$$

$$\frac{\partial w}{\partial t_b} = \frac{\partial\Phi_1}{\partial\xi} = 3c\,\frac{\partial}{\partial\xi}\,\mathrm{sech}^2\left[\sqrt{c/2}\ \xi\ \right]\quad . \tag{18}$$

Equations (17) and (18) are easily solved for the particle position using energy conservation,

$$E = \frac{1}{2}\,w^2 - \Phi_1\ ,$$

yielding

$$\xi(\xi_0,w_0,t_b) = \begin{cases} \sqrt{2/c}\ \sinh^{-1}\{\sinh(\eta_0)\cosh(\kappa\nu t_b)\ + \\ \quad \mathrm{sgn}(w)\ \left(1/\kappa^2+\cosh^2(\eta_0)\right)^{1/2}\sinh(\kappa\nu t_b)\} & 0<\kappa<\infty \quad (19a) \\ \sqrt{2/c}\ \sinh^{-1}\{\sinh(\eta_0)\cos(\kappa\nu t_b)+ \\ \quad \mathrm{sgn}(w)\ (1/k^2-\cosh^2(\eta_0)s)^{1/2}\sin(\kappa\nu t_b)\} & 0<k<\mathrm{sech}\eta_0 \quad (19b) \end{cases}$$

•

We have defined

$$\nu = \sqrt{3}\, c \quad , \qquad \eta_0 = \sqrt{c/2}\ \xi_0 \quad . \tag{20}$$

Note $\nu t_b = \omega_b t$ where ω_b is the bounce frequency at the bottom of the soliton well. (Recall ω_b is scaled with ω_{pi}).

Equation (19a) represents an untrapped orbit with energy $E = 3c\kappa^2$ while Eq. (19b) represents a trapped particle orbit with energy $E= -3c\kappa^2$. Substitution of Eqs. (19) into Eq. (15) constitutes a solution of the Vlasov equation.

IV. Conservation Laws and Generalized Damping

It is well-known that Eq. (11) with the source term set to zero possesses an infinite sequence of conservation laws.[17] Since the source term is of the form $\partial S/\partial\xi$, the lowest conservation law is maintained; i.e., $dI_0/d\tau = 0$ for

$$I_0 = \int \Phi_1 d\xi \quad . \tag{21}$$

Physically this corresponds to mass conservation. The next two conservation laws in the sequence are the momentum,

$$I_1 = \int \frac{1}{2} \Phi_1^2 \, d\xi \ , \tag{22}$$

and the energy,

$$I_2 = \int \left(\frac{1}{3} \Phi_1^3 - \frac{1}{2} \left(\frac{\partial \Phi_1}{\partial \xi}\right)^2 d\xi \right. . \tag{23}$$

The addition of the source term results in the following:

$$\frac{dI_1}{d\tau} = - \int \frac{\partial \Phi_1}{\partial \xi} S \, d\xi \tag{24}$$

and

$$\frac{dI_2}{d\tau} = 2 \int \frac{\partial \Phi_1}{\partial \tau} S \, d\xi \quad . \tag{25}$$

The method of perturbed conservation laws, mentioned in the Introduction, amounts to the substitution of the solution to the KdV Eq. (13) into a conservation law [e.g. Eq. (24) or (25)] and allowing c to vary on the slow soliton time scale, τ. This variation allows for and is determined by the source term.

Let us begin with conservation of momentum (Subsequently we will discuss mass and energy). Using Eqs. (15), (16) and the form of the source term in Eq. (25), yields

$$\frac{dI_1}{d\tau} = - \frac{1}{2\varepsilon^{3/2}} \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} dw \, \frac{\partial \Phi_1}{\partial \xi} \, f[\xi_0(\xi,w,t_b), \, w_0(\xi,w,t_b), \, 0]$$

$$= - \frac{1}{2\varepsilon^{3/2}} \int_{-\infty}^{\infty} d\xi_0 \int_{-\infty}^{\infty} dw_0 \, \frac{\partial \Phi_1}{\partial \xi} \, [\xi(\xi_0,w_0,t_b)] \; f(\xi_0,w_0,0) \; . \tag{26}$$

To obtain Eq. (26) we have used the fact that the motion is area preserving and so the Jacobian $\partial(\xi,w)/\partial(\xi_0,w_0) = 1$.

The electron orbits, Eqs. (19), have velocity excursions of at most $v_T \sim O(\varepsilon^{1/2})$. This is also the width of the integrand $\partial\Phi_1/\partial\xi$ about $w_0 = 0$ when $t_b \gtrsim O(1)$. Thus to lowest order in ε we expand f about $w_0 = 0$, keeping the first non-vanishing term. Furthermore, according to the discussion following Eq. (15), the lowest order contribution from f is independent of ξ_0:

$$\frac{dI_1}{d\tau} = -\frac{1}{2\varepsilon^{3/2}} \frac{\partial f}{\partial w_0}\Big|_{w_0=0} \int_{-\infty}^{\infty} d\xi_0 \int_{-\infty}^{\infty} dw_0 \; w_0 \frac{\partial \Phi_1}{\partial \xi}[\xi(\xi_0,w_0,t_b)] \; . \tag{27}$$

Substitution of the soliton form, Eq. (13), and the orbits, Eq. (19) into Eq. (27) gives

$$\frac{dI_1}{d\tau} = \frac{\delta^{1/2}}{\varepsilon} \frac{36}{\sqrt{2\pi}} c^2 \, (J_u + J_T) \tag{28a}$$

where

$$J_u = \int_0^{\infty} d\kappa \; \kappa^3(1+\kappa^2) \int_{-\infty}^{\infty} d\phi \; \frac{\cosh\phi}{[(1+\kappa^2)\cosh^2\phi - 1]^{1/2}} \; \frac{\sinh(\phi-\kappa\omega_b t)}{[(1+\kappa^2)\cosh^2(\phi-\kappa\omega_b t)-1]^{3/2}} \tag{28b}$$

$$J_T = \int_0^{1} d\kappa \; \kappa^3(1-\kappa^2) \int_{-\pi}^{\pi} d\phi \; \frac{\cos\phi}{[1-(1-\kappa^2)\cos^2\phi]^{1/2}} \; \frac{\sin(\phi-\kappa\omega_b t)}{[1-(1-\kappa^2)\cos^2(\phi-\kappa\omega_b t)]^{3/2}} \; . \tag{28c}$$

The integrals J_u and J_T represent the untrapped and trapped particle contributions, respectively.

Inserting the soliton form, Eq. (13), into the left hand side of Eq. (27) results in the following equation for the variation of c on the slow soliton time scale:

$$\frac{dc}{d\tau} = - c^{3/2}\, \gamma(t_b) \ , \tag{29}$$

where the generalized damping rate is given by

$$\gamma(t_b) = \frac{-2\delta^{1/2}}{\sqrt{\pi}\varepsilon} (J_u + J_T) \quad . \tag{30}$$

Observe that if we rewrite Eq. (29) in terms of the variable t, we obtain

$$\frac{dc}{dt} = - \varepsilon^{3/2}\, c^{3/2}\, \gamma(\omega_b t). \tag{31}$$

If $\varepsilon \sim O(\delta^{1/2})$ then we are within the valid region of our ordering (c.f. Fig. 1); Eq. (31) verifies consistency in that our assumption that c varies on the τ time scale is borne out. In the next section we investigate the solution of this equation.

In concluding this section we remark on the conservation laws I_2 and I_0. It is an interesting fact that both conservation laws, I_1 and I_2, yield exactly Eq. (29) when the soliton form Eq. (13) is assumed; this lends confidence to our analysis. Physically this arises because, within our ordering, soliton energy is lost at a rate that is proportional to the product of the sound speed and the momentum loss.

More rigorous perturbation theories (Keener and McLaughlin[18], Karpman and Maslov[15], Watanabe[19]) show that in addition to the slow modulation , a tail is typically produced behind the soliton. These theories, however, lead to an equation that is identical to the result obtained by substituting Eq. (13) into either Eq. (24) or (25). As Watanabe shows, an estimate for the size of the tail generated may be obtained from the conservation law

$$\frac{dI_0}{d\tau} = \frac{d}{d\tau} \int \Phi_1 d\xi = 0$$

by assuming $\Phi_1 = \Phi_S + \delta\Phi = 0$ and using Eq. (25) to describe the evolution of Φ_S. This shows that if the soliton damps, a positive amplitude tail must form.

V. Results

Unlike the O'Neil calculation, the dependence of c on the slow time scale τ results in the variation of the soliton width and speed, as well as its amplitude. We will see that asymptotically this variation tends to zero.

In Fig. 2 we plot numerical computations of the time variation of $\gamma(\omega_b t)$ and separately its contributions from J_u and J_T. Observe that J_T quickly tends to zero, while J_u oscillates and does so more slowly. This variation arises because the initial condition, Eq. (15), is not a BGK equilibrium. If the wave form is frozen then untrapped particles that are uniformly fed into the system at $|x| = \infty$ will require a characteristic transit time before temporal variation monotonically tends to zero, due to the uniformity of particle phase space density upon an untrapped trajectory. This explains the time dependence of J_u. The damped oscillatory behavior of J_T can be explained by the usual phase space smearing effect of particles in a potential well. In Appendix B we show by integration by parts that

$$J_T \sim \frac{2\pi}{(\omega_b t)^2} \sin \omega_b t + \text{h.o.t.} \quad .$$

This asymptotic behavior is indicated by the dotted curve of Fig. 2. Apparently one expects c to approach a finite saturated state.

In Fig. 3 the results obtained by numerically integrating Eq. (29) are presented. To lowest order it is consistent to suppress the c dependence of γ; therefore, we set ω_b to its value at t=0: $\omega_b = \sqrt{3}c(0)\varepsilon/\delta^{1/2}$. Observe that as in the O'Neil calculation, c oscillates on roughly the bounce time scale as it approaches its asymptotic state. As mentioned, physically this arises because of phase mixing of electrons inside the soliton trough. Unlike the O'Neil calculation there is no oscillation due to untrapped particles. The graph is plotted as a function of $\omega_b t$; hence, since ω_b depends upon c(0), the initial soliton speed, the scales for the various cases are different.

The asymptotic values shown in Fig. 3 can be explicitly obtained from Eq. (29). Integrating, we obtain

$$c(\infty) = \frac{c(0)}{(1 + \Gamma/\sqrt{c(0)}\,)^2} \tag{32}$$

where

$$\Gamma \equiv \Gamma_u + \Gamma_T \equiv -\left(\frac{\bar{\varepsilon}^3}{\pi}\right)^{1/2} \int_0^\infty (J_u + J_T)dt \quad . \tag{33}$$

Consider Γ_u:

$$\Gamma_u = \left(\frac{\bar{\varepsilon}^3}{\pi}\right)^{1/2} \int_0^\infty dt \int_0^\infty d\kappa \int_{-\infty}^\infty d\phi\, \kappa^3(1+\kappa^2)\; G(\phi,\kappa^2)\; G'(\phi-\kappa\omega_b t,\kappa^2) \tag{34}$$

where

$$G(\phi,\kappa^2) = \frac{\cosh\phi}{\left((1+\kappa^2)\cosh^2\phi - 1\right)^{1/2}} \; , \tag{35}$$

and the prime is used to indicate differentiation with respect to the first argument. If we replace $G(\phi,k^2)$ by

$$\hat{G}(\phi,\kappa^2) = G(\phi,\kappa^2) - \frac{1}{(1+\kappa^2)^{1/2}} , \tag{36}$$

then the value of Γ_u is unchanged. This is true since $G' = \hat{G}'$ is odd in its first argument. Performing the time integration of Eq. (34) yields

$$\Gamma_u = \frac{-2}{\omega_b} \left(\frac{\varepsilon^3}{\pi}\right)^{1/2} \int_0^\infty d\kappa \int_0^\infty d\phi \; \kappa^2(1+\kappa^2) \left[\hat{G}(\phi,\kappa^2)\right]^2 . \tag{37}$$

The remaining integrations of Eq. (37) are of standard form; we obtain

$$\Gamma_u = -\frac{\sqrt{\pi}}{6\omega_b} (\delta\varepsilon)^{1/2} . \tag{38}$$

Similarly, the contribution from Γ_T can be shown to be

$$\Gamma_T = -\frac{\sqrt{\pi}}{3\omega_b} (\delta\varepsilon)^{1/2} . \tag{39}$$

The asymptotic values seen in Fig. 3 are predicted by the formula

$$c(\infty) = c(0) \left[1 + \left(\frac{\pi\delta^2}{12\varepsilon c(0)}\right)^{1/2}\right]^{-2} . \tag{40}$$

In Fig. 4 we plot $c(\infty)/c(0)$ as a function of $c(0)$. The horizontal axis is $\varepsilon c(0)/\delta^2$ where $\varepsilon c(0)$ is physically the dimensionless soliton speed in excess of the ion sound speed. Observe that for small $\varepsilon c(0)$, $c(\infty)/c(0)$ deviates significantly from unity. Values of $\varepsilon c(0) \lesssim \delta^2$ are beyond the region of validity of our theory, since $\omega_b \lesssim \gamma_L$ in this region. In this case it is not sufficient to assume the soliton amplitude is constant when solving the Vlasov equation, and one expects linear damping to dominate the saturation due to phase mixing.

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Appendix A Linear Limit

As pointed out in the text, the theory of Ott and Sudan is valid for times $t \ll \omega_b^{-1}$. If one takes the limit $\omega_b t \to 0$ while keeping $\kappa\omega_b t$ finite, then the electron orbits as described by Eq. (19) reduce to uniform motion. We will show, in this limit, that the right-hand-side of Eq. (28a) reduces to the appropriate expression for linear electron Landau damping of the ion acoustic soliton.

Recall that J_T arises from the integration over the region of phase space that corresponds to trapped electrons. Since in our limit no such particles exist, evidently

$$\lim_{\omega_b t \to 0} J_T = 0 \quad .$$

Consider now the contribution due to untrapped particles.

$$J_u = \int_0^\infty d\kappa \int_{-\infty}^\infty \frac{1+\kappa^2}{\kappa} F\left(\frac{1}{\kappa^2}, \phi, \kappa\omega_b t\right) d\phi \tag{A-1}$$

where

$$F = \frac{\sinh(\phi-\kappa\omega_b t)\cosh\phi}{[(1+1/\kappa^2)\cosh^2\phi - 1/\kappa^2]^{1/2}[(1+1/\kappa^2)\cosh^2(\phi-\kappa\omega_b t) - 1/\kappa^2]^{3/2}} . \tag{A-2}$$

Expanding F in a Taylor series in its first argument yields

$$J_u = \int_0^\infty d\kappa \int_{-\infty}^\infty d\phi \left[\kappa F(0,\phi,\kappa\omega_b t) + \frac{1}{\kappa} F(0,\phi,\kappa\omega_b t) + \frac{1}{\kappa}\frac{\partial F}{\partial(1/\kappa^2)}(0,\phi,\kappa\omega_b t) + \text{h.o.t.}\right]$$

(A-3)

The first two terms in the integrand of Eq. (A-3) can be shown to vanish. The only nonvanishing contribution from the third term is

$$J_u = \int_0^\infty d\kappa \int_{-\infty}^{\infty} \frac{\sinh(\phi-\kappa\omega_b t)\, d\phi}{2\kappa \cosh^2\phi \cosh^3(\phi-\kappa\omega_b t)} + \text{h.o.t.} \quad . \tag{A-4}$$

Continuing this procedure, the next non-vanishing contribution is $O(\omega_b t)$. Substituting $\phi' = \phi - \kappa\omega t$ and making use of the parity of the integrand yields the following equivalent form for J_u:

$$J_u = \frac{1}{8} P \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\operatorname{sech}^2\phi}{\phi'-\phi} \frac{\partial}{\partial\phi'} \operatorname{sech}^2\phi' \, d\phi\, d\phi' + O(\omega_b t) \ . \tag{A-5}$$

Here P is used to mean principle part. In this limit Eq. (28a) produces the result of Ref. 1.

To conclude this Appendix we point out that upon Fourier transforming $\operatorname{sech}^2\phi$ and making use of the identity

$$P \int_{-\infty}^{\infty} \frac{\exp(i\kappa\phi)}{\phi-\phi'} d\phi = i\pi \ \operatorname{sgn} \kappa \ \exp(i\kappa\phi')$$

one obtains for the integral of Eq. (A-5)

$$P \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\operatorname{sech}^2\phi}{\phi'-\phi} \frac{\partial}{\partial\phi'} \operatorname{sech}^2\phi' \, d\phi\, d\phi' = \frac{24}{\pi^2} \zeta(3) = 2.92 \quad ,$$

where ζ is the Riemann zeta function.[20]

Appendix B Fourier Representation

In this appendix the Fourier representation of the integrals of Eq. (20) analogous to those obtained in Ref. 3 are presented. This form is used to obtain the long time asymptotic limit of γ.

Consider J_T:

$$J_T = -\int_0^1 d\kappa \int_{-\pi}^{\pi} \kappa^2(1-\kappa^2)\; G(\phi,\kappa)\, \frac{\partial G}{\partial \phi}\, (\phi-\kappa\omega_b t,\kappa)\, d\phi \tag{B-1}$$

where

$$G(\phi,k) \equiv \frac{\cos\phi}{[1-(1-k^2)\cos^2\phi]^{1/2}} \; .$$

Expanding $G(\phi,\kappa)$ in a Fourier series,

$$G(\phi,\kappa) = \sum_{n=-\infty}^{\infty} g_n(\kappa)\, \exp(-in\phi) \; , \tag{B-2}$$

results in the following form

$$J_T = -4\pi \sum_{n=1}^{\infty} n \int_0^1 \kappa^3(1-\kappa^2)\,|g_n(\kappa)|^2\, \sin(n\kappa\omega_b t)\; d\kappa \; . \tag{B-3}$$

The Fourier coefficients are given by

$$g_n = \frac{2}{\pi}\int_0^{\pi/2} \frac{\cos\phi\,\cos n\phi\;\, d\phi}{[1-(1-\kappa^2)\cos^2\phi]^{1/2}} \qquad n=1,3,5\ldots \quad , \tag{B-4}$$

while for even n the coefficients vanish. The integral of Eq. (B-4) can be evaluated by Taylor expanding the integrand in powers of $1-\kappa^2$. The integrals of the resulting series are of standard form, and upon integration the series can be identified as the following:

$$g_{2m+1}(\kappa) = \frac{1}{2\sqrt{\pi}} \frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+1)} F(m+\frac{1}{2}, m+\frac{3}{2}, 2m+2; 1-\kappa^2) \left(\frac{1-\kappa^2}{4}\right)^m , \tag{B-5}$$

where $\Gamma(x)$ is the usual gamma function and F is the hypergeometric function.[20] Insertion of Eq. (B-5) into Eq. (B-3) yields

$$J_T = -\sum_{m=0}^{\infty} \frac{(2m+1)}{2^{4m}} \int_0^1 \kappa^3 (1-\kappa^2)^{2m+1} \left[\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+1)} F(m+\frac{1}{2}, m+\frac{3}{2}, 2m+2; 1-\kappa^2)\right]^2$$

$$\times \sin\left[(2m+1)\kappa\omega_b t\right] d\kappa \quad . \tag{B-6}$$

Similarily, the Fourier integral representation for the untrapped particle contribution can be shown to be

$$J_u = -\int_0^{\infty} \frac{pdp}{\pi} \int_0^{\infty} \frac{\kappa^3}{(1+\kappa^2)^2} \left| \Gamma(1+\frac{ip}{2}) \, \Gamma(1-\frac{ip}{2}) \, F\left(1+\frac{ip}{2}, 1-\frac{ip}{2}, 2; \frac{1}{1+\kappa^2}\right) \right|^2$$

$$\times \sin(p\kappa\omega_b t) \, dp \quad . \tag{B-7}$$

As mentioned in the text the dominant contribution to the time asymptotic behavior of $\gamma(t)$ comes from J_T [see Fig. (2)]. This behavior can be extracted by the integration by parts procedure.[21] Writing J_T in the form

$$J_T = \sum_{m=0}^{\infty} \int_0^1 F_m(\kappa) \sin\left[(2m+1)\kappa\omega_b t\right] d\kappa \ , \tag{B-8}$$

where $F_m(\kappa)$ is defined by comparing Eqs. (B-8) and (B-6). Integrating by parts twice yields

$$J_T = \sum_{m=0}^{\infty} \frac{F_m'(1)\sin[(2m+1)\omega_b t]}{(2m+1)^2(\omega_b t)^2} + 0\left[\frac{1}{(\omega_b t)^3}\right] . \qquad \text{(B-9)}$$

Since $F_m'(1)$ vanishes unless m=0, we obtain

$$J_T = \frac{2\pi \sin \omega_b t}{(\omega_b t)^2} + 0\left[\frac{1}{(\omega_b t)^3}\right] . \qquad \text{(B-10)}$$

Figure Captions

1. Plot of relevant characteristic frequencies vs. $\varepsilon \equiv e\Phi/T_e$.
2. Plots of trapped, J_T and untrapped, J_u contributions to the generalized damping coefficient, γ. $J_u + J_T$ is indicated by dash dot. Dashed curve indicates the asymptotic behavior of J_u.
3. Plots of $c(t)/c(0)$ vs. time. (a) $\gamma_L/\omega_b = .057$ (b) $\gamma_L/\omega_b = .183$ (c) $\gamma_L/\omega_b = .57$. Note that ω_b depends on $c(0)$.
4. Asymptotic soliton amplitude vs. initial amplitude.

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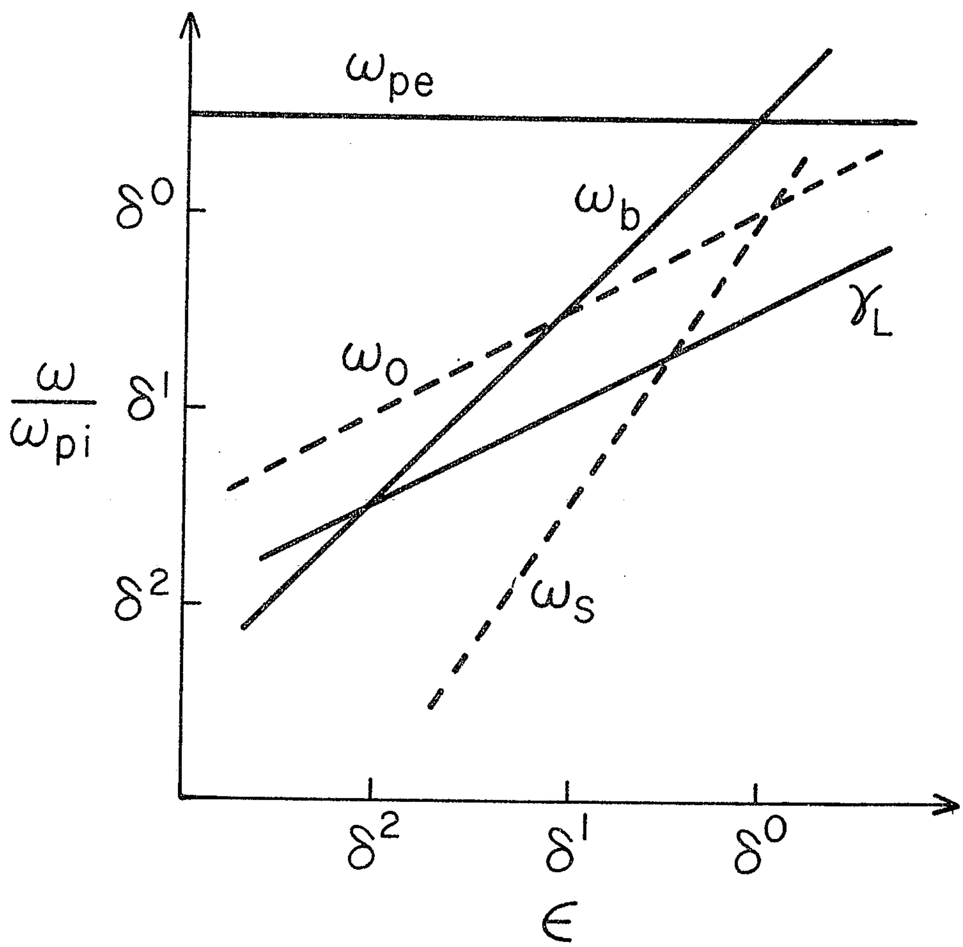

ω_{pe}
ω_b
δ^0
ω_0
γ_L
$\frac{\omega}{\omega_{pi}}$
δ^1
δ^2
ω_s
δ^2
δ^1
δ^0
ϵ

FIG. 1

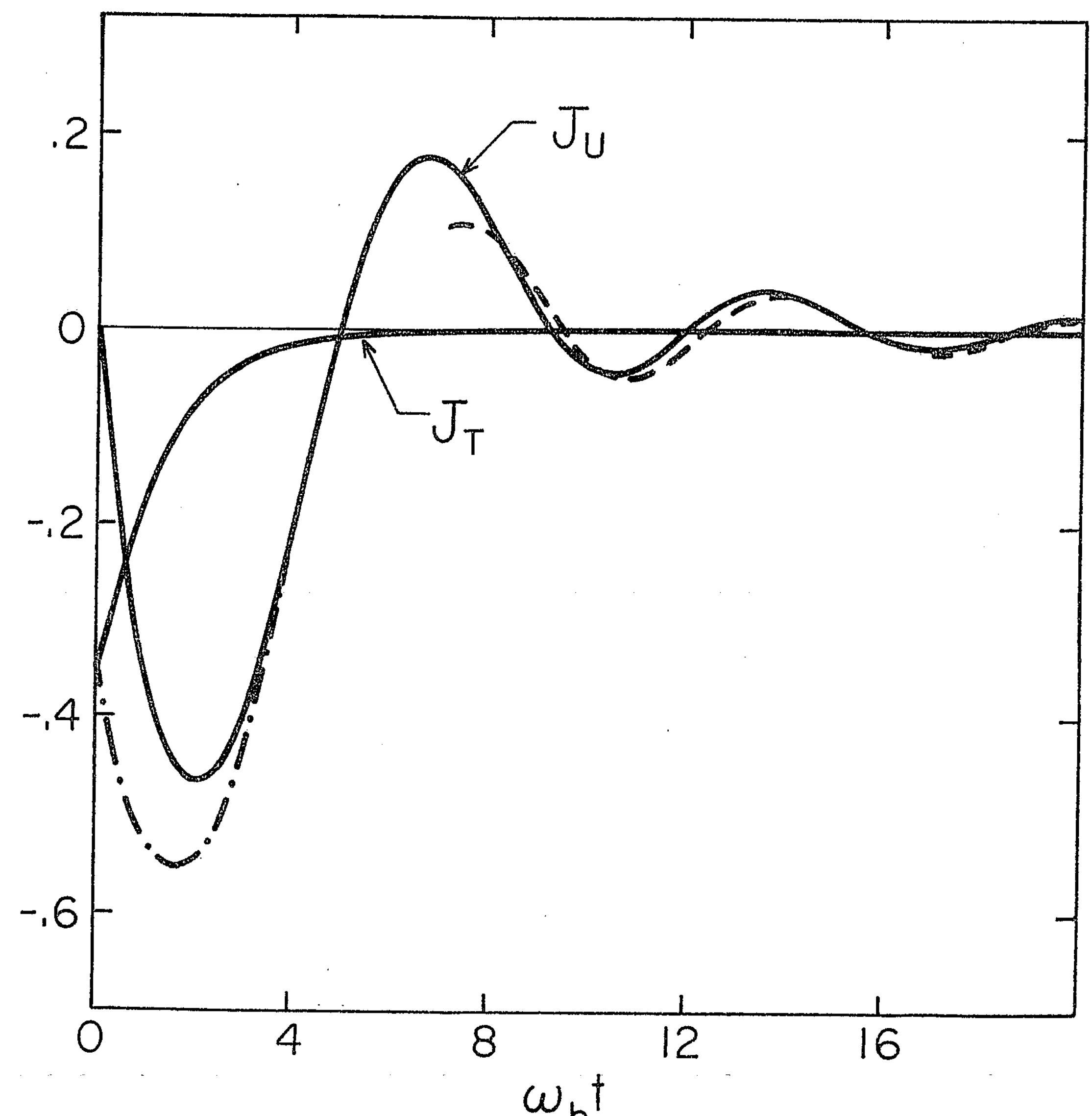

FIG. 2

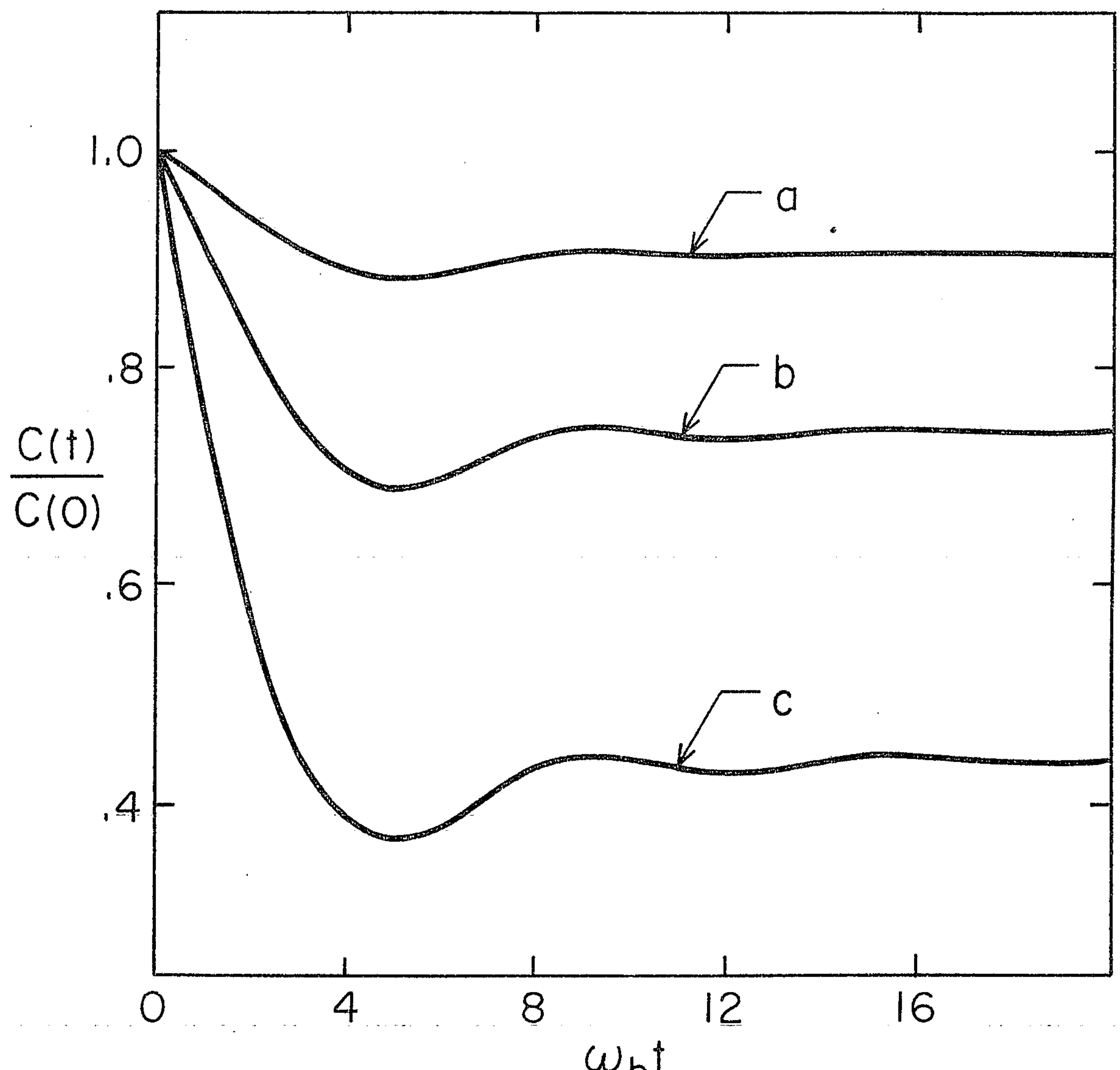

FIG. 3

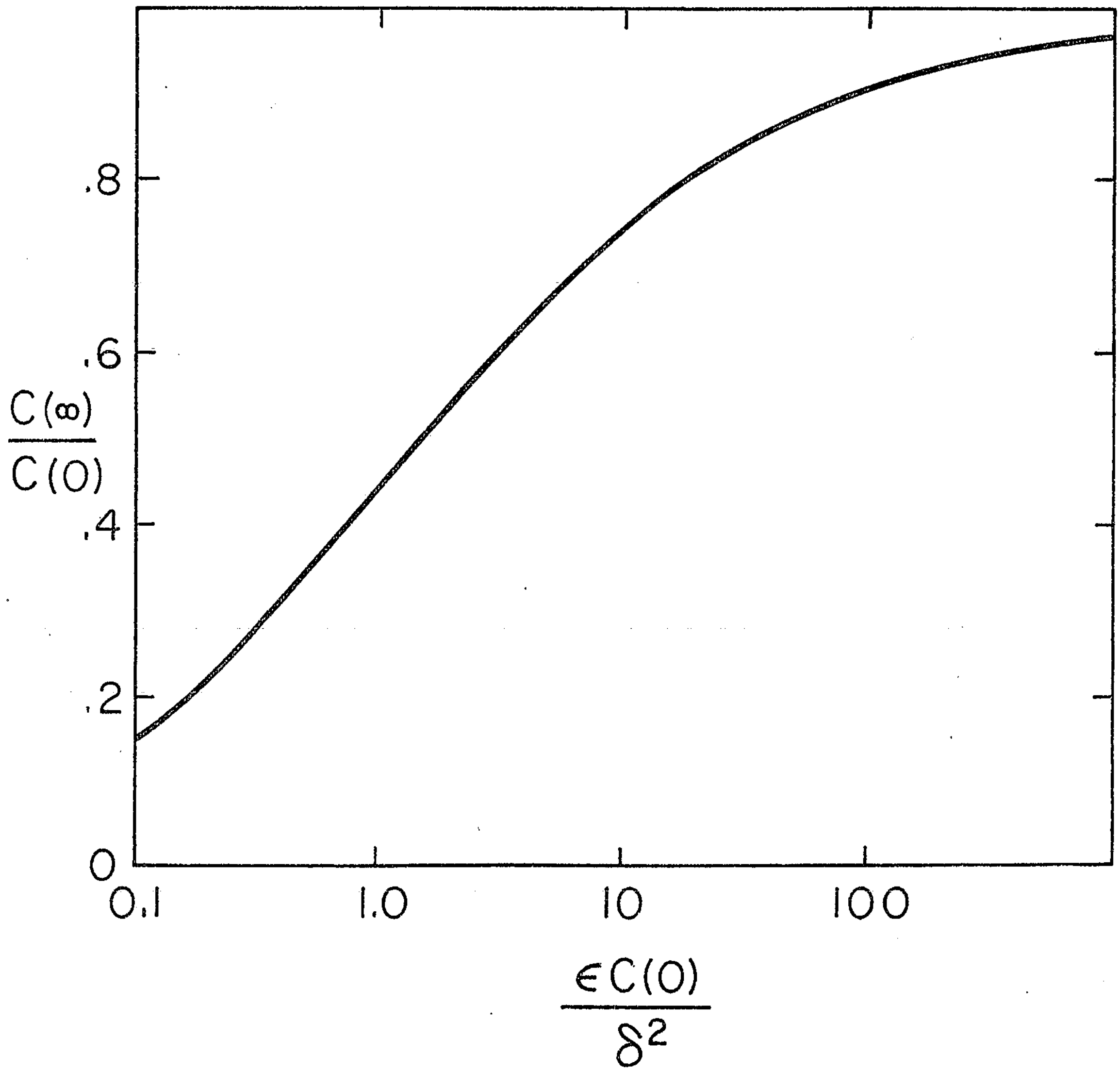

FIG. 4