DOI/ET-53088-610 IFSR #610

Concept of Statistical Attractor for Solitons in Nonintegrable Hamiltonian Wave Systems

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June 1993

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Abstract

Soliton, or solitary wave, is a particular solution even among stationary solutions of nonlinear wave equations. The special cases of solutions represent a nontypical behavior. Particular exact solutions cannot be taken as representative solutions unless they are *attractors*. The new unusual feature of solitons in nonintegrable systems is that they are the *Hamiltonian* attractors.

How can it be seen? First, and most important, is that a soliton realizes min or max of energy under the conservation of others invariants. Energy is an invariant and can not relax to an extremum. The energy relaxes to a min only in macroscopic degrees of freedom. This means that we must consider the *thermodynamics* of nonlinear waves.

The classical thermodynamics are science for ordinary Hamiltonian differential equations. We attempt to extend this approach for *partial* differential equations. The fundamental problem here is the *absence of measure* in a functional space. Fortunately, the solutions of wave equations can be presented as a sum of solitons and weakly interacting linear waves. This model of the two thermodynamic phases leads to the concept of the statistical attractor for Hamiltonian nonintegrable partial Hamiltonian equations. Stable solitons are attractors for wave equations. Thermodynamics give an adequate language for this purpose. Equilibrium between solitons and free waves is calculated analytically. There is a numerical evidence confirming the theoretical predictions.

1 Introduction

The Hamiltonian equations of strong turbulence are nonintegrable. Only the presence of attractors simplifies the analysis and clarifies the situation. It is known how useful simple

*To be presented at the International Workshop on Transport, Chaos and Plasma Physics, Marseille, France, 5-9 July 1993.

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and strange attractors for the investigation of nonlinear dissipative dynamics are [1]. On the other hand, Hamiltonian systems have no attractors, because they are prohibited by the conservation of phase volume (Liouville theorem). It is important to note, however, that dissipative equations appear in physics as an approximate description of many degrees of freedom, where only few energetic degrees of freedom are considered explicitly, and the phase volume is hidden in the remaining degrees. For partial differential Hamiltonian equations the number of the degrees of freedom is very large, they can be considered as large Poincare systems [2] with nonintegrable and irreversible behaviour and, perhaps, statistical attractors.

Here we will consider the Hamiltonian equations of nonlinear waves possessing several conservation laws: the energy E, the momentum P, and, sometimes, the wave action N. If the conservation laws allow wave collapses, they generally take place. A description of collapses for nonlinear systems where the collapses exist is presented in review [3]. Otherwise, stable solitons exist, long-time evolution without dissipation is possible, and a thermodynamic description is appropriare.

All conservation laws influence the thermodynamic equilibria. For linear waves the Gibbs-Boltzmann distribution takes place. If the Fourier representation for wave action n_k is used, then

$$f = \exp\left(-\frac{E_k}{T} - \frac{P_k}{T_p} - \frac{n_k}{T_n}\right) = \exp\left(-\frac{(\omega_k + \omega_0 + v_0 k)n_k}{T}\right), \tag{1}$$

where ω_k is the wave frequency and T, ω_0, v_0 are thermodynamic constants. We have three constants in accordance with the number of conservation laws. A dozen of conservation laws would lead to a dozen of constants. Consider the mean value of the wave action at a given wavenumber N_k

$$N_k \equiv \frac{\int n_k f(n_k) dn_k}{\int f(n_k) dn_k}. \tag{2}$$

Then the modified Rayleigh-Jeans distribution for the linear waves is

$$N_k = \frac{T}{\omega_0 + \omega_k + v_0 k}. \tag{3}$$

For example, consider high-frequency plasma waves in the frame where the mean momentum is zero, $v_0 = 0$. If the denominator is zero, $\omega_0 + \omega_k = 0$, then $N_k = \infty$. This is impossible for positive frequency ω_0 and for linear waves which have positive frequencies. However, this is possible for soliton, because the soliton frequency shift,

$$\omega_{nl} \equiv dE/dN, \tag{4}$$

is negative. If

$$\omega_{nl} = -\omega_0, \tag{5}$$

then the waves condensate in a soliton. This is the principal quantitative result of this paper, based upon the thermodynamic equilibrium of the two phases: solitons (drops) and free waves (vapour), and the condition of maximum entropy.

The following are confirmation, interpretation, and instruction for user. Below we present interpretation and numerical evidence of the discussed effect and provide a general recipe of dealing with nonintegrable Hamiltonian systems.

2 Thermodynamic description of nonlinear waves

Thermodynamics of linear waves were developed for the problem of black body radiation. A classical field has an infinite number of degrees of freedom and the energy is infinite at equilibrium (the ultraviolet catastrophe). This paradox was solved by quantum mechanics, but we consider classical fields and need to cut the spectra on short scale. The other distinction is that we consider nonlinear wave and wave-wave interaction instead of wave-particle interaction. For weak wave turbulence kinetic description of spectra is possible, this direction was developed by Zakharov *et al.* [4]. Sometimes these spectra describe thermodynamic equilibria.

The concept of equipartition does not exist for a functional space but can be introduced by a finite-dimensional approximation. Both Fourier components and Wiener measure for functional integral can be used [5, 6, 7]. Meiss and Horton [6] considered a soliton gas model, Tasso [7] considered linearized model. Most papers are concerned with the general theory of canonical Gibbs distribution with the application to correlation functions and spectra [8, 9, 10, 11].

Here we will concentrate on the attracting properties of solitons or, more exactly, solitary waves. These properties are absent from integrable equations [12, 13, 14], but we will use the term "soliton" because this is essentially the same solution, which gets attracting properties in the consequence of non-integrability caused by small Hamiltonian perturbations. The approach taken in [15, 16, 17, 15, 18] differs from other papers by the explicit use of two-phase model, based on the small ratio of the amplitude of linear waves to the amplitudes of solitons. This assumption is confirmed by result. The thermodynamic approach includes the following steps:

1. Writing equations in a Hamiltonian form.

2. Hypothesis of equipartition on the hypersurface specified by all conservation laws. Liouville's property of Hamiltonian systems, long time of evolution, and non-integrability are the background of the equipartition.

3. The equipartition gives $F = const$ for total distribution function. Integration over all coordinates but several leads to a several-particle distribution, namely the Gibbs-Boltzmann distribution.

4. Analyzing the Gibbs-Boltzmann distribution for a particular system.

In most cases it is sufficient to analyse item 4 only. However, classical thermodynamics are developed for ordinary differential equations (ODE), not for partial (PDE) ones. Introducing measure for PDE by a finite-dimensional approximation is a principal difference from ODE. For this reason we dwell below upon all four steps.

2.1 Hamiltonian form of equations

Wave equations can be presented in different Hamiltonian forms. For linear waves Fourier representation is more adequate and we will use canonical variables (n_k, α_k) coresponding to the action and the angle for linear case. In addition, the Fourier representation gives an effective finite-dimensional approximation. For three-wave interaction the Hamiltonian is

$$H = \sum n_k \omega_k + \sum G_{k_1 k_2} \sqrt{n_{k_1} n_{k_2} n_{k_3}} \sin(\alpha_{k_1} - \alpha_{k_2} - \alpha_{k_3}) \tag{6}$$

For example, one dimensional Zakharov equations citeZakharov84,

$$iE_t + E_{xx} = nE, \quad n_{tt} - n_{xx} = |E^2|_{xx}, \tag{7}$$

can be presented in this form. These equations describe coupled plasma (Langmuir) and acoustic waves. For stationary solutions they lead to the well-known soliton of nonlinear Schrödinger (NLS) equation and give a simple model considered in [19, 17, 18].

2.2 Liouvillianity and mixing

Canonical form of Hamiltonian equations leads to the conservation of the phase-space volume (Liouville theorem). Stable solitons and wave collapses are alternatives [3], which gives a possibility for long evolution of a soliton system. The danger of ultraviolet catastrophe depends on particular spectra of weak turbulence [4, 10, 11]. For one-dimensional case within the framework of weak turbulence a cascade into small scales is forbidden. Indeed, a plasma wave can decay into a plasma and an acoustic wave, the second plasma wave can also decay, and so on until a wave is obtained that cannot decay into anything. Reverse coalescenses do not introduce new waves into this system, and the system finally decays into a set of noninteracting subsystems (Fig. 1).

Allowing for nonlinear frequency shifts in solitons leads to the coupling of the subsystems. If, however, the wavelengths are several times shorter than the soliton size, the spectra are cut off. The higher order nonlinear processes lead to a slow cascade which can be neglected. Zakharov equations are nonintegrable, an therefore present a typical large Poincare system amenable to a thermodynamic approach [2].

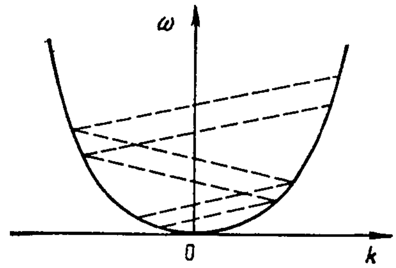
2.3 Equipartition and the Gibbs-Boltzmann formula

Consider linear waves assuming the equipartition on the hypersurface of energy $\epsilon = \sum \epsilon_k = \sum n_k \omega_k$ and introduce the single-particle distribution

$$f(\epsilon_k) = v(\epsilon_k), \tag{8}$$

where $v(\epsilon_k)$ is the area of the hypersurface defined by the condition

$$\epsilon = N_0 T - \epsilon_k, \tag{9}$$



Decay of a system of one-dimensional waves into noninteracting subsystems (two subsystems are shown): solid line – Langmuir-waves dispersion curve; dashed line – ω and *k* for sound waves.

Figure 1: Noninteracting subsystems of plasma and ion acoustic waves.

where N_0 being the number of harmonics. The shapes of the hypersurface for different ϵ_k are similar, only the scale along each axis is decreased by a factor of $(1-\epsilon_k/N_0T)$. Hence, using the celebrated Euler formula

$$f(\epsilon_k) \propto (1-\epsilon_k/N_0T)^{N_0} \longrightarrow \exp(-\epsilon_k/T), \tag{10}$$

as $N_0 \longrightarrow \infty$, we obtain the celebrated Gibbs-Boltzmann formula. The exponential is the result of "a linearization in multidimensional space". The number of harmonics N_0 has dropped out of the result. In the general nonlinear case we can not act in this simple manner, but the answer also involves an exponential, with T^{-1} the coefficient of linearization. Since the number of degrees of freedom is large and is not significantly decreased by additional integrals of motion, the latters can be taken into account independently.

2.4 Consequence of the Gibbs-Boltzmann distribution

Now we use the particular features of plasma waves, qescribed by Zakharov equations. Using the same reasoning as for the energy, we take into account the momentum P and the wave action N of the waves,

$$f(n_k) = \exp-(\epsilon_k/T + p_k/T_P + n_k/T_N) = \exp[-(\omega_k + kv_0 + \omega_0)T^{-1}]n_k, \tag{11}$$

where T, v_0, and ω_0 are three thermodinamic constants in accordance with the number of conservation laws. The RHS is written for linear waves. Transforming to the average wave action,

$$N_k \equiv \frac{\int n_k f(n_k) dn_k}{\int f(n_k) dn_k}, \tag{12}$$

for linear waves we have the modified Rayleigh-Jeans distribution

$$N_k = \frac{T}{\omega_0 + kv_0 + \omega_k}. \tag{13}$$

Since the wave action of acoustic waves is not conserved, we have for acoustic waves

$$N_k^{(ac)} = \frac{T}{kv_0 + \omega_k}. \tag{14}$$

Assume that the total momentum is zero, i.e. $v_0 = 0$. We have the dispersion relation of plasma waves $\omega_k = k^2$. If $\omega_0 > 0$, then N_k is finite and ω_0 is the characteristic frequency of the free waves,

$$N_k = \frac{T}{\omega_0 + k^2}. \tag{15}$$

If $\omega_k = -\omega_0$, then $N_k = \infty$. This is possible for a soliton, because soliton's frequency shift is negative. What does this mean? The following is the central part of our analysis. How to use the Gibbs-Boltzmann formula for nonlinear waves, i.e. for a complex dependence of energy on canonical variables? I do not know general methods to solve this problem. Fortunately, waves evolve to two slightly interacting phases, i.e. solitons plus linear waves, both easily described analytically (solitons in x-representation, linear waves in Fourier representation). A soliton is a structureless formation, fully described by a few integrals of motion, therefore, if the distance between solitons is much larger than their size, the number of soliton degrees of freedom is much less than the number of linear wave degrees of freedom ones, so that the soliton entropy can be neglected in comparison that in the linear waves. One can say that a soliton is a macroscopic degree of freedom, a condensate, the store of energy and wave action. The equilibrium maximizes the entropy of linear waves, while the dependence of the energy of linear waves on wave action is given by the soliton solution. For Zakharov equations the soliton solutions coincide exactly with the NLS solitons,,

$$E = k\sqrt{2}\exp(i\omega_{nl}t)\cosh^{-1}(kx), \tag{16}$$

where $\omega_{nl} = -k^2 \equiv d\epsilon/dN = -N^2$ is the nonlinear shift of frequency in solitons, the only parameter essential for an equilibrium. The expression

$$N_k = \frac{T}{\omega_0 + \omega_k} \tag{17}$$

has a pole at

$$\omega_{nl} = -\omega_0. \tag{18}$$

We have considered the thermodynamic equilibrium of the two phases: solitons (drops) and linear waves (vapour). The simple formula $\omega_{nl} = -\omega_0$ represents the condition of maximum entropy. An absorption of linear waves by soliton decreases the wave action of linear waves and entropy, but heating of linear waves because of the negative frequency shift of soliton increases the entropy.

This can also be seen from the Boltzmann exponential. Consider the phase volume of linear waves, (entropy $s \propto \ln V$),

$$V = \exp \frac{\epsilon(N) + N\omega_0}{T} \tag{19}$$

The condition of the maximum of entropy is $d\epsilon/dN + \omega_0 = 0$. Here we the use conservation of energy, namely the variation of the linear wave energy plus the variation of the soliton energy is zero.

The knowledge of the existence of two different phases is useful even far from equilibrium. For example, a typical weather humidity 40—80%, is pretty far from the equilibrium 100%. Nevertheless, pedestrians distinguish between water and vapour very easily. So we may take into consideration only the solitons and the linear waves. (The stationary solutions of wave equation, knoidal waves, form an infinite family, but only the two solutions of this family show up under the evolution.)

In the absence of equilibrium the thermodynamics indicate the direction of processes: a soliton collision transfers part of the energy to the more intense soliton, which is accompanied by the emission of linear waves, while the solitons increase in intensity because their number decreases.

The thermodynamics of waves were developed not for the investigation of equilibria but rather for understanding the character of the evolution.

A similar analysis was done for acoustic-type equations [16] and for NLS with wave repulsion [20].

3 Simulations

The first simulation designed to check the predictions of nonlinear wave thermodynamics was done by Krylov [17]. He considered the "dew" effect, that is, merger of solitons without collisions, through evaporation of weaker soliton under the agitation of linear waves. The physical reason of the dew effect is found in the increase of the frequency shift (i.e., the binding energy) with the growth of solitons. The transfer of wave action from weaker to stronger solitons releases energy, heats linear waves, and thereby increases the entropy. Krylov simulated 1D Zakharov equations, where two nearly equal solitons plus linear waves were taken as an initial condition. Two additional small local attracting potentials were introduced in equations to fix the positions of the solitons and avoiding their collision merger. After several hundreds of acoustic periods the strong soliton eats the poor (Fig. 2).

A simulation on a finer grid is presented in [21]. The nonlinear Schrödinger equation,

$$i\psi_t + \nabla^2\psi = f(|\psi|^2)\psi, \tag{20}$$

was investigated for different nonlinear functions $f(|\psi|^2)$, when this equation is nonintegrable. Initial conditions were constant ψ plus a small noise. From two examples of evolution presented in Figs. 3 and 4, the attraction to a soliton is evident. Two-dimensional simulation

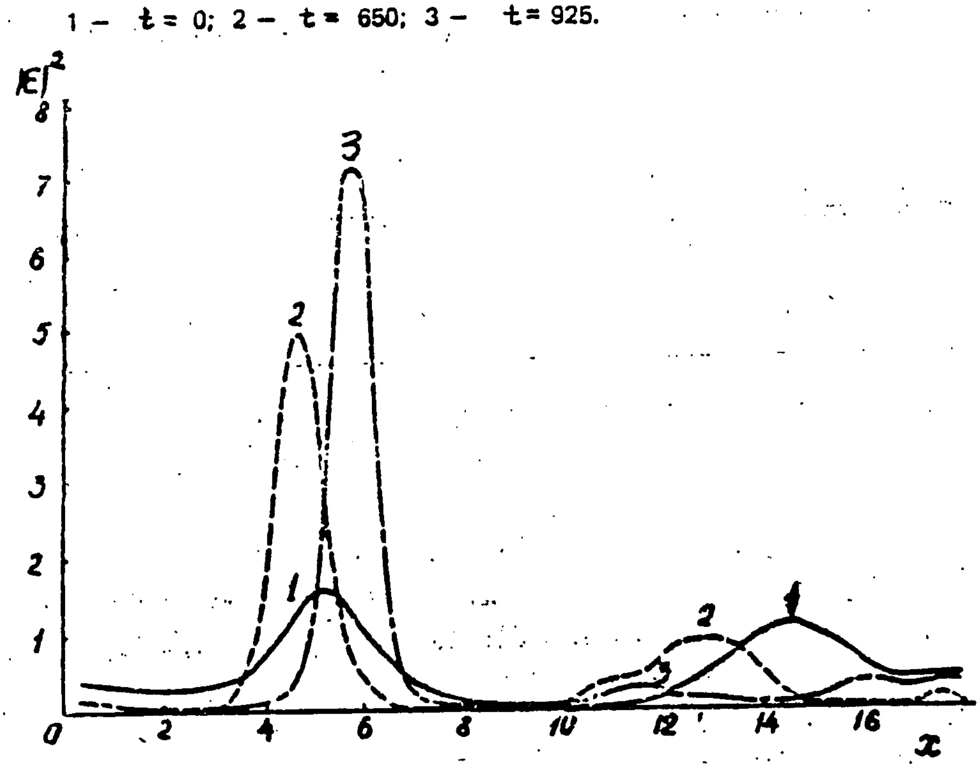


Figure 2: Fragment of the solution of the 1D Zakharov equations [17].

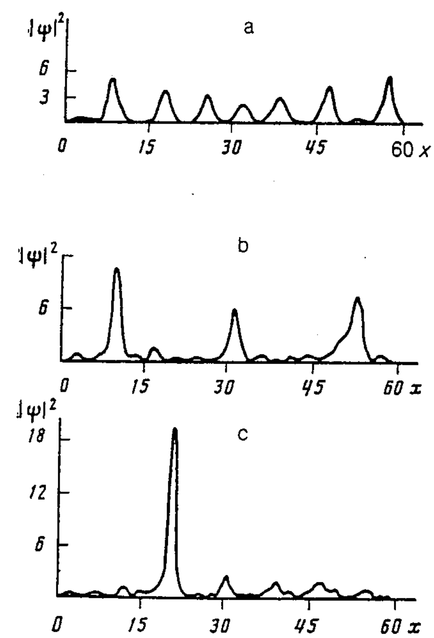


Figure 3: Fragment of solution of the 1D equation $i\psi_t + \psi_{xx} = |\psi|\psi$ [21].

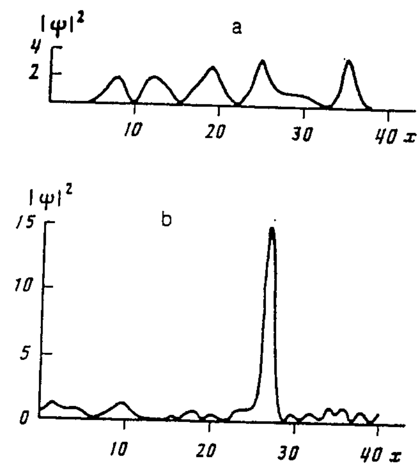


Figure 4: Fragment of the solution of the 1D NLS equation $i\psi_t+\psi_{xx} = |\psi|^2\psi(1+0.1|\psi^2|)/(1+0.5|\psi^2|)$, [21].

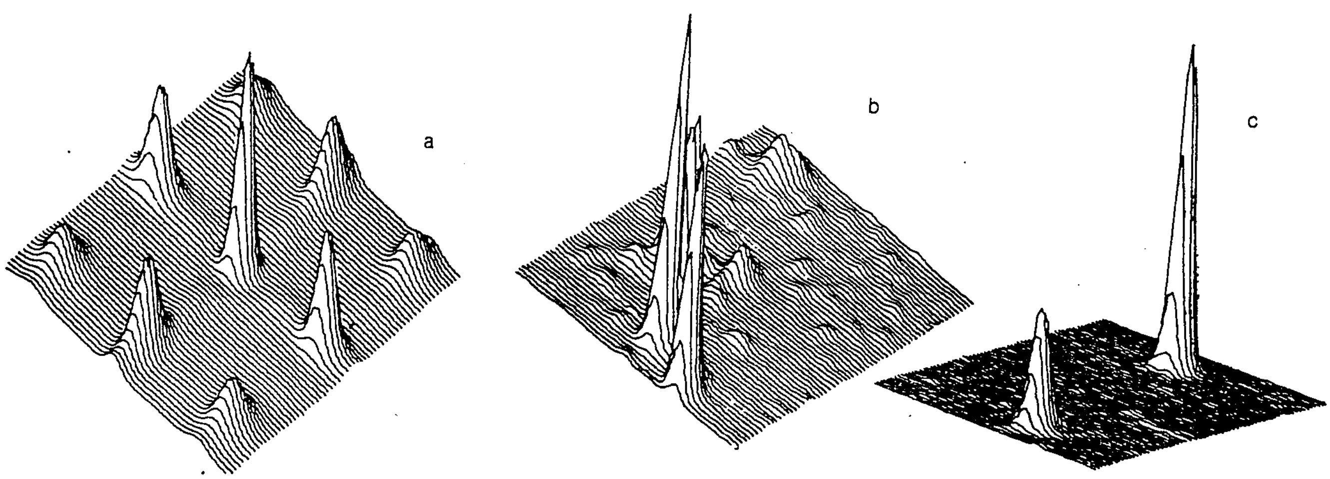


Figure 5: Fragment of the solution of the equation $i\psi_t + \nabla^2\psi = |\psi^{0.5}|\psi$ [22].

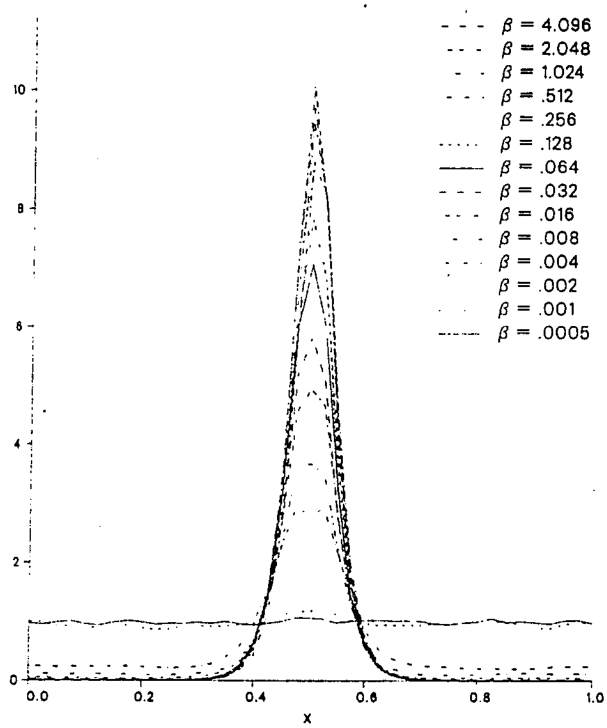


Figure 6: Typical solution of the 1D NLS [23].

leads to analogous results, as shown in fig. 5, [22]. Thermodynamics of NLS were introduced analytically in [23] using the Wiener measure. Conservation of both the energy and the wave action was taken into account, the momentum was ignored. This work did not use the model of two phases (and, consequently, the small parameter) and analyzed equilibria numerically. The result presented in Fig. 6 led the authors to conclude that

> ...one might suspect the existence of measures not satisfying translation invariance, corresponding to concentration of the measure on fields near a particular soliton-like structure.

This means that the small parameter was observed numerically.

The Fourier measure of this paper and the Wiener measure based on a lattice version of the ensemble [22] led to the analagous results. Perhaps, result will be different for nonlinear generalizations of the lattice in Fig. 7, which is an approximation of linear acoustic waves. The masses of the points and the distances between the points on the right of the lettice are one half those on to the left. This means that the sound velocity is the same, but the energy equipatition leads to twice larger concentration of energy to the right of the lattice. The long scale noise on both sides are the same. The thermodynamic behaviour for nonlinear case is not known. Further computer work for this situation is needed.

4 Instruction for user

If you want to check your Hamiltonian wave equations with respect to attracting properties of solitons, you should:

1. Write the integrals of motion.

Figure 7: Acoustic waves in lattice.

2. Find out whether stable solitons or collapse take place [3].

3. Check nonintegrable behaviour.

   Items 1 through 3 are sufficient for the existence of an attractor. If you want to describe the attractor, then

4. Write the distribution of linear waves in Fourier representation, Eq. (3).

5. Analytically in r-representation or numerically find the soliton.

6. For this soliton calculate the dependence of the energy on others integrals and use Eq. (4) as in equation similar to (5).

5 Future directions

The concept of strange attractors was brought about by computational science. For statistical attractors the same methods are useful. The observation of the attraction of solutions to solitons for different wave systems is the first step of simulations. Others steps are:

- Investigation of the rate of attracting, for example, for small Hamiltonian perturbation integrable equations (KdV, etc.).

- Numerical check of thermodynamic spectra.

- Investigation of the fluctuation in the amplitude and the velocity of solitons.

A subtle subject is using of finite-dimensional approximations. Measures of Fourier, Wiener, and particle measure may give different results. For example, the thermodynamics introduced by point 2D vortices depend on the strength of vortices. The exists an evidance that localized vortices emerge in turbulent relaxation, [24], analogously to solitons [15].

6 Place in the Nonlinear Sun

The theory of strong turbulence is a part of nonlinear physics. The three "religious rules" of nonlinear physics present a heuristic viewpoint that can be used to qualitatively predict the evolution of nonlinear systems. These rules are as follows.

1. The basic results can be obtained from the conservation laws. If some kind of process is not forbidden by these laws, it generally occurs. If it doesn't, this means that we are missing another conserved quantity imposing the constraint.

2. The universal law of "20/80" takes place: 20% of people drink 80% of beer. In other words, interesting processes usually take place in localized structures occupying a small share of volume. The localized structures interact weakly and therefore maintain their identity. For this reason they are universal and can be investigated.

3. The "general situation" is nonintegrable. The special cases of exact solutions in integrable models represent a degenerate (nontypical) behavior. Particular exact solutions cannot be taken as representative solutions unless they are attractors. The presence of attractors simplifies the analysis and clarifies the situation.

In plasma and fluid physics we deal with infinite-dimensional (PDE) systems distributed in space. The application of religious rules 1 and 2 then leads to the following. If the conservation laws do not prohibit the development of singularities they do occur. If the singularities are prohibited, then stable localized structures take place.

We were mostly concerned with the case of a finite number of conservation laws and stable structures (solitons or solitary waves). The case of an infinite number of conservation laws is typical for 2D hydrodynamics [25, 26, 27, 24, 28, 29], where localized vortices are examples of stable structures. Wave collapse [3], wavebreaking, shock waves represent the case of the finite number of conservation laws with fallwed singularities. The case of an infinite number of cognservation laws and singularities can be illustrated on magnetic reconnection [30] and singularities in ideal liquid [31, 17]. Perhaps, works on the appearance of current sheets [32, 33] can be used as a bridge between thermodynamics of systems with finite (waves) and infinite (vortices) integrals of motion.

It is interesting to see how solitons appearing as solutions of integrable equations [13, 14] become even more essential in general, nonintegrable, case.

Acknowledgements

I am grateful to M. Isichenko for help and stimulating suggestions. This work was supported, in part, by a Soros Foundation Grant awarded by the American Physical Society. I acknowledge the hospitality of the Institute for Fusion Studies, where this work was completed. The visit to the IFS was supported by the U.S. Department of Energy.

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