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Non-Radiative Collisions of Langmuir Solitons

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Abstract

Collisions of Langmuir solitons are described in terms of equations of motion for equivalent point particles. The description is valid in the limit when the eigenfrequencies of bound plasmons are much higher than the inverse interaction time, and the velocities of the solitons are much less than the ion acoustic velocity. It is shown that the velocities of the solitons do not change due to binary collisions, but they generally change when more than two solitons collide simultaneously.

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I Introduction

There is a widespread trend in studies of one-dimensional Langmuir turbulence to describe a strongly turbulent plasma as an ensemble of interacting plasmons, ion acoustic waves and solitons.[1, 2, 3, 4] Although many authors use this approach on a qualitative level, no accurate formal procedure has been developed that would allow the reduction of the basic nonlinear equations for the coupled modes to equations of motion for interacting quasiparticles. The major difficulty in implementing such a reduction arises from the necessity of identifying weakly interacting objects in a strongly nonlinear system. This usually requires having some small physical parameters, in addition to small wave amplitudes.

In this paper, we will present an example of a situation in which solitons can actually be consistently described as interacting quasiparticles. The simplifying small parameters in our problem are associated with the existence of different time scales corresponding to "fast" and "slow" degrees of freedom. A natural tool for handling such problems is the *adiabatic approximation.* We will apply the idea of adiabaticity in two forms. First, we assume that the evolution of a soliton is slow compared with the bounce period (the inverse eigenfrequency) of the bound plasmons, so that bound plasmons adjust adiabatically to the changing shape of the soliton with no state-to-state transitions. Second, we restrict our consideration to the case of slow solitons whose velocities are much less than the ion acoustic velocity. This means that colliding solitons actually do not radiate ion-acoustic waves, for the same reason that colliding nonrelativistic charged particles do not produce substantial electromagnetic radiation. However, unlike charged particles, solitons do not interact at a distance; the interaction only occurs when solitons overlap. This brings up the problem of how to introduce coordinates for the overlapping objects of finite size and evolving shape.

The solution for this problem follows from the results presented in Ref. 5 in which multisoliton stationary solutions (compound solitons) were constructed with arbitrary distances between the components. We will show that the coordinates we need are nothing else than the free parameters of the compound solitons. By selecting these coordinates and using the adiabatic approximation we reduce the Lagrangian of the continuous system of interacting modes to a Lagrangian of the finite number of point particles, each of them corresponding to a soliton. We then apply energy and momentum conservation laws to analytically solve the problem of binary collisions. Binary collisions are shown to be trivial: they do not change the velocities of the solitons. We also present a numerical solution for a triple collision, which demonstrates that, generally, the velocities of the solitons do change due to the interaction. This suggests the idea of extending the technique described here to the studies of kinetic phenomena in a gas of solitons.

II Basic Equations

We will start from the one-dimensional Zakharov equations[6] that describe nonlinear coupling between high frequency Langmuir waves and low frequency ion-acoustic waves. The coupling is due to 1) the ponderomotive force and 2) the dependence of the electron plasma frequency on density perturbations. We write these equations in the following dimensionless form:

$$\frac{\partial^2 n}{\partial t^2} - \frac{\partial^2 n}{\partial x^2} = \frac{\partial^2}{\partial x^2} |E|^2 \tag{1}$$

$$ig \frac{\partial E}{\partial t} + \frac{\partial^2 E}{\partial x^2} = nE , \tag{2}$$

where n is the perturbed plasma density and E is the amplitude of the high frequency electric field. The parameter g, which is herein assumed to be small, measures the ratio of the acoustic frequency (at a wavelength equal to the typical soliton size) to the eigenfrequency of the bound plasmons. The condition that g is small means that the "potential energy" n in the Schrödinger equation (2) is a slowly varying function of time compared with the oscillating time-dependent eigenfunctions of the bound plasmons. Therefore, the solution of Eq. (2) can be written in adiabatic form; i.e.

$$E = \sum_j e^{i\kappa_j^2 t/g} E_j \ ,$$

where E_i is a slowly varying adiabatic eigenfunction and κ_i^2 is an eigenvalue of the equation

$$-\kappa_i^2 E_i + \frac{\partial^2 E_i}{\partial x^2} = n E_i \ , \tag{3}$$

in which all quantities depend on time parametrically through n. In the adiabatic limit, Eq. (2) conserves the quantities

$$N_i = \int |E_i|^2 dx \ , \tag{4}$$

which represent the occupation numbers of the adiabatic states of plasmons. This shows that transitions of plasmons from state to state are forbidden in this limit.

In what follows we assume that all the populated states belong to the discrete spectrum. With g being small, the characteristic beat frequencies between different eigenmodes appear to be much larger than the acoustic frequency. We therefore neglect rapidly oscillating cross-terms in the right-hand side of Eq. (1) and only retain the diagonal terms which create the major ion response. Then Eq. (1) takes the form[5]

$$\frac{\partial^2 n}{\partial t^2} - \frac{\partial^2 n}{\partial x^2} = \frac{\partial^2}{\partial x^2} \sum_{i=1}^{\nu} |E_i|^2 \ , \tag{5}$$

where ν is the total number of populated states.

A particular solution of the system (3) and (5) is a single soliton moving with a constant velocity s:

$$n = -\frac{1}{1-s^2} |E|^2 \ ,$$

$$E = \frac{\kappa\sqrt{2(1-s^2)}}{\cosh(\kappa(x - st - x_0))} \ . \tag{6}$$

As we restrict ourselves to considering only slow solitons with $s \ll 1$, we neglect the s^2 contributions to formula (6), and rewrite this formula in the following simplified form:

$$n = -|E|^2 ,$$

$$E = \frac{\kappa\sqrt{2}}{\cosh(\kappa(x - x^*))} , \tag{7}$$

where the position of the soliton x^* changes linearly with time, as the soliton moves with a constant speed $s \ll 1$. Note that, in this approximation, the width of the soliton (defined by the parameter κ) is uniquely related to the number of plasmons:

$$N = 4\kappa .$$

A trivial generalization of the solution (6) is a set of several solitons with different values of κ and different velocities. This generalization is good as long as solitons do not overlap, which of course is not the case when solitons collide.

To quantitatively describe collisions, we use the following approach: To zeroth order, we neglect second-time derivative in Eq. (5), since all the solitons under consideration are very slow. To this order, at any moment of time during the collision, the profiles of n and E_i must satisfy the following set of equations:

$$-\frac{\partial^2 n}{\partial x^2} = \frac{\partial^2}{\partial x^2} \sum_{i=1}^{\nu} |E_i|^2 \tag{8}$$

$$-\kappa_i^2 E_i + \frac{\partial^2 E_i}{\partial x^2} = n E_i . \tag{9}$$

It was shown in Ref. 5 that, for any given number of occupied states ν with given occupation numbers N_i, Eqs. (8), (9) have a multisoliton solution with ν free constants. We choose these free parameters to be generalized coordinates of the solitons. We then consider these coordinates to be slowly varying functions of time so that second-time derivative in Eq. (5) can be treated as a perturbation. From the perturbed equations, one can find a small correction to the multisoliton solution and (what is more important) the solvability conditions for the first order problem. These conditions can be formulated as the equations of motion for the solitons. The procedure described above is actually an asymptotic expansion based on the small parameters g and s.

III Generalized Coordinates

We now introduce generalized coordinates in a formal way. Similar to a single solution, a multisoliton solution corresponds to a non-reflective potential well in the Schrödinger equation (5). These potentials are known from the solution of the inverse scattering problem.[7, 8, 9] The corresponding profile of the perturbed plasma density has the form

$$n = -2\frac{d^2}{dx^2}\log\left(\det\left(\mathbf{I}+\mathbf{C}\right)\right) , \tag{10}$$

where $\mathbf{I}$ is the unit matrix and $\mathbf{C}$ is a symmetric matrix defined as

$$\mathbf{C}_{ik} = \frac{C_i C_k}{\kappa_i + \kappa_k} \exp\left(-\left(\kappa_i + \kappa_k\right)x\right) .$$

Here, C_i are free parameters related to the positions of the solitons. The normalized eigenfunctions of the bound states in the potential (10) satisfy a set of linear algebraic equations:

$$\varphi_i(x) + \sum_{k=1}^{\nu} \varphi_k(x)\frac{C_i C_k}{\kappa_i + \kappa_k} \exp(-(\kappa_i + \kappa_k)x) = C_i \exp(-\kappa_i x) . \tag{11}$$

It is also interesting to note that potential (10) can be presented as a combination of squares of its eigenfunctions[?]:

$$n(x) = -4\sum_{i=1}^{\nu} \kappa_i \varphi_i^2(x) .$$

Equations (10) and (11) generalize the trivial solution for a set of the well-separated solitons to the case when the solitons can overlap and lose their individual profiles.

We choose the parameters C_i to be generalized coordinates of the solitons. These parameters can be expressed through soliton positions that are well defined for the separated solitons. The positions obviously become uncertain when solitons overlap. Nevertheless, the quantities C_i do not have any uncertainty. Therefore, by following the time behavior of C_i, one can relate final positions and velocities of the solitons to those before collision.

In order to find C_i for well-separated solitons we introduce new variables x_i^* so that

$$C_i = \sqrt{2\kappa_i} e^{\kappa_i x_i^*} \ . \tag{12}$$

For a single soliton ($\nu = 1$), x_i^* is exactly the position. For $\nu > 1$, the position of i-th soliton, as we will see, may differ from x_i^* by a certain shift, which is roughly of the order of the soliton width. To determine the shift more precisely we assume, without loss of generality, that

$$x_1^* < x_2^* \cdots < x_n^* \tag{13}$$

and also that

$$|x_i^* - x_k^*| \gg \max\left(\frac{1}{\kappa_i}; \frac{1}{\kappa_k}\right) \tag{14}$$

for all $i \neq k$. The latter inequality is the separation condition for the solitons.

It follows from Eqs. (13) and (14) that, in the vicinity of x_j^*, we have $C_{ii} \gg 1$ for $i < j$ and $C_{ii} \ll 1$ for $i > j$. These inequalities allow us to simplify matrix $\mathbf{I} + \mathbf{C}$ as

$$\mathbf{I} + \mathbf{C} =$$

$$\begin{pmatrix} e^{-2\kappa_1(x-x_1^*)} & .. & \frac{2\sqrt{\kappa_1\kappa_j}}{\kappa_1+\kappa_j} e^{-\kappa_1(x-x_1^*)-\kappa_j(x-x_j^*)} & .. & \frac{2\sqrt{\kappa_1\kappa_n}}{\kappa_1+\kappa_n} e^{-\kappa_1(x-x_1^*)-\kappa_n(x-x_n^*)} \\ .. & .. & .. & .. & .. \\ \frac{2\sqrt{\kappa_1\kappa_j}}{\kappa_1+\kappa_j} e^{-\kappa_1(x-x_1^*)-\kappa_j(x-x_j^*)} & .. & 1 + e^{-2\kappa_j(x-x_j^*)} & .. & \frac{2\sqrt{\kappa_j\kappa_n}}{\kappa_j+\kappa_n} e^{-\kappa_j(x-x_j^*)-\kappa_n(x-x_n^*)} \\ .. & .. & .. & .. & .. \\ \frac{2\sqrt{\kappa_1\kappa_n}}{\kappa_1+\kappa_n} e^{-\kappa_1(x-x_1^*)-\kappa_n(x-x_n^*)} & .. & \frac{2\sqrt{\kappa_n\kappa_j}}{\kappa_n+\kappa_j} e^{-\kappa_j(x-x_j^*)-\kappa_n(x-x_n^*)} & .. & 1 \end{pmatrix} . \tag{15}$$

By taking out common multipliers from rows and columns with numbers less than j, and by neglecting exponentially small off-diagonal terms in rows and columns with numbers larger than j, we find that the determinant of this matrix is

$$\det(\mathbf{I} + \mathbf{C}) = \left(\prod_{i=1}^{j-1} e^{-2\kappa_i(x-x_i^*)}\right)$$

$$\det\begin{pmatrix} 1 & \frac{2\sqrt{\kappa_1\kappa_2}}{\kappa_1+\kappa_2} & \cdot\cdot & \frac{2\sqrt{\kappa_1\kappa_j}}{\kappa_1+\kappa_j}e^{-\kappa_j(x-x_j^*)} & 0 & \cdot\cdot & 0 \\ \frac{2\sqrt{\kappa_1\kappa_2}}{\kappa_1+\kappa_2} & 1 & \cdot\cdot & \frac{2\sqrt{\kappa_2\kappa_j}}{\kappa_2+\kappa_j}e^{-\kappa_j(x-x_j^*)} & 0 & \cdot\cdot & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{2\sqrt{\kappa_1\kappa_j}}{\kappa_1+\kappa_j}e^{-\kappa_j(x-x_j^*)} & \frac{2\sqrt{\kappa_2\kappa_j}}{\kappa_2+\kappa_j}e^{-\kappa_j(x-x_j^*)} & \cdot\cdot & 1+e^{-2\kappa_j(x-x_j^*)} & 0 & \cdot\cdot & 0 \\ 0 & 0 & \cdots & 0 & 1 & \cdot\cdot & \cdot\cdot \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdot\cdot & 0 & 0 & \cdot\cdot & 1 \end{pmatrix} \tag{16}$$

Because of the structure of Eq. (10), the common exponential multiplier in this expression does not contribute to the form of the density profile. We now substitute Eq. (16) into Eq. (10) to obtain

$$n = -2\frac{d^2}{dx^2}\log\left(1 + A_j\, e^{-2\kappa_j(x-x_j^*)}\right) \ , \tag{17}$$

where A_j depends on $\kappa_1 \ldots \kappa_{j-1}$ but not on x. Equation (17) describes a soliton centered at

$$x_j = x_j^* + \log(A_j)/(2\kappa_j) \ . \tag{18}$$

The explicit expression for A_j can be found by calculating the determinant in Eq. (16). In the case of two solitons we obtain:

$$A_1 = 1 \ ; \qquad A_2 = \left(\frac{\kappa_1-\kappa_2}{\kappa_1+\kappa_2}\right)^2 \ . \tag{19}$$

For three well-separated solitons, the result is

$$A_1 = 1 \ ; \ A_2 = \left(\frac{\kappa_1-\kappa_2}{\kappa_1+\kappa_2}\right)^2 \ ; \ A_3 = \frac{(\kappa_1-\kappa_3)^2(\kappa_2-\kappa_3)^2}{(\kappa_1+\kappa_3)^2(\kappa_2+\kappa_3)^2} \ . \tag{20}$$

Equations (12) and (18) and quantities A_i give the required relationship between the generalized coordinates C_i and the positions of the separated solitons. Also, these equations show that it is preferable to use x_j^* as generalized coordinates. This will make generalized velocities of the separated solitons $\dot{x}_j^*$ equal to their actual velocities $\dot{x}_j$. Therefore, in what follows, we will discuss the equations of motion for x_j^* rather than for C_j.

IV Equations of Motion

To derive the equations of motion for x_i^* we use the following procedure. First, we introduce a new function $\xi(t,x)$, which is the displacement of ions from their equilibrium positions. Hence, the perturbation of plasma density is

$$n = -\frac{\partial \xi}{\partial x} \, ,$$

and Eq. (5) takes the form

$$\frac{\partial^2 \xi}{\partial t^2} - \frac{\partial^2 \xi}{\partial x^2} = -\frac{\partial}{\partial x} \sum_{i=1}^{\nu} |E_i|^2 \, . \tag{21}$$

Second, we note that Eqs. (3) and (4) define a functional dependence of $|E_i|^2$ on the instantaneous density profile. This allows us to rewrite Eq. (21) as

$$\frac{\partial^2 \xi}{\partial t^2} = \widehat{A}(\xi) \tag{22}$$

where

$$\widehat{A}(\xi) \equiv \frac{\partial^2 \xi}{\partial x^2} - \frac{\partial}{\partial x} \sum_{i=1}^{\nu} |E_i|^2$$

is a time-independent operator (however the right-hand side of Eq. (22) depends on time parametrically through ξ).

In the case of slow solitons we can treat the left-hand side of Eq. (22) as a perturbation and seek the solution of this equation in the form:

$$\xi(t,x) = \xi_0(x, x_i^*(t)) + \xi_1(t,x) \, . \tag{23}$$

We assume that $\xi_1 \ll \xi_0$, where $\xi_0(x, x_i^*(t))$ is the solution of the zeroth order equation

$$\widehat{A}(\xi_0) = 0 \, , \tag{24}$$

which is given by

$$\xi_0 = 2 \frac{d}{dx} \log \det (\mathbf{I} + \mathbf{C}) \tag{25}$$

where

$$\mathbf{C}_{ik} = \frac{2\sqrt{\kappa_i \kappa_j}}{\kappa_i + \kappa_j} e^{-\kappa_i(x-x_i^*)-\kappa_j(x-x_j^*)} . \tag{26}$$

The first order terms in Eq. (22) give the following linear equation for ξ_1:

$$\frac{\partial^2}{\partial t^2} \xi_0 = \widehat{A}_L \xi_1 \tag{27}$$

where $\widehat{A}_L$ is linearized operator $\widehat{A}$, that depends on ξ_0.

It can be proved straightforwardly that $\widehat{A}_L$ is a self-adjoint operator. Also, by differentiating Eq. (24) with respect to x_i^*, and taking into account that ξ_0 satisfies this equation regardless of x_i^*, we conclude that

$$\widehat{A}_L \frac{\partial \xi_0}{\partial x_i^*} = 0 ,$$

i.e. functions $\frac{\partial \xi_0}{\partial x_i^*}$ are eigenfunctions of the operator $\widehat{A}_L$ with zero eigenvalues. Now, by integrating Eq. (27) over x with the weight functions $\frac{\partial \xi_0}{\partial x_i^*}$, we obtain the solvability conditions,

$$\int_{-\infty}^{+\infty} dx \frac{\partial \xi_0}{\partial x_i^*} \frac{\partial^2 \xi_0}{\partial t^2} = 0 , \tag{28}$$

which are actually the equations of motion for x_i^*. These equations can also be written as

$$\sum_{k=1}^{\nu} \ddot{x}_k^* \int_{-\infty}^{+\infty} dx \frac{\partial \xi_0}{\partial x_k^*} \frac{\partial \xi_0}{\partial x_i^*} + \sum_{k,l=1}^{\nu} \dot{x}_k^* \dot{x}_l^* \int_{-\infty}^{+\infty} dx \frac{\partial^2 \xi_0}{\partial x_k^* \partial x_l^*} \frac{\partial \xi_0}{\partial x_i^*} = 0 . \tag{29}$$

Note that Eq. (29) is derived in the small velocity limit $\left(\dot{x}_i^* \ll 1\right)$. However, by shrinking the time scale we can formally make $\dot{x}_i^* \approx 1$, which is convenient for solving the equation numerically. The physical solution can be obtained from our numerical results by an appropriate stretching of the time unit.

Another derivation of Eq. (29) is based on the minimum action principle for Eqs. (3) and (5). The corresponding Lagrangian has the form

$$L = \frac{1}{2} \int \left(\frac{\partial \xi}{\partial t}\right)^2 dx - \frac{1}{2} \int \left(\frac{\partial \xi}{\partial x}\right)^2 dx + \int \frac{\partial \xi}{\partial x} \sum_{i=1}^{\nu} |E_i|^2 dx - \int \sum_{i=1}^{\nu} \left|\frac{\partial E_i}{\partial x}\right|^2 dx , \tag{30}$$

wherein the following constraint is imposed on E_i:

$$\int |E_i|^2 dx = 4\kappa_i = \text{const} . \tag{31}$$

In accordance with the above described procedure, we choose a trial function ξ_0 given by Eq. (25) with x_i^* being the generalized coordinates that are to be determined from the minimum action principle. As before, $|E_i|^2$ and $\left|\frac{\partial E_i}{\partial x}\right|^2$ are functionals of ξ_0. It was shown in Ref. 5 that ξ_0 makes the second term in the Lagrangian independent of x_i^*. This remarkable feature is related to the fact that x_i^* are free parameters of the the solution of Eq. (24). Also, the sum of the last two terms in the Lagrangian does not depend on x_i^*, since if we multiply Eq. (3) by E_i^* and integrate over x, using Eq. (31), we obtain that

$$\int \frac{\partial \xi}{\partial x} |E_i|^2 dx - \int \left|\frac{\partial E_i}{\partial x}\right|^2 dx = 4\kappa_i^3 .$$

Hence, Lagrangian reduces to

$$L = \frac{1}{2} \sum_{i,k=1}^{\nu} \dot{x}_i^* \dot{x}_k^* \int_{-\infty}^{+\infty} \frac{\partial \xi_0}{\partial x_i^*} \frac{\partial \xi_0}{\partial x_k^*} dx . \tag{32}$$

The corresponding Euler-Lagrange equations coincide with Eq. (29). This equation can also be interpreted as the equation for the geodesic motion of a particle in a space of ν dimensions with a metric tensor of the form

$$g_{ij} = \int_{-\infty}^{+\infty} \frac{\partial \xi_0}{\partial x_i^*} \frac{\partial \xi_0}{\partial x_j^*} dx ,$$

where ν is the number of solitons.

As Lagrangian (32) is translationally invariant and does not have an explicit time dependence, we conclude that Eqs. (29) conserve the total momentum,

$$P = \sum_{i,k=1}^{\nu} \dot{x}_i^* \int_{-\infty}^{+\infty} \frac{\partial \xi_0}{\partial x_i^*} \frac{\partial \xi_0}{\partial x_k^*} dx , \tag{33}$$

and total energy of the solitons,

$$W = \frac{1}{2} \sum_{i,k=1}^{\nu} \dot{x}_i^* \dot{x}_k^* \int_{-\infty}^{+\infty} \frac{\partial \xi_0}{\partial x_i^*} \frac{\partial \xi_0}{\partial x_k^*} dx . \tag{34}$$

These conservation laws are sufficient for the analysis of binary collisions. In order to describe this particular case we introduce new variables,

$$\Delta = \frac{x_1^* + x_2^*}{2} ,$$

$$\delta = x_1^* - x_2^* ,$$

and rewrite Eqs. (33) and (34) in form the

$$P = \dot{\Delta} \int \left(\frac{\partial \xi_0}{\partial \Delta} \right)^2 dx + \dot{\delta} \int \frac{\partial \xi_0}{\partial \Delta} \frac{\partial \xi_0}{\partial \delta} \, dx , \tag{35}$$

$$W = \frac{\dot{\Delta}^2}{2} \int \left(\frac{\partial \xi_0}{\partial \Delta} \right)^2 dx + \frac{\dot{\delta}^2}{2} \int \left(\frac{\partial \xi_0}{\partial \delta} \right)^2 dx + \dot{\Delta}\dot{\delta} \int \frac{\partial \xi_0}{\partial \Delta} \frac{\partial \xi_0}{\partial \delta} dx . \tag{36}$$

By combining Eqs. (35) and (36) we obtain

$$W = \frac{1}{2 \int \left(\frac{\partial \xi_0}{\partial \Delta} \right)^2 dx} \left\{ P^2 + \dot{\delta}^2 \left[\int \left(\frac{\partial \xi_0}{\partial \Delta} \right)^2 dx \int \left(\frac{\partial \xi_0}{\partial \delta} \right)^2 dx - \left(\int \frac{\partial \xi_0}{\partial \Delta} \frac{\partial \xi_0}{\partial \delta} \, dx \right)^2 \right] \right\} . \tag{37}$$

It is clear from the structure of Eqs. (25) and (26) that

$$\left(\frac{\partial \xi_0}{\partial \Delta} \right)_{\delta;x} = - \left(\frac{\partial \xi_0}{\partial x} \right)_{\Delta;\delta}$$

and, using the fact that $\int \left(\frac{\partial \xi_0}{\partial x} \right)^2 dx$ is independent of the soliton coordinates, we conclude that

$$\int \left(\frac{\partial \xi_0}{\partial \Delta} \right)^2 dx = \text{const} .$$

In evaluating this integral we can assume, without loss of generality, that solitons are well separated. This gives

$$\int \left(\frac{\partial \xi_0}{\partial \Delta} \right)^2 dx = \frac{16}{3} \left(\kappa_1^3 + \kappa_2^3 \right) .$$

The other two integrals in Eq. (37) are independent of Δ but they generally depend on δ. Thus, Eq. (37) defines $\dot{\delta}$ as a function of δ. It should be noted that $\dot{\delta}$ does not change it's sign during the collision, which means that solitons pass through each other without reflection.

It follows from Eq. (37) that asymptotic values for $\dot{\delta}$ at $\delta \to -\infty$ and $\delta \to +\infty$ coincide, both of them being given by

$$\dot{\delta}_\infty^2 = \left(\frac{3}{8}\left(\kappa_1^3 + \kappa_2^3\right) W - \frac{9}{256} P^2\right) / \left(\kappa_1 \kappa_2\right)^3 .$$

Therefore, final velocities of the colliding solitons are equal to the initial velocities, although the velocities do not remain constant during the interaction.

V Three-Soliton Collision

When more than two solitons interact simultaneously, the conservation laws are insufficient to fully describe the collision. This situation is similar to point particle collisions in classical mechanics. In order to solve the problem of a three-soliton collision, we integrate Eq. (29) numerically. The issue is whether the final velocities of the solitons differ from the initial ones in this case , as opposed to binary collisions. The results presented in this section give a positive answer to this question.

We solve an initial value problem for (29) by using the fourth-order Runge-Kutta method. The integrals in (29) are evaluated on the spatial interval $[-10, 10]$, which provides sufficient accuracy, since the derivatives of ξ_0 are exponentially localized in space.

The binary collision presented in Fig. 1 illustrates the results of the previous section and allows us to test the numerical scheme. As shown in Fig. 1b, the velocities of the solitons change considerably during the interaction. However, the asymptotic values of the velocities are well conserved. Also, the total energy and momentum are conserved, with an accuracy of 10^{-4} (this refers to both two-soliton and three-soliton collisions).

An example of a three-soliton collision is shown in Figs. 2 and 3. Here, two of the three solitons are initially identical to those in Fig. 1. However, with the third soliton, the collision becomes nontrivial. The collision presented in Figs. 2 and 3 looks almost like a velocity exchange between the first and the second solitons in presence of the third party.

This only happens due to particular choice of initial conditions. Generally, the interaction changes the velocities of all three solitons.

It is very likely that multiple collisions in a gas of solitons will bring this gas to thermal equilibrium. An important feature of this process is that the relaxation time for a low density gas must be much longer than the inverse frequency of binary collisions. It would be rather interesting to derive kinetic equation for this relaxation process. Another interesting extension of the results presented here would be the accurate calculation of various thermodynamic functions for the gas of solitons in explicit form, at least in a low density limit when nonideal effects are relatively small.

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Figure Captions

1. Two-soliton collision. The inverse widths of the solitons are $\kappa_1 = 2.1$ and $\kappa_2 = 2.0$ for the first and second soliton, respectively. (a) Time evolution of soliton positions x_i^*. (b) Time evolution of soliton velocities $\dot{x}_i^*$.

2. Three-soliton collision. Inverse widths of the solitons are $\kappa_1 = 2.1$, $\kappa_2 = 2.0$, and $\kappa_3 = 1.9$ for the first, second, and third soliton, respectively. (a) Time evolution of soliton positions x_i^*. (b) Time evolution of soliton velocities $\dot{x}_i^*$.

3. Instanteneous density profiles during the three-soliton collision. Arrows indicate the directions of the soliton motion. (a) Initial profile. (b) Intermediate profile. (c) Final profile.

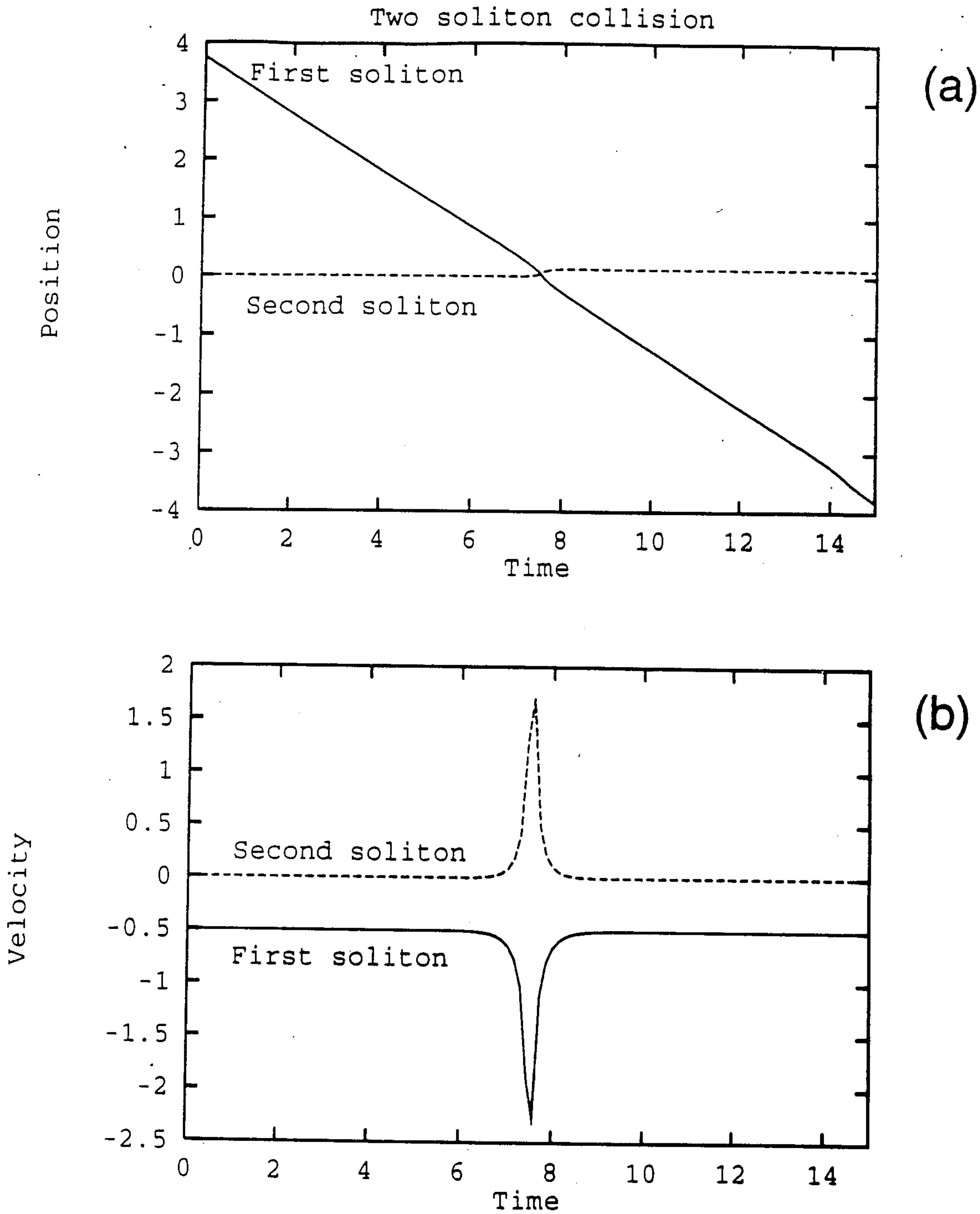

Two soliton collision
First soliton
Second soliton
Position
Time
(a)
Second soliton
First soliton
Velocity
Time
(b)

Figure 1

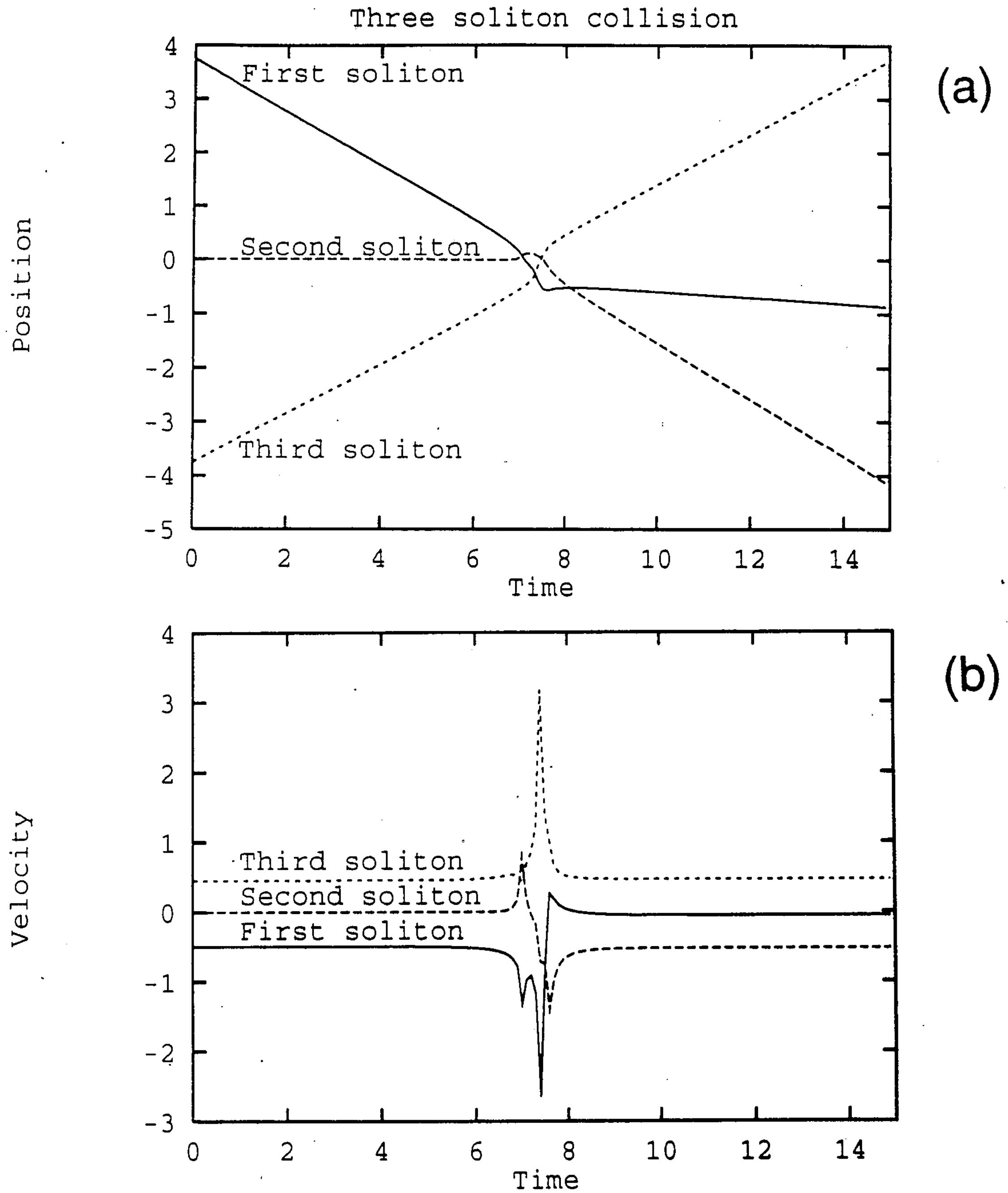
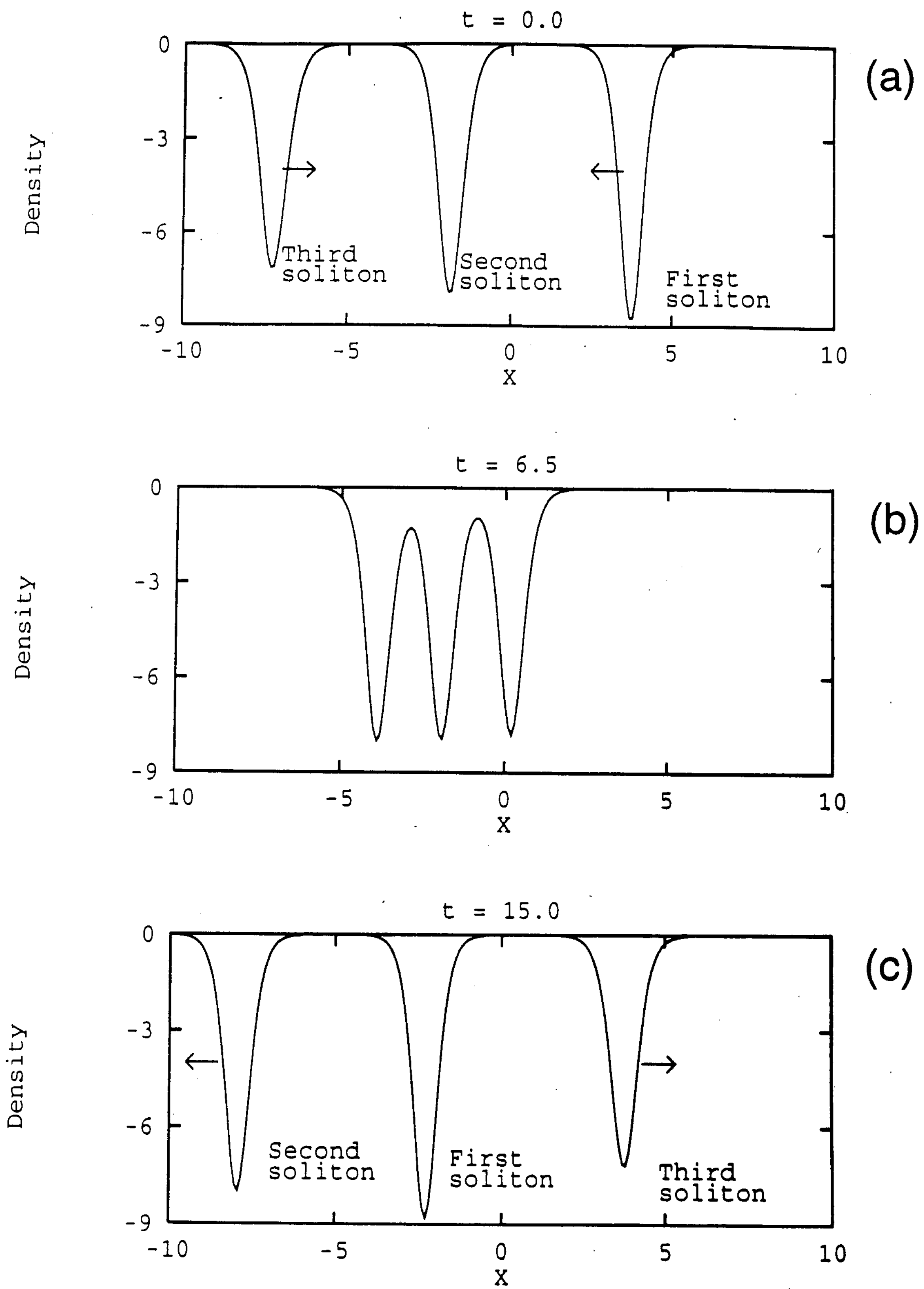

Three soliton collision
First soliton
Second soliton
Third soliton
Position
Time
(a)
Third soliton
Second soliton
First soliton
Velocity
Time
(b)

Figure 2



t = 0.0
(a)
Density
0
-3
-6
-9
-10
-5
0
5
10
X
Third soliton
Second soliton
First soliton
t = 6.5
(b)
Density
0
-3
-6
-9
-10
-5
0
5
10
X
t = 15.0
(c)
Density
0
-3
-6
-9
-10
-5
0
5
10
X
Second soliton
First soliton
Third soliton

Figure 3