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Reduced Braginskii Equations

M. YAGI[a] and W. HORTON
Institute for Fusion Studies
The University of Texas at Austin
Austin, Texas 78712

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[a] *Plasma Analysis Div., Naka Fusion Research Estab., JAERI, Japan*

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Institute for Fusion Studies
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Abstract

A set of reduced Braginskii equations is derived without assuming flute ordering and the Boussinesq approximation. These model equations conserve the physical energy. It is crucial at finite β that we solve the perpendicular component of Ohm's law to ensure $\nabla \cdot \mathbf{j} = 0$ for energy conservation.

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[a]Plasma Analysis Div., Naka Fusion Research Estab., JAERI, Japan

I Introduction

The reduced magnetohydrodynamics (MHD) equations are useful to investigate the nonlinear phenomena such as resistive tearing, ballooning or microturbulence, such as ion temperature gradient modes. Many authors had constructed such model equations.[1, 2, 3, 4, 5, 6] As pointed out in Ref. 3, the Boussinesq approximation[7] and neglecting $v_{\parallel}\nabla_{\parallel}$ in total derivative d/dt (flute ordering), result in the loss of the direct physical energy conservation law. In most cases an alternate energy-like invariance can be found which was discussed in Refs. 4 and 5. But when we take into account the transport effect in the temperature evolution equation, we need an additional assumption, as pointed out in Refs. 3 and 6, to have an energy-like invariant. Here, generalizing the method in Ref. 3, we derive the reduced model for Braginskii equations[8] which conserves the direct physical energy integral. These model equations are simplified especially for the slab and cold ion limit with low β ordering. These are different from those given by Ref. 3 in that the polarization drift term in the total time derivatives is not necessary, which makes the system more attractive for numerical simulation.

The remaining sections of this paper are organized as follows. In Sec. II we derive the reduced Braginskii equations and energy conservation relation in the cold ion limit. We have shown that there are two choices in constructing the energy conservation relation in the cold ion limit. In the appendix, we discuss a class of more general model equations.

II Model Equations

In this section we discuss the reduced Braginskii equations in cold ion limit for simplicity. The complete set of equations is given in the appendix. We start with the following equations:

$$\frac{\partial n}{\partial t} = -\nabla \cdot (n\mathbf{v}) \tag{1}$$

$$nm_i\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla\right)\mathbf{v}+\nabla p_e=\mathbf{j}\times\mathbf{B} \tag{2}$$

$$\mathbf{E}+\mathbf{v}_e\times\mathbf{B}+\frac{1}{en}\nabla p_e=\eta_\parallel\mathbf{j}_\parallel-\frac{0.71}{e}\nabla_\parallel T_e+\eta_\perp\mathbf{j}_\perp+\frac{3}{2}\frac{1}{e\Omega_e\tau_e}\widehat{\mathbf{z}}\times\nabla T_e \tag{3}$$

$$\frac{3}{2}\frac{\partial p_e}{\partial t}+\frac{3}{2}\nabla\cdot(p_e\mathbf{v}_e)+p_e(\nabla\cdot\mathbf{v}_e)=-\nabla\cdot\mathbf{q}_e^u-\nabla\cdot\mathbf{q}_e^T+Q_{e\parallel}+Q_{e\perp} \tag{4}$$

$$\mathbf{v}_e=\mathbf{v}-\frac{1}{en}\mathbf{j} \tag{5}$$

$$\frac{\partial\mathbf{B}}{\partial t}=-\nabla\times\mathbf{E} \tag{6}$$

$$\nabla\cdot\mathbf{B}=0 \tag{7}$$

$$\mathbf{j}=\nabla\times\mathbf{B} \tag{8}$$

where

$$\mathbf{q}_e^u=-\frac{0.71}{e}T_e\mathbf{j}_\parallel+\frac{3}{2}\frac{T_e}{e\Omega_e\tau_e}\widehat{\mathbf{z}}\times\mathbf{j}_\perp \tag{9}$$

$$\mathbf{q}_e^T=-\kappa_{e\parallel}\nabla_\parallel T_e-\kappa_{e\perp}\nabla_\perp T_e+\frac{5}{2}\frac{nT_e}{\Omega_e m_e}\widehat{\mathbf{z}}\times\nabla_\perp T_e \tag{10}$$

$$Q_{e\parallel}=\eta_\parallel j_\parallel^2-\frac{0.71}{e}j_\parallel\nabla_\parallel T_e \tag{11}$$

$$Q_{e\perp}=\eta_\perp j_\perp^2+\frac{3}{2}\frac{1}{e\Omega_e\tau_e}\mathbf{j}_\perp\cdot\widehat{\mathbf{z}}\times\nabla T_e \tag{12}$$

and $\Omega_e = -eB/m_e < 0$, $\tau_e = (3/4)\sqrt{m_e/2\pi}(T_e^{3/2}/\lambda e^4 Z^2 n)$, $\chi_{e\parallel} = 0.36(nT_e\tau_e/m_e)$, and $\chi_{e\perp} = 4.66(nT_e\tau_e/m_e\Omega_e^2\tau_e^2)$. Equation (1) is the ion continuity equation. Equation (2) is the momentum balance equation for the center of mass. In Eq. (2), $\mathbf{v}$ is the center mass velocity where we use the small electron to ion mass ratio ordering to derive this equation. This velocity agrees with ion velocity in the lowest order. Equation (3) is the electron momentum balance equation. In Eq. (3), we neglect the electron inertia term for simplicity. Equation (4) is the electron pressure balance equation. In Eq. (4), we neglect the ion-electron energy equipartition term but retain the other transport terms denoted in Eqs. (9)–(12). The electron velocity is related to ion velocity through Eq. (5). To close this system we use Maxwell equations (6)–(8). This system consists of $\{n, p_e, \mathbf{v}, \mathbf{v}_e, \mathbf{E}, \mathbf{B}\}$.

A. Energy conservation law

First, we construct the energy conservation relation in this system. Then we discuss the energy conservation relation for the reduced equations. We define $\langle \cdots \rangle \equiv \int_V \cdots d^3x$ where V is the volume and we impose the condition that the surface integral like $\int_{\partial V} \mathbf{K} \cdot d\mathbf{S}$ where ∂V is boundary of V zero to escape the complicated surface integral expressions. Volume averaging Eq. (4) we obtain

$$\frac{d}{dt}\left\langle \frac{3}{2} p_e \right\rangle + \left\langle p_e \nabla_\parallel \left(v_\parallel - \frac{1}{en} j_\parallel \right) \right\rangle + \left\langle p_e \nabla_\perp \cdot \left(\mathbf{v}_\perp - \frac{1}{en} \mathbf{j}_\perp \right) \right\rangle = \left\langle Q_{e\parallel} \right\rangle + \left\langle Q_{e\perp} \right\rangle \ . \quad (13)$$

Calculating $\nabla \cdot \mathbf{j} = 0$ using Eq. (2), we obtain

$$\nabla \cdot \left\{ \frac{\mathbf{B}}{B^2} \times \left[n m_i \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v}_\perp + \nabla_\perp p_e \right] \right\} + \mathbf{B} \cdot \nabla \left(\frac{j_\parallel}{B} \right) = 0 \ . \quad (14)$$

Multiplying α into Eq. (14), and volume averaging, we obtain

$$- \left\langle \nabla \alpha \cdot \frac{\mathbf{B}}{B^2} \times \left[n m_i \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v}_\perp + \nabla_\perp p_e \right] \right\rangle - \left\langle j_\parallel \nabla_\parallel \alpha \right\rangle = 0 \ . \quad (15)$$

Identifying $\mathbf{v}_\perp = \mathbf{B} \times \nabla \alpha / B^2$, we then write

$$\left\langle n m_i \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \frac{v_\perp^2}{2} \right\rangle + \left\langle \mathbf{v}_\perp \cdot \nabla p_e \right\rangle - \left\langle j_\parallel \nabla_\parallel \alpha \right\rangle = 0 \ . \quad (16)$$

Using Eq. (1), we obtain

$$\frac{d}{dt}\left\langle\frac{nm_i}{2}v_\perp^2\right\rangle-\left\langle j_\parallel\nabla_\parallel\alpha\right\rangle+\langle\mathbf{v}_\perp\cdot\nabla p_e\rangle=0\ . \tag{17}$$

Similarly, from the parallel component of Eq. (2), we obtain

$$\frac{d}{dt}\left\langle\frac{nm_i}{2}v_\parallel^2\right\rangle+\left\langle v_\parallel\nabla_\parallel p_e\right\rangle=0\ . \tag{18}$$

If we decouple the Ohm's law Eq. (3) into parallel and perpendicular components, we obtain, using Eqs. (5)–(8) as

$$-\nabla_\parallel\phi-\frac{\partial A_\parallel}{\partial t}+\frac{1}{en}\nabla_\parallel p_e=\eta_\parallel j_\parallel-\frac{0.71}{e}\nabla_\parallel T_e \tag{19}$$

$$-\nabla_\perp\phi-\frac{\partial\mathbf{A}_\perp}{\partial t}+\nabla_\perp\alpha-\frac{1}{en}\mathbf{j}_\perp\times\mathbf{B}+\frac{1}{en}\nabla_\perp p_e=\eta_\perp\mathbf{j}_\perp+\frac{3}{2}\frac{1}{e\Omega_e\tau_e}\widehat{\mathbf{z}}\times\nabla T_e \tag{20}$$

$$j_\parallel=\nabla_\parallel(\nabla\cdot\mathbf{A})-\nabla^2A_\parallel \tag{21}$$

$$\mathbf{j}_\perp=\nabla_\perp(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}_\perp \tag{22}$$

where we use $\mathbf{v}_\perp=\mathbf{B}\times\nabla_\perp\alpha/B^2$. Operating with $j_\parallel$ into Eq. (19) and with $\mathbf{j}_\perp\cdot$ into Eq. (20) and volume averaging, we obtain

$$-\left\langle j_\parallel\nabla_\parallel\phi\right\rangle-\left\langle j_\parallel\frac{\partial A_\parallel}{\partial t}\right\rangle+\left\langle\frac{1}{en}j_\parallel\nabla_\parallel p_e\right\rangle=\left\langle Q_{e\parallel}\right\rangle \tag{23}$$

$$-\langle\mathbf{j}_\perp\cdot\nabla_\perp\phi\rangle-\left\langle\mathbf{j}_\perp\cdot\frac{\partial\mathbf{A}_\perp}{\partial t}\right\rangle-\left\langle j_\parallel\nabla_\parallel\alpha\right\rangle+\left\langle\frac{1}{en}\mathbf{j}_\perp\cdot\nabla_\perp p_e\right\rangle=\langle Q_{e\perp}\rangle \tag{24}$$

where we use the relation $\langle\mathbf{j}_\perp\cdot\nabla_\perp\alpha\rangle=-\left\langle j_\parallel\nabla_\parallel\alpha\right\rangle$ in Eq. (24). From Eq. (24), we can see the third term in Eq. (16) cancels with the third term in Eq. (24). Combining Eqs. (23) and (24), and using the relation $\left\langle j_\parallel\nabla_\parallel\phi\right\rangle+\langle\mathbf{j}_\perp\cdot\nabla_\perp\phi\rangle=0$, we obtain

$$\frac{d}{dt}\left\langle\frac{1}{2}|\nabla\times\mathbf{A}|^2\right\rangle+\left\langle j_\parallel\nabla_\parallel\alpha\right\rangle-\left\langle\frac{1}{en}\mathbf{j}\cdot\nabla p_e\right\rangle=-\left\langle Q_{e\parallel}\right\rangle-\langle Q_{e\perp}\rangle \tag{25}$$

for the rate of change of the total magnetic energy. Substituting Eqs. (17), (18), and (25) into Eq. (13) we obtain

$$\frac{d}{dt}\left(\left\langle\frac{3}{2}p_e\right\rangle+\left\langle\frac{nm_i}{2}v^2\right\rangle+\left\langle\frac{1}{2}|\nabla\times\mathbf{A}|^2\right\rangle\right)=0\ . \tag{26}$$

Thus, we show that this dynamical system conserves the total of the thermal, kinetic and magnetic energy defined by the direct volume integrals of these energy densities.

Now, we discuss the reduced system. If we assume the low β ordering with $\nabla\phi \sim O(\epsilon)$, $\partial A/\partial t \sim O(\epsilon^2)$, $v_\perp \sim v_\parallel \sim O(\epsilon)$, and $\beta \sim O(\epsilon^2)$, then we neglect the finite β effect in Eq. (13) which is derived from $\mathbf{j}_\perp$. We should then neglect $\langle Q_{e\perp}\rangle$ consistently. In this case we have $\alpha = \phi$ and $j_\parallel = -\nabla_\perp^2 A_\parallel$. So the third term in Eq. (16) cancels the first term in Eq. (23) in this case, and $\nabla\cdot\mathbf{j} = 0$ is not ensured due to the neglect of $\mathbf{j}_\perp$ and setting $j_\parallel = \nabla_\perp^2 A_\parallel$. Therefore in the cold ion limit we have two choices to construct the energy conservation law for finite β plasma, which corresponds to the high β ordering. In the first case we solve $A_\parallel$, $\mathbf{A}_\perp$, and in the second we permit the deviation from the relation $\nabla\cdot\mathbf{j} = 0$ and accord to the conventional modeling (but we retain $v_\parallel\nabla_\parallel$ in total time derivative). We choose the second case and set $\alpha = \phi$, obtaining from Eq. (20)

$$\frac{1}{en}\mathbf{j}_\perp = -\frac{\mathbf{B}}{B^2}\times\left(-\frac{1}{en}\nabla_\perp p_e + \frac{\partial\mathbf{A}_\perp}{\partial t} + \eta_\perp\mathbf{j}_\perp + \frac{3}{2}\frac{1}{e\Omega_e\tau_e}\hat{\mathbf{z}}\times\nabla T_e\right) . \tag{27}$$

We calculate the following quantity in Eq. (13)

$$\left\langle p_e\nabla_\perp\cdot\left(-\frac{1}{en}\mathbf{j}_\perp\right)\right\rangle - \langle Q_{e\perp}\rangle = -\left\langle\frac{\mathbf{B}}{B^2}\times\left(\frac{\partial\mathbf{A}_\perp}{\partial t} + \eta_\perp\mathbf{j}_\perp + \frac{3}{2}\frac{1}{e\Omega_e\tau_e}\hat{\mathbf{z}}\times\nabla T_e\right)\cdot\nabla p_e\right\rangle$$
$$- \left\langle\eta_\perp j_\perp^2\right\rangle - \left\langle\frac{3}{2}\frac{1}{e\Omega_e\tau_e}\mathbf{j}_\perp\cdot\hat{\mathbf{z}}\times\nabla T_e\right\rangle . \tag{28}$$

To cancel out the dissipation, we must set $\mathbf{j}_\perp = \mathbf{B}\times\nabla p_e/B^2$. If we write $\mathbf{B}\cdot\hat{\mathbf{z}} = B_0 + B_\beta$ we can write this energy transfer term as

$$\left\langle p_e\nabla_\perp\cdot\left(-\frac{1}{en}\mathbf{j}_\perp\right)\right\rangle - \langle Q_{e\perp}\rangle = \left\langle\mathbf{j}_\perp\cdot\frac{\partial\mathbf{A}_\perp}{\partial t}\right\rangle = \frac{d}{dt}\left\langle\frac{1}{2}B_\beta^2\right\rangle . \tag{29}$$

If we take into account toroidal effects which come from $\nabla_\perp\cdot(\mathbf{B}\times\nabla_\perp p_e/B^2)$ in vorticity equation (14), we must retain curvature terms related to $\nabla_\perp\cdot\mathbf{v}_E$ in the ion continuity equation (1) to conserve the energy. Furthermore, we need an artifice in the pressure evolution equation. That is, in Eq. (4), we substitute $(\mathbf{B}\times\nabla p_e)/B^2$ into the $\mathbf{j}_\perp$ term which is derived from $(3/2)\nabla \cdot (p_e \mathbf{v}_e)$ term not to solve $\mathbf{A}_\perp$ directly, but use Eq. (27) for the $\mathbf{j}_\perp$ term, which is derived from $p_e(\nabla \cdot \mathbf{v}_e)$ to cancel out the $Q_{e\perp}$ term. Briefly, we make ordering explicitly for Eq. (4). According to the above consideration, we order $\mathbf{j}_\perp = \epsilon \mathbf{j}_\perp^{(1)} + \epsilon^2 \mathbf{j}_\perp^{(2)}$, where $\mathbf{j}_\perp^{(1)} = \mathbf{B} \times \nabla p_e / B^2$ and $\mathbf{j}_\perp^{(2)}$ represents the other terms in Eq. (27). We order $\partial/\partial t \sim O(\epsilon)$, $p_e \sim O(\epsilon)$, $\nabla \cdot \mathbf{q}_e^u \sim O(\epsilon^2)$, $\nabla \cdot \mathbf{q}_e^T \sim O(\epsilon^2)$, $Q_{e\parallel} \sim O(\epsilon^2)$, $Q_{e\perp} \sim O(\epsilon^3)$. Therefore the term $p_e(\nabla \cdot \mathbf{v}_e)$ contains the higher order terms ($O(\epsilon^3)$). Essentially the finite β effect related to the perpendicular resistivity comes into play as the higher order correction based on this ordering. An alternative method is discussed in Ref. 3 where the extra polarization drift term and the term related to the perpendicular resistivity appear in the electron perpendicular velocity although we only have the $\mathbf{E} \times \mathbf{B}$ term in it. The alternative method is related to the first choice in our paper where we handle the relation $\nabla \cdot \mathbf{j} = 0$ with more accuracy. We will discuss it briefly in the appendix.

For this case we can write down the model equations with $\alpha = \phi$.

$$\nabla_\perp \cdot \left(\frac{nm_i}{B} \frac{d}{dt} \frac{1}{B} \nabla_\perp \phi \right) = \nabla_\parallel j_\parallel + \nabla \cdot \left(\frac{\mathbf{B}}{B^2} \times \nabla p_e \right) \tag{30}$$

$$\frac{dn}{dt} = -n \left(\nabla_\parallel v_\parallel + \nabla_\perp \cdot \mathbf{v}_E \right) \tag{31}$$

$$nm_i \frac{dv_\parallel}{dt} + \nabla_\parallel p_e = 0 \tag{32}$$

$$-\nabla_\parallel \phi - \frac{\partial A_\parallel}{\partial t} + \frac{1}{en} \nabla_\parallel p_e = \eta_\parallel j_\parallel - \frac{0.71}{e} \nabla_\parallel T_e \tag{33}$$

$$\frac{3}{2} \frac{\partial p_e}{\partial t} + \frac{3}{2} \nabla_\perp \cdot (p_e \mathbf{v}_E) + \frac{3}{2} \nabla_\parallel (p_e v_\parallel) - \frac{3}{2} \nabla_\parallel \left(\frac{p_e}{en} j_\parallel \right)$$

$$+ \frac{\partial}{\partial t} \frac{B_\beta^2}{2} + p_e \nabla_\perp \cdot \mathbf{v}_E + p_e \nabla_\parallel \left(v_\parallel - \frac{1}{en} j_\parallel \right) =$$

$$\frac{0.71}{e}\nabla_\parallel(T_e j_\parallel) + \nabla_\parallel(\kappa_{e\parallel}\nabla_\parallel T_e) + \nabla_\perp\cdot(\kappa_{e\perp}\nabla_\perp T_e)$$

$$+ \eta_\parallel j_\parallel^2 - \frac{0.71}{e} j_\parallel \nabla_\parallel T_e - \frac{5}{2}\nabla_\perp\cdot\left(\eta_\perp\nabla_\perp\frac{B_\beta^2}{2}\right) + \nabla_\perp\cdot\left(\eta_\perp B_\beta T_e\nabla_\perp\frac{n}{B}\right)$$

$$+ \nabla_\perp\cdot\left(\frac{5}{2}\frac{T_e}{eB}\hat{\mathbf{z}}\times\nabla_\perp p_e\right) + \nabla_\perp\cdot\left(\frac{5}{2}\frac{p_e}{eB}\hat{\mathbf{z}}\times\nabla T_e\right) \tag{34}$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v}_E\cdot\nabla_\perp + v_\parallel\nabla_\parallel \tag{35}$$

where $B_\beta = -p_e/B$ and $j_\parallel = -\nabla_\perp^2 A_\parallel$. Notice that in this second model, we can not regard $d\overline{n}/dr =$ const to conserve the energy where $n = \overline{n} + \tilde{n}$ and the terms related to B_β are higher order correction.

B. Slab model reduction

In the limit of slab geometry with low β ordering ($\beta \sim O(\epsilon^2)$) we retain only the $\mathbf{E}\times\mathbf{B}$ drift term in the electron velocity and neglect the terms related to the perpendicular resistivity). This model reduces to a more compact form. Using the following normalization

$$(c_s/L_n)t \to t\ ,\ r/\rho_s \to r\ ,\ z/L_n \to z\ ,\ \kappa_\parallel/(L_n c_s n_0) \to \chi_\parallel\ ,$$

$$L_n\kappa_\perp/(c_s\rho_s^2 n_0) \to \chi_\perp\ ,\ L_n\eta/(\rho_s^2 c_s) \to \eta\ ,\ (L_n/\rho_s n_0)\overline{n} \to \overline{n}\ , \tag{36}$$

we obtain

$$\frac{dn}{dt} = \frac{d\overline{n}}{dx}\frac{\partial\phi}{\partial y} - \xi(\overline{n}+n)\nabla_\parallel v \tag{37}$$

$$\nabla_\perp\cdot\left\{\xi(\overline{n}+n)\frac{d}{dt}\nabla_\perp\phi\right\} = \nabla_\parallel j_\parallel \tag{38}$$

$$\xi(\overline{n}+n)\frac{dv}{dt} = -\nabla_\parallel p \tag{39}$$

$$\frac{\partial A}{\partial t} = -\nabla_\parallel \phi + \frac{1}{\xi(\overline{n}+n)} \nabla_\parallel p - D j_\parallel + \alpha \nabla_\parallel T \tag{40}$$

$$\frac{3}{2}\frac{dp}{dt} - \frac{3}{2}\frac{d\overline{p}}{dx}\frac{\partial \phi}{\partial y} - \frac{3}{2}\frac{j_\parallel}{\overline{n}+n}\nabla_\parallel p + \frac{5}{2}\xi(\overline{p}+p)\nabla_\parallel \left(v - \frac{j_\parallel}{\xi(\overline{n}+n)} \right) =$$

$$+ 0.71 \nabla_\parallel j_\parallel + \chi_\parallel \nabla_\parallel^2 T + \chi_\perp \nabla_\perp^2 T + \xi D j_\parallel^2 - 0.71 \xi j_\parallel \nabla_\parallel T \tag{41}$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + [\phi, \quad] + \xi v \nabla_\parallel \tag{42}$$

$$T = \frac{\overline{p}+p}{\overline{n}+n} \tag{43}$$

$$j_\parallel = -\frac{1}{\beta}\nabla_\perp^2 A \tag{44}$$

here $\xi = \rho_s/L_n$, $D = \eta\beta$, $\beta = p/B^2$. Integrating $\phi\times$ Eq. (37), $v\times$ Eq. (38), $j_\parallel\times$ Eq. (39), and Eq. (40) over the volume, and assuming appropriate boundary conditions , we obtain the energy conservation relation as

$$\frac{d}{dt}\left(\frac{3}{2}\langle p \rangle + \frac{\xi^2}{2}\left\langle (\overline{n}+n)v^2 \right\rangle + \frac{\xi^2}{2}\left\langle (\overline{n}+n)|\nabla_\perp \phi|^2 \right\rangle + \frac{\xi}{2\beta}\left\langle |\nabla_\perp A|^2 \right\rangle \right) = 0 \ . \tag{45}$$

These model equations are similar to those derived by Horton *et al.*[9] and are relevant to the collisional drift wave problem. In this system, the electron internal energy is dominant and ion kinetic energy is a small contribution (order $O(\xi^2)$).

Finally we mention another choice of $\mathbf{A}_\perp$ for constructing the energy conservation relation. In this case we should solve Eqs. (30)–(33), (35), (20), (21), and (22), and the following pressure evolution equation

$$\frac{3}{2}\frac{\partial p_e}{\partial t} + \frac{3}{2}\nabla_\perp \cdot \left(p_e \left(\mathbf{v}_E - \frac{1}{en}\mathbf{j}_\perp \right) \right) + p_e \nabla_\perp \cdot \left(\mathbf{v}_E - \frac{1}{en}\mathbf{j}_\perp \right)$$

$$+\frac{5}{2}\nabla_\parallel \left(p_e \left(v_\parallel - \frac{1}{en} j_\parallel \right) \right) = -\nabla \cdot \mathbf{q}_e^u - \nabla \cdot \mathbf{q}_e^T + Q_{e\parallel} + Q_{e\perp} \ . \tag{46}$$

III Conclusion

We have derived new reduced Braginskii equations. To construct the energy conservation relation in the cold ion limit, we have two choices for the finite β plasma corresponding to the high β ordering ($\beta \sim O(\epsilon)$) given in Sec. II. We now describe the two choices.

The one choice is based on the conventional reduced MHD procedure, i.e. we retain the lowest order term in the perpendicular electron velocity ($\mathbf{E} \times \mathbf{B}$ and the electron diamagnetic drift velocity) but in this case we need an artifice in the pressure evolution equation to conserve the energy. That is, we substitute the lowest order perpendicular current term $\mathrm{B} \times \nabla p_e/B^2$ ($O(\epsilon)$) into the $\mathbf{j}_\perp$ term in Eq. (4), which is derived from the $(3/2)\nabla \cdot (p_e \mathbf{v}_e)$ in the electron pressure evolution equation, and do not solve for the time evolution of $\mathbf{A}_\perp$ directly. But we substitute the perpendicular current term Eq. (27) with higher order (where $\partial \mathbf{A}_\perp/\partial t \sim O(\epsilon^2)$, $\eta_\perp \mathbf{j}_\perp \sim O(\epsilon^2)$, $(3/2e\Omega_e\tau_e)\hat{\mathbf{z}} \times \nabla T_e \sim O(\epsilon^2)$) into the $\mathbf{j}_\perp$ term which is derived from $p_e(\nabla \cdot \mathbf{v}_e)$, to cancel out the higher order term $Q_{e\perp}$ ($O(\epsilon^3)$). Therefore the relation $\nabla \cdot \mathbf{j} = 0$ is not ensured for this approximation and the finite β effect related to the perpendicular resistivity enters as a higher order correction ($O(\epsilon^3)$). The merit of this modeling is that equations are simplified especially for the slab and cold ion limit. These model equations are relevant to the nonlinear collisional drift wave problem.

The other choice is to retain higher order terms in the conventional resistive MHD ordering, such as the polarization drift term and/or the term related to the perpendicular resistivity (where we reorder $\eta_\perp \mathbf{j}_\perp \sim O(\epsilon)$ or $\eta_\perp \sim O(1)$) in the perpendicular electron velocity according to the method discussed in Ref. 3. More generally, we solve the perpendicular component of Ohm's law for the time evolution of $\mathbf{A}_\perp$ to ensure $\nabla \cdot \mathbf{j} = 0$ with more accuracy, as discussed in the appendix of this article. If we need to treat the toroidal effect accurately, then it is better to choose the second method although this model is computationally very time consuming. It is difficult to construct energy conservation without solving the perpendicular component of Ohm's law for $\mathbf{j}_\perp$, when we take into account finite ion pressure. This model is relevant to the nonlinear finite β toroidal drift wave problem.

To check the validity of the Boussinesq approximation and the flute ordering we plan to compare the prediction of this new model and the conventional model equations[9] for the collisional drift wave problem in a future work.

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Appendix — General Equation

In this appendix, we derive a more complete form of the reduced Braginskii equations by solving perpendicular Ohm's law. When we take into account the finite ion pressure effect, it is difficult to construct an energy conservation relation without solving perpendicular Ohm's law, since the third term in Eq. (16) is not cancelled to the first term in Eq. (23) in this case. Therefore the conventional approach is only useful in the cold ion limit. This model consist of 8 fields $\{n, p_e, p_i, v_\parallel, \phi, A_\parallel, \mathbf{A}_\perp\}$.

$$\nabla_\perp \cdot \left(\frac{nm_i}{B} \frac{d}{dt} \frac{1}{B} \nabla_\perp \alpha \right) = \nabla_\parallel j_\parallel + \nabla_\perp \cdot \left(\frac{\mathbf{B}}{B^2} \times \nabla_\perp (p_e + p_i) \right) + \nabla_\perp \cdot \left(\frac{\mathbf{B}}{B^2} \times \nabla \cdot \mathbf{\Pi}_i \right) \quad \text{(A1)}$$

$$\frac{d}{dt} n = -n \nabla \cdot \mathbf{v} \quad \text{(A2)}$$

$$nm_i \frac{d}{dt} v_\parallel + \nabla_\parallel (p_e + p_i) + \hat{\mathbf{z}} \cdot \nabla \cdot \Pi_i = 0 \quad \text{(A3)}$$

$$-\nabla_\parallel \phi - \frac{\partial A_\parallel}{\partial t} + \frac{1}{en} \nabla_\parallel p_e = \eta_\parallel j_\parallel - \frac{0.71}{e} \nabla_\parallel T_e \quad \text{(A4)}$$

$$-\nabla_\perp \phi - \frac{\partial \mathbf{A}_\perp}{\partial t} + \nabla_\perp \alpha - \frac{1}{en} \mathbf{j}_\perp \times \mathbf{B} + \frac{1}{en} \nabla_\perp p_e = \eta_\perp \mathbf{j}_\perp + \frac{3}{2} \frac{1}{e\Omega_e \tau_e} \hat{\mathbf{z}} \times \nabla_\perp T_e \quad \text{(A5)}$$

$$\nabla \cdot \mathbf{A} = 0 \quad \text{(A6)}$$

$$j_\parallel = -\nabla^2 A_\parallel \quad \text{(A7)}$$

$$\mathbf{j}_\perp = -\nabla^2 \mathbf{A}_\perp \quad \text{(A8)}$$

$$\frac{3}{2}\frac{\partial p_i}{\partial t}+\frac{3}{2}\nabla_\perp\cdot(p_i\mathbf{v})+p_i\nabla\cdot\mathbf{v}=-\nabla\cdot\mathbf{q}_i+Q_i-\Pi_i\colon\nabla\mathbf{v} \tag{A9}$$

$$\frac{3}{2}\frac{\partial p_e}{\partial t}+\frac{3}{2}\nabla\cdot\left(p_e\left(\mathbf{v}-\frac{1}{en}\mathbf{j}\right)\right)+p_e\nabla\cdot\left(\mathbf{v}-\frac{1}{en}\mathbf{j}\right)=-\nabla\cdot\mathbf{q}_e^u-\nabla\cdot\mathbf{q}_e^T+Q_e \tag{A10}$$

$$\mathbf{v}=\frac{\mathbf{B}}{B^2}\times\nabla_\perp\alpha+v_\parallel\hat{\mathbf{z}} \tag{A11}$$

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla \tag{A12}$$

where $Q_e=Q_{e\parallel}+Q_{e\perp}-(3/2\tau_\epsilon^{ei})(T_e-T_i)$, $Q_i=(3/2\tau_\epsilon^{ei})(T_e-T_i)$, $\mathbf{q}_i=-\kappa_{i\parallel}\nabla_\parallel T_i-\kappa_{i\perp}\nabla_\perp T_i+(5nT_i/2\Omega_i m_i)\hat{\mathbf{z}}\times\nabla T_i$, $\Omega_i>0$, $\kappa_{i\parallel}=3.9n_iT_i\tau_i/m_i$, $\kappa_{i\perp}=2n_iT_i/m_i\Omega_i^2\tau_i$, Π_i is the stress tensor given in Eqs. (2.19)–(2.27) in Ref. 8, and other parameters are given by Eqs. (9)–(12). In this case, the energy conservation relation is given by

$$\frac{d}{dt}\left(\left\langle\frac{3}{2}p_e\right\rangle+\left\langle\frac{3}{2}p_i\right\rangle+\left\langle\frac{nm_i}{2}\frac{|\nabla_\perp\alpha|^2}{B^2}\right\rangle+\left\langle\frac{nm_i}{2}v_\parallel^2\right\rangle+\left\langle\frac{1}{2}|\nabla\times\mathbf{A}|^2\right\rangle\right)=0\ . \tag{A13}$$

We should note that to ensure $\nabla\cdot\mathbf{j}=0$ means to retain higher order terms in the electron velocity $\mathbf{v}-\mathbf{j}/en$, therefore there occurs the similar situation in Ref. 3 where the polarization drift and the term related to perpendicular resistivity appear in it. It is difficult to construct the energy conserving system with finite β effect only by means of rigorous ordering. In such a sense the parameter α is arbitrary so that we can choose it up to the order we like. But if we retain the lowest order terms($\mathbf{E}\times\mathbf{B}$ and the ion diamagnetic drift velocity) in the ion velocity, then α is given by

$$\nabla_\perp\alpha=\nabla_\perp\phi+\frac{1}{en}\nabla_\perp p_i\ . \tag{A14}$$

This model doesn't have the symmetry for the lowest order ion and electron velocity like $\mathbf{v}_{e\perp}=\mathbf{B}\times\nabla\phi/B^2-\mathbf{B}\times\nabla p_e/(enB^2)$ and $\mathbf{v}_{i\perp}=\mathbf{B}\times\nabla\phi/B^2+\mathbf{B}\times\nabla p_i/(enB^2)$.

Next we discuss the effect of perpendicular Ohm's law more briefly. To do so, it is convenient to normalize these equations using Eq. (36). For simplicity, we take the dissipationless and cold ion limit. Then (A4) and (A5) are rewritten as follows:

$$-\nabla_{\parallel}\phi - \frac{\partial A_{\parallel}}{\partial t} + \frac{1}{n}\nabla_{\parallel}p_e = -0.71\nabla_{\parallel}T_e \tag{A15}$$

$$-\frac{\rho_s^2}{L_n^2}\beta_e\frac{\partial \mathbf{A}_{\perp}}{\partial t} + \frac{1}{n}\left(\nabla_{\perp}^2 + \frac{\rho_s^2}{L_n^2}\nabla_{\parallel}^2\right)\mathbf{A}_{\perp}\times\mathbf{b} + \frac{\beta_e}{n}\nabla_{\perp}p_e = 0 \tag{A16}$$

where $\mathbf{b}$ is a unit vector of the direction of the magnetic field line and $\beta_e = p_{e0}/B^2$, electron plasma beta. From this, we can see that advancing the time step for $\mathbf{A}_{\perp}$ is very expensive. The linear stability of the mode will be changed through the term $-(3/2e)\mathbf{j}_{\perp}\cdot\nabla T_e + (T_e/en)\mathbf{j}_{\perp}\cdot\nabla n$ is the electron pressure evolution equation, although the correction is order $O(\beta_e\rho_s^2/L_n^2)$.

References

[1]H.R. Strauss, Phys. Fluids **19**, 134 (1976).

[2]H.R. Strauss, Phys. Fluids **20**, 1354 (1977).

[3]J.F. Drake and T.M. Antonsen, Jr., Phys. Fluids **27**, 898 (1983).

[4]R.D. Hazeltine, Phys. Fluids **26**, 3242 (1983).

[5]R.D. Hazeltine, M. Kotschenreuther, and P.J. Morrison, Phys. Fluids **28**, 903 (1985).

[6]F. Romanelli and F. Zonca, Plasma Phys. **31**, 1365 (1989).

[7]S. Chandrasekhar, in *Hydrodynamic and Hydromagnetic Stability*, (Dover, New York, 1961), p. 16.

[8]S.I. Braginskii, in *Reviews of Plasma Physics*, ed. by M.A. Leontovich (Consultants Bureau, New York, 1965), Vol. I, p. 205.

[9]W. Horton, Jr. and R.K. Varma, Phys. Fluids **15**, 620 (1972).