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On the Stability of Shear Alfvén Vortices

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Abstract

Linear stability of shear-Alfvén vortices is studied analytically using the Lyapunov method. Instability is demonstrated for vortices belonging to the drift mode, which is a generalization of the standard Hasegawa-Mima vortex to the case of large parallel phase velocities. In the case of the convective-cell mode, short perpendicular-wavelength perturbations are stable for a broad class of vortices. Eventually, instability of convective-cell vortices may occur on the perpendicular scale comparable with the vortex size, but it is followed by a simultaneous excitation of coherent structures with better localization than the original vortex.

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I Introduction

Reduced magnetohydrodynamic (RMHD) equations[1, 2] are widely used in numerical and analytical studies of nonlinear processes in large aspect ratio tokamaks, in cases when both magnetic shear and curvature effects can be neglected. RMHD equations are relatively simple, their derivation is consistent, and they offer a reliable description of a broad class of nonlinear processes ranging from major plasma disruptions, to nonlinear kink modes, to the destruction of flux surfaces, and other low-beta magnetohydrodynamic (MHD) processes. Studies of the shear-Alfvén weak turbulence in the RMHD regime reveal the possibility of spectrum cascade towards both larger and smaller perpendicular wavenumbers[3] which is the usual "signature" of self-organization processes leading to the formation of localized nonlinear structures. Although the existence of such coherent structures was not explicitly demonstrated, it was anticipated by the application of the Hamiltonian formalism to the RMHD system.[4]

Due to the dominance of the convective nonlinear terms, arising from the $\mathbf{E} \times \mathbf{B}$ drift and the twisting of the magnetic field lines, the reduced MHD equations belong to the class of equations with vector product (or Poisson bracket) nonlinearities, which in many cases possess coherent nonlinear stationary solutions in the form of double vortices. For shear-Alfvén perturbations vortex solutions have the form of double vortex tubes, rotating in opposite directions, which are slightly tilted relative to the magnetic field.[5] The perpendicular scale of such a vortex is of the order of the collisionless skin depth, which requires a two-fluid, rather than MHD description. In the MHD limit the collisionless skin depth vortex becomes singular, i.e. reduces to a point- or filament- vortex.

The simplest vortex equation is the Hasegawa-Mima equation describing nonlinear drift waves in plasmas and Rossby waves in rotating shallow fluids. Although it is not fully integrable[6] and the drift-wave vortices may not be regarded as solitons in the mathematical sense, many numerical[7] and laboratory[8, 9] studies indicate that they are remarkably robust objects, appearing to be stable even in the case of large perturbations, such as collisions with other vortices. Thus, relevant plasma transport coefficients in the fully developed drift-wave turbulence can be calculated modelling the turbulent plasma state as a superposition of coherent vortices and weakly correlated wave-like fluctuations.[10]

However, a rigorous analytic proof of the stability of Hasegawa-Mima drift-wave vortices still remains elusive, in spite of more than a decade of efforts. Shortcomings of a number of analytical attempts are summarized in Ref. 11. In a recent work[12] linear stability of vortices propagating in the electron drift direction was proven using a numerical experiment. First, performing the expansion on an appropriate basis, all types of perturbations which may cause the vortex instability were identified analytically by the Lyapunov method, and then the vortex evolution in the presence of such critical perturbations was studied numerically.

The situation with shear-Alfvén vortices is far less clear. Their evolution is described by fully three-dimensional, nonlinear equations, and due to the complexity of the problem, no numerical results on their evolution and stability have been published. Some analytical considerations indicated their structural instability in the presence of kinetic effects associated with the electron motion along the magnetic field lines. Namely, shear-Alfvén vortices may efficiently exchange energy and enstrophy with resonant electrons,[13] leading to their adiabatic perturbation in which the vortices preserve their shape, but gradually slow down and spread in space. On the electron bounce time scale, when the electron trapping takes place, a new type of coherent structures becomes possible,[14] which is a hybrid of a three-dimensional electron hole and a shear-Alfvén vortex.

This paper presents the first attempt to study analytically the linear stability of shear-Alfvén vortices, using the Lyapunov stability analysis. There are two distinct vortex types: one which asymptotically behaves as the $k_\perp = 0$ quasimode, and the second type as an evanescent linear shear-Alfvén wave. These are called drift- and convective-cell vortex modes, respectively. While the first is linearly unstable, the latter is stable in the presence of a broad spectrum of perpendicular wavenumbers of perturbations. There remains one possible "window" of instability, corresponding to large-scale perturbations with the typical wavelength comparable to the vortex size. It is demonstrated that their eventual instability would be followed by a simultaneous growth of short space scale perturbations, with a better localization than the original vortex.

The paper is organized as follows. In Sec. II we derive our basic set of equations describing nonlinear shear-Alfvén perturbations, and construct the drift- and convective-cell type vortex solutions. Sec. III is devoted to the stability analysis of the shear-Alfvén equations linearized around the coherent nonlinear vortex solution. Conclusions are given in Sec. IV.

II Basic equations

We study electromagnetic perturbations in a homogeneous plasma with the unperturbed density n_0, immersed in a homogeneous magnetic field $B_0\mathbf{e}_z$. We assume that both ions and electrons are cold, with the electron pressure being much smaller than the magnetic pressure

$$\beta = \frac{2n_o T_e}{c^2\epsilon_0 B_0^2} \ll \frac{m_e}{m_i} \,. \tag{1}$$

Here m_e and m_i are electron and ion masses, and T_e is the electron temperature. For perturbations which are slowly varying in time compared to the ion gyrofrequency Ω_i, and weakly z-dependent

$$\frac{\partial}{\partial t} \ll \Omega_i \quad , \qquad \frac{\partial}{\partial z} \ll \nabla_\perp \,, \tag{2}$$

with the accuracy to the first order in m_e/m_i and in the small parameters defined above, electrons are three dimensional, $v_{||e} \sim v_{\perp e}$, while the ions can be regarded as strictly two-dimensional, $v_{||i} \ll v_{||e}$. Thus, the ion and electron hydrodynamic velocities are, with the accuracy described above, given by

$$\mathbf{v}_i = \mathbf{v}_E + \mathbf{v}_P$$

$$\mathbf{v}_e = \mathbf{v}_E + \frac{\mathbf{B}}{B} v_{\|e} \ . \qquad (3)$$

Here subscripts $\|$, $\perp$ denote vector components parallel and perpendicular to the zero-order magnetic field, and $\mathbf{v}_E$, $\mathbf{v}_P$ are the $\mathbf{E} \times \mathbf{B}$ and ion polarization drift velocities, respectively

$$\mathbf{v}_E = \frac{\mathbf{E} \times \mathbf{B}}{B^2}$$

$$\mathbf{v}_P = \frac{1}{\Omega_i} \left(\frac{\partial}{\partial t} + \mathbf{v}_E \cdot \nabla \right) \frac{\mathbf{E}_\perp}{B} \ . \qquad (4)$$

We assume small perturbations of the electron density and of the magnetic field

$$\delta n_e \equiv n_e - n_0 \ll n_0$$

$$| \, \delta \mathbf{B} \, | \quad \equiv \quad | \, \mathbf{B} - B_0 \mathbf{e}_z \, | \quad \ll B_0 \ , \qquad (5)$$

and neglect the compressional component of the magnetic field perturbation, $\delta B_z = 0$, which is justified in the low β regime, Eq. (1). Then, subtracting the electron and ion continuity equations, and making use of the quasineutrality condition, $n_e \approx n_i$, we readily obtain to the leading order in the small parameters, from Eqs. (2), (4), and (5)

$$\nabla_\perp \cdot \mathbf{v}_P - \frac{\mathbf{B}}{B} \cdot \nabla v_{\|e} = 0 \ , \qquad (6)$$

while the parallel electron momentum equation, with the same accuracy, takes the form

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_E \nabla \right) v_{\|e} = - \frac{e}{m_e} \frac{\mathbf{B} \cdot \mathbf{E}}{B} \ . \qquad (7)$$

Here e and m_e are the electron charge and mass, respectively.

Finally, we express the electric and magnetic fields in terms of the electrostatic potential ϕ and the z-component of the vector potential $\mathbf{A}$

$$\mathbf{E} = -\nabla \phi - \mathbf{e}_z \frac{\partial A_z}{\partial t}$$

$$\mathbf{B} = B_0 \mathbf{e}_z - \mathbf{e}_z \times \nabla A_z \ , \tag{8}$$

and making use of the parallel component of Ampére's law

$$j_{\|} = -en_0 v_{\|e} = -c^2 \epsilon_0 \nabla_{\perp}^2 A_z \tag{9}$$

we may rewrite Eqs. (6) and (7) in the following convenient form

$$\left[\frac{\partial}{\partial t} + \frac{1}{B_0}(\mathbf{e}_z \times \nabla\phi)\cdot\nabla\right]\nabla_{\perp}^2\phi = -c_A^2\left[\frac{\partial}{\partial z} - \frac{1}{B_0}(\mathbf{e}_z \times \nabla A_z)\cdot\nabla\right]\nabla_{\perp}^2 A_z \tag{10}$$

$$\left[\frac{\partial}{\partial t} + \frac{1}{B_0}(\mathbf{e}_z \times \nabla\phi)\cdot\nabla\right]\left(1 - \lambda_s^2\nabla_{\perp}^2\right)A_z = -\frac{\partial\phi}{\partial z} \ . \tag{11}$$

Here $c_A = c\Omega_i/\omega_{pi}$ is the Alfvén speed, $\lambda_s = c/\omega_{pe}$ is the collisionless skin depth, and ω_{pi}, ω_{pe} are the ion and electron plasma frequencies, respectively. Equations (10) and (11) constitute our basic system describing nonlinear shear-Alfvén perturbations. In the limit $\lambda_s \rightarrow 0$ they reduce to the low-β RMHD system, Ref. 1.

The dipole vortex[5, 13] is found as a time stationary, z-independent solution of Eqs. (10) and (11), in the reference frame travelling with the velocity

$$\mathbf{u} = u_y \mathbf{e}_y + u_z \mathbf{e}_z \quad , \qquad | u_z | \gg | u_y | \ , \tag{12}$$

i.e. we assume that we have a two-dimensional elongated structure moving with the velocity u_y along the y axes, and making a pitch angle u_y/u_z to the z-axis. Using

$$\frac{\partial}{\partial t} = -u_y \frac{\partial}{\partial y} \quad , \qquad \frac{\partial}{\partial z} = \frac{u_y}{u_z}\frac{\partial}{\partial y} \ , \tag{13}$$

and the properties of the Poisson brackets, we can integrate Eqs. (10) and (11) each one time, yielding the two-dimensional nonlinear partial differential equations

$$\left(1 - \lambda_s^2\nabla_{\perp}^2\right)(u_z A_z - B_0 u_y x) = H(\phi - B_0 u_y x) \tag{14}$$

$$\frac{c_A^2}{u_z^2} H'(\phi - B_0 u_y x)\nabla_{\perp}^2 u_z A_z - \nabla_{\perp}^2\phi = G(\phi - B_0 u_y x) \ , \tag{15}$$

where $H(\zeta)$, $G(\zeta)$ are arbitrary functions of their arguments, and the prime denotes the derivative of these functions. In the standard vortex scenario, $H(\zeta)$, and $G(\zeta)$ are taken to be linear functions, $H(\zeta) = \zeta H$, $G(\zeta) = \zeta G$, allowing for different slopes G^{out}, H^{out} and G^{in}, H^{in} outside and inside the vortex core, respectively. The core is taken to be a circle in the x, y plane, with the radius R.

Obviously, a localized solution is possible only if

$$H^{\text{out}} = 1 \quad , \qquad G^{\text{out}} = 0 \; . \tag{16}$$

Now, Eqs. (14) and (15) are decoupled to give

$$\left(\nabla_\perp^2 + \kappa_1^2\right)\left(\nabla_\perp^2 + \kappa_2^2\right)\left(\phi - B_0 u_y x\right) = 0 \; , \tag{17}$$

where the wavenumbers κ_1, κ_2 are related with H, G through

$$\begin{aligned} \kappa_1^2 \kappa_2^2 &= -\frac{G}{\lambda_s^2} \\ \kappa_1^2 + \kappa_2^2 &= \frac{1}{\lambda_s^2}\left(\frac{c_A^2}{u_z^2} H^2 - 1\right) + G \; . \end{aligned} \tag{18}$$

From Eq. (16) in the outer region $r > R$ we have

$$\begin{aligned} \kappa_1^{\text{out}} &= 0 \\ \kappa_2^{\text{out}\,2} &= -\frac{1}{\lambda_s^2}\left(1 - \frac{c_A^2}{u_z^2}\right) \; . \end{aligned}$$

The fourth order wave equation (17) separates variables in the cylindrical frame. Every localized solution must contain the first cylindrical harmonic due to the presence of the term $B_0 u_y x$. Thus, we can readily write the simplest localized solution in the standard form of a double vortex

$$\phi = B_0 u_y R \cos\theta \begin{cases} \dfrac{r}{R} + \alpha_1 J_1\left(r\kappa_1^{\text{in}}\right) + \alpha_2 J_1\left(r\kappa_2^{\text{in}}\right) \; , & r < R \\[2ex] \beta_1 \dfrac{R}{r} + \beta_2 K_1(r\rho) \; , & r > R \end{cases} \tag{19}$$

where $r = (x^2 + y^2)^{1/2}$, $\theta = \arctan(y/x)$, and J_1, K_1 are Bessel functions of the first order. For simplicity, we introduced

$$\rho^2 \equiv -(\kappa^{\mathrm{out}})^2 = \frac{1}{\lambda_s^2}\left(1 - \frac{c_A^2}{u_z^2}\right) .$$

Obviously, the continuity of the function $G\,(\phi - B_0 u_y x)$ requires that at the core edge we have

$$\phi - B_0 u_y x \,\big|_{r=R} = 0 .$$

Coefficients α_1, α_2, β_1, β_2 are determined from the continuity of the potentials ϕ, A_z and their radial derivatives at the core edge, which after a somewhat lengthy, but straightforward algebra (for details see Ref. 13), reveals the existence of the following two distinct vortex modes:

1. Drift-type vortex mode

This mode (called also the Larichev-Reznik mode[13]) can be identified as the generalization of the vortex associated with nonlinear drift-wave perturbations to the case of large parallel phase velocities u_z, i.e. to the regime when the electrons are hydrodynamic, rather than Boltzmann distributed. For this mode, parameter $H = 1$ is constant on the whole x, y plane, while the constant of integration G^{in} is determined from the following dispersion relation

$$\frac{(\lambda_s \kappa_1^{\mathrm{in}})^2}{1 + (\lambda_s \kappa_1^{\mathrm{in}})^2} \frac{\frac{\partial}{\partial R} J_1\left(R\kappa_1^{\mathrm{in}}\right)}{J_1\left(R\kappa_1^{\mathrm{in}}\right)} = \frac{(\lambda_s \kappa_2^{\mathrm{in}})^2}{1 + (\lambda_s \kappa^{\mathrm{in}})^2} \frac{\frac{\partial}{\partial R} J_1\left(R\kappa_2^{\mathrm{in}}\right)}{J_1\left(R\kappa_2^{\mathrm{in}}\right)} \tag{20}$$

and the coefficients α_1, α_2, β_1, β_2 are given by

$$\alpha_m = \frac{2}{R\frac{\partial}{\partial R} J_1\left(R\kappa_m^{\mathrm{in}}\right)} \frac{c_A^2}{u_z^2} \frac{{\kappa_n^{\mathrm{in}}}^2}{{\kappa_m^{\mathrm{in}}}^2 - {\kappa_n^{\mathrm{in}}}^2} \frac{1}{1 + \lambda_s^2 {\kappa_n^{\mathrm{in}}}^2} \quad , \qquad m = 1, 2$$

$$\beta_1 = 1 - \frac{2J_1\left(R\kappa_1^{\mathrm{in}}\right)}{R\frac{\partial}{\partial R} J_1\left(R\kappa_1^{\mathrm{in}}\right)} \frac{c_A^2}{c_A^2 + u_z^2} \frac{\lambda_s^2 {\kappa_2^{\mathrm{in}}}^2}{1 + \lambda_s^2 {\kappa_2^{\mathrm{in}}}^2}$$

$$\beta_2 = \frac{2J_1\left(R\kappa_2^{\mathrm{in}}\right)}{R\frac{\partial}{\partial R} J_1\left(R\kappa_2^{\mathrm{in}}\right)} \frac{c_A^2}{c_A^2 + u_z^2} \frac{\lambda_s^2 {\kappa_1^{\mathrm{in}}}^2}{1 + \lambda_s^2 {\kappa_1^{\mathrm{in}}}^2} \frac{1}{K_1\left(R\rho\right)} . \tag{21}$$

Note that vortices of this type are well localized ($\rho^2 > 0$) only if they propagate faster than the Alfvén speed, $u_z^2 > c_A{}^2$. The vortex is produced by nonlinearities acting within the vortex core, while the outer solution is an evanescent linear Alfvén wave, whose dispersion relation is given by

$$k_\perp^2 \left(\lambda_s^2 k_\perp^2 + 1 - \frac{c_A^2 k_z^2}{\omega^2} \right) = 0 \ . \tag{22}$$

In the MHD limit $\lambda_s \to 0$, the drift-vortex becomes infinitesimally thin, $1/\kappa_1^{\text{in}} \sim 1/\kappa_2^{\text{in}} \sim R \sim \lambda_s$, i.e. it reduces to a line or filament. One can easily verify that finite size MHD drift vortices are forbidden. Namely, Eqs. (18) with $\lambda_s^2 {\kappa_1^{\text{in}}}^2 \ll 1$ yield $\lambda_s^2 {\kappa_2^{\text{in}}}^2 = c_A^2/u_z^2 - 1 < 0$, and consequently it is impossible to have a finite solution at $r = 0$.

2. Convective-cell mode

This vortex type arises from the coupling between electrostatic and magnetostatic convective-cell modes. It is characterized by the discontinuity at $r = R$ of both constants of integration G and H. In the outer region it decays as $\cos\theta/r$, and physically it is just a $k_\perp = 0$ linear quasimode. The "inside" wavenumbers are related through the following dispersion relation

$$\alpha_1 J_1 \left(R\kappa_1^{\text{in}} \right) = \alpha_2 J_1 \left(R\kappa_2^{\text{in}} \right) = 0 \ . \tag{23}$$

Coefficients α_1, α_2 in Eq. (34), when different than zero, are given by

$$\alpha_m = -\frac{2\left(1 + \lambda_s^2 {\kappa_m^{\text{in}}}^2\right)^{\frac{1}{2}}}{R\frac{\partial}{\partial R} J_1 \left(R\kappa_m^{\text{in}}\right)} \times$$

$$\left\{ \left[\left(1 + \lambda_s^2 {\kappa_1^{\text{in}}}^2\right)^{\frac{1}{2}} + \left(1 + \lambda_s^2 {\kappa_2^{\text{in}}}^2\right)^{\frac{1}{2}} \right]^{-1} + \frac{c_A - u_z}{u_z} \frac{\left(1 + \lambda_s^2 {\kappa_n^{\text{in}}}^2\right)^{\frac{1}{2}}}{\lambda_s^2 \left({\kappa_n^{\text{in}}}^2 - {\kappa_m^{\text{in}}}^2 \right)} \right\} \ ,$$

$$m = 1, 2 \tag{24}$$

while the "outside" coefficients β_1, β_2 are given by

$$\beta_1 = 1 \quad , \qquad \beta_2 = 0 \ . \tag{25}$$

Particularly interesting is the convective-cell vortex given by only one Bessel function inside

$$\alpha_1 = \frac{-2}{R\frac{\partial}{\partial R} J_1\left(R\kappa_1^{\text{in}}\right)} \quad , \qquad \alpha_2 = 0 \tag{26}$$

which corresponds to

$$\kappa_1^{\text{in}} = \frac{j_{1,m}}{R}$$

$$\kappa_2^{\text{in}} = \frac{1}{\lambda_s}\left[\frac{c_A^2}{u_z^2}\left(1 + \lambda_s^2 \kappa_1^{\text{in}^2}\right) - 1\right]^{\frac{1}{2}} \tag{27}$$

where $j_{1,m}$ is the m-th zero of the Bessel function J_1. In this special case we have $\phi = u_z A_z$, its z-component of the electric field is equal to zero, and consequently this structure can not be Landau damped by electrons propagating along the magnetic field lines [14].

In the MHD limit $\lambda_s \to 0$, the convective-cell type vortex becomes singular, i.e. reduces to a point vortex or filament, similarly to the drift-vortex. It is obvious from Eq. (24) that a vortex with $u_z \neq c_A$ has an infinite amplitude if $\lambda_s \to 0$. In the special case $u_z = c_A$, solution of Eqs. (14) and (15) with a finite perpendicular scale length $\lambda_s^2 \nabla_\perp^2 \sim \lambda_s^2/R^2 \ll 1$ is possible only if $G^{\text{out}} = G^{\text{in}} = 0$, $H^{\text{out}} = H^{\text{in}} = 1$. However, in that case we have essentially linear Alfvén modes, and localized vortices are not possible.

III Linear Stability Analysis

Now we study the stability of the vortex solution, Eq. (19) in the presence of small perturbations to the vortex potentials ϕ, A_z. A stationary solution is considered to be linearly stable if any infinitesimally small perturbation remains finite at $t \to \infty$. Stability in a broader sense may be defined allowing even infinite growth of perturbations in a finite number of isolated singular points. In that case it is required that the perturbations are square integrable across the singularities.

Linearizing our starting system of equations (10) and (11) around the stationary solution $\phi^{(0)}$, $A_z^{(0)}$, given by Eq. (19), and using Eqs. (14) and (15) we obtain the following set of equations for the perturbations

$$u_z\frac{\partial}{\partial z}\delta\phi+\frac{\partial}{\partial t}\left(1-\lambda_s^2\nabla_\perp^2\right)\delta\psi=\mathbf{V}\cdot\nabla\left[H\delta\phi-\left(1-\lambda_s^2\nabla_\perp^2\right)\delta\psi\right] \tag{28}$$

$$\frac{\partial}{\partial t}\nabla_\perp^2\delta\phi+\frac{c_A^2}{u_z^2}\frac{\partial}{\partial z}\nabla_\perp^2\delta\psi=-\mathbf{V}\cdot\nabla\left[\left(\nabla_\perp^2+G\right)\delta\phi-H\frac{c_A^2}{u_z^2}\nabla_\perp^2\delta\psi\right]-$$
$$\mathbf{W}\cdot\nabla\left[H\delta\phi-\left(1-\lambda_s^2\nabla_\perp^2\right)\delta\psi\right]\ . \tag{29}$$

Here we used the notations

$$\delta\psi=u_z\delta A_z$$
$$\mathbf{V}=\frac{1}{B_0}\mathbf{e}_z\times\nabla\left(\phi^{(0)}-B_0u_yx\right)=\mathbf{v}_E^{(0)}-\mathbf{e}_yu_y$$
$$\mathbf{W}=\frac{1}{B_0}\mathbf{e}_z\times\nabla\frac{u_z^2}{c_A^2}\frac{\lambda_s^2}{H}\left(\nabla_\perp^2+G\right)\left(\phi^{(0)}-B_0u_yx\right)=-u_z\lambda_s^2\nabla_\perp^2\delta\mathbf{B}^{(0)} \tag{30}$$

where $\mathbf{v}_E^{(0)}$ and $\delta\mathbf{B}^{(0)}$ are the leading order fluid velocity and the magnetic field perturbation, associated with the zero-order vortex.

The linear partial differential equations (28) and (29) for the perturbations separate Fourier components only in the ω and k_z domains, while the $\mathbf{k}_\perp$ components are coupled via the inhomogeneous flows $\mathbf{V}$ and $\mathbf{W}$. Thus, any initial perturbation with a localized $\mathbf{k}_\perp$ spectrum will eventually, given enough time, spread over the entire range of perpendicular wavenumbers k_x, k_y.

A. Local stability analysis

One can easily verify that the drift-type vortex, Eqs. (19)–(21) is unstable. For perturbations whose characteristic perpendicular wavelength is much shorter than the vortex radius R

$$R^2\nabla_\perp^2\gg 1\ , \tag{31}$$

in the vicinity of an arbitrary point in the x, y plane we may solve Eqs. (28) and (29) in the local approximation, using $\mathbf{V}=$ const., $\mathbf{W}=$ const. Thus, in the outer region $x^2+y^2>R^2$, using $H^{\rm out} = 1$, $G^{\rm out} = 0$, we obtain from Eqs. (28) and (29) the following linear dispersion relation

$$\frac{(1+\lambda_s^2 k_\perp^2)\,(\omega - \mathbf{k}_\perp\cdot\mathbf{V})}{\lambda_s^2 k_\perp^2\,(k_z u_z - \mathbf{k}_\perp\cdot\mathbf{V}) + (1+\lambda_s^2 k_\perp^2)\,\mathbf{k}_\perp\cdot\mathbf{W}} = \frac{k_z u_z - \mathbf{k}_\perp\cdot\mathbf{V}}{\lambda_s^2 k_\perp^2\,(\omega - \mathbf{k}_\perp\cdot\mathbf{V}) + \frac{c_A^2}{u_z^2}\mathbf{k}_\perp\cdot\mathbf{W}}\ , \tag{32}$$

which can be solved for the frequency ω

$$\left(\omega - \mathbf{k}_\perp\cdot\mathbf{V} + \frac{c_A^2}{u_z^2}\frac{\mathbf{k}_\perp\cdot\mathbf{W}}{2\lambda_s^2 k_\perp^2}\right)^2 = \frac{1}{1+\lambda_s^2 k_\perp^2}\left[k_z u_z - \mathbf{k}_\perp\cdot\mathbf{V} + \frac{\mathbf{k}_\perp\cdot\mathbf{W}}{2\lambda_s^2 k_\perp^2}\left(1+\lambda_s^2 k_\perp^2\right)\right]^2 - \left(\frac{\mathbf{k}_\perp\cdot\mathbf{W}}{2\lambda_s^2 k_\perp^2}\right)^2\left(1+\lambda_s^2 k_\perp^2 - \frac{c_A^4}{u_z^4}\right)\ . \tag{33}$$

For drift-type vortices we have $\mathbf{W}^{\rm out} \neq 0$ and $c_A^4/u_z^4 < 1$. Thus, the last term on the right-hand side of Eq. (33) is always negative, and an instability will develop in the region of the x, y plane where the first (positive) term is small enough. Particularly simple is the case of short wavelength perturbations $\lambda_s^2 k_\perp^2 \gg 1$ (which is in a good agreement with the assumption Eq. (31) used in the derivation of the above dispersion relation) when from Eq. (33) we obtain the following condition for instability

$$\frac{\mathbf{k}_\perp\cdot\mathbf{W}}{\mathbf{k}_\perp\cdot\mathbf{V} - k_z u_z} > 1\ , \tag{34}$$

or, using the definitions of the flows $\mathbf{V}$, $\mathbf{W}$

$$-u_z\frac{\mathbf{k}_\perp\cdot\lambda_s^2\nabla_\perp^2\delta\mathbf{B}^{(0)}}{\mathbf{k}\cdot\left(\mathbf{v}_E^{(0)} - \mathbf{u}\right)} > 1\ .$$

Obviously, perturbations with an arbitrary wave vector $\mathbf{k}$ are unstable in the spatial region where the fluid velocity associated with the vortex, calculated in the reference frame moving with the vortex, is perpendicular to the wave vector

$$\mathbf{k}\cdot(\mathbf{v}_E^{(0)} - \mathbf{u}) \approx 0\ .$$

For a given k_z this condition may not be fulfilled for all values of the perpendicular wave vector $\mathbf{k}_\perp$. However, as we have discussed above, the perpendicular wavenumber spectrum of perturbations is broad, $(k_x, k_y) \in (-\infty, \infty)$ due to the presence of the inhomogeneous flows $\mathbf{V}$ and $\mathbf{W}$. A similar tendency of a spectrum cascade towards both smaller and larger values of $k_\perp$, arising in the weak turbulence of shear-Alfvén waves was demonstrated in Ref. 5. As a consequence, for an arbitrary value of the parallel wavenumber k_z, the spectrum will always contain some unstable modes, whose wave vector satisfies Eq. (34). Thus, with an arbitrary initial perturbation, the maximum characteristic time of the vortex destruction is determined by the rate of the spectral cascade towards these unstable modes.

For convective-cell vortices, Eqs. (19), (23), and (24), we have $\mathbf{W}^{\rm out} = 0$, and the dispersion relation, Eq. (33) does not indicate a short-wavelength instability in the exterior region. Thus, their stability analysis requires a more detailed inspection of the interior ($r < R$) solution, and of the perturbation with longer scale lengths, $k_\perp R \leq 1$.

B. Lyapunov stability analysis

We will restrict ourselves to the "one Bessel function" convective cell vortex, Eqs. (19), (26), and (27), in order to exclude the Landau damping. Then, our evolution equations (28) and (29) for small perturbations $\delta\phi$, $\delta\psi$ simplify to

$$u_z \frac{\partial}{\partial z}\delta\phi + \frac{\partial}{\partial t}\left(1 - \lambda_s^2 \nabla_\perp^2\right)\delta\psi = \mathbf{V}\cdot\nabla\left[\left(1 + \lambda_s^2\kappa_1^2\right)\delta\phi - \left(1 - \lambda_s^2\nabla_\perp^2\right)\delta\psi\right] \tag{35}$$

$$\frac{\partial}{\partial t}\nabla_\perp^2\delta\phi + \frac{c_A^2}{u_z^2}\frac{\partial}{\partial z}\nabla_\perp^2\delta\psi = -\mathbf{V}\cdot\nabla\left(\nabla_\perp^2 + \kappa_1^2\right)\left(\delta\phi - \frac{c_A^2}{u_z^2}\delta\psi\right) . \tag{36}$$

Here κ_1 is the wavenumber of the zero-order vortex, Eq. (26), whose values are $\kappa_1^{\rm out} = 0$, $\kappa_1^{\rm in} = j_{1,m}/R$.

We proceed by applying the Lyapunov stability analysis to the above system. In the construction of the Lyapunov functional we need to have at least one quadratic quantity conserved by Eqs. (35) and (36). For that purpose we introduce the following integrals.

First, we multiply Eq. (35) by $[(1 + \lambda_s^2\kappa_1^2)\,\delta\phi - (1 - \lambda_s^2\nabla_\perp^2)\,\delta\psi]$, and integrate within the vortex core $r = (x^2 + y^2)^{1/2} < R$. Using the fact that $\phi^{(0)} - B_0 u_y x\,|_{r=R} = 0$, and that consequently the flow $\mathbf{V}$ is tangential to the core edge, we have

$$\int_{r<R} dxdy \left[u_z \frac{\partial}{\partial z}\delta\phi + \frac{\partial}{\partial t}\left(1-\lambda_s^2\nabla_\perp^2\right)\delta\psi\right]\left[\left(1+\lambda_s^2\kappa_1^2\right)\delta\phi - \left(1-\lambda_s^2\nabla_\perp^2\right)\delta\psi\right] = 0 \ . \tag{37}$$

Similarly, multiplying Eqs. (35) and (36) respectively by $(\nabla_\perp^2 + \kappa_1^2)\left(\delta\phi - \frac{c_A^2}{u_z^2}\delta\psi\right)$ and $[(1+\lambda_s^2\kappa_1^2)\,\delta\phi - (1-\lambda_s^2\nabla_\perp^2)\,\delta\psi]$, and adding, we obtain after the integration for the whole x, y plane

$$\int_{-\infty}^{\infty} dxdy \left\{\left[u_z \frac{\partial}{\partial z}\delta\phi + \frac{\partial}{\partial t}\left(1-\lambda_s^2\nabla_\perp^2\right)\delta\psi\right]\left(\nabla_\perp^2+\kappa_1^2\right)\left(\delta\phi - \frac{c_A^2}{u_z^2}\delta\psi\right) - \right.$$
$$\left.\left(\frac{\partial}{\partial t}\nabla_\perp^2\delta\phi + \frac{c_A^2}{u_z^2}\frac{\partial}{\partial z}\nabla_\perp^2\delta\psi\right)\left[\left(1+\lambda_s^2\kappa_1^2\right)\delta\phi - \left(1-\lambda_s^2\nabla_\perp^2\right)\delta\psi\right]\right\} = 0 \ . \tag{38}$$

Finally, for the zero-order vortices which satisfy

$$(\lambda_s\kappa_1^{\rm in})^2 \equiv \frac{\lambda_s^2}{R^2} j_{1,m}^2 = \frac{u_z^2}{c_A^2} - 1 \tag{39}$$

we may write down the sought for quadratic conserved quantity, simply multiplying Eq. (37) by $\lambda_s^2\kappa_1^2/\left(1+\lambda_s^2\kappa_1^2\right)$ and subtracting from Eq. (38)

$$\int_{-\infty}^{\infty} dxdy \left(\frac{\partial P}{\partial t} - u_z\frac{\partial Q}{\partial z}\right) = 0 \tag{40}$$

where

$$P = \left(\nabla_\perp^2\delta\phi\right)\left(1+\lambda_s^2\kappa_1^2\right)\delta\phi + \frac{c_A^2}{u_z^2}\left(1+\lambda_s^2\kappa_1^2\right)\left(\nabla_\perp^2\delta\psi\right)\left(1-\lambda_s^2\nabla_\perp^2\right)\delta\psi -$$
$$2\left(\nabla_\perp^2\delta\phi\right)\left(1-\lambda_s^2\nabla_\perp^2\right)\delta\psi$$

$$Q = \delta\phi\nabla_\perp^2\delta\phi + \frac{c_A^2}{u_z^2}\left(\nabla_\perp^2\delta\psi\right)\left(1-\lambda_s^2\nabla_\perp^2\right)\delta\psi - 2\delta\phi\frac{c_A^2}{u_z^2}\nabla_\perp^2\delta\psi \ . \tag{41}$$

The above expression is further simplified by the integration in z, t, when after some straightforward algebra we obtain the conserved quantity L in the form

$$L = \text{const.} = \int_{-\infty}^{\infty} dxdydz \left\{\left[\nabla_\perp\left(\delta\phi - \frac{c_A}{u_z}\delta\psi\right)\right]^2 + \lambda_s^2\left[\nabla_\perp^2\left(\delta\phi - \frac{c_A}{u_z}\delta\psi\right)\right]^2 + \right.$$

$$\frac{c_A^2}{u_z^2}\left(1+\lambda_s^2\kappa_1^2\right)\lambda_s^2\left(\nabla_\perp^2\delta\psi\right)^2-\delta\phi\lambda_s^2\nabla_\perp^2\left(\nabla_\perp^2+K\right)\delta\phi-\frac{c_A^2}{u_z^2}\delta\psi\lambda_s^2\nabla_\perp^2\left(\nabla_\perp^2+K\right)\delta\psi\Big\} \tag{42}$$

where the parameter K is defined by

$$\lambda_s^2 K=\frac{c_A}{u_z}\left(1+\lambda_s^2\kappa_1^2\right)-1 \tag{43}$$

or, with the use of Eq. (39)

$$\lambda_s^2 K^{\text{in}}=\frac{u_z}{c_A}-1>0 \quad , \quad \lambda_s^2 K^{\text{out}}=\frac{c_A}{u_z}-1<0\ . \tag{44}$$

Obviously, the first three terms in the integrand of Eq. (42) are positive definite, and in the case of an instability they can grow only at the expense of the last two. Thus, the sufficient and necessary stability condition is that the (possibly negative) sum of the last two terms remains bounded from below for all times, and for all possible realizations of $\delta\phi, \delta\psi$, which satisfy the evolution equations (35) and (36). If this were fulfilled, then the above defined integral of motion L would be the Lyapunov functional for our system.

For a detailed analysis of the functional L, Eq. (42), we will expand the potentials $\delta\phi, \delta\psi$ on the complete basis of orthogonal and normalized functions g, in the following way:

$$\delta\phi\left(t,z,\mathbf{r}_\perp\right)=\sum_n e^{in\theta}\left[\sum_k \widehat{\delta\phi}_{n,k}\left(t,z\right)g_{n,k}\left(r\right)+\int_0^\infty dk\widehat{\delta\phi}_n\left(k,t,z\right)g_n(k,r)\right] \tag{45}$$

with an analogous expression for $\delta\psi$.

Eigenfunctions $g_{n,k}\left(r\right)$ and $g_n(k,r)$ are obtained from the Schrödinger equation describing a particle in a two-dimensional, cylindrical potential well, whose radius is equal to the vortex core radius R, and whose depth is equal to s^2. The discrete spectrum $g_{n,k}\left(r\right)$ is determined from the following eigenvalue problem

$$\left[\nabla_\perp^2+S^2\left(r\right)\right]e^{in\theta}g_{n,k}\left(r\right)=q_{n,k}^2 e^{in\theta}g_{n,k}\left(r\right)\ , \tag{46}$$

with

$$S^2\left(r\right)=\begin{cases} s^2\ , & r<R \\ 0\ , & r>R \end{cases}\ . \tag{47}$$

Solving Eq. (46), functions $g_{n,k}(r)$ can be readily expressed in terms of Bessel-, and modified Bessel functions of the order n, J_n, K_n, respectively

$$g_{n,k}(r) = a_{n,k} \begin{cases} \dfrac{J_n\left(r\xi_{n,k}\right)}{J_n\left(R\xi_{n,k}\right)} \;, & r < R \\ \\ \dfrac{K_n\left(rq_{n,k}\right)}{K_n\left(Rq_{n,k}\right)} \;, & r > R \end{cases} \tag{48}$$

where $\xi_{n,k} = \left(s^2 - q_{n,k}^2\right)^{1/2}$. Eigenvalues $q_{n,k}$ are determined from the smoothness at $r = R$, yielding

$$\xi_{n,k}\frac{J_{n-1}\left(R\xi_{n,k}\right)}{J_n\left(R\xi_{n,k}\right)} = -q_{n,k}\frac{K_{n-1}\left(Rq_{n,k}\right)}{K_n\left(Rq_{n,k}\right)} \;. \tag{49}$$

Similarly, the continuous spectrum $g_n(k,r)$ is obtained from

$$\left[\nabla_\perp^2 + S^2\left(r\right)\right] e^{in\theta} g_n(k,r) = -k^2 e^{in\theta} g_n(k,r) \;, \tag{50}$$

which yields

$$g_n(k,r) = a_n(k) \begin{cases} \dfrac{J_n(r\xi)}{J_n(R\xi)} \;, & r < R \\ \\ \dfrac{J_n(rk) + d_n Y_n(rk)}{J_n(Rk) + d_n Y_n(Rk)} \;, & r > R \end{cases} \;. \tag{51}$$

Here $\xi = (s^2 + k^2)^{1/2}$, and the coefficients d_n are determined from the smoothness at $r = R$

$$d_n = -\frac{\xi J_{n-1}(R\xi)J_n(R\xi) - kJ_{n-1}(Rk)J_n(Rk)}{\xi J_{n-1}(R\xi)J_n(R\xi) - kY_{n-1}(Rk)Y_n(Rk)} \;. \tag{52}$$

One can easily verify that the eigenfunctions $g_n(k,r)$, and $g_{n,k}(r)$ are orthogonal. We will adopt the parameters $a_{n,k}$, $a_n(k)$ from the normalization, and consequently we have

$$\int_0^\infty drr g_{n,k}(r) g_{n,k'}(r) = \frac{\delta_{k,k'}}{2\pi}$$
$$\int_0^\infty drr g_n(k,r) g_n(k',r) = \frac{\delta(k-k')}{2\pi} \;. \tag{53}$$

The depth of the potential well s^2 will be conveniently adopted as

$$s^2 = \frac{1}{2}\left(K^{\rm in} - K^{\rm out}\right) = \frac{1}{2\lambda_s^2}\frac{c_A}{u_z}\left(\frac{u_z^2}{c_A^2} - 1\right) \;, \tag{54}$$

which permits us to write down the following identities

$$\nabla_\perp^2\left(\nabla_\perp^2+K\right)e^{in\theta}g_{n,k}(r) =$$

$$\left[q_{n,k}^2\left(q_{n,k}^2+K^{\text{out}}\right)-\frac{1}{4}\left(K^{\text{in}2}-K^{\text{out}2}\right)h(R-r)\right]e^{in\theta}g_{n,k}(r)$$

$$\nabla_\perp^2\left(\nabla_\perp^2+K\right)e^{in\theta}g_n(k,r) =$$

$$\left[k^2\left(k^2-K^{\text{out}}\right)-\frac{1}{4}\left(K^{\text{in}2}-K^{\text{out}2}\right)h(R-r)\right]e^{in\theta}g_n(k,r)\ , \tag{55}$$

where h denotes the Heaviside unit step function.

Finally, using the expansions, Eq. (45), orthogonality, Eq. (53), and the above identities, we may rewrite the functional L in Eq. (42) in the following form

$$L = L^{(+)}(t) - L^{(-)}(t) = \text{const.}\ , \tag{56}$$

where

$$L^{(+)}(t) = \int_{-\infty}^{\infty} dxdydz\left\{\left[\nabla_\perp\left(\delta\phi-\frac{c_A}{u_z}\delta\psi\right)\right]^2+\lambda_s^2\left[\nabla_\perp^2\left(\delta\phi-\frac{c_A}{u_z}\delta\psi\right)\right]^2+\right.$$

$$\left.\frac{c_A^2}{u_z^2}\left(1+\lambda_s^2\kappa_1^2\right)\lambda_s^2\left(\nabla_\perp^2\delta\psi\right)^2+\frac{\lambda_s^2}{4}\left(K^{\text{in}2}-K^{\text{out}2}\right)\left(\delta\phi^2+\frac{c_A^2}{u_z^2}\delta\psi^2\right)h\left(R-r\right)\right\}-$$

$$\int_{-\infty}^{\infty} dz\sum_{n,k}\left(\mid\widehat{\delta\phi}_{n,k}\mid^2+\frac{c_A^2}{u_z^2}\mid\widehat{\delta\psi}_{n,k}\mid^2\right)q_{n,k}^2\left(q_{n,k}^2+K^{\text{out}}\right)\ , \tag{57}$$

and

$$L^{(-)}(t) = \int_{-\infty}^{\infty} dz\sum_n\int dk\left(\mid\widehat{\delta\phi}_n(k)\mid^2+\frac{c_A^2}{u_z^2}\mid\widehat{\delta\psi}_n(k)\mid^2\right)k^2\left(k^2-K^{\text{out}}\right)\ . \tag{58}$$

Noting that, according to Eq. (39) our zero-order vortex is propagating faster than the Alfvén speed, $u_z^2 > c_A^2$, we have $K^{\text{out}} < 0$, and consequently the functional $L^{(-)}(t)$, Eq. (58) is positive definite. Similarly, from

$$K^{\text{in}2}-K^{\text{out}2} = \frac{1}{\lambda_s^4}\left(1-\frac{c_A^2}{u_z^2}\right)\left(\frac{u_z}{c_A}-1\right)^2 > 0\ .$$

follows that the functional $L^{(+)}(t)$ will be positive definite if the following inequality is satisfied

$$q_{n,k}^2 + K^{\text{out}} < 0 \ , \tag{59}$$

or equivalently, making use of Eqs. (50) and (55)

$$\frac{u_z}{c_A}\left(\frac{u_z}{c_A}+1\right) > \left(\frac{u_z}{c_A}-1\right) D_{n,k} \qquad , \qquad D_{n,k} = \frac{{\kappa_1^{\text{in}}}^2}{2\xi_{n,k}^2} \ . \tag{60}$$

We will study the lowest order stationary vortex solution, Eq. (27), i.e. we adopt $\kappa_1^{\text{in}} = j_{1,1}/R$ (where $j_{n,m}$ denotes the m-th zero of the Bessel function J_n). The "inside" eigenvalues $\xi_{n,k}$ may be estimated from the dispersion relation, Eq. (49), whose smallest solution is somewhat below $j_{0,1}/R$, but very close to it:

$$\xi_{n,k} \geq \xi_{0,1} \sim \frac{j_{0,1}}{R} \ . \tag{61}$$

This sets the numerical value of the parameter $D_{n,k}$ in Eq. (60) below 1.3:

$$D_{n,k} \leq \frac{{\kappa_1^{\text{in}}}^2}{2\xi_{0,1}^2} \sim 1.2688 \ . \tag{62}$$

Inequality Eq. (60) is always satisfied with the above value of $D_{n,k}$, which implies that the functional $L^{(+)}(t)$ is positive definite.

In order to complete our stability proof, it would be necessary to show that either $L^{(+)}(t)$, or $L^{(-)}(t)$ is bounded from above (then, from Eq. (56) the other functional automatically remains finite). Noting that $L^{(+)}(t)$ and $L^{(-)}(t)$ are defined as integrals for the entire space of some positive definite functions, their boundedness implies that their integrands, and consequently also the perturbations $\delta\phi$, $\delta\psi$, remain infinitesimally small for all times, and in the whole space. Possible exceptions of finite size spatial regions with finite perturbations, and of "well behaved" (i.e. square integrable) isolated singularities of $\delta\phi$, $\delta\psi$ are not regarded as instabilities in the usual sense, as defined in the beginning of this section.

It is important to note at this point that only the continuum spectrum of the perturbations is contributing to the functional $L^{(-)}(t)$. Furthermore, it can be shown that modes belonging to the high n, k end of the continuum spectrum are stable, and that consequently their contribution to the functional $L^{(-)}(t)$ is bounded. Namely, if the characteristic scale length is much shorter than the vortex size

$$n \sim kr \sim kR \gg 1 \tag{63}$$

the Bessel functions in Eq. (51) have the following simple asymptotics

$$\begin{aligned} J_n(kr) &= \sqrt{\frac{2}{\pi}} \left(k^2r^2 - n^2\right)^{-\frac{1}{4}} \cos\left[\left(k^2r^2 - n^2\right)^{\frac{1}{2}} - \varphi_n\right] \\ Y_n(kr) &= \sqrt{\frac{2}{\pi}} \left(k^2r^2 - n^2\right)^{-\frac{1}{4}} \sin\left[\left(k^2r^2 - n^2\right)^{\frac{1}{2}} - \varphi_n\right] \\ \varphi_n &= \frac{\pi}{4} + n \arccos\left(\frac{n}{kr}\right) , \end{aligned} \tag{64}$$

and the eigenfunctions $e^{in\theta}g_n(kr)$ may be replaced by simple Fourier components. In this regime we also consider $\mathbf{V} \approx const.$, and apply the local approximation to Eqs. (46) and (47), which permits us to calculate the frequency from the corresponding linear dispersion relation. As we have already shown, in the outer region, setting $\kappa_1^{\text{out}} = 0$ in Eqs. (46) and (47) we obtain linearly stable Fourier components

$$(\omega - \mathbf{k}_\perp \cdot \mathbf{V})^2 = \frac{(k_z u_z - \mathbf{k}_\perp \cdot \mathbf{V})^2}{1 + \lambda_s^2 k_\perp^2} . \tag{65}$$

Within the vortex core the linear dispersion relation, obtained from Eqs. (46) and (47) has the form

$$\frac{u_z^2}{c_A^2}\left(1 + \lambda_s k_\perp^2\right) \Omega \left(\Omega + \frac{\zeta}{\lambda_s^2 k_\perp^2}\right) - (1 - \zeta)\left(1 + \frac{\zeta}{\lambda_s^2 k_\perp^2}\right) = 0 , \tag{66}$$

where we used the notation

$$\Omega = \frac{\omega - \mathbf{k}_\perp \cdot \mathbf{V}}{k_z u_z - \mathbf{k}_\perp \cdot \mathbf{V}} \quad , \quad \zeta = \lambda_s^2 \kappa_1^{\text{in}\,2} \frac{\mathbf{k}_\perp \cdot \mathbf{V}}{k_z u_z - \mathbf{k}_\perp \cdot \mathbf{V}} . \tag{67}$$

In the short-wavelength limit

$$\lambda_s^2 k_\perp^2 \sim R^2 k_\perp^2 \gg 1 \tag{68}$$

the above dispersion relation becomes

$$\frac{u_z^2}{c_A^2}\lambda_s^2 k_\perp^2 \Omega^2 = 1 - \zeta \ ,$$

and we have linear stability, Im $\Omega = 0$, provided $\zeta < 1$ or equivalently

$$\frac{k_z u_z}{\mathbf{k}_\perp \cdot \mathbf{V}} < 1 \qquad \text{or} \qquad \frac{k_z u_z}{\mathbf{k}_\perp \cdot \mathbf{V}} > 1 + (\lambda_s \kappa_1^{\text{in}})^2 = \frac{u_z^2}{c_A^2} \ . \tag{69}$$

For a broad range of parallel wavenumbers k_z these conditions are satisfied by short wavelength perturbations, Eq. (68). A simple example are the perturbations with a small enough value of the parallel wavenumber

$$\lambda_s k_z \leq \frac{u_y}{u_z} \tag{70}$$

which are unstable, according to Eq. (69), only if $\lambda_s k_\perp \leq 1$. However, this is outside of the validity region of the dispersion relation Eq. (66), and the existence of the instability remains uncertain. For a conclusive proof of either stability, or instability, extensive numerical investigation is required, solving three-dimensional evolution equations in the presence of initial perturbations belonging to the continuum spectrum $g_n(k,r)$, whose the scale length is comparable to the vortex size

$$n \sim kR \sim 1 \ . \tag{71}$$

From Eqs. (56)–(58) it should be noted that an eventual instability of the continuum modes is followed by a simultaneous growth of the discrete modes $\widehat{\delta\phi}_{n,k}(t,z)\ g_{n,k}(r)$, which are well localized in space, having a better localization than the zero-order vortex. Such a behavior bears a close resemblance with the dual cascade in the weak shear-Alfvén turbulence[3, 5] towards both smaller and larger wavenumbers. This may possibly indicate a transition to a new nonlinear structure, different from the vortex Eqs. (19), (24), and (25).

IV Conclusion

In this work we studied the linear stability of drift- and convective-cell type shear-Alfvén vortices. Solving the linearized evolution equations, it was shown that the first type is destabilized by short wavelength perturbations, Eq. (34) in the spatial region where the perturbations are propagating (almost) perpendicularly to the vortex fluid velocity, calculated in the reference frame moving with the vortex. Since the scattering of the perturbations on the zero-order vortex gives rise to broadening of the perturbation spectrum, these unstable modes would evolve, given enough time, for any arbitrary initial perturbation. Thus, the upper limit of the drift-vortex lifetime can be estimated from the rate of spectrum cascade towards the unstable modes.

In case of the convective-cell nonlinear mode, we restricted our study to the "one Bessel function" vortex, Eqs. (19), (26), and (27), which is not Landau damped by resonant electrons. Lyapunov stability analysis was applied after the expansion of the perturbations on an appropriate basis. Such a procedure indicated stability for a broad range of vortex parameters, Eqs. (69) and (70), in the case when only short wavelength perturbations were present. The remaining part of the continuum spectrum of perturbations, corresponding to long-range, wave-like perturbations with the wavelength comparable to the vortex size is considered to be critical for the vortex stability. In order to prove the stability, without solving the evolution equations (35) and (36), it would be necessary to find constraints for the amplitudes of all these critical modes. However, only a finite number of conserved quantities seems to be available,[13] and such a general proof can not be constructed. Alternatively, a numerical experiment of the vortex evolution under critical perturbations would provide the information about stability, similarly to the procedure applied to the Hasegawa-Mima equation in.[14]

Although stability could not be proven decisively by the present analytical approach, it was concluded from Eqs. (56)–(58) that any possible destabilization by the critical long range perturbations, with the characteristic wavelength comparable to the vortex size, must be accompanied by a simultaneous growth of short range perturbations, having better localization than the original vortex. Such a behavior is resembling the dual cascade, towards both smaller and larger perpendicular wavenumbers in the weak shear-Alfvén turbulence, which is the characteristic "signature" of the self-organization processes leading to the creation of coherent structures. One may speculate that in our case it indicates transition to another coherent solution. However, to prove this a detailed numerical solution is required of the fully nonlinear, three-dimensional evolution equations describing convective-cell vortices.

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