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Abstract

The variational structure of the plasma linear response function is used to demonstrate the relation of magnetohydrodynamic and trapped-particle instabilities. Though in most systems, where bending energy stabilizes ballooning modes, trapped-particle instabilities have a low growth rate, in tandem mirrors with thermal barriers the trapped-particle instability growth rate approaches that of MHD instabilities. In addition, the kinetic theory yields stabilizing effects due to the difference in electron and ion orbits, and destabilizing effects due to the variation of the $\underset{\sim}{E}\times\underset{\sim}{B}$ drifts along a field line.

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I. INTRODUCTION

It has been proposed that tandem mirror fusion reactors be designed with thermal barriers in order to substantially reduce the ion energy and density required in the end cells.[1] One such design is the axicell tandem, in which the thermal barrier is produced by interposing axisymmetric auxiliary mirror cells (axicells) between quadrupole mirror end cells and the central solenoidal cell. The use of axicells possesses the dual virtues of reducing radial particle transport and allowing for higher magnetic fields.

Because of the effective barrier produced by the axicell, particle densities can be very low in the transition region between the axicell and the end cell where the magnetic surfaces are transformed from circular to elliptical. While such configurations may be stable against MHD perturbations due to the anchoring of the field lines by the quadrupole end cells, they can be unstable to the trapped-particle mode.[2] Furthermore, we find that the existence of the low density transition region results in growth rates of the unstable trapped-particle mode which are considerably enhanced above that of the usual estimates; in extreme cases, the growth rates can approach MHD growth rates. The most unstable perturbations appear to be those which are flute-like in the central cell and axicells, but which decrease to zero in the transition region. The instability is driven by the average unfavorable curvature in the central region. Since the trapped-particle modes are primarily electrostatic, the field lines are not significantly perturbed and no bending energy stabilization occurs. Some stabilization is, however, possible as a result of the charge separation which arises because of finite Larmor radius effects and of differing ion and electron trajectories. This latter stabilizing feature is due to the presence of equilibrium electrostatic potentials essential for improved particle confinement in tandem mirrors; its counterpart in tokamak trapped-particle instabilities is negligible.

II. VARIATIONAL PRINCIPLE AND GROWTH RATES

In order to make a quantitative study of the trapped-particle instability in tandem mirrors, we start with the low-frequency gyrokinetic equations derived in the high mode number limit using the eikonal representation. The appropriate set of field variables are: ϕ the electrostatic potential, $B_\parallel$ the parallel component of the magnetic field, and $A_\parallel$ the parallel component of the vector potential or, equivalently, χ where $A_\parallel = c/i\omega(\underset{\sim}{b}\cdot\underset{\sim}{\nabla}\chi)$.

In the limit of low pressure ($\beta < 1$) and small Larmor radius, $k_\perp^2 v_{th}^2/\Omega^2 = k_\perp^2 r_L^2 < 1$, the eigenmode equations for ϕ , $B_\parallel$, and χ may be derived from the following quadratic variational form (which differs from those derived by Antonsen and Lee[3] only in the inclusion of finite Larmor radius terms):

$$\int \frac{d\ell}{B_0}\left[\frac{k_\perp^2 c^2 \sigma}{4\pi\omega^2}(\underset{\sim}{b}\cdot\nabla\chi)^2 + D\chi^2 + \psi^2 S_0 + \frac{Q^2}{4\pi}\tau_* - \sum\int d\Gamma \frac{\partial F_0}{\partial \varepsilon}\frac{(\omega - \omega_*)}{(\omega - \overline{\omega}_D)}\overline{G}^2\right] = 0 \qquad (1)$$

where the new field variables ψ and Q are given by

$$\psi = \phi + \frac{\partial\rho/\partial B_0}{\partial\rho/\partial\phi_0}\left(B_\parallel + \frac{\chi c}{\omega}\,\hat{\underset{\sim}{e}}\cdot\underset{\sim}{\nabla}B_0\right) - \chi\left\{1 - \frac{\omega_E}{\omega} - \frac{1}{\partial\rho/\partial\phi_0}\left[\left(1 - \frac{\omega_E}{\omega}\right)S_1 + S_2\right]\right\}$$

$$Q = B_\parallel + \chi\,\frac{c}{\omega}\,\hat{\underset{\sim}{e}}\cdot\left(\underset{\sim}{\nabla}B_0 - \frac{\sigma B_0}{\tau_*}\,\underset{\sim}{\kappa}\right)$$

$$G = q\psi\left(1 - \frac{k_\perp^2 v_\perp^2}{4\Omega^2}\right) + \mu_* Q + q\chi d$$

$$\hat{\underset{\sim}{e}} = \frac{\underset{\sim}{k}_\perp \times \underset{\sim}{b}}{B_0}$$

$$\underset{\sim}{\kappa} = (\underset{\sim}{b}\cdot\underset{\sim}{\nabla})\underset{\sim}{b}$$

and

$$\int d\Gamma = \frac{4\pi}{m^2}\int B_0\,\frac{d\varepsilon\,d\mu}{|v_\parallel|}$$

$$\mu = \frac{mv_\perp^2}{2B_0}$$

$$\varepsilon = \frac{mv_\parallel^2}{2B_0} + \mu B_0 + q\phi_0$$

$$\frac{\partial\rho}{\partial\phi_0} = \sum\int d\Gamma q^2\,\frac{\partial F_0}{\partial\varepsilon}$$

$$\frac{\partial\rho}{\partial B_0} = \sum\int d\Gamma q\mu\,\frac{\partial F_0}{\partial\varepsilon}$$

$$\omega_* = -\frac{c}{q}\left(\frac{\hat{\underset{\sim}{e}}\cdot\underset{\sim}{\nabla}F_0}{\partial F_0/\partial\varepsilon}\right)$$

$$\omega_D = \omega_E + \frac{c}{q}\mu\hat{\underset{\sim}{e}}\cdot\underset{\sim}{\nabla}B_0 + \frac{c}{q}mv_\parallel^2\hat{\underset{\sim}{e}}\cdot\underset{\sim}{\kappa}$$

$$\omega_E = c\hat{\underset{\sim}{e}}\cdot\underset{\sim}{\nabla}\phi$$

$$\sigma = 1 + \frac{4\pi}{B_0^2}\left(p_\perp - p_\parallel\right)$$

$$\tau_* = 1 + \frac{4\pi}{B_0}\left(\frac{\partial p_\perp}{\partial B_0}\right) - 4\pi\left[\frac{(\partial\rho/\partial B_0)^2}{(\partial\rho/\partial\phi_0)}\right]$$

$$p_\perp = \sum\int d\Gamma F_0 \mu B_0$$

$$p_\parallel = \sum\int d\Gamma F_0 m v_\parallel^2$$

$$S_0 = \frac{\partial\rho}{\partial\phi_0} + S_1$$

$$S_1 = \sum\int d\Gamma q^2 \frac{k_\perp^2 v_\perp^2}{2\Omega^2}\left(\frac{1}{B_0}\right)\frac{\partial F_0}{\partial\mu}$$

$$S_2 = \sum\int d\Gamma q^2 \frac{k_\perp^2 v_\perp^2}{2\Omega^2}\left(1 - \frac{\omega_*}{\omega}\right)\frac{\partial F_0}{\partial\varepsilon}$$

$$\Omega = \frac{qB_0}{mc}$$

$$\mu_* = \mu - q\left[\frac{(\partial\rho/\partial B_0)}{(\partial\rho/\partial\phi_0)}\right]$$

$$D = \sum\int d\Gamma q^2 \frac{\partial F_0}{\partial\varepsilon}\left(\frac{\omega_*\hat{\omega}_c}{\omega^2}\right)$$

$$+ \left(1 - \frac{\omega_E}{\omega}\right)^2 S_1 + \left(1 - \frac{\omega_E}{\omega}\right)S_2$$

$$d = \frac{\hat{\omega}_c}{\omega} - \frac{1}{(\partial\rho/\partial\phi_0)}\left[\left(1 - \frac{\omega_E}{\omega}\right)S_1 + S_2\right]$$

$$\hat{\omega}_c = \frac{c}{q}\left(mv_\parallel^2 + \frac{\sigma\mu_* B_0}{\tau_*}\right)\hat{\underset{\sim}{e}}\cdot\underset{\sim}{\kappa} \quad .$$

The summation $\sum$ is over particle species and the distribution function F_0 is considered to be symmetric in $v_\parallel$. The particle bounce frequencies ω_b are assumed to be large compared with the mode frequency ω $(\omega_b > \omega)$; $\overline{G} = \left(\int d\ell G/|v_\parallel|\right)/\int\left(d\ell/|v_\parallel|\right)$ and implies averaging over a trapped-particle trajectory.

If $\omega > \omega_* > \omega_D$, and we neglect Q in the interest of brevity, Eq. (1) can be solved to obtain the following variational expression for the growth rate $(\omega = i\gamma)$:

$$\gamma^2 = \frac{\int d\ell/B_0\left[(-\partial\rho/\partial\phi_0)\langle\omega_*\omega_c(\chi + \overline{\psi})^2\rangle - k_\perp^2 c^2(\underset{\sim}{b}\cdot\nabla\chi)^2/4\pi\right]}{\int d\ell/B_0(-\partial\rho/\partial\phi_0)\left[\langle\psi^2 - \overline{\psi}^2\rangle + \langle\langle k_\perp^2 v_\perp^2/2\Omega^2\rangle\rangle(\chi^2 + \psi^2 + \langle 2\chi\overline{\psi}\rangle)\right]} \tag{2}$$

where

$$\langle\alpha\rangle = \frac{\sum\int d\Gamma q^2(\partial F_0/\partial\varepsilon)\alpha}{\sum\int d\Gamma q^2(\partial F_0/\partial\varepsilon)}$$

$$\langle\langle\alpha\rangle\rangle = \frac{\sum\int d\Gamma q^2\left[(\partial F_0/\partial\varepsilon) + 1/B_0(\partial F_0/\partial\mu)\right]\alpha}{\sum\int d\Gamma q^2(\partial F_0/\partial\varepsilon)}$$

$$\omega_c = \frac{c}{q}\left(mv_\parallel^2 + \mu B_0\right)\hat{\underset{\sim}{e}}\cdot\underset{\sim}{\kappa} .$$

In strict accordance with the derivation presented, ω_c, as written above is only valid at very low beta. However, Eq. (2) with the above definition of ω_c, is valid for small but finite β and flute-like perturbations where we include finite Q in the analysis.[4] That is to say, Q adjusts itself so as to cancel the drifts due to the diamagnetic well. We also note that when $\gamma \lesssim \omega_{*i}$, $\chi \ll \psi$, and if the equilibrium is only weakly dependent upon electric drift fields, Eq. (2) is still valid, and is the variational expression for the trapped-particle mode.

As the denominator in Eq. (2) is positive definite, perturbations for which the numerator is positive are unstable. In normal orderings when the passing and trapped particles are comparable, the fastest-growing modes (i.e., MHD modes) arise by making the denominator as small as possible, i.e., $\mathcal{O}(k_\perp^2 r_\perp^2)$, This is effected by choosing perturbations such that $\chi \neq 0$ but $\psi \to 0$ (equivalent to the MHD assumption $E_\parallel \to 0$). The resulting variational expression for γ is identical to the MHD variational principle and the growth rates can be large $\gamma \sim (\omega_* \omega_c / k_\perp^2 r_L^2)^{1/2}$. The flute interchange mode corresponds to $\chi \neq 0$, $\underset{\sim}{b}\cdot\underset{\sim}{\nabla}\chi = 0$, and it is unstable unless

$$-\int \frac{d\ell}{B_0}\left(\frac{\partial \rho}{\partial \phi_0}\right)\langle \omega_* \omega_c \rangle < 0 .$$

If the flute interchange is stable, rapidly growing ballooning modes may still arise and they correspond to perturbations in χ which are localized in regions of average unfavorable curvature. However, since these perturbations involve field-line bending (that is $\underset{\sim}{b} \cdot \underset{\sim}{\nabla} \chi \neq 0$), they are stable in low β plasmas, $\beta < \beta_{crit}$ where β_{crit} is the MHD ballooning mode limit for stability. However, instability can still be found if we allow $\chi \to 0$ to eliminate the bending energy and choose $\psi \neq 0$. These perturbations correspond to the trapped-particle instabilities which in conventional orderings produce low growth compared to an MHD time scale. The mode is predominantly electrostatic and localized in regions of average unfavorable curvature. Since the term proportional to $\langle \psi^2 - \overline{\psi}^2 \rangle$ is typically of order one for many confinement systems, an estimate for the trapped-particle growth rate is $\gamma \sim (\omega_* \omega_c)^{1/2}$, much smaller than a typical MHD growth rate. For axicell tandem mirrors, however, where the perturbations can be flute-like in the central cell and axicells [hence, for trapped particles $(\psi = \overline{\psi})$] and decrease to zero in the low density transition region, $\langle \psi^2 - \overline{\psi}^2 \rangle$ is proportional to the fraction of transiting particles and, therefore, much less than unity. Thus trapped-particle growth rates are considerably enhanced above the usual estimates. If $T_0/T_t(n_t/N_0)L_t/L_0 \sim k_\perp^2 r_L^2$, the growth rates approach MHD growth rates where $n_t(L_t , T_t)$ and $N_0(L_0 , T_0)$ are the density (length, temperature) of the transition region and central region respectively.

In Eq. (2) we have assumed $\gamma \ll \omega_b$. In practice this assumption is marginal for ions. However, we have been able to show that if finite ion bounce frequencies are included in the theory, with $\chi = 0$, the growth rate will be larger than predicted by the variational form of Eq. (2).

III. CHARGE SEPARATION

We will now discuss the case of $\omega \sim \omega_* \gtrsim \omega_E$ when Eq. (2) is no longer valid. We restrict our discussion to $\beta < \beta_{crit}$ and thus neglect perturbations in χ. We first consider the limit $\omega_* > \omega_E$ and describe a model calculation in which the stabilizing influence of charge separation is assessed. We then derive the dispersion relation for the limit $\omega_* \sim \omega_E$.

If $\omega \sim \omega_* > \omega_E$, the quadratic variational form obtained from Eq. (1) and relevant to trapped-particle modes can be written in the form:

$$\omega^2 \mathcal{A} + \omega \mathcal{B} + \mathcal{C} = 0 \tag{3}$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are quadratic functionals of the field variable ψ,

$$\mathcal{A} = \int \frac{d\ell}{B_0}\left(-\frac{\partial\rho}{\partial\phi_0}\right)\left[\langle\psi^2 - \bar{\psi}^2\rangle + \left\langle\!\left\langle \frac{k_\perp^2 v_\perp^2}{2\Omega^2}\,\psi^2\right\rangle\!\right\rangle\right]$$

$$\mathcal{B} = \int \frac{d\ell}{B_0}\left(-\frac{\partial\rho}{\partial\phi_0}\right)\left[\left\langle(\omega_* - \omega_D)(\bar{\psi}^2 - \psi^2)\right\rangle - \left\langle\bar{\psi}^2(\omega_* - \bar{\omega}_D)\,\frac{k_\perp^2 v_\perp^2}{2\Omega^2}\right\rangle\right]$$

$$\mathcal{C} = \int \frac{d\ell}{B_0}\left(-\frac{\partial\rho}{\partial\phi_0}\right)\left\langle(\omega_* - \bar{\omega}_D)\omega_D\left(1 - \frac{k_\perp^2 v_\perp^2}{2\Omega^2}\right)\bar{\psi}^2\right\rangle$$

and

$$\omega_D \simeq \omega_E + \omega_c \ .$$

The previous expression for the growth rate of the trapped-particle mode was obtained by neglecting the linear term in ω .

The importance of the linear term in ω is that it introduces a stabilizing effect. Equation (3) may be formally solved for ω :

$$\omega = -\frac{\mathcal{B}}{2\mathcal{A}} \pm \left(\frac{\mathcal{B}^2}{\mathcal{A}^2} - 4\frac{\mathcal{C}}{\mathcal{A}}\right)^{1/2} .$$

A finite value of $\mathcal{B}$ contributes a positive term (ψ is taken to be real) to the discriminant and is stabilizing. It represents the stabilizing effect of charge separation on the trapped-particle mode.

The coefficient $\mathcal{B}$ is a sum of two terms. The first term in $\mathcal{B}$, $\langle(\omega_* - \omega_D)(\overline{\psi}^2 - \psi^2)\rangle$, is usually zero for confinement systems like tokamaks where the distribution functions and trapped particle trajectories of ions and electrons are self-similar. This follows, since by definition $\langle(\omega_* - \omega_D)\rangle = 0$ is just the expression of charge neutrality. Tandem mirrors, however, have finite electrostatic potentials in equilibrium, and this manifests itself not only in the presence of electric drifts but also in trajectories which are different for ions and electrons. Thus, $\langle(\omega_* - \omega_D)(\overline{\psi}^2 - \psi^2)\rangle \neq 0$. The second term in $\mathcal{B}$ is the usual finite Larmor radius term and is usually non-zero, $\langle\overline{\psi}^2(\omega_* - \overline{\omega}_D)(k_\perp^2 v_\perp^2/2\Omega^2)\rangle \neq 0$.

To estimate the stabilizing effect of $\mathcal{B}$, we consider a model calculation in which the equilibrium magnetic (B_0) and electrostatic (ϕ_0) fields of the axicell tandem are represented by step-functions as shown in Fig. 1.

The perturbed field variable ψ is

$$\psi = \begin{cases} 1 & L_1 > \ell > -L_1 \\ 0 & |\ell| > L_1 \ . \end{cases}$$

ψ is constant in the central region where the majority of the particles are trapped and undergo average unfavorable curvature drifts, and is zero in the low-density transition region. This perturbation tends to maximize the trapped-particle growth rate.

For particles trapped in the central region:

$$\overline{\psi} = 1 \ .$$

For the passing particles which leave the central region and enter the transition region:

$$\overline{\psi} = \frac{t_1}{t_1 + t_2} \ ,$$

where t_1 and t_2 are the transit times in the central and transitional regions respectively and $t_1 >> t_2$.

The distribution functions for ions and electrons are taken to be Maxwellian:

$$F_0 = \left(\frac{m}{2\pi T}\right)^{3/2} N_0 \exp\left(-\frac{(\varepsilon - q\phi_0)}{T}\right) \ .$$

The original system of equations is invariant to a fixed $\underset{\sim}{E}\times\underset{\sim}{B}$ drift and it is convenient to choose a frame of reference moving with the plasma in the central cell. Hence, if ω_{E1} is the electric field rotation frequency of the plasma in the central region and $\omega' = \omega - \omega_{E1}$, then Eq. (3) can be replaced by:

$$\omega'^2 \mathcal{A} + \omega' \mathcal{B} + \mathcal{C}' = 0 \tag{4}$$

where $\mathcal{C}' = \mathcal{C} - \omega_{E1}\mathcal{B}$ and

$$\mathcal{A} = \frac{(L_2 - L_1)}{B_1} e^2 \left(\frac{n_{it}}{T_i} + \frac{n_{et}}{T_e} \right) + \frac{L_1}{B_1} \left(\frac{k_\perp^2 e^2 N_0}{m_i \Omega_i^2} \right)$$

$$\mathcal{B} = \frac{(L_2 - L_1)}{B_1} \left[2(\omega_{E1} - \omega_{E2}) e^2 \left(\frac{n_{it}}{T_i} + \frac{n_{et}}{T_e} \right) - \frac{ce}{B_1} \underset{\sim}{k}_\perp \cdot \underset{\sim}{b} \times \underset{\sim}{\nabla} (n_{it} - n_{et}) \right]$$

$$- \frac{L_1}{B_1} \left(\frac{k_\perp^2}{m_i \Omega_i^2} \right) \frac{ce}{B_1} \underset{\sim}{k}_\perp \cdot \underset{\sim}{b} \times \underset{\sim}{\nabla} N_0 T_i$$

$$\mathcal{C}' = e^2 N_0 \left(\frac{1}{T_i} + \frac{1}{T_e} \right) \int \frac{d\ell}{B_0} \left\langle (\omega_* - \bar{\omega}_{E1}) \omega_c \bar{\psi}^2 \right\rangle + \frac{(L_2 - L_1)}{B_1} (\omega_{E2} - \omega_{E1})$$

$$\times \frac{ce}{B_1} \underset{\sim}{k}_\perp \cdot \underset{\sim}{b} \times \underset{\sim}{\nabla} (n_{it} - n_{et}) + \frac{(L_2 - L_1)}{B_1} (\omega_{E2} - \omega_{E1})^2 e^2 \left(\frac{n_{it}}{T_i} + \frac{n_{et}}{T_e} \right).$$

n_{it} and n_{et} are the density of the passing particles in the transition region.

The stabilizing effect of charge separation becomes dominant when $\mathcal{B}^2 > 4\mathcal{A}\mathcal{C}'$ and this inequality reduces to:

$$\left| \frac{c}{e} \left[\frac{\hat{\underset{\sim}{e}} \cdot \underset{\sim}{\nabla}(n_{it} - n_{et})}{(n_{it}/T_i) + n_{et}/T_e)} \right] \right| > 2\gamma_0 \tag{5}$$

when the Larmor radius terms are negligible;

$$\left| \frac{c}{e} \left(\frac{\hat{\underset{\sim}{e}} \cdot \underset{\sim}{\nabla} N_0 T_i}{N_0} \right) \right| > 2\gamma_0 \tag{6}$$

when the Larmor radius terms are dominant.

γ_0 is defined to be the trapped-particle growth rate when $\mathcal{B}$ can be neglected.

$$\gamma_0^2 = \frac{(1 + T_i/T_e) \int d\ell/B_0 \langle(\omega_* - \omega_{E1})\omega_c\rangle}{[(L_2 - L_1)/B_1]\{[n_{it} + n_{et}(T_i/T_e)]/N_0\} + (L_1/B_1)k_\perp^2 r_{L_i}^2} \, . \tag{7}$$

It should be remarked, of course, that in the case when the trapped particle mode is stabilized in this way by charge separation, the resulting negative energy modes may still be destabilized residually by collisions or bounce resonances.

IV. AXIAL ELECTRIC FIELD VARIATION

In the more realistic case where $\omega_* \sim \omega_E \gg \omega_c$, the quadratic variational form with $\chi = Q = 0$ is:

$$\int \frac{d\ell}{B_0} \left(\frac{\partial \rho}{\partial \phi_0}\right) \left[\left\langle \psi^2 \left(\frac{\omega - \omega_* + \omega_c}{\omega - \omega_E} \right) - \bar{\psi}^2 \left(\frac{\omega - \omega_* + \bar{\omega}_c}{\omega - \bar{\omega}_E} \right) \right\rangle + \left\langle\left\langle \frac{k_\perp^2 v_\perp^2}{2\Omega^2} \psi^2 \right\rangle\right\rangle \right.$$

$$- \left\langle \frac{k_\perp^2 v_\perp^2}{2\Omega^2} \psi^2 \right\rangle + \left\langle \frac{k_\perp^2 v_\perp^2}{2\Omega^2} \left(\frac{\omega - \omega_*}{\omega - \bar{\omega}_E} \right) \bar{\psi}^2 \right\rangle$$

$$\left. + \left\langle \frac{(\omega_* - \bar{\omega}_D)\omega_c}{(\omega - \bar{\omega}_E)^2} \bar{\psi}^2 \right\rangle \right] = 0 \qquad (8)$$

where $|\omega - \bar{\omega}_E| > \omega_c$ is assumed.

As before, an estimate of the mode frequency ω may be obtained from Eq. (8) by substitution of a suitable trial function for ψ.

To this end, it is convenient to separate the axicell tandem into distinct regions of localized trapped-particles. If we again consider the limit where the fraction of particles which are able to pass from one region to another is small, then within each region quasi-neutrality tends to constrain ψ to be flute-like: $\psi \sim \hat{\psi}_j(1 - \omega_{Ej}/\omega)$ with $\hat{\psi}_j$ a constant. The subscript j denotes the j^{th} region.

It therefore appears reasonable to consider a trial function for ψ:

$$\psi = \hat{\psi}_j\left(1 - \frac{\omega_{Ej}}{\omega}\right) \qquad j^{th} \text{ region} ,$$

where $\hat{\psi}_j$ are adjustable constants to be chosen so that Eq. (8) is stationary with respect to variations in $\hat{\psi}_j$.

For locally trapped particles:

$$\overline{\psi} = \hat{\psi}_j\left(1 - \frac{\overline{\omega}_{Ej}}{\omega}\right) \qquad j^{th} \text{ region} \quad .$$

For the passing particles:

$$\overline{\psi} = \sum_j \hat{\psi}_j\left(1 - \frac{\overline{\omega}_{Ej}}{\omega}\right) \alpha_j \quad ,$$

where

$$\overline{\omega}_{Ej} = \frac{\int_j (d\ell/|v_\parallel|)\omega_E}{\int_j d\ell/|v_\parallel|}$$

$$\alpha_j = \frac{\int_j d\ell/|v_\parallel|}{\int d\ell/|v_\parallel|} \quad .$$

α_j is the fraction of time spent by a passing particle in the j^{th} region.

If we substitute the trial function for ψ in Eq. (8) and we impose the requirement that Eq. (8) be stationary with respect to variations of $\hat{\psi}_j$, we obtain the following linear equations for $\hat{\psi}_j$:

$$\int_j \frac{d\ell}{B_0}\left(\frac{\partial\rho}{\partial\phi_0}\right)\left\langle\left(1-\frac{\omega_{Ej}}{\omega}\right)\left(1-\frac{\omega_*}{\omega}\right)\left\{\hat{\psi}_j - \frac{\sum_k \hat{\psi}_k(1-\overline{\omega}_{Ek}/\omega)\alpha_k}{[1-\sum_k(\overline{\omega}_{Ek}/\omega)\alpha_k]}\right\rangle\right.$$

$$+\int_j \frac{d\ell}{B_0}\left(\frac{\partial\rho}{\partial\phi_0}\right)\left[\left\langle\left\langle\hat{\psi}_j\left(1-\frac{\omega_{Ej}}{\omega}\right)^2\frac{k_\perp^2 v_\perp^2}{2\Omega^2}\right\rangle\right\rangle - \left\langle\hat{\psi}_j\left(1-\frac{\omega_{Ej}}{\omega}\right)\frac{(\omega_*-\omega_E)}{\omega}\frac{k_\perp^2 v_\perp^2}{2\Omega^2}\right\rangle\right.$$

$$\left.\left.+\left\langle\hat{\psi}_j\frac{(\omega_*-\overline{\omega}_D)\omega_c}{\omega^2}\right\rangle\right]\right\} . \qquad (9)$$

We have neglected terms of order $\omega_c k_\perp^2 v_\perp^2/2\omega\Omega^2$. It will be noted that in Eq. (9) only the passing particles contribute to the first term, whereas the locally-trapped particles are assumed to dominate the contributions to the other three terms.

The dispersion relation is obtained by equating the determinant of the set of equations given by Eq. (9) to zero.

Solutions of this dispersion relation are currently being investigated. We may note that in the case where the axial variation of ω_E considerably exceeds ω_* and n_t is not too much smaller than N_0 , that a flute mode may be driven unstable by axial variation of ω_E .

In this limit, the first term in Eq. (9) will be the largest, requiring that all $\hat{\psi}_j$'s be equal (<u>i.e.</u>, a flute displacement). A solubility condition is then found by summing over j leading to the dispersion relation:

$$\int \frac{d\ell}{B} \frac{m_i N_i}{B^2} k_\perp^2 \left[(\omega - \omega_E)(\omega - \omega_E) - \frac{c}{qN_i} \hat{\underset{\sim}{e}} \cdot \underset{\sim}{\nabla} p_{\perp i} \right]$$

$$= - \int \frac{d\ell}{B} (\hat{\underset{\sim}{e}} \cdot \underset{\sim}{\kappa}) [\hat{\underset{\sim}{e}} \cdot \underset{\sim}{\nabla} (p_\perp + p_\parallel)] \ . \qquad (10)$$

and $\underset{\sim}{\kappa} = (\underset{\sim}{b} \cdot \underset{\sim}{\nabla}) \underset{\sim}{b}$.

For large $k_\perp$, the left-hand side of Eq. (10) will dominate yielding a quadratic equation for ω . Instability will result when $\overline{(\omega_E - \overline{\omega}_E)^2} > \overline{\omega}_{*i}^2$ where the raised bar indicates field line average. A similar result has also been obtained by X. S. Lee *et al.*[5]

V. SUMMARY

We have generalized the variational principle for the trapped-particle mode and applied it to the case of tandem mirror stability. This case has several novel features.

1) Due to the potential barriers the equilibrium may contain regions of very low density in the transition between central cell and stabilizing anchor. In this case we have shown that the growth rate γ_0 , of electrostatic trapped-particle modes localized away from the anchor may approach that of the unanchored MHD system, even though MHD interchange and ballooning stability criteria are met.

2) However, because of the equilibrium parallel electric fields, ion and electron trajectories differ. This leads to a stabilization (effective even for m = 1) roughly when $\omega_* > \gamma_0$, although residual collisional or resonance growth remains.

3) The equilibrium may also be characterized by large axial variations of the $\underset{\sim}{E} \times \underset{\sim}{B}/B^2$ drift frequency. If these are very large compared to ω_* , a novel type of flute instability is predicted.

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Figure Caption

Fig. 1

Profiles along a field line of idealized equilibrium magnetic field, B_0 , electrostatic field, ϕ_0 , and trial function, ψ , used to model an Axicell Tandem. System has reflection symmetry.

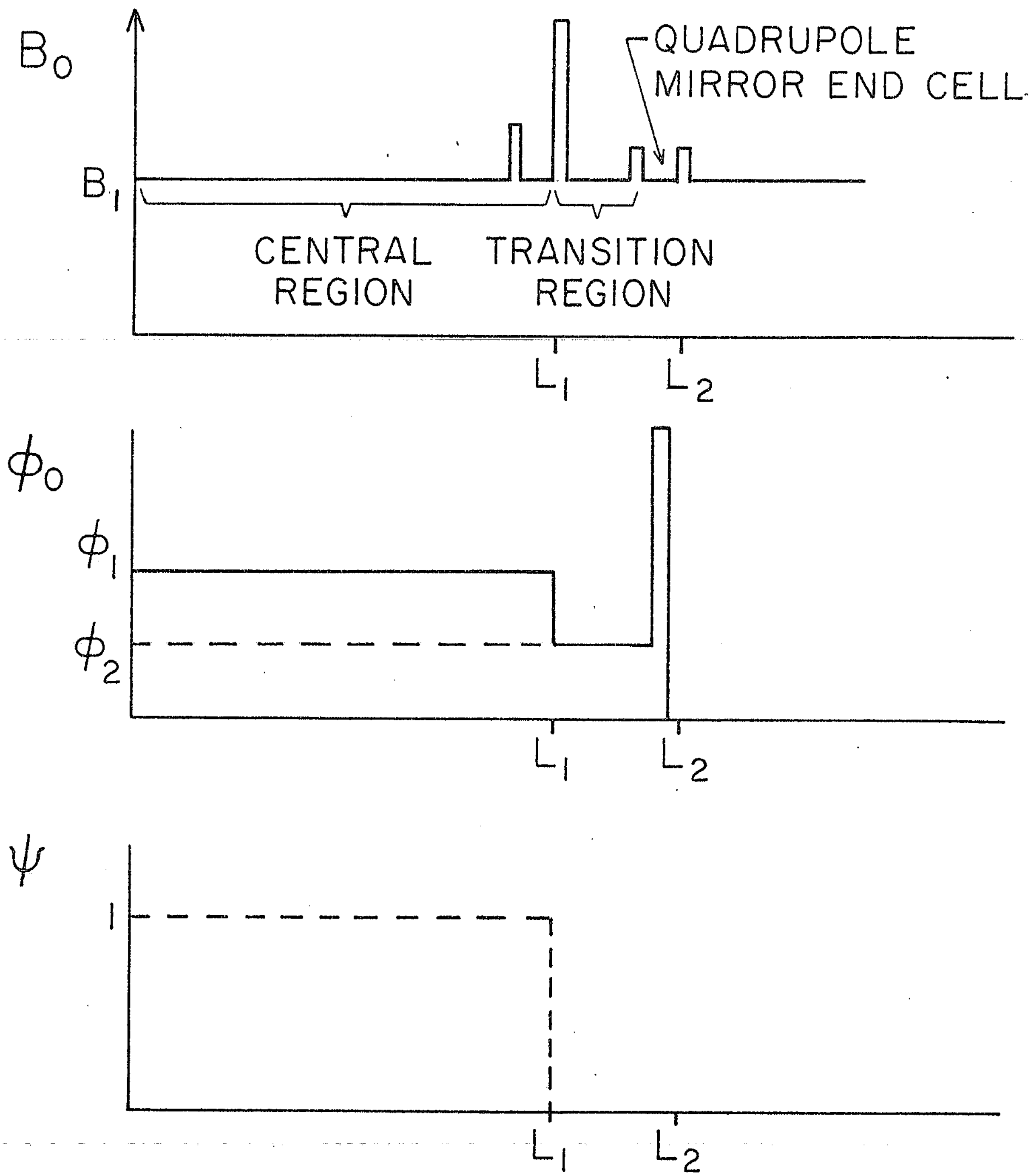
B0
QUADRUPOLE
MIRROR END CELL
B1
CENTRAL
REGION
TRANSITION
REGION
L1
L2
φ0
φ1
φ2
L1
L2
ψ
1
L1
L2

Fig. 1