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TOPICS IN PLASMA INSTABILITIES: TRAPPED-PARTICLE MODES AND MHD

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I. CLASSIFICATION OF PLASMA INSTABILITIES

The bulk of this paper, concerning the trapped-ion modes in tandem mirror machines is an elaboration of work done by T. M. Antonsen, D. E. Baldwin, H. L. Berk, M. N. Rosenbluth, and H. V. Wong[1] submitted for publication to the Leontovich Memorial issue of the Soviet Journal of Plasma Physics. Before proceeding to this technical discussion, I would like to briefly describe the present understanding of instabilities relevant to fusion plasma confinement. Table I is a crude attempt to schematize the different types of low-frequency plasma instabilities, although, of course, there are intermediate cases for which problems arise. All these modes are energetically driven, either by the unfavorable curvature of the confining field, or, in the case of tearing and kink modes, by the magnetic energy associated with parallel plasma currents.

The simplest of these modes are described by ideal magnetohydrodynamic (MHD) theory. Here the constraint of frozen-in field lines provides great simplification as well as stabilization, and allows the use of a powerful variational expression, δW, for the determination of the stability of static equilibria. The importance of positive δW is clearly seen in the violent behavior of some pinches and axisymmetric mirrors. With the aid of large computers a very accurate estimate may be made of the stability of axisymmetric systems like tokamaks, and at least some understanding is emerging for more complex geometries like stellarators. From a practical point of view, the highest stable MHD β permitted by various confinement schemes is of great importance.

Perhaps the greatest theoretical interest lies in kinetic modifications of the MHD modes, the simplest of which are finite gyroradius effects which greatly improve stability at short wavelengths. An area of great recent activity has been the MHD stability of plasmas containing a high-energy component which may be decoupled from the fluid motion. Examples of such systems are the stable field-reversed hot electron rings of Fleischmann at Cornell and the stable Elmo Bumpy Torus (EBT) at Oak Ridge and Nagoya with its relativistic electron layer. An interesting theoretical picture has been developed showing that MHD stability is achieved due to the negative compressibility of field lines in the presence of such high-energy components.[2] The predicted, but not yet tested, Lee-Van Dam breakdown of this stabilization could be crucial for such systems.

The simplest modification of the MHD equations results from the consideration of fluid resistivity. This permits reconnection at rational surfaces ($q = m/n$ in tokamak parlance). The past decade has seen a rather spectacular growth of understanding of the non-linear behavior of such modes, e.g., non-linear energetics, the effects of stochastic fields arising from island overlap, etc., such that theory, computer simulation, and experiments all seem in agreement in descriptions of disruptions in tokamaks. Not yet investigated is the behavior of Coppi's second stability region in tokamaks, which has such favorable ideal MHD behavior at high β. Another class of instabilities, the resistive g-modes, and its relatives, resistive ballooning and the kinetic trapped-particle modes, have not yet been firmly identified experimentally but are crucial in determining the quality of confinement not only for tokamaks, but even more importantly for reverse-field pinches and compact torii.

When we pass to the microinstabilities, those exceedingly complex motions not susceptible of a fluid description but depending upon microscopic plasma quantities like gyroradii and requiring consideration of diverse particle trajectories, much less is understood. Volumes have been written, and large computers strained to the limit, in the detailed calculation of the linear theory of such modes under simplifying assumptions. They have been identified in simple laboratory experiments such as Q-machines and Sen's recent collisional trapped-particle mode experiment at Columbia, but their existence is mostly inferred from the low-frequency, short-wavelength turbulence seen in all tokamaks. The range of frequencies and wavelengths is what is to be expected, but positive identification is difficult since what is seen is always the saturated non-linear state.

The non-linear theory is now the principal frontier in stability research and has made great advances in the past few years, although it is far from having predictive capability. Of particular interest are Direct Interaction Approximation (DIA) mode-coupling calculations and introduction of "clump effects" which go a long way toward explaining the broad-frequency spectra which are seen experimentally. Perhaps the biggest question mark as to whether our qualitative understanding is at all correct is the recent stellarator experiments on Wendelstein, which showed that, at least in that geometry, drift waves were closely associated with plasma currents. This is in complete contrast to all the linear theory predictions which have, however, only been made for tokamaks.

The most dangerous of these predicted microinstabilities, at least in the collisionless limit, is the trapped-particle mode of Kadomtsev and Pogutse[3], driven by unfavorable curvature. In this paper we point out that in some proposed tandem mirror schemes these microinstabilities become very similar in their properties to macroscopic MHD flute modes, thereby providing a link between the two classes of instabilities. A yet-to-be developed collisional theory could modify the discussion.

II. A PHYSICAL PICTURE OF TRAPPED-PARTICLE MODES IN TANDEM MIRRORS

Magnetic mirrors have traditionally been considered as minimum B geometries, immune to such low-frequency modes. However recently, in an effort to obtain improved high-frequency microinstability and enhanced classical confinement, the concept of tandem mirrors has evolved, consisting of a long central-cell axisymmetric mirror, MHD-stabilized by quadrupole end plugs. It has been assumed that their macroscopic stability would be governed by ideal MHD. Standard MHD stability criteria for ballooning and interchange modes indeed permits favorable high-β confinement (up to 50 per cent). Further reduction of end losses is obtained by erecting "thermal barriers" which are electrostatic potentials produced by high-energy populations of ECRH relativistic electrons and "pumped" sloshing ion beams. See Fig. 1 for a schematic plot of magnetic fields and potentials. The existence of these large potentials makes the situation essentially different from tokamak confinement.

An important feature of these systems is that because of the insulation, only a relatively small number of particles actually traverse both the central and transition regions with their unfavorable curvature, and the stabilizing end plugs. Hence, in the transition regions, only a few particles are affected by the development of electrostatic sheaths with parallel electric field $E_{\parallel} \neq 0$. This can lead to the breakdown of validity of the MHD equations and, hence, to development of trapped-particle modes. It will appear from our calculations that extreme thermal barriers, in which there are very few transiting particles ($<10^{-1}$ of the central density), can suffer trapped-particle instabilities with growth rates comparable to those of unstable MHD interchange modes. At the same time, the electrostatic equilibrium potentials imply a difference between ion and electron orbits which gives rise to a stabilizing perturbed-charge separation similar to, but larger than, conventional finite Larmor radius effects which can lead to stabilization at higher transiting density. Discussion of these matters is the principal content of this paper.

Recent methods used in the development of collisionless drift-kinetic theory make it possible to discuss this complicated situation quantitatively.

Before presenting this quantitative calculation, it is useful to give a simple discussion for a purely electrostatic trapped-particle mode applicable at very low β . Figure 2 shows a cross-section in the central cell. Particles in the equilibrium are drifting azimuthally with their curvature drift $v_0 \sim v_{th}(\rho/R)$ with $\rho = mv_{th}/q_jB$ the gyroradius and R the appropriate orbit-averaged field-line curvature radius. We consider a simple model in which the particles are of two types--those trapped in the central cell, denoted by subscript 0, and a small number which transit the entire system, denoted by subscript t.

The equilibrium is perturbed by an azimuthally-varying electrostatic potential $\phi \sim \exp[i(\omega t - m\theta)]$ as indicated by the dashed lines. It also will be taken to vary along the field line, having the constant value ϕ_0 in the central cell and zero outside it. The potential induces a radial $\underset{\sim}{E}\times\underset{\sim}{B}/B^2$ velocity. We may determine the perturbed density of the j^{th} trapped species by

$$\frac{\partial n_j^1}{\partial t} + \frac{v_j^0}{r}\left(\frac{\partial n_j^1}{\partial \theta}\right) = -v_j^1 \cdot \frac{\partial n_j^0}{\partial r}$$

$$i(\omega - \omega_{cj}^0)n_j^1 = -v_j^1\left(\frac{\partial n_j^0}{\partial r}\right)$$

$$n_j^1 = i\,\frac{v_j^1 n_j^0}{(\omega - \omega_{cj})}\left(\frac{1}{n_j^0}\right)\frac{\partial n_j^0}{\partial r} .$$

with $\omega_{cj} = mv_j^0/r$.

For the trapped species, $v_0^1 = im\phi_0/rB$. However, the transiting particles are only moving in the perturbing field for a fraction, f , of their orbit. Defining the average potential $f\Phi_0 = \overline{\phi}$, we have for the transiting particles $v_t^1 = im\overline{\phi}/rB$. Furthermore, the above discussion only accounts for the transverse motion of the transiting particles. In addition, they will order themselves along the field line with density similar to $\exp(-q\phi/T)$. An additional density of transiting particles in the central cell will thus arise equaling $n_t^0[-q(\phi_0 - \overline{\phi})/T_t]$. For simplicity, we assume that $1/n^0(dn^0/dr)$ and temperature are the same for all species. Using $\omega > \omega_{cj}$, we then have for the charge density

$$\rho = \sum q_j n_j^1 = \sum_j \frac{m\phi_0 n_j^0}{rB} \times \frac{\omega_{cj} q_j}{\omega^2}\left(\frac{1}{n}\frac{\partial n}{\partial r}\right) - n_t^0(\phi_0 - \overline{\phi})\left[\frac{q^2}{T} - \frac{qm}{\omega rB}\left(\frac{1}{n}\right)\frac{\partial n}{\partial r}\right].$$

Here we neglect the small correction proportional to $n^t\omega_c$. Defining, in the usual way, $\omega_* = (mTdn/dr)/(rqBn)$, we have

$$\rho = \left[\frac{n^0 q^2}{T}\left(\frac{\omega_*\omega_c}{\omega^2}\right) - \frac{n_t^0 q^2}{T}(1 - f)\left(1 - \frac{\omega_{*t}}{\omega}\right)\right]\phi_0 \ .$$

We obtain the dispersion relation from $\varepsilon m^2\phi/r^2 = \rho$, with ε the plasma dielectric constant $\varepsilon = n^0 m_i/B^2(1 - \omega_{*i}/\omega)$; the latter term including the conventional finite gyroradius modification of the ion response. Thus, we obtain the dispersion relation

$$\omega(\omega - \omega_{*i})\frac{m^2}{r^2}\rho_i^2 + \frac{n_t}{n_0}(1 - f)\omega(\omega - \omega_{*i}) + \omega_*\omega_c = 0 \ . \qquad (1)$$

with n_0 the total ion density in the central cell. Thus, we see that the growth rate of the trapped-particle mode goes to the usual MHD growth rate $(\gamma^2 = v_{thi}^2/rR)$ in the unfavorable curvature of the central cell as $n_t/n_0 \to 0$. We also note that the transiting particles have two effects: To modify the dielectric constant and also produce a charge separation linear in ω.

While this simple picture reproduces the essential features of the situation, it is deficient in a number of ways--it does not show how charge neutrality obtains along all regions of the field line and is not directly applicable to finite β .

III. THE DRIFT-KINETIC EQUATION AND VARIATIONAL PRINCIPLE

In this section we summarize the drift-kinetic equation as developed by Antonsen et al.[4]

We may characterize our equilibrium in terms of ion and electron distributions, f_0 , as functions of energy $\varepsilon = mv^2/2 + q\Phi_0$, magnetic moment $\mu = mv_\perp^2/2B_0$, and guiding center coordinates. For short wavelength perturbations, we may use an eikonal representation, supposing all quantities to vary slowly along the magnetic field and to vary as exp(iS) across it. Here, $\underset{\sim}{k}_\perp \equiv \underset{\sim}{\nabla} S$ is large compared to the equilibrium variation.

The perturbed fields are described in terms of a scalar potential ϕ and a vector potential $\underset{\sim}{A} = \underset{\sim}{b}\overline{\psi} - i\hat{\sigma}\underset{\sim}{b}\times\underset{\sim}{\nabla}S$, with $\underset{\sim}{b}$ a unit vector along the unperturbed field. The perturbed distribution function satisfies

$$f_1 = q\phi\left(\frac{\partial f_0}{\partial \varepsilon} + \frac{1}{B}\frac{\partial f_0}{\partial \mu}\right) - q\psi\left(\frac{v_\parallel}{c}\right)\frac{1}{B}\left(\frac{\partial f_0}{\partial \mu}\right) + g\ \exp(iL) \quad ,$$

with $L = \underset{\sim}{v}\times\underset{\sim}{B}\cdot\underset{\sim}{\nabla}S/\Omega$, and $g(x', \varepsilon, \mu)$ is to be determined where $\underset{\sim}{x}'$ is the guiding center position. In turn,

$$g = h - \frac{1}{B}\frac{\partial f_0}{\partial \mu}\left[J_0\left(\frac{v_\perp|\nabla S|}{\Omega}\right)(q\hat{\phi} - v_\parallel q\hat{\psi}) + q\hat{\sigma}\,\frac{v_\perp|\nabla S|}{\Omega}\,J_1\left(\frac{v_\perp|\nabla S|}{\Omega}\right)\right],$$

and h satisfies the collisionless drift-kinetic equation

$$-i(\omega - \omega_d + iv_\parallel \mathbf{b}\cdot\nabla)h$$

$$= i\omega\left(\frac{\partial f_0}{\partial \varepsilon} - \frac{\mathbf{B}_0\times\nabla S\cdot\nabla f_0}{B_0 m\Omega\omega}\right)\left[J_0\left(\frac{v_\perp|\nabla S|}{\Omega}\right)\times(q\hat{\phi} - v_\parallel q\hat{\psi}) + q\hat{\sigma}|\nabla S|v_\perp J_1\left(\frac{v_\perp|\nabla S|}{\Omega}\right)\right].$$

Here, the guiding center drift,

$$\omega_d = \frac{\nabla S\cdot\mathbf{B}_0\times(mv_\parallel^2\mathbf{b}\cdot\nabla\mathbf{b} + \mu\nabla B_0 + q\nabla\Phi_0)}{mB_0\Omega}.$$

We will work in the limit of high-bounce frequency where the term $v_\parallel\nabla$ is dominant on the left-hand side of Eq. (1). This approximation is appropriate in all cases for high-temperature electrons, but may be marginal for ions in some applications. In this limit, the drift-kinetic equation may be solved for h in terms of bounce-averaged potentials.

Having solved for f in terms of the potentials, we may write down integro-differential equations for the potentials using the quasi-neutrality condition and the components of Ampere's Law in the directions of $\mathbf{B}_0$ and $\mathbf{B}_0\times\nabla S$. Rather than writing these down, it is simpler to use a variational expression derived from $\gamma\delta W = \int \mathbf{j}\cdot\mathbf{E}d\mathbf{x}$, from which they may be deduced. This variational expression is useful in making further simplifications and assumptions. In the eikonal limit, the variational expression reduces to an integral along a field line. For completeness we will present the general variational form, an extension including finite gyroradius effects of that given previously by Antonsen et al.[4] (The reader is advised not to study the details, but more the general structure.)

$$\delta W =$$

$$\int \frac{d\ell}{B}\left[\frac{|\nabla S|^2\sigma^*(\underset{\sim}{b}\cdot\underset{\sim}{\nabla}\chi)^2}{\omega^2} + D\chi^2 + \psi^2 S_0 + Q^2\tau^* - \sum\int d^3\underset{\sim}{v}\,\frac{\partial f_0}{\partial\varepsilon}\,\frac{(\omega - \omega_*)}{(\omega - \overline{\omega}_D)}\,\overline{G}^2\right] . \quad (2)$$

The $\sum$ is a sum over particle species.

Here, the new potential variables (χ, ψ, Q) are given by (keeping only first-order terms in $|\nabla S|^2 v_\perp^2/\Omega^2$),

$$\hat{\psi} = -\frac{i}{\omega}(\underset{\sim}{b}\cdot\underset{\sim}{\nabla})\chi$$

$$\psi = \hat{\phi} + \frac{(\partial\rho/\partial B)}{(\partial\rho/\partial\phi)}\left(\hat{\sigma}|\nabla S|^2 + \frac{\chi}{\omega B_0}\nabla S\cdot\underset{\sim}{B}\times\nabla B_0\right)$$

$$- \chi\left\{1 - \frac{\omega_E}{\omega} - \frac{1}{(\partial\rho/\partial\phi)}\left[\left(1 - \frac{\omega_E}{\omega}\right)S_1 + S_2\right]\right\}$$

$$Q = \hat{\sigma}|\nabla S|^2 + \chi\frac{c}{\omega B_0}\,\nabla S\cdot\underset{\sim}{b}\times\left(\nabla B_0 - \frac{\sigma^* B_0}{\tau^*}\,\underset{\sim}{b}\cdot\nabla\underset{\sim}{b}\right)$$

$$G = q\psi\left(1 - \frac{|\nabla S|^2 v_\perp^2}{4\Omega^2}\right) + \mu^* Q + q\chi d \quad ,$$

and

$$\int d^3\underset{\sim}{v} = \frac{B_0}{m^2}\int\frac{d\varepsilon\, d\mu}{|v_\parallel|}$$

$$\frac{\partial\rho}{\partial\phi_0} = \sum\int d^3\underset{\sim}{v} q^2 \frac{\partial\phi_0}{\partial\varepsilon}$$

$$\frac{\partial\rho}{\partial B_0} = \sum\int d^3\underset{\sim}{v} q\mu \frac{\partial f_0}{\partial\varepsilon}$$

$$\omega_* = -\frac{1}{qB_0}\left[\frac{\underset{\sim}{\nabla}S\cdot\underset{\sim}{b}\times\underset{\sim}{\nabla}f_0}{(\partial f_0/\partial\varepsilon)}\right]$$

$$\sigma^* = 1 + \frac{1}{B_0^2}\left(p_{\perp 0} - p_{\parallel 0}\right)$$

$$\tau^* = 1 + \frac{4\pi}{B_0}\left(\frac{\partial p_\perp}{\partial B_0}\right) - \frac{\left(\partial\rho/\partial B_0\right)^2}{\left(\partial\rho/\partial\phi_0\right)}$$

$$S_0 = \frac{\partial\rho}{\partial\phi_0} + S_1$$

$$S_1 = \sum\int d^3\underset{\sim}{v} q^2\left(\frac{|\underset{\sim}{\nabla}S|^2 v_\perp^2}{2\Omega^2}\right)\frac{1}{B}\left(\frac{\partial f_0}{\partial\mu}\right)$$

$$S_2 = \sum\int d^3\underset{\sim}{v} q^2 \frac{|\underset{\sim}{\nabla}S|^2 v_\perp^2}{2\Omega^2}\left(1 - \frac{\omega_*}{\omega}\right)\frac{\partial f_0}{\partial\varepsilon}$$

$$\mu^* = \mu - q\,\frac{\left(\partial\rho/\partial B_0\right)}{\left(\partial\rho/\partial\phi_0\right)}$$

$$D = \sum\int d^3\underset{\sim}{v} q^2\left(\frac{\partial f_0}{\partial\varepsilon}\right)\frac{\omega_*\hat{\omega}_c}{\omega^2} + \left(1 - \frac{\omega_E}{\omega}\right)^2 S_1 + \left(1 - \frac{\omega_E}{\omega}\right)S_2$$

$$d = \frac{\hat{\omega}_c}{\omega} - \frac{1}{\left(\partial\rho/\partial\phi_0\right)}\left[\left(1 - \frac{\omega_E}{\omega}\right)\left(S_1 + S_2\right)\right]$$

$$\omega_E = \frac{1}{B_0} \underset{\sim}{\nabla} S \cdot \underset{\sim}{b} \times \underset{\sim}{\nabla} \Phi_0$$

$$\hat{\omega}_c = \frac{1}{qB_0}\left(mv_\parallel^2 + \mu^* B_0\right) \underset{\sim}{\nabla} S \cdot \underset{\sim}{b} \times \underset{\sim}{b} \cdot \underset{\sim}{\nabla} \underset{\sim}{b} \quad .$$

The summation is over particle species, and $\overline{\alpha} \equiv \left(\int d\ell\alpha/|v_\parallel|\right)/\int d\ell/|v_\parallel|$ denotes the bounce-averaged value of α .

Equation (2) is very complicated, particularly in that it involves three unknown functions; ψ, essentially the electrostatic potential, χ , related to the MHD displacement, and Q , the change in $|B|$. However, we might expect on physical grounds that if $\beta = 2p/B^2 << 1$ that the perturbed magnetic field parallel to B_0 would be small. Indeed, we may proceed by assuming Q to be small, solving for it in terms of χ and ψ , and substituting back into Eq. (2). We find upon doing this a great simplification. The primary effect is simply to subtract from the drift ω_d that portion arising from the diamagnetic well so that ω_d reduces to the sum of the equilibrium $\underset{\sim}{E} \times \underset{\sim}{B}/B^2$ and curvature drifts.

Equation (2), however, remains complicated because of the denominators $\left(\omega - \overline{\omega}_D\right)^{-1}$ which occur. To see what further simplifications are possible we rewrite Eq. (2) in the conventional limit $\omega > \omega_* > \omega_D$. While the diamagnetic drift ω_* is indeed large compared to the curvature drifts, it may not be large compared to the radial electric field-induced drifts which exist in tandem mirrors. We will return to examine this point later.

IV. THE TRAPPED-PARTICLE MODE AND MHD

However, with these simplifications Eq. (2) becomes a variational expression for the growth rate γ :

$$-\gamma^2 = \frac{\int (d\ell/B)\{\int d^3\underset{\sim}{v}[\omega_*\omega_c(\chi+\overline{\psi})^2(\partial f_0/\partial\varepsilon)] + (|\underset{\sim}{\nabla}S|^2/q^2)(\underset{\sim}{b}\cdot\underset{\sim}{\nabla}\chi)^2\}}{\int (d\ell/B)\int d^3\underset{\sim}{v}\{(\partial f/\partial\varepsilon)(\psi^2-\overline{\psi}^2)+[(\partial f/\partial\varepsilon) + 1/B(\partial f/\partial\mu)][v_\perp^2(\underset{\sim}{\nabla}S)^2/2\Omega^2](\chi^2+\psi^2+2\chi\overline{\psi})\}} \tag{3}$$

Here, ω_c is the usual magnetic curvature drift, $\omega_c = (mv_\parallel^2 + \mu B_0)\underset{\sim}{\nabla}S\cdot\underset{\sim}{b}\times(\underset{\sim}{b}\cdot\underset{\sim}{\nabla})\underset{\sim}{b}/qB_0$, and ω_* has already been defined as the conventional diamagnetic drift. Barred quantities are bounce averaged.

The denominator, for distribution functions satisfying $\partial f_0/\partial\varepsilon < 0$, $\partial f_0/\partial\varepsilon + (1/B_0)\partial f_0/\partial\mu < 0$, is negative definite. This variational principle shows neatly the relationship between MHD and trapped-particle mode. We wish to maximize γ^2 . Since the first term in the denominator is not proportional to the gyroradius, the obvious thing to do is to let $\psi \equiv 0$. Recalling the definition of ψ , this is equivalent to the MHD assumption $E_\parallel = 0$. Equation (2) then becomes

$$-\gamma^2 = \frac{\int (d\ell/B)\{\int d^3\underset{\sim}{v}\omega_*\omega_c\chi^2(\partial f_0/\partial\varepsilon) + |\underset{\sim}{\nabla}S|^2/q^2[(\underset{\sim}{b}\cdot\underset{\sim}{\nabla})\chi]^2\}}{\int d^3\underset{\sim}{v}\int (d\ell/B)\{[\partial f/\partial\varepsilon + 1/B(\partial f/\partial\mu)](|\underset{\sim}{\nabla}S|^2 v_\perp^2/2\Omega^2)\ \chi^2\}} \tag{4}$$

If a trial function χ can be found which makes the numerator positive, the system will be unstable with growth rate $\gamma^2_{MHD} \approx \omega_* \omega_c / k^2_\perp \rho^2_L = v^2_{th}/rR$ with R the effective radius of curvature.

We are interested, however, in the opposite case when the line-averaged curvature is favorable and β is low enough so that no line bending can be tolerated, i.e. $\beta < \beta_{crit}$. In this case, the numerator of Eq. (4) is negative definite. We conclude that to maximize γ^2 we must choose $\chi \ll \psi$ and that indeed a good approximation is to set it equal to zero. This is the trapped-particle limit:

$$\gamma^2 = \frac{\int d\ell/B \int d^3\underset{\sim}{v}(\partial f_0/\partial \varepsilon)\omega_* \omega_c \overline{\psi}^2}{\int d\ell/B \int d^3\underset{\sim}{v}\{\partial f_0/\partial \varepsilon(\psi^2 - \overline{\psi}^2) + [\partial f_0/\partial \varepsilon + 1/B(\partial f_0/\partial \mu)](|\underset{\sim}{\nabla} S|^2 v^2_\perp/2\Omega^2)\psi^2\}} \tag{5}$$

The second term in the denominator is ordinarily small. Since we have assumed flute-mode stability, i.e., the numerator to be negative for constant ψ, we conclude that ψ must be peaked in the central cell and an order of magnitude estimate for the growth rate is

$$\gamma^2_t = \omega_* \omega_c = k^2_\perp \rho^2_i \gamma^2_{MHD} ,$$

i.e., very small compared to MHD growth rates. This estimate would apply to tokamaks, for example.

At this point, however, the particular features of the tandem mirror become of importance--namely that it consists of a central cell with unfavorable curvature, and end anchor with favorable curvature, and a transition region with relatively few connecting particles (see

Fig. 1). Thus, if we imagine as a trial function $\psi = 1$ in the central cell, and $\psi = 0$ outside the central cell, only the transiting particles will have $\psi \neq \overline{\psi}$. We may thus estimate the growth rate to be:

$$\gamma_0^2 = \frac{\omega_* \omega_c}{(n_t/n_0) + k_\perp^2 \rho^2} .$$

Clearly, in this case growth rates may approach MHD levels for $n_t/n_0 << 1$. Without going through the analysis it is easy to see what the consequences of low-bounce frequency would be. Solving the drift-kinetic equation in terms of bounce harmonics

$$\psi_n = \frac{\int d\ell/v_\parallel (\psi) \exp\left(\cdot in\omega_b \int d\ell/v_\parallel\right)}{\int d\ell/v_\parallel} ,$$

we find that the first term in the denominator of Eq. (5)

$$\langle \psi^2 - \overline{\psi}^2 \rangle = \sum_{n\neq 1} |\psi_n|^2 ,$$

is replaced by

$$\sum_n \frac{|\psi_n|^2 n^2 \omega_b^2}{\gamma^2 + n^2 \omega_b^2} .$$

With this substitution, Eq. (5) may be rewritten as a new variational principle for a parameter $\lambda(\gamma)$

$$\lambda = \frac{\int d\ell/B\int d^3\underset{\sim}{v}(\partial f_0/\partial\varepsilon)\omega_*\omega_c\overline{\psi}^2}{\gamma^2\int d\ell/B(\int d^3\underset{\sim}{v}\{\partial f_0/\partial\varepsilon\sum[n^2\omega_b^2\psi_n^2/(\gamma^2+n^2\omega_b^2)]+|\underset{\sim}{\nabla}S|^2v_\perp^2/2\Omega^2[\partial f_0/\partial\varepsilon + 1/B(\partial f/\partial\mu)]\psi^2\})} \; . \qquad (6)$$

If λ is maximized with respect to ψ at a fixed γ, the eigenvalue equation for ψ is identical to the desired equation if $\lambda_{MAX} = 1$. It is clear from the above discussion that this will happen at a larger value of γ than that given by Eq. (5). We hence conclude that taking into account the finite bounce frequency of particles increases the growth rate over that obtained in the large bounce frequency limit. It is, of course, reasonable that if the connecting particles have insufficient velocity to pass from unfavorable to favorable regions in a growth time that connection with the ends is effectively lost, and MHD growth rates ensue.

Let us now abandon the approximation $\omega \gg \omega_*$, ω_D and study the effects of charge separation stabilization with $\omega \sim \omega_*$. Having seen that the perturbed parallel magnetic field Q may be neglected except for removal of the diamagnetic drift velocity, and that the perturbed parallel vector potential, X_r, must be set equal to zero for β less than the ballooning limit, then Eq. (2) in this limit, with $\chi = Q = 0$ becomes:

$$\delta W =$$

$$\int \frac{d\ell}{B} \sum \int d^3\underset{\sim}{v} \left[\psi^2 \left(\frac{\partial f}{\partial \varepsilon} + \frac{1}{B}\frac{\partial f}{\partial \mu} \frac{v_\perp^2 |\underset{\sim}{\nabla} S|^2}{2\Omega^2} \right) - \frac{\partial f}{\partial \varepsilon} \overline{\psi}^2 \left(\frac{\omega - \omega_*}{\omega - \overline{\omega}_D} \right) \left(1 - \frac{|\underset{\sim}{\nabla} S|^2 v_\perp^2}{2\Omega^2} \right) \right] , \tag{7}$$

where $\overline{\omega}_D = \overline{\omega}_E + \overline{\omega}_c$. We wish to study the solutions of the eigenvalue equations derivable from Eq. (6). This is made complicated by the denominators which appear. Two approximations suggest themselves. We may go to a frame rotating with the $\underset{\sim}{E}\times\underset{\sim}{B}/B^2$ drift velocity of the central cell. If there is small axial variation of ω_E this would leave only the curvature drift which is indeed small compared to ω or ω_* .

Thus, the primary case of interest is $\omega \sim \omega_* \dot{>} \omega_d$, in which case the denominator may be expanded in the usual way. Using the equilibrium charge neutrality condition

$$\sum \int d^3\underset{\sim}{v} \frac{\partial f_0}{\partial \varepsilon}(\omega_* - \omega_D) = 0 \tag{8}$$

Eq. (8) becomes

$$\delta W = \int \frac{d\ell}{B} \sum_j d^3\underset{\sim}{v} \left\{ (\psi^2 - \overline{\psi}^2)\frac{\partial f}{\partial \varepsilon}\left(1 - \frac{\omega_*}{\omega}\right) + \overline{\psi}^2 \frac{\partial f}{\partial \varepsilon}\left(\frac{\omega_* \overline{\omega}_D}{\omega^2}\right) \right.$$

$$\left. + \frac{|\underset{\sim}{\nabla} S|^2 v_\perp^2}{2\Omega^2} \psi^2 \left[\frac{\partial f_0}{\partial \varepsilon}\left(1 - \frac{\omega_*}{\omega}\right) + \frac{1}{B}\left(\frac{\partial f_0}{\partial \mu}\right) \right] \right\} \tag{9}$$

where we have treated the finite gyroradius terms as small.

Equation (8) is a variational, but non-minimal quadratic equation for ω. Hence, quantitative results would demand solution of the integro-differential equation. For a qualitative discussion, however, we may again expect that the stationary function would be flute-like in the central cell and fall rapidly to zero outside it, so that $\psi = \overline{\psi}$ only for the small number n_t of transiting particles. If these are primarily of one of the plasma species, the term $\sum\omega_*/\omega(\psi^2 - \overline{\psi}^2)$ will not vanish.

The approximate eigenvalue equation is thus Eq. (1) which we rewrite here.

$$\omega^2\left(\frac{n_t}{n_0} + k_\perp^2\rho_i^2\right) - \omega\omega_{*i}\left(k_\perp^2\rho_i^2 + \frac{n_t}{n_0}\frac{\omega_{*t}}{\omega_{*i}}\right) + \omega_*\omega_c = 0 .$$

The physical origin of the linear in ω stabilizing terms is the charge separation arising from the particular orbits of the transiting particles, and from the ion FLR, which is out of phase with the charge drive by curvature drifts. Note that if the transiting particles are primarily ions, the stabilizing terms add. If they are primarily electrons, the term will vanish at some critical wavenumber. Note also, that while the ion FLR terms similar to $k_\perp^2\rho^2$ are known to vanish for $m = 1$, i.e., $k_\perp \sim 1/r$, there is no reason the new charge separation terms should do so. Choosing $m = 1$, we find a stability criterion

$$\frac{n_t}{n_0} > 4\frac{\omega_c}{\omega_*} = 4\frac{r}{R} \qquad (10)$$

where R is the average unfavorable radius of curvature. For the presently designed MFTFB system at Livermore it would appear that the geometric factors are such that $n_t/n_0 > 5\text{-}10\%$ is required for stability. In the case where the transiting particles are ions with low bounce frequency, the stabilizing effect is decreased, the effective n_t/n_0 being reduced as before.

It should be noted that the resulting real-frequency waves may be negative energy waves, subject to collisional or resonance destabilization. For example, we have looked at collisional effects for the case where the transiting particles are electrons. The dominant effect is collisional trapping and detrapping of transiting electrons. Assuming the electrons to be trapped by the electrostatic potential of the thermal barrier, we may treat collisional effects by introducing a bounce-averaged energy-scattering operator into the electron drift-kinetic equation. The calculation has only been done crudely, and we will not present the details here, but indicate the results for the residual growth rate in terms of γ_0 , the growth rate neglecting charge separation, and ν the effective collision rate for diffusing energy by order of T_e . We find

$$\gamma = \gamma_0 \sqrt{\frac{\nu}{\gamma_0}} , \qquad \text{if } \nu < \gamma_0$$

$$\gamma = \frac{\gamma_0^2}{\nu} , \qquad \text{if } \nu > \gamma_0 .$$

Generally, the second case would apply and the growth rates would be greatly reduced. We have not yet investigated ion collisions or resonant destabilization.

We return now to Eq. (7), and note that in the presence of large axially-varying electric fields a new type of instability may occur. For this discussion we consider $k_\perp^2\rho_i^2$ as the small parameter and also neglect curvature drifts in order to focus on the rotational drive. While we are now not permitted to expand the denominators in Eq. (7), we may note, again using the charge neutrality condition, that a flute displacement $\psi = 1 - \omega_E/\omega$ leads to exact cancellation of the terms independent of $k_\perp^2\rho^2$. Hence, terms proportional to $k_\perp^2\rho^2$ determine the dispersion relation for a flute mode:

$$\int \frac{d\ell}{B}\left(\frac{m_i n_i}{B^2}\right) |\nabla S|^2\left[(\omega - \omega_E)(\omega - \omega_E - \hat{\omega}_{*i})\right] = 0 \qquad (11)$$

with

$$\hat{\omega}_{*i} = \frac{1}{qN_i}\left(\frac{\nabla S\times b}{B}\right)\cdot \nabla P_{0\perp i} \quad .$$

Again, we have a quadratic equation for ω and may conclude that if the density-weighted variation of ω_E is large compared to ω_*, a flute instability can occur. For radial potentials of order kT we have $\omega_E \sim \omega_*$ so that a more detailed knowledge of equilibrium than presently exists would be required to assess the importance of these modes. However, in cases where density is high only in the central cell, it would appear likely that the flute modes are in fact stable. Other solutions of Eq. (7), including the curvature terms, are responsible for the trapped-particle modes previously discussed.

V. SUMMARY

Our principal conclusion, then, is that, from a theorist's point of view, the tandem mirror experiment may provide an ideal means of assessing the reality of microinstability theory. By adjusting the configuration and the height of the potential barriers, the density of particles which link the good and bad curvature regions may be varied. For extremely high barriers when connection is severed, MHD would predict unstable flutes in the central cell.

When the density is finite but very low, MHD modes with $E_{\parallel} = 0$ are stabilized, but the trapped-particle modes grow at essentially MHD growth rates and at arbitrary wavelength. While only linear theory has been done, it seems clear that the non-linear growth will proceed essentially as for MHD flutes, until massive detrapping occurs and the condition in Eq. (10) is satisfied. This will happen when the perturbed potentials reach $q\phi \sim T$. Using $V = \phi'/B \simeq \phi/rB \sim \gamma_{MHD}\xi \sim [v_{th}/\sqrt{(rR)}]\xi$ we estimate a limiting non-linear displacement $\xi = \rho_i\sqrt{R/r}$ with consequent radial diffusion $D = D_{Bohm}(\rho_i/r)\sqrt{R/r}$, probably too large to be tolerated. In addition we would expect end loss enhancement.

As the equilibrium is adjusted to allow the density of connecting particles to increase to satisfy the stability condition $n_t/n_0 > 4r/R$, the trapped-particle mode is stabilized by charge separation and negative energy modes exist. In this regime its behavior approximates that of drift waves in tokamaks and, in the absence of a detailed understanding of non-linear behavior, we can only guess that some small level of radial diffusion will result.

Acknowlegments

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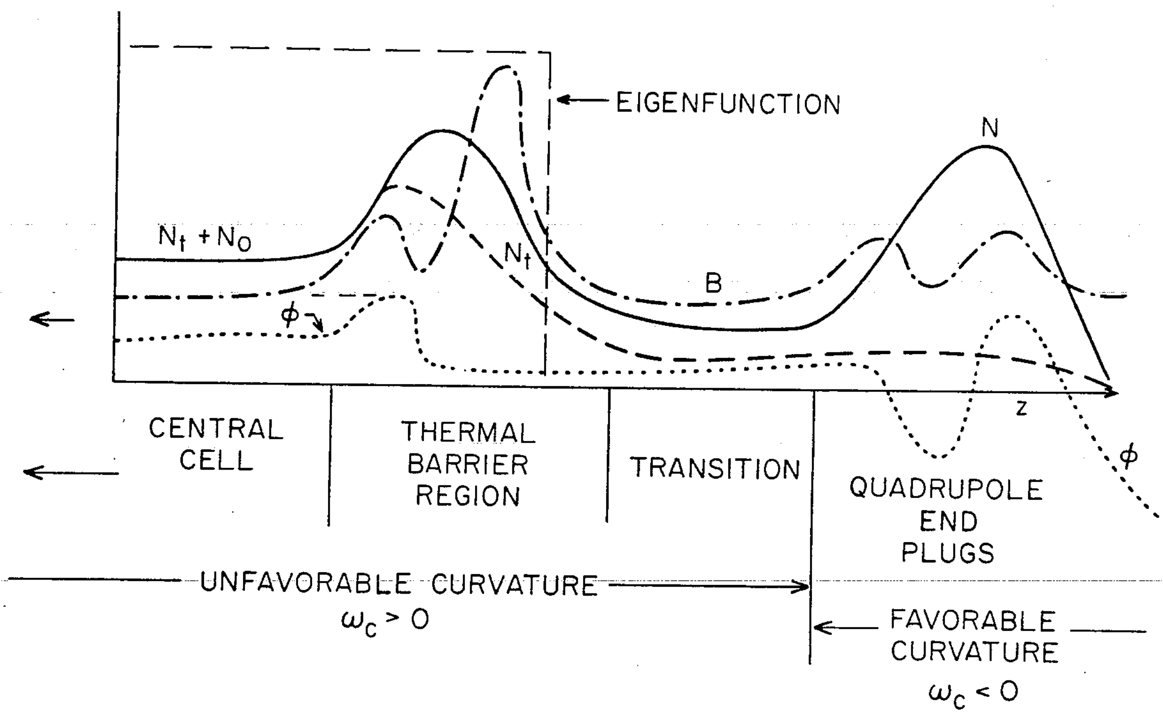
	Characteristic	Experimental Evidence	Linear Understanding	Non-linear Understanding
Ideal MHD	$\underset{\sim}{\vec{E}} + \underset{\sim}{\vec{V}} \times \underset{\sim}{B} = 0$	kinks flutes ballooning?	δW - very good; (non-axisymmetric); kinetic modifications, EBT?	adequate for large scale thermal systems
Resistive MHD	$\underset{\sim}{\vec{E}} + \underset{\sim}{\vec{V}} \times \underset{\sim}{\vec{B}} = \eta \underset{\sim}{\vec{j}}$	tearing resistive-g? ballooning?	good in simple geometries kinetic modifications? second stability region?	disruptions understood! stochastic field effects; minimum energy Taylor states, resistive ballooning?
Microscopic (Low Frequency)	$E_{\parallel} \neq 0$ (electrostatic double layers) $\hat{\omega} \sim \omega_*$	drift wave turbulence?	axisymmetric systems, complex equilibria? stellarator results?	broad frequency spectrum same qualitative understanding predictive ability?

Types of stabilities important for fusion plasmas. The symbol (?) indicates some degree of lack of understanding.

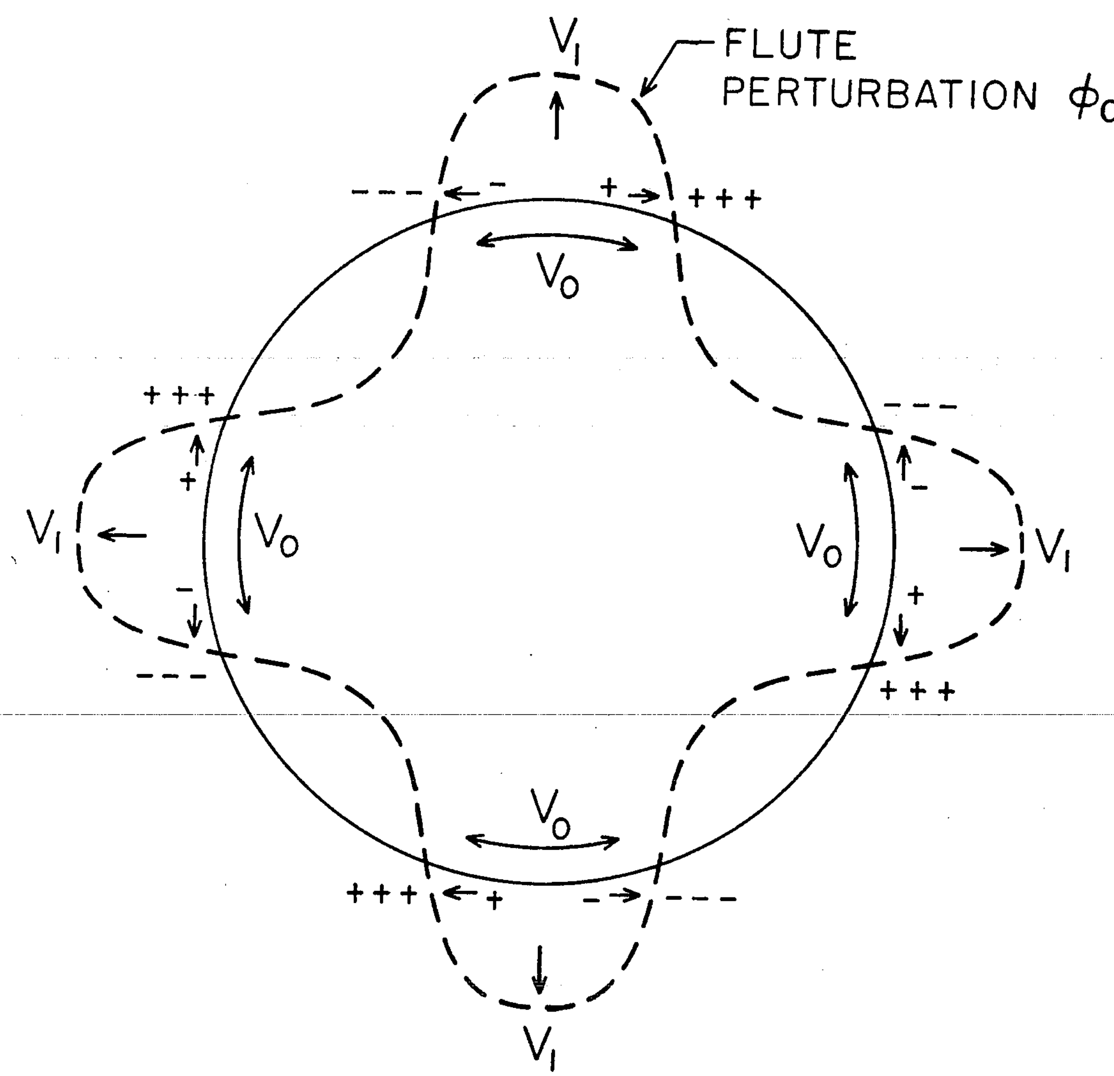
TABLE I

TANDEM MIRROR (AXICELL) SCHEMATIC



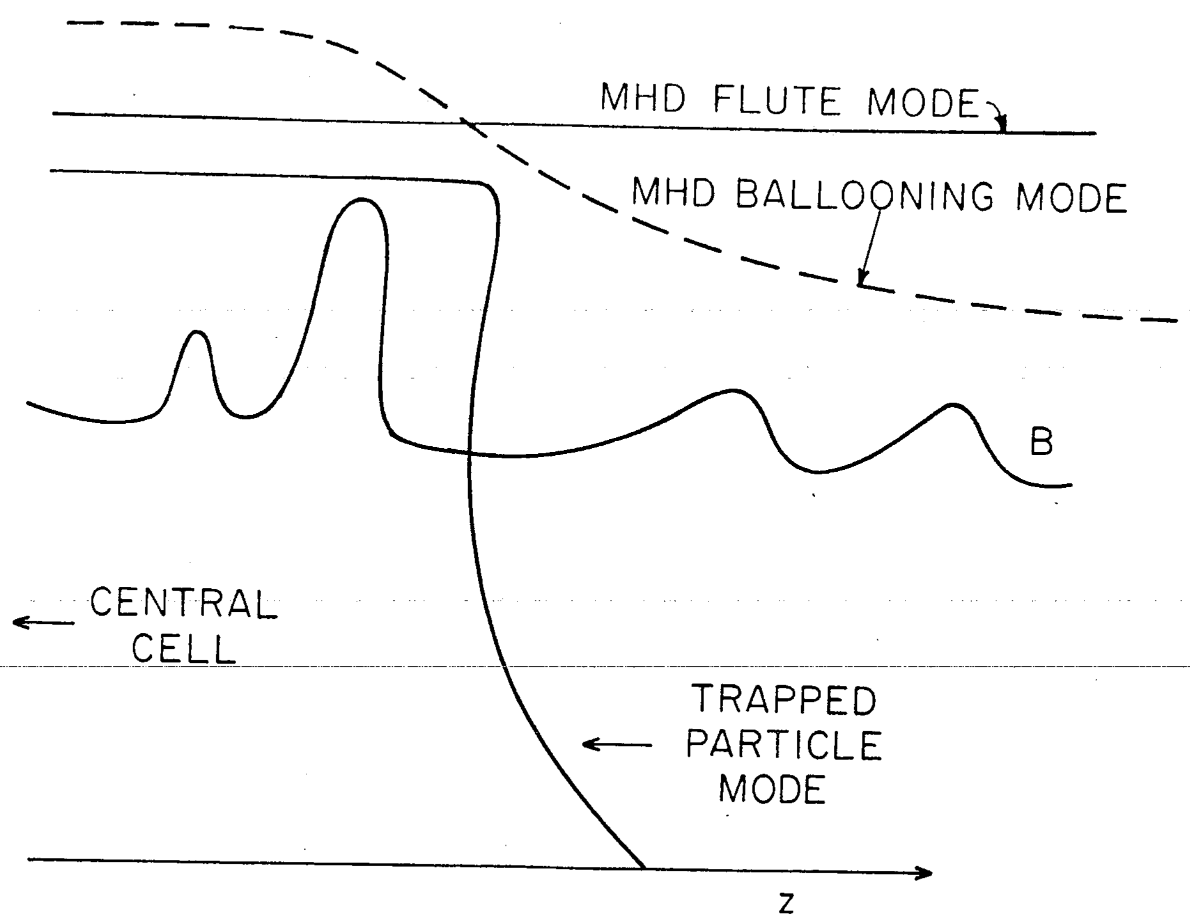
POTENTIALS CREATED BY "SLOSHING" IONS AND ECRH ELECTRONS

CENTRAL CELL CROSS SECTION



V_0 - CURVATURE DRIFT

V_1 - PERTURBED E x B DRIFT



MACROSCOPIC INSTABILITY MODE

EIGENFUNCTIONS