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Conservation of Generalized Vorticity and Collisionless Relaxation and Reconnection in Two-Fluid MHD

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Abstract

The Canonical Hamiltonian formulation of fluid particles equation of motion leads to the frozen-in law for generalized vorticity for electrons and ions. In one-fluid magnetohydrodynamics (MHD) the frozen-in law for generalized ion vorticity disappears when the Hall term in Ohm's law is neglected. If the field lines of ion and electron generalized vorticities have different topology, the conservation of both generalized vorticities strongly affects plasma relaxation even for large-scale configurations. For example, electron inertia and finite ion Larmor radius does not lead to collisionless reconnection, when magnetic field lines lie on nested tori, and the field lines of generalized ion vorticity are chaotic. The topology of the generalized vorticity field lines breakdown only if the thermal drifts of electrons and ions are both taken into account.

1 Introduction

Integrals of motion play an important role in analyzing the behavior of a complex nonlinear systems. The conserved quantities are used for nonlinear stability and relaxation analyses of hydrodynamic and magnetohydrodynamic (MHD) systems. In many cases simplified equations are used to approximate the behavior of these systems. For an approximation to be consistent, we should conserve the number of integrals during the simplification of the equations, otherwise even small corrections may have strong effects on long time evolution.

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Stability analyses of plasma configurations are usually based on the one-fluid MHD approximation, which is a simplification of more complete two-fluid MHD for large scale motions $\lambda \gg c/\omega_{pi}$ (here ω_{pi} is ion plasma frequency). It was noted in [1, 2, 3, 4, 5] that two fluid MHD has two infinite series of frozen-in integrals of motion: the curls of generalized electron and ion momenta are frozen-into the respective fluid [1, 2]. Under the assumptions leading to one-fluid MHD, the frozen-in law for generalized ion vorticity disappears, the generalized electron vorticity becomes the magnetic field B, which is then frozen-into the plasma. In this letter we show that considering two infinite series of frozen-in laws leads to a qualitative change in the relaxation process of an initial state.

In Sec. 2 we derive the frozen-in law for the curl of both generalized momenta from the canonical Hamiltonian structure of fluid particle equations of the motion [6]. In mechanics these integrals are known as Poincaré's relative integral invariants [7]. Existence of such invariants, also known as Casimirs, depends on the Poisson bracket structure and is independent of specific Hamiltonian. Two families of frozen-in laws for two-fluid MHD are presented. Their transformations, during the transition to one-fluid MHD are analyzed. In Sec. 3 we apply the simplified two-fluid MHD equations to the problem of plasma relaxation.

We show that the analysis of plasma relaxation, based on the two infinite families of integrals, predicts the formation of discontinues, whereas taking into account only the frozen magnetic field (one-fluid MHD) predicts a smooth equilibrium.

In Sec. 4 we draw our conclusions. Specifically, in the contest of "collisionless reconnection" we claim that taking into account electron inertia and finite Larmor radius of ions do not change the topology of generalized vorticity field lines. However, allowing for thermal drift of both electrons and ions does lead to the reconnection.

2 MHD Integrals of Motion

In this section we present two families of frozen-in integrals of motion of two-fluid MHD equations and trace their relation to the canonical form of Hamilton equations. We show how one of these families is lost when two-fluid MHD equations are simplified to one-fluid MHD.

Two integral families in two-fluid MHD appear as a consequence of canonical Hamiltonian structure of the initial system [6]. In classical mechanics such invariants are known as Poincaré's relative integral invariant. Namely, equations of motion of a Hamiltonian system have the canonical form:

$$\frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{x}} , \qquad \frac{d\mathbf{x}}{dt} = \frac{\partial H}{\partial \mathbf{p}} . \tag{1}$$

Then, the integral $I = \oint \mathbf{p} \cdot d\mathbf{x}$, taken around a contour moving with the phase-space flow, is constant [7]. The Hamiltonian $H(\mathbf{x}, \mathbf{p}, t)$ need not be conservative. For fluid particle in a continuous medium we may assume $\mathbf{p} = \mathbf{p}(\mathbf{x})$, then this integral may be transformed into a surface integral by Stokes's lemma,

$$I = \oint \mathbf{p} \cdot d\mathbf{x} = \int \nabla \times \mathbf{p} \cdot d\mathbf{S} . \tag{2}$$

As $\mathbf{x}$ and $\mathbf{p}$ appear in Eq. (1) symmetricaly, we may also interchange them in Eq. (2) and this leads a conservation law that is not commonly recognized. In hydrodynamics Eq. (2) is usually written in the equivalent differential form [1]:

$$\partial_t \boldsymbol{\Omega} = \nabla\times (\mathbf{v} \times \boldsymbol{\Omega}) , \tag{3}$$

$$\boldsymbol{\Omega} = \nabla\times\mathbf{p} .$$

These equations are the consequence of the canonical Hamiltonian structure of initial equations and do not depend on the particular form of the Hamiltonian, whenever the fluid particle equations of motion can also be written in the canonical form (1). The Hamiltonian of each species in a plasma is

$$H = P/m + e\varphi + \left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 /2m \ . \tag{4}$$

Here P is the normalized pressure, e is the charge of the particle, m is the particle mass. φ and $\mathbf{A}$ are the scalar and the vector potentials of the electromagnetic field, respectively, and

$$\mathbf{p} \equiv m\mathbf{v} + \frac{e}{c}\mathbf{A} \tag{5}$$

is the generalized momentum of particle. If the fluid density, ρ, is a function of the pressure, P, only, equations of fluid particle motion are canonical with the Hamiltonian given by Eq. (4), and the canonical momentum given by Eq. (5).

Now consider a more general case, namely $P = P(\rho, s)$, where s is the entropy per unit mass, and the local adiabaticity law is satisfied: $d_t s = 0$. The fluid particle Hamiltonian still has form (4), but the effective pressure (normalized to a unit mass of fluid particle) varies for different trajectories and Poincaré's relative integral invariant (2) is generally not conserved. If, however, the integration contour is on the surface of constant entropy, the Hamiltonian is the same for all fluid particles on the contour, and this integral is constant. The integral of the component of generalized vorticity normal to the surface of constant entropy is conserved.

$$d_t \frac{\mathbf{\Omega} \cdot \nabla s}{\rho} = 0 \ .$$

Here s is entropy, and ρ density of the plasma. The non-collinearity of pressure and density gradients destroys only part of an infinite number of frozen-in conservation laws. For hydrodynamics this result was obtained by Ertel (1942), as quoted in Landau and Lifshitz [8] but our method is more transparent.

The frozen-in integrals can also be obtained by using a traditional technique. Equations of motion for the electron and ion fluids with negligible collisions are

$$m_\alpha \left[\partial_t \mathbf{v}_\alpha + (\mathbf{v}_\alpha \cdot \nabla)\mathbf{v}_\alpha\right] = -\frac{\nabla P_\alpha}{n_\alpha} + e_\alpha \mathbf{E} + \frac{e_\alpha}{c}\mathbf{v}_\alpha \times \mathbf{B} \ . \tag{6}$$

Here the subscript α denotes the species, $\mathbf{v}$ the velocity, P the pressure, n the density, e_α the charge, which, by assumption, has the same asolute value and opposite sign for the electrons and ions, m_α is the particle mass, and $\mathbf{E}$ and, $\mathbf{B}$ electric and magnetic field, respectively.

If the density is a function of pressure, $n_\alpha = n_\alpha(P_\alpha)$, we may take the curl of Eq. (6) to obtain that the generalized vorticity,

$$\mathbf{\Omega}_\alpha = \nabla \times \mathbf{v}_\alpha + \frac{e_\alpha \mathbf{B}}{m_\alpha c} , \tag{7}$$

is frozen-into the corresponding fluid:

$$\partial_t \mathbf{\Omega}_\alpha = \nabla \times (\mathbf{v}_\alpha \times \mathbf{\Omega}_\alpha) , \tag{8}$$

where we have used the Maxwell equation

$$\nabla \times \mathbf{E} = -\frac{1}{c}\, \partial_t \mathbf{B} .$$

In order to obtain one-fluid MHD equations from Eqs. (8) the following four assumptions are made:

First: we neglect the electron mass.

$$m_e \ll m_i , \quad \text{or} \quad m_e = 0 . \tag{9}$$

This assumption allows us to omit the inertia term, $\nabla \times \mathbf{v}_e$ in the electron generalized vorticity $\mathbf{\Omega}_e$.

Second: we assume that the displacement current $\partial_t \mathbf{E}$ in the Maxwell equation

$$\nabla \times \mathbf{B} = (\partial_t \mathbf{E} + 4\pi \mathbf{j})/c$$

is negligible.

The third assumption is the quasi-neutrality

$$|n_e - n_i| \ll n_e = n .$$

These assumptions allow us to express electron velocity $\mathbf{v}_e$ as

$$\mathbf{v}_e = \mathbf{v}_i - \frac{c}{4\pi n e} \nabla \times \mathbf{B} , \tag{10}$$

and then system (7), (8) takes the form

$$\partial_t \left(\nabla \times \mathbf{v}_i + \frac{e\mathbf{B}}{m_i c} \right) = -\nabla \times \left[\left(\nabla \times \mathbf{v}_i + \frac{e\mathbf{B}}{m_i c} \right) \times \mathbf{v}_i \right] , \tag{11}$$

$$\partial_t \mathbf{B} = \nabla \times \left[\left(\mathbf{v}_i - \frac{c}{4\pi n e} \nabla \times \mathbf{B} \right) \times \mathbf{B} \right] . \tag{12}$$

This system still has two families of integrals of motion. Namely, the magnetic field $\mathbf{B}$ is frozen into the electron flow (10), and the generalized ion vorticity,

$$\Omega_i = \nabla \times \mathbf{v}_i + \frac{e\mathbf{B}}{m_i c} , \tag{13}$$

is frozen into the ion flow $\mathbf{v}_i$. This approximation is also known as ideal one-fluid MHD with the Hall effect.

The forth assumption is

$$\mathbf{v}_i \gg \frac{c}{4\pi n e} \nabla \times \mathbf{B} \equiv (\mathbf{v}_i - \mathbf{v}_e) . \tag{14}$$

This assumption allows us to write $\mathbf{v}_i = \mathbf{v}_e$, and the equation for electron vorticity (12) may be written in the well-known form of Faraday law:

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{v}_i \times \mathbf{B}) . \tag{15}$$

Now the formal replacement of $\mathbf{v}_i$ by $\mathbf{v}$ and the quasi-neutrality condition complete our reduction of two-fluid to one-fluid MHD. After all the assumptions are made, the ion equation of motion (11) takes the form

$$\partial_t \nabla \times \mathbf{v} = \nabla \times [\mathbf{v} \times (\nabla \times \mathbf{v})] + \nabla \times \left[\left(\frac{c}{4\pi n e} \nabla \times \mathbf{B} \right) \times \frac{e\mathbf{B}}{mc} \right] \tag{16}$$

which is unlikely to be a form of a frozen-in law. This does not yet mean that we lose the Hamiltonian structure by neglecting the advection of the magnetic field by plasma current. This only changes the form of the Poisson bracket, which becomes noncanonical. The noncanonical Hamiltonian representation for one-fluid MHD was worked out in [9].

However, the conservation of the cross helicity,

$$I = \int \mathbf{v} \cdot \mathbf{B} d^3x \; , \tag{17}$$

may be treated as a remnant of the ion generalized vorticity conservation [5]. If both $\mathbf{v}$ and $\mathbf{B}$ lie on toroidal surfaces, the cross helicity is conserved on each surface individually. In a two-dimensional geometry such surfaces always exist, leading to the existence of an infinite familiesof frozen-in integrals [10, 11].

3 Effects of the Second set of Frozen-in Integrals

In this section we show that the freezing of the ion generalized vorticity, even though it is very close to the electron generalized vorticity, can strongly affect the relaxation process as the two vorticities have different toplogy which lead to the formation of discontinuities, absent from the predictions of one-fluid MHD.

In general, plasma relaxation occurs in two stages. At the first stage all ideal integrals are well conserved, however, the process may lead to the formation of current sheets. The time scale of dissipative processes is proportional to the square of the spatial scale of fields, hence, at the second stage of relaxation, we may consider dissipative effects only in regions where there are discontinuities. More robust helicity and, perhaps, cross helicity remain constant during the second stage of relaxation [12]. However the topological structure of fields may change. The second stage of relaxation subject to the helicity or cross helicity conservation has been studied, [12, 5], but here we are focus on the other issues.

Consider the first stage of the magnetized plasma relaxation on spatial scale $L \gg c/\omega_{pi}$.

In this case the fields $\Omega_i \equiv \nabla\times\mathbf{v} + \frac{e\mathbf{B}}{m_i c}$ and $\Omega_e \equiv \mathbf{B}$ are very close, but the topology of their field lines may differ. We assume that the initial magnetic filed lines lie on nested tori. One-fluid MHD with frozen-in magnetic field predicts the relaxation to a smooth equilibrium state with the same magnetic field topology.

The situation can drastically change if we take into account the existence of the second set of frozen-in integrals, namely the generalized ion vorticity

$$I = \oint \Omega_i \cdot dl \equiv \oint \left(\nabla\times\mathbf{v} + \frac{e\mathbf{B}}{m_i c}\right) \cdot dl \ .$$

Under the assumption $L \gg c/\omega_{pi}$, Ω_i is close to $e\mathbf{B}/(m_i c)$, but may have a different topology. On the one hand, a generic solenoidal vector field does not lie on surfaces, as it is chaotic, and fills 3 D regions. This is what we assume for the field $\nabla\times\mathbf{v}$. Further if

$$\nabla\times\mathbf{v} \ll \frac{e\mathbf{B}}{mc} \ ,$$

then, according to KAM theorem [7], the topology of Ω_i is basically the same as that of $\mathbf{B}$. However, there may exist small regions where Ω_i is chaotic and these regions are filled with one field line of Ω_i, [13].

On the other hand, we know that a stationary equilibrium configuration satisfies the equation

$$\mathbf{v} \times \Omega_i = \nabla\psi \ , \tag{18}$$

where ψ is a scalar function. This means that $\mathbf{v}$ and Ω_i lie on a family of nested, generally toroidal, surfaces $\psi = \text{const}$. But the relaxation of a chaotic Ω_i to toroidal one is impossible without changing the field topology, as discussed by Arnold [14].

This contradiction leads to two possibilities.

First: relaxation to the final state with the $\mathbf{B}$ field lying on nested toroidal surfaces and the Ω_i field filling 3D regions leads to discontinuities in the frozen-in field $\mathbf{\Omega}_i$. Therefore the current sheets, resulting from separatrices in the framework of one-fluid MHD [15], may now appear everywhere.

Second: it is also possible that the system will relax to the generalized "force-free" configuration,

$$\mathbf{v} = \mu \left(\nabla \times \mathbf{v} + \frac{e\mathbf{B}}{m_i c} \right) \tag{19}$$

and, if $\nabla \cdot \mathbf{v} = 0$, $\mu = \text{const}$ in this configurations.

The current sheets, which appear according to the described mechanism, will trigger reconnection leading to relaxation to configurations with a different field topology. [12, 5].

Note that even though the ion and the electron generalized vorticities have a symmetric form and obey similar equations (7), the frozen-in condition for the ion vorticity breakdown much easier. The reason is the large ion to electron mass ratio. In the ion generalized vorticity the magnetic field $\mathbf{B}$ is strongly coupled to the vorticity $\nabla \times \mathbf{v}_i$. The hydrodynamic description of ion velocity relaxation is not accurate because of kinetic wave-breaking along $\mathbf{B}$ and large ion Larmor radius. A better conservation of the electron helicity compared to the ion one was noted in [5]. A recent review [16] clarifies the physical meaning of the electron generalized vorticity.

4 On the Collisionless Reconnection

Reconnection and, therefore, relaxation also requires considering the second families of frozen-in invariants. Such an analysis for electron MHD has been done [15]. The problem of collisionless magnetic field reconnection is important for plasma confinement in tokamaks and space plasma physics.

Usual analysis of collisionless reconnection studies the change of a magnetic field topology due to small effects like electrion inertia on characteristic spatial scale $\lambda \gg c/\omega_{pe}$. It is our opinion that generalized electron vorticity Ω_e, which is initially close to the magnetic field $\mathbf{B}$, remains close to $\mathbf{B}$ because of the smallnes of $c/(\lambda\omega_{pe})$. However even small perturbations of the magnetic field can cause drastic topological change for a small part of the magnetic field lines. This change is not important, since electrons are attached to the forzen-in field lines of Ω_e.

This may explain the strong decrease in the reconnection rate observed in simulations [17]. The finite ion Larmor radius can lead to a current sheet instability [18]; however, the frozen-in generalized electron vorticity guarantees the preservation of magnetic field global topology. It is our opinion that the main mechanism of collisionless reconection is ion and electron thermal drift. The reconnection time scale can be estimated as the characteristic thermal drift time, e.g., Bohm time $\tau \approx L/v_{dr} \approx L^2 eB/cT$. This problem has been studied [18, 19, 20] but not in the terms of frozen-in generalized vorticities, which is of great interest. The problem of helicity conservation on collisionless reconnection time scale remains unclear [5].

5 Conclusion

Small corrections to one-fluid MHD, namely the advection of magnetic field lines by plasma current, recovers two families of frozen-in integrals, which can be incompatible with the existence of smooth equilibria. The importance of these integrals increase with increasing ratio of current velocity to local Alfvén velocity. This may strongly affect plasma relaxation in confinement devices. For example, it was found [21] that the magnetic field advection by plasma current terminates the Z-pinch neck compression. Analogous effects strongly affect the operation of plasma opening switches [22, 23]. In tokamaks the current velocity is generally small and parallel to the invariant toroidal direction. However, near the plasma boundary the density rapidly decreases and the considered effects may be important. Our analysis suggests that electron inertia leads to a small change in the frozen-in field, whereas the field line reconnection of generalized electron vorticity is still forbiden. Thermal motion of both electrons and ions that destroy the frozen-in laws for generalized vorticities and allows reconnection. The approach based on Poincaré's relative integral invariant has a methodological importance, because it simplifies arguments and allows one to find frozen-in laws for new systems.

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