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A Two-Dimensional Kinetic Model of the Scrape-Off Layer

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Abstract

A two-dimensional (radius and poloidal angle), analytically tractable kinetic model of the ion (or energetic electron) behavior in the scrape-off layer of a limiter or divertor plasma in a tokamak is presented. The model determines the boundary conditions on the core ion density and ion temperature gradients, the power load on the limiter or divertor plates, the energy carried per particle to the walls, and the effective flux limit. The self-consistent electrostatic potential in the quasi-neutral scrape-off layer is determined by using the ion kinetic model of the layer along with a Maxwell-Boltzmann electron response that occurs because most electrons are reflected by the Debye sheaths (assumed to be infinitely thin) at the limiter or divertor plates.

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I Introduction

Tokamaks employ either a limiter or a divertor to control contact between the hot plasma and the container wall. The balance between rapid ion flow to the limiter or divertor plates (usually estimated at the sound speed C_s) and perpendicular diffusion across the last closed flux surface, results in a narrow boundary layer or scrape-off layer (SOL). Equating these two ion flows,

$$\frac{nC_s}{L} \approx \frac{D\partial^2 n}{\partial r^2} ,$$

yields a SOL width of order $(LD/C_s)^{1/2}$, where D is the spatial diffusion coefficient and L the connection length. Because of the importance of the SOL region to tokamak operation[1] in general and the International Thermonuclear Experimental Reactor (ITER)[2] in particular, a more careful kinetic treatment is warranted; such a treatment is presented here. The SOL model that is considered might also be useful for developing two- and three-dimensional kinetic codes which would ultimately yield a more quantitative understanding of power and particle exhaust in reactor relevant machines such as ITER. It is the simplest meaningful two-dimensional kinetic model possible, and is appropriate for ions or energetic electrons.

The model focuses on the boundary layer caused by spatial diffusion and free streaming to the limiter or divertor plates. It neglects the additional geometric complications of a spatially varying magnetic field or drifts of the particles from flux surfaces or field lines. Such effects have been considered previously by Hinton and Hazeltine.[3] It also ignores the complications that arise from atomic physics processes such as charge exchange, impurities, ionization, recombination, line radiation, and imperfectly absorbing walls. Finally it is assumed that diffusion and streaming outweigh Coulomb collisions in determining the SOL structure. Including collisional effects would require a major enlargement of the analysis, mainly because of increased dimensionality.

Section II expresses the SOL model in terms of kinetic equations and gives the appropriate boundary conditions. The resulting mathematical problem is solved by a Wiener-Hopf procedure. The velocity space diffusion results of Baldwin, Cordey and Watson[4] are modified, summarized, corrected, and extended in Sec. III to make them appropriate for the SOL case. The influence of the core is considered in Sec. IV, where the edge boundary conditions on the core density and temperature gradients are evaluated for a constant spatial diffusivity and for diffusion in a stochastic magnetic field. The power load and the energy carried per particle to the limiter or divertor plates is evaluated in Sec. V for the same two models, yielding effective flux limit coefficients. Section VI formulates and approximately solves a self-consistent model of the SOL using a Wiener-Hopf model for the ions and Maxwell-Boltzmann electrons and thereby obtains the electrostatic potential Φ from quasineutrality in the SOL.

II Model

To model the scrape-off layer (SOL) the full kinetic equation is not required. A steady-state solution exists if spatial diffusion feeds particles into the layer to replace those lost to the limiter or divertor plates. Consequently, the collision operator and any sources and sinks may be neglected. Only the terms

$$v_\parallel \hat{n}\cdot\nabla f - \left(\frac{Ze}{M}\right) v_\parallel \hat{n}\cdot\nabla\Phi \frac{\partial f}{\partial\varepsilon} = \frac{1}{r}\frac{\partial}{\partial r}\left(rD\frac{\partial f}{\partial r}\right) \tag{1a}$$

need be retained, where $f = f(\mathbf{r},\varepsilon,\mu)$ is the species distribution function, and Φ is the electrostatic potential. Also, $\varepsilon = \frac{1}{2}v^2$ is the kinetic energy, μ the magnetic moment with $v_\parallel^2 = v^2 - 2\mu B$, $\hat{n} \equiv \mathbf{B}/B$, r the minor radius, and D a spatial diffusion coefficient which models either classical or anomalous diffusion processes.

Equation (1a) can be simplified by suppressing the geometrical complications of the magnetic field, assuming $L \equiv 2\pi B/\mathbf{B}\cdot\nabla\theta \approx 2\pi qR$ is a constant, where θ is the poloidal angle variable, R is the major radius, and q is the safety factor. For a divertor geometry q is logarithmically singular at the last closed flux surface but the logarithmic behavior only dominates in an extremely narrow layer.[5] As a result, the model considered here is expected to be appropriate for both belt limiter and divertor configurations, although, since it does not account for the private flux region of diverted machines, it is probably a somewhat better model of limiter devices. To begin, introduce L and let $y = \theta/2\pi$ be the normalized poloidal variable. Then Eq. (1a) becomes

$$\frac{v_\parallel}{L}\left(\frac{\partial f}{\partial y} - \frac{Ze}{M}\frac{\partial \Phi}{\partial y}\frac{\partial f}{\partial \varepsilon}\right) = \frac{\partial}{\partial r}\left(D\frac{\partial f}{\partial r}\right) , \tag{1b}$$

where

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rD\frac{\partial f}{\partial r}\right) \approx \frac{\partial}{\partial r}\left(D\frac{\partial f}{\partial r}\right)$$

is employed since the SOL width is much narrower than the minor radius. The core plasma begins many SOL widths inside the last closed flux surface.

To simplify Eq. (1b) further it is convenient to assume that Φ is a slowly varying function of y by neglecting the poloidal potential variation. This assumption corresponds to making a square well model of the potential along the magnetic field lines connecting the limiter or divertor plates located at $y = 0$ and 1. Note that arbitrary radial variation of Φ is allowed. In addition, neglecting the radial and poloidal variation of D in the SOL allows the new radial variable

$$x = (r - a)\left(\frac{|v_\parallel|}{LD}\right)^{1/2} \tag{2}$$

to be introduced, where a is the minor radius of the last closed flux surface. With these additional assumptions Eq. (1b) reduces to

$$\sigma\frac{\partial f_\sigma}{\partial y} = \frac{\partial^2 f_\sigma}{\partial x^2} , \tag{3}$$

where $\sigma \equiv v_\parallel / |v_\parallel|$ and

$$f = \begin{cases} f_+(x,y) & \sigma = +1 \\ f_-(x,y) & \sigma = -1\ . \end{cases} \tag{4}$$

The v and μ dependence of f_σ and D is suppressed for notational simplicity. If D or L depend on y but not x, Eq. (3) could be modified by introducing a y dependent coefficient and the procedure to be employed to solve (3) would be appropriate with a mild generalization.

Equation (3) can be solved for a model consisting of two perfectly conducting limiter plates, one at $y = 0$ and the other at $y = 1$, and both extending from $x = 0$ to $+\infty$. The plates at $y = 0, 1$ model the two sides of a belt limiter or divertor plates located at $\theta = 0, 2\pi$. The solution for $x < 0$ must be periodic in y (or θ) to model the solution in the closed flux surface region inside the last closed flux surface.

The solutions for $x > 0$ are assumed to satisfy absorbing boundary conditions: no particles are emitted or reflected from the plates. This assumption means that the Debye sheath present at the plates can at most accelerate particles into the plates. These boundary conditions restrict the solution to either ions, which are accelerated into the plates by the Debye sheath, or suprathermal electrons, which are not significantly influenced by the electric field of the sheath region. In either case, the Debye sheaths at the plates are assumed to be infinitely thin; that is, the Debye length is presumed smaller than any length associated with the SOL.

Because of the geometrical symmetry about $y = 1/2$,

$$f_{-\sigma}(x,y) = f_\sigma(x, 1-y)\ , \tag{5}$$

so once f_+ is found f_- is easily formed as well. For f_+ the boundary conditions to be imposed are as follows:

$\underline{x > 0}$: $\quad f_+(x, y=0) = 0 \quad$ (absorbing plates) (6a)

$f_+(x \to +\infty, y) \to 0 \quad$ (all particles lost to plates) (6b)

$$\underline{x<0}: \qquad f_+(x, y=1) = f_+(x, y=0) \qquad \text{(core periodicity)} \tag{6c}$$

$$f_+(x \to -\infty, y) \to \alpha + \beta x \qquad \text{(input flux from core)}\ . \tag{6d}$$

The x and y independent coefficients α and β are to be determined and may depend on v and μ. It is important to emphasize that the boundary condition (6d) is necessary in order that new particles be resupplied by the core to replace those lost to the plates.

III Solution

The solution of the mathematical problem posed by Eq. (3) and the boundary conditions of Eqs. (6) has been found, using a Wiener-Hopf technique, by Baldwin, Cordey, and Watson[4] for a velocity space diffusion problem of the scattering of particles into the loss cone of a mirror machine. It is not necessary to repeat the details of that analysis here. However, certain parts of the argument are reconsidered, for the sake of clarity, to explicate the modifications necessary to treat the SOL problem, and to correct some of the mathematical details laid forth in Ref. 4.

In the notation used here, the Baldwin, Cordey, and Watson[4] solution to Eqs. (3) and (6) is given by

$$f_+(x,y) = \begin{cases} (\beta/2\pi) \displaystyle\int_\Gamma dk\, \exp\left[-ikx - k^2 y + k q_L(k)\right] & (k \text{ below } \Gamma) \\ (\beta/2\pi) \displaystyle\int_\Gamma dk\, \exp\left[-ikx - k^2 y + k q_U(k)\right] \Big/ [1 - \exp(-k^2)] & (k \text{ above } \Gamma)\ , \end{cases} \tag{7}$$

where Γ is a contour just below and parallel to the real k axis,

$$\frac{1}{2\pi i} \int_L \frac{dz\, q(z)}{z-k} = \begin{cases} q_L(k) & (k \text{ below } L) \\ q_U(k) & (k \text{ above } L)\ , \end{cases} \tag{8}$$

$$q(k) = k^{-1} \ell n \left[1 - \exp(-k^2)\right] = q_U(k) - q_L(k)\ , \tag{9}$$

and

$$\beta = -\int_0^\infty dx\, f_+(x, y=1)\ . \tag{10}$$

The contour L extends along the real z axis from $-\infty$ to $+\infty$ but passes below the branch cut extending from the branch point at $z=0$ to $z=-\infty$ along the negative real axis, and q_L and q_U are analytic in the half planes below and above L, respectively.

Notice that $\exp[-ikx + kq_L(k)]$ is analytic below the contour Γ for $x>0$. Hence, closing the contour below at infinity is allowed, and gives $f_+(x>0, y=0)=0$, as required by boundary condition (6a).

To see that the periodicity condition of Eq. (6c) is satisfied for $x<0$, use

$$\frac{\exp(-k^2)}{1-\exp(-k^2)} = \frac{1}{1-\exp(-k^2)} - 1$$

in $f(x, y=1)$ and note that it reduces to $f(x, y=0)$ since $\exp[-ikx + kq_U(k)]$ is analytic above Γ for $x<0$, giving

$$\int_\Gamma dk\, \exp[-ikx + kq_U(k)] = 0\ .$$

To rewrite Eq. (8), integrate it along the contour L which follows the Re $z=x$ axis but is indented into a semi-circle below $z=0$, remembering to account for the phase change of the logarithm on passing from $x>0$ to $x<0$. For q_L, these steps lead to the alternate form

$$q_L(k) = \frac{1}{2\pi i} P \int_{-\infty}^{\infty} \frac{dx\, \ell n|1-\exp(-x^2)|}{x(x-k)} + \frac{1}{2\pi i}\int_{-\infty}^{-\epsilon} \frac{dx}{x-k}\left(\frac{-2\pi i}{x}\right)$$
$$+ \frac{2i}{2\pi i}\int_{-\pi}^{0} \frac{d\theta(\ell n\, \epsilon + i\theta)}{[\epsilon\exp(i\theta) - k]}$$

where $z = \epsilon\exp(i\theta)$ on the indentation, P denotes principle value, and k is below the real axis. Taking $\epsilon \to 0$, the $\ell n\, \epsilon$ terms cancel, leaving

$$kq_L(k) = \frac{k}{2\pi i} P\int_{-\infty}^{\infty} \frac{dx\, \ell n|1-\exp(-x^2)|}{x(x-k)} - \ell n\, k + i\frac{\pi}{2} \tag{11}$$

where, for $|k| \gg 1$, the principal value integral is of order k^{-1} and may be neglected. Equation (11) corrects the result given in Eq. (47) of Ref. 4. Using the $|k| \gg 1$ form,

$kq_L \to -\ell n\, k + i\pi/2$, in Eq. (7) shows that the saddle at $k_s = -x/2y$ is insensitive to kq_L as long as $x/y^{1/2} \gg 1$. Carrying out the saddle point evaluation gives

$$f_+(x,y) \longrightarrow -\frac{\beta y^{1/2}}{\pi^{1/2} x} \exp\left(\frac{-x^2}{4y}\right) \tag{12}$$

for $x/y^{1/2} \gg 1$ [which generalizes Eq. (55) of Ref. 4 to all y.]

Finally, Eqs. (7) can be used to confirm that the final boundary condition (6d) is satisfied. First, deform the contour Γ upward to find that the lowest singularity (a double pole at $k = 0$) gives the following $-x \gg 1$ behavior:

$$f_+(x,y) \longrightarrow i\,\beta \frac{d}{dk} \exp\left[-ikx - k^2 y + kq_U(k)\right]\Big|_{k=0} \tag{13}$$

where contributions from the simple poles at $(i \pm 1)\pi^{1/2}$ give corrections of order $\exp(-\pi|x|)$.

To evaluate Eq. (13), the small k form of kq_U is required. Using $k/(z-k) = \sum (k/z)^n$ and integrating by parts gives

$$kq_U(k) = \sum_{n=1}^{\infty} \frac{k^n}{n\pi i} \int_L \frac{d\,z\,z}{z^n[\exp(z^2) - 1]} \,.$$

The preceding integral can be related to the Riemann Zeta function $\zeta(z)$ by recalling the definition of L and Hankel's contour[6] and letting $t = z^2 \exp(i2\pi)$ to obtain

$$kq_U(k) = -\sum_{n=1}^{\infty} \frac{[k\exp(i\pi/2)]^n \zeta(1-\frac{1}{2}n)}{n\Gamma(n/2)} \tag{14}$$

and

$$\frac{\partial}{\partial k} kq_U(k)\Big|_{k=0} = -i\pi^{-1/2}\zeta(1/2)$$

where $\zeta(1/2) = -1.46$. Consequently, Eq. (13) becomes

$$f_+(x,y) \longrightarrow \beta\left[x + \pi^{-1/2}\zeta(1/2)\right] \tag{15}$$

for $-x \gg 1$ and all y [generalizing Eq. (53) of Ref. 4]. Notice that the SOL solution is independent of y as $x \to -\infty$ and that the α of Eq. (6d) has been determined by the

Wiener-Hopf technique in terms of $\beta = \beta(\mathbf{v})$, which remains the only x and y independent quantity to be determined.

The preceding demonstrates that Eq. (7) is the solution of Eq. (13) satisfying the boundary conditions of Eqs. (6). The full solution of f is easily constructed using the symmetry property of Eq. (5):

$$f = \begin{cases} f_+(x,y) & v_\parallel > 0 \\ f_+(x, 1-y) & v_\parallel < 0 \end{cases} \tag{16}$$

with f_+ as given by Eqs. (7)–(10).

IV Core Influence on SOL

The velocity-space dependence of β is determined by the core solution. As one moves from the SOL into the core, collisions can no longer be neglected and streaming dominates over radial diffusion. As a result, $\partial f/\partial y = 0$ to lowest order and the kinetic equation, with the collision operator C and spatial diffusion retained, can be poloidally averaged to annihilate the next order streaming term and obtain

$$\overline{C}\{f\} + \frac{\partial}{\partial r}\left(D\frac{\partial f}{\partial r}\right) = 0 \ , \tag{17}$$

where the bar denotes the average, D is still assumed to be independent of y, and a slab approximation remains appropriate in this transition region.

Rewriting Eq. (15) in terms of the original r and θ variables and noting that $f_+ = f = f_-$, since f_+ is independent of y, gives

$$f \longrightarrow \beta(\mathbf{v})\left[\pi^{-1/2}\zeta(1/2) + (r-a)\left(\frac{|v_\parallel|}{LD}\right)^{1/2}\right] \ . \tag{18}$$

This form is indeed consistent with Eq. (17); the collision operator operating on $\beta(\mathbf{v})$ and $\beta(|v_\parallel|/D)^{1/2}$ generates corrections to Eq. (17) of order x^2L/λ and $|x|^3L/\lambda$, where λ is the mean free path. Since $-x \gg 1$ in this region, these corrections are small in the long mean free path limit $\lambda \gg L$. In this $|x|^3 L/\lambda \ll 1$ limit β may be taken proportional to a Maxwellian such that

$$\pi^{-1/2}\zeta(1/2)\beta = f_M \equiv n_a \left(\frac{M}{2\pi T_a}\right)^{3/2} \exp\left(-\frac{Mv^2}{2T_a}\right) , \tag{19}$$

with the edge density n_a and edge temperature T_a constants and $C\{f_M\} = 0$. This choice correctly matches the first term of Eq. (18). To match the second term would require a more elaborate analysis of the SOL-core transition.

More generally, the matching of Eq. (18) to the solution of Eq. (17) is sensitive to the $\mathbf{v}$ dependence of D. In particular, if $D = |v_\parallel| D_m$ with D_m a constant magnetic diffusivity,[7] as is often appropriate for energetic electrons,[7] then the choice of Eq. (19) is exact. However, for all other $D(\mathbf{v})$ a detailed matching of the core and SOL solutions is more complicated and Eq. (19) must be viewed as a reasonable assumption.

Equations (18) and (19) can be employed to evaluate the density and temperature gradients at the edge of the SOL:

$$n \equiv \int d^3v\, f \longrightarrow n_a \left[1 + \frac{(r-a)}{n_a \zeta(1/2)} \int d^3v \left(\frac{\pi |v_\parallel|}{LD}\right)^{1/2} f_M\right]$$

and

$$\frac{3}{2} nT \equiv \int d^3v \frac{1}{2} Mv^2 f \longrightarrow \frac{3}{2} n_a T_a \left[1 + \frac{2(r-a)}{3 n_a T_a \zeta(1/2)} \int d^3v \left(\frac{\pi |v_\parallel|}{LD}\right)^{1/2} \frac{1}{2} Mv^2 f_M\right] . \tag{20}$$

Assuming D is independent of $\mathbf{v}$, we then find

$$n \longrightarrow n_a \left[1 + \frac{\Gamma(3/4)(2T_a/M)^{1/4}(r-a)}{\zeta(1/2)(LD)^{1/2}}\right]$$

and

$$\frac{3}{2} nT \longrightarrow \frac{3}{2} n_a T_a \left[1 + \frac{7\Gamma(3/4)(2T_a/M)^{1/4}(r-a)}{6\zeta(1/2)(LD)^{1/2}}\right] , \tag{21}$$

where $2^{1/4}\Gamma(3/4)/\zeta(1/2) \approx -1$. Consequently, for D independent of $\mathbf{v}$, Eqs. (21) provide the boundary conditions on the core solution. If D depends on $\mathbf{v}$ then Eqs. (20) must be evaluated. For the special case of diffusion due to stochastic magnetic fields $D = |v_\parallel| D_m$ and Eqs. (20) give a density gradient but no temperature gradient.

V Heat Load

The power load P on the plates is determined by evaluating the parallel heat flux $\int d^3v \frac{1}{2} M v^2 v_\parallel f$ at the plates, then integrating it over both plates from $r = a$ to $+\infty$, and finally multiplying by the plate circumference $2\pi R$ to find

$$P = 4\pi R \left(\frac{B_P}{B_T}\right) \int_a^\infty dr \int d^3v \, \frac{1}{2} M v^2 v_\parallel \, f_+(x, y = 1) \ , \tag{22}$$

where $f(x, y = 1) = f_+(x, y = 1)$ is employed and the integration is over $v_\parallel > 0$ particles since the plates cannot emit particles. The ratio of the poloidal (B_P) to the toroidal (B_T) magnetic field is inserted to account for oblique incidence of the field lines on the plates. Using $dr = (LD/|v_\parallel|)^{1/2} dx$ and Eqs. (10) and (19), Eq. (22) becomes

$$P = -\frac{4\pi^{3/2} R}{\zeta(1/2)} \left(\frac{B_P}{B_T}\right) \int d^3v \, \frac{1}{2} M v^2 (v_\parallel \, LD)^{1/2} f_M \Big|_{v_\parallel > 0} \ . \tag{23}$$

Assuming D is independent of $\mathbf{v}$, the integrals can be performed to find

$$P_0 = \frac{7\pi \Gamma(3/4)}{2^{3/4} |\zeta(1/2)|} \left(\frac{B_P}{B_T}\right) R (DL)^{1/2} \left(\frac{T_a}{M}\right)^{1/4} n_a T_a \ , \tag{24}$$

where the subscript 0 is used to denote that D is independent of $\mathbf{v}$ and $(DL)^{1/2}(M/T_a)^{1/4}$ is a measure of the SOL width. A similar procedure can be employed to evaluate the number of particles collected by the plates per unit time

$$N = 4\pi R \int_a^\infty dr \int d^3v \, v_\parallel \, f_+(x, y = 1) = -\frac{4\pi^{3/2} R}{\zeta(1/2)} \left(\frac{B_P}{B_T}\right) \int d^3v (v_\parallel LD)^{1/2} f_M \Big|_{v_\parallel > 0} \ . \tag{25}$$

For D independent of $\mathbf{v}$, Eq. (25) gives

$$N_0 = \frac{2^{5/4} \pi \, \Gamma(3/4)}{|\zeta(1/2)|} \left(\frac{B_P}{B_T}\right) R (DL)^{1/2} \left(\frac{T_a}{M}\right)^{1/4} n_a \ . \tag{26}$$

Therefore, the energy carried to the plates per particle is

$$\frac{P_0}{N_0} = \frac{7 T_a}{4} \tag{27}$$

for D a constant. For a half-Maxwellian flowing into a SOL, the energy carried per particle is $P_M/M_M = 2T_a$. As a result, the value of the effective flux limit $\mathcal{F}$ for the constant D case is $\mathcal{F} = 7/8$.

The preceding expressions give the correct power load on the plates and obviate the need for an artificial flux limit.

For diffusion in a stochastic magnetic field, $D = |v_\parallel| D_m$ with D_m the velocity space independent magnetic diffusivity of Rechester and Rosenbluth.[7] In this case the power load on the plates, P_1, the number of particles collected N_1, and the energy carried to the plates per particle are as follows:

$$P_1 = \frac{2^{5/2}\pi}{|\zeta(1/2)|} \left(\frac{B_P}{B_T}\right) R(LD_m)^{1/2} \left(\frac{T_a}{M}\right)^{1/2} n_a T_a \ , \tag{28a}$$

$$N_1 = \frac{2^{3/2}\pi}{|\zeta(1/2)|} \left(\frac{B_P}{B_T}\right) R(LD_m)^{1/2} \left(\frac{T_a}{M}\right)^{1/2} n_a \ , \tag{28b}$$

and

$$\frac{P_1}{N_1} = 2T_a \ . \tag{28c}$$

In this case $(LD_m)^{1/2}$ is a measure of the SOL width and the effective flux limit coefficient is $\mathcal{F} = 1$ since the velocity space dependence of D results in half-Maxwellian flows to the wall.

VI Layer Potential Variation

A Debye sheath is established at the limiter or divertor plates to prevent the faster moving electrons from escaping more rapidly than the ions. The Debye sheath reflects all but the most energetic electrons and as r increases from $r = a$ to $+\infty$ the electron reflecting potential barrier decreases so that less energetic electrons can escape. Recall that the analysis in Sec. III considers only the region inside this barrier, ignoring the sheath structure.

To study the spatial behavior of the electrostatic potential in the SOL, the electrons are assumed to be Maxwell-Boltzmann to lowest order. Then the local electron density n_e is given by

$$n_e = Zn_a \exp\left[\frac{e(\Phi - \Phi_a)}{T_e}\right] \tag{29}$$

where Φ_a is the edge potential. The Maxwell-Boltzmann assumption describes the electrons that are bouncing back and forth between the plates to neutralize the slower moving, radially diffusing ions, but neglects the relatively few electrons energetic enough to leak out through the Debye potential.

To evaluate the spatial behavior of the potential the ion density, n_i, is required:

$$n_i = \int d^3v\, f = \int d^3v\, f_+(x,y)\bigg|_{v_\parallel>0} + \int d^3v\, f_+(x,1-y)\bigg|_{v_\parallel<0} \tag{30}$$

where the velocity space integrals are to be performed holding r fixed (not x). A closed-form evaluation of n_i is not possible in general, but various limits can be examined by specifying the velocity space dependence of D. Here D is assumed to be independent of $\mathbf{v}$ for simplicity, but other models, such as $D = |v_\parallel| D_m$, can also be evaluated.

The $-x \gg 1$ limit is particularly simple. Using Eq. (21) for n_i, quasineutrality ($n_e = n_i$) gives

$$\frac{e(\Phi - \Phi_a)}{T_e} \approx \ell n\left[1 + \frac{\Gamma(3/4)(2T_a/M)^{1/4}(r-a)}{\zeta(1/2)(LD)^{1/2}}\right] , \tag{31}$$

which is independent of $y = \theta/2\pi$ as assumed. Recall that Eq. (21) follows from Eq. (18) or (15) which is obtained for $1 \ll -x = (a-r)(|v_\parallel|/LD)^{1/2}$, so Eq. (31) cannot be expanded about $r = a$.

For $x \gtrsim 0$ analytic results are more difficult to obtain. From Eq. (12) it is seen that for $x \gg y^{1/2}$ the ion density will depend on y so that quasineutrality would result in a y dependent potential. Since the y dependence of Φ has been neglected, an evaluation of

$\Phi(x, y)$ is needed to ascertain the validity of this assumption. Thus a more general evaluation of $f_+(x, y)$ than given by Eq. (12) is required.

In the k above Γ form of Eq. (7) deform the contour Γ downward to $|k| \gg 1$ such that the principal value term in Eq. (11) [which is of $\mathcal{O}(1/k)$] may be neglected. Equation (7) then becomes

$$f_+(x, y) \longrightarrow \left(\frac{i\beta}{2\pi}\right) \int_\Gamma dk\, k^{-1} \exp(-k^2 y - ikx) \;, \tag{32}$$

where Eq. (12) is recovered if Γ passes through the saddle at $k = -ix/2y$ and $x/y^{1/2} \gg 1$ is employed. However, a more general evaluation is possible by locating the contour at $|k| \gg 1$ such that

$$\exp(-k^2 y - ikx) = \exp\left\{-\left(\frac{x^2}{4y}\right) - y\left[k + \left(\frac{ix}{2y}\right)\right]^2\right\} \ll 1 \tag{33}$$

everywhere along Γ. The condition (33) is satisfied when $x/y \gg |\operatorname{Im} k| \gg 1$, where Im denotes the imaginary part. Once the deformed contour satisfies Eq. (33) the principal value term can be neglected. Then the contour can be pulled back to the real axis, but with an indentation about $k = 0$, to obtain

$$\begin{aligned} f_+(x, y) &\longrightarrow -\frac{1}{2}\beta + \frac{i\beta}{2\pi} P \int_{-\infty}^{\infty} dk\, k^{-1} \exp(-k^2 y - ikx) \\ &\longrightarrow -\frac{1}{2}\beta + \left(\frac{\beta}{\pi}\right) \int_0^\infty d\tau\, \tau^{-1} \exp(-\tau^2) \sin\left(\frac{x\tau}{y^{1/2}}\right) \\ &\longrightarrow -\frac{1}{2}\beta\left[1 - \operatorname{erf}\left(\frac{x}{2y^{1/2}}\right)\right] = -\frac{1}{2}\beta \operatorname{erfc}\left(\frac{x}{2y^{1/2}}\right) \;, \end{aligned} \tag{34}$$

where $\operatorname{erf}(z) = 2\pi^{1/2} \int_0^z dt\, \exp(-t^2) = 1 - \operatorname{erfc}(z)$, P denotes principal value, and we must have $x \gg y > 0$.

Inserting Eq. (34), with β given by Eq. (19), into Eq. (30) gives integrals of the form[9]

$$I(p) \;\equiv\; \frac{-1}{2\zeta(1/2)p^2} \int_0^\infty dt\, t \left[\exp\left(\frac{-t^4}{4p^4}\right)\right] \operatorname{erfc}(t) \tag{35}$$

$$\longrightarrow \begin{cases} \dfrac{-1}{8\zeta(1/2)p^2}\left(1-\dfrac{15}{16p^4}+\cdots\right) & p \gg 1 \\ \dfrac{-\pi^{1/2}}{4\zeta(1/2)}\left[1-2^{3/2}\pi^{-1}\Gamma(3/4)p+\cdots\right] & p \ll 1 \end{cases}$$

where

$$p = \frac{(2T_a/M)^{1/4}(r-a)}{2(2LDy)^{1/2}} \equiv p_+ \quad \text{and} \quad p_- \equiv py^{1/2}(1-y)^{-1/2} \ . \tag{36}$$

It is convenient to define $I_\pm \equiv I(p_\pm)$. Then the ion density and quasineutrality condition are approximately

$$n_i \approx n_a(I_+ + I_-) \qquad (x > 1) \tag{37}$$

and

$$e\frac{(\Phi - \Phi_a)}{T_e} \approx \ell n(I_+ + I_-) \qquad (x > 1) \ . \tag{38}$$

A Padé approximation to I, using the $p \gg 1$ and $p \ll 1$ forms, gives

$$I(p) \approx \frac{-\pi^{1/2}/4\zeta(1/2)}{1+2\pi^{1/2}p^2} \ , \tag{39}$$

whence

$$I_+ + I_- \approx -\frac{\pi^{1/2}}{4\zeta(1/2)}\left[\frac{\rho^2+2y(1-y)}{(\rho^2+y)(\rho^2+1-y)}\right] \ , \qquad (\rho > 1) \tag{40}$$

where $\rho = (r-a)/w$ is r-a normalized to the layer width $w \equiv 2(LD)^{1/2}/(2\pi T/M)^{1/4}$ and $p^2 = \rho^2/2\pi^{1/2}y$. The Padé approximation to $I_+ + I_-$, given by Eq. (40), and its logarithm are plotted versus ρ and y in Figs. 1a and b. Figures 1a and b are only expected to be correct for $\rho > 1$ where the y dependence is indeed weak. However, the characteristic feature of Fig. 1 is that the y dependences of the density and potential are indeed very weak, consistent with our assumption, except in a narrow region near the plates. The next few paragraphs show that this feature persists even for small x.

An evaluation of the y dependence of $f_+(x, y)$ for general x is a formidable task because of the complexity of the Wiener-Hopf solution of Eqs. (7)–(9). However, some insight is obtained by the following procedure. To begin, an alternate form of $kq_U(k)$ can be found by extending the procedure that led to Eq. (11) for Im $k < 0$ to real k. For q_U, k must be above the contour L, which is therefore indented below both $z = 0$ and $z = k$ to obtain

$$kq_U(k) = -\ell n\, k + i\frac{\pi}{2} + \frac{1}{2}\ell n|1 - \exp(-k^2)| - i\lambda(k) \tag{41}$$

where for k real

$$\lambda(k) \equiv \frac{k}{2\pi} P \int_{-\infty}^{\infty} \frac{dx\, \ell n|1 - \exp(-x^2)|}{x(x-k)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dx\, \ell n|1 - \exp(-x^2)|}{x - k} \,. \tag{42}$$

In obtaining Eqs. (41) and (42) it is important to only account once for the phase change of $\ell n[1 - \exp(-z^2)]$ about $z = 0$ by inserting the absolute value signs as shown. Notice that since k must remain above the contour L its phase changes from zero to π as k moves from $k > 0$ to $k < 0$. Consequently, it is convenient to introduce the step function $\theta(k)$, which is unity for $k > 0$ and vanishes for $k < 0$, to rewrite kq_U as

$$kq_U(k) = \frac{1}{2}\ell n\left\{k^{-2}\left[1 - \exp(-k^2)\right]\right\} + i\Lambda(k) \,, \tag{43}$$

where the logarithm no longer manifests a phase change about $k = 0$ and

$$\Lambda(k) \equiv \pi\theta(k) - \frac{\pi}{2} - \lambda(k) \,, \qquad (k\,\text{real}) \,. \tag{44}$$

Using the power series representation of kq_U given by Eq. (14) it can be shown that the even powers of k correspond to the logarithm term in Eq. (43). Moreover, it can be verified that the odd powers of k in the sum of Eq. (14) are the small k expansion of the odd function $\Lambda(k)$. It follows that $\Lambda(k)$ is smooth at $k = 0$; the step from $\pi\theta(k)$ exactly canceling the step from $\lambda(k)$.

The preceding representation is useful on portions of the Γ contour that lie along the real k axis. To avoid the double pole at $k = 0$ use Eqs. (43) and (44) to compute $\partial f_+/\partial y$, for which the contour Γ can be taken to be the real k axis:

$$\begin{aligned}\frac{\partial f_+(x,y)}{\partial y} &= \frac{-\beta}{2\pi} \int_{-\infty}^{\infty} \frac{dk\, k \exp[-yk^2 - ikx + i\Lambda(k)]}{[1 - \exp(-k^2)]^{1/2}} \\ &= -\frac{\beta}{\pi} \int_0^{\infty} \frac{dk\, k \exp(-yk^2)}{[1 - \exp(-k^2)]^{1/2}} \cos[\Lambda(k) - kx]\end{aligned} \tag{45}$$

where the phases are such that as $k \to 0$, $[1-\exp(-k^2)]^{1/2} \to k$ to make it an odd function of k.

To determine whether the y variation is weak at $x = 0$, $\partial f/\partial y$ can be formed and the result integrated over velocity space to obtain the y derivative of the ion density

$$Y \equiv n_a^{-1} \left.\frac{\partial n_i}{\partial y}\right|_{x=0} = \frac{1}{\pi^{1/2}\zeta(1/2)} \int_0^\infty \frac{dk\, k \exp(-k^2/2)}{[1-\exp(-k^2)]^{1/2}} \sinh\left[k^2\left(y - \frac{1}{2}\right)\right] \cos\Lambda(k) \qquad (46)$$

where Eqs. (16) and (19) are employed.

As long as $|Y|$ is small, the y dependence of the ion density and the potential will be weak. To make a simple estimate, the Padé approximation to Λ given by

$$\Lambda \approx \frac{\pi k}{2|k| - [\pi^{3/2}/\zeta(1/2)]} \qquad (47)$$

is employed. Since this Padé approximation is smaller than the actual Λ, the first zero of the cosine occurs at larger k than it should, and the y dependence that results is actually somewhat stronger than the actual y dependence. Nonetheless, inserting Eq. (47) into Eq. (46) and using Mathematica to evaluate the integral gives $Y(0) = 0$, $Y(y = 1/4) \approx .2$ and $Y(y = 0.1) \approx 0.5$. Thus, the y dependence at $x = 0$ remains weak until close to the plates, as indicated qualitatively in Figs. 1. It is likely that the y dependence is weaker still for the more realistic case of partially emitting walls.

Because of the poloidal behavior of Φ indicated in Fig. 1, the self-consistent potential accelerates ions to the limiter or divertor plates and, therefore, into the Debye sheath where they are accelerated further. The radial change in $e(\Phi - \Phi_a)/T_e$ is of order unity across the SOL but this does not affect the validity of the results since an arbitrary radial variation of Φ is allowed.

The preceding justifies the neglect of the $\partial\Phi/\partial y$ term in Eq. (3) over most of the SOL. Without this assumption neither Eq. (1b) or its $E = \frac{1}{2}v^2 + (Ze\Phi/M)$ independent variable version

$$\frac{v_\parallel}{L}\frac{\partial f}{\partial y}\bigg|_E = \frac{\partial}{\partial r}\left(D\frac{\partial f}{\partial r}\right)$$

with $v_\parallel^2 = 2\left[E - (Ze/M)\Phi(r,y) - \mu B\right]^{1/2}$, can be solved by Fourier transformation.

VII Discussion

A model for the ions (or energetic electrons) in the SOL for a limiter or divertor plasma is constructed. The model reduces to a Wiener-Hopf problem considered previously in the literature. The relevant portions of the solution are employed and extended to obtain a realistic kinetic description of the radial boundary layer and the electrostatic potential caused by a limiter or divertor. The narrow boundary layer is generated by the steady-state balance of the flow into the limiter or divertor plates and the slow radial diffusion into the SOL.

Extending the Wiener-Hopf procedure to model a SOL leads to several interesting results as well as rigorous treatment of the SOL. First, it provides the proper boundary conditions on both the core density and temperature gradients. In addition, the power load on the limiter or divertor plates, the energy carried per particle, and the effective flux limit are evaluated for a constant diffusion coefficient D and for diffusion in a stochastic magnetic field ($D \propto |v_\parallel|$). Finally, a nearly self-consistent evaluation of the electrostatic potential Φ is made over much of the quasineutral SOL. In the model considered all but the most energetic electrons are reflected by the infinitely thin Debye sheaths at the walls so a Maxwell-Boltzmann electron response is combined with a Wiener-Hopf ion response to obtain the behavior of Φ in the SOL. The potential varies logarithmically in the SOL; the poloidal variation of Φ is weak except in the near vicinity of the plates' inner edge.

The calculation presented here is clearly idealized, omitting the details of the Debye sheath at the plates, recycling, charge exchange, and the large variation in collisionality from the tokamak mid-plane to the divertor plates. However, the estimates for the particle and energy fluxes to the plates may be somewhat better than they first appear. The increase in these fluxes by ion acceleration in the Debye sheath is somewhat off-set by recycling since instead of ions being perfectly absorbed at the plates they tend to recombine and come back as slightly less energetic neutrals which charge exchange and then ionize.[1] Recycling also acts to widen the SOL, thereby spreading out the power load on the plates. Work in progress[10] attempts to more quantitatively model the effect of recycling.

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Figure Captions

1. The particle density (a) and the electrostatic potential (b) in the scrape-off layer.

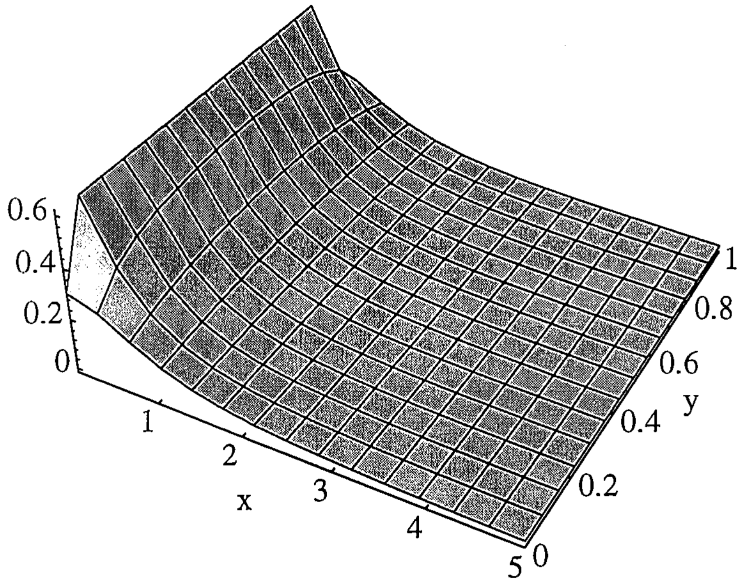

Fig 1a