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Linear Stability of Stationary Solutions of the Vlasov-Poisson System in Three Dimensions

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Abstract

Rigorous results on the stability of stationary solutions of the Vlasov-Poisson system are obtained in both the plasma physics and stellar dynamics contexts. It is proven that stationary solutions in the plasma physics (stellar dynamics) case are linearly stable if they are decreasing (increasing) functions of the local, i.e. particle, energy. The main tool in the analysis is the free energy of the system, a conserved quantity. In addition, an appropriate global existence result is proven for the linearized Vlasov-Poisson system and the existence of stationary solutions that satisfy the above stability condition is established.

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1 Introduction

The evolution considered in this paper is governed by the Vlasov-Poisson system,

$$\partial_t f + v \cdot \partial_x f - \partial_x U \cdot \partial_v f = 0 \ ,$$

$$\triangle U = 4\pi \left(\rho^+ + \gamma\rho\right) \ ,$$

$$\rho(t,x) := \int f(t,x,v)\, dv \ ,$$

where $t \geq 0$ denotes time, $x \in \mathbb{R}^3$ position, and $v \in \mathbb{R}^3$ velocity. For $\gamma = -1$ the system describes a collisionless plasma of electrons, which move in an electrostatic field that arises self-consistently from the electron spatial charge density $\rho(t,x)$ and a fixed ion background with spatial charge density $\rho^+ = \rho^+(x) \geq 0$. The case where $\gamma = 1$ and ρ^+ is set to zero describes a collisionless ensemble of self-gravitating point masses, e.g. stars in a galaxy or galaxies in a galactic cluster. In this case $\rho(t,x)$ represents the spatial mass density. The function $f = f(t,x,v)$ denotes the phase space density of either the electrons or stars, while $-U$ or U denotes the electrostatic or gravitational potential respectively.

The initial value problem for this system, where the initial phase space density $f(0,x,v) = \mathring{f}(x,v)$ is prescribed, is now well understood, and the existence of global, classical solutions for C^1 data with appropriate decay at infinity is established [10, 20, 25, 30, 32].

However, the above rigorous results lend limited information about the qualitative behavior of the solutions. The purpose of the present investigation is to clarify the question of stability of certain stationary solutions. Two main stability concepts have to be distinguished: A stationary solution is *nonlinearly stable* if solutions of the nonlinear Vlasov-Poisson system remain arbitrarily close to the stationary solution in some norm for all times, provided the Vlasov-Poisson solutions start sufficiently close to the stationary solution. The stationary solution is *linearly stable* if the solutions of the nonlinear system are replaced by the solutions of the corresponding linearized problem in the above "definition." Obviously, a global existence result — at least for initial data close to the steady state under consideration — is an integral part of both stability concepts.

If the solution is written as $f_0 + f(t)$, where f_0 is the distribution function of the steady state, and the term that is quadratic in $f(t)$ is neglected, one obtains the linearized Vlasov-Poisson system:

$$\partial_t f + v \cdot \partial_x f - \partial_x U_0 \cdot \partial_v f = \partial_x U_f \cdot \partial_v f_0 \ ,$$

$$\triangle U_f = 4\pi\gamma\rho_f \ ,$$

$$\rho_f(t,x) := \int f(t,x,v)\, dv \ ,$$

where the steady state (f_0, U_0) satisfies the stationary Vlasov-Poisson system:

$$v \cdot \partial_x f_0 - \partial_x U_0 \cdot \partial_v f_0 = 0 \ ,$$

$$\triangle U_0 = 4\pi(\rho^+ + \gamma\rho_0) \ ,$$

$$\rho_0(x) := \int f_0(x,v)\, dv \ .$$

Stability conditions are often expressed in terms of how f_0 depends on the local or particle energy $E(x,v) := \frac{1}{2}v^2 + U_0(x)$. Since E and, for spherical symmetry where $U_0(x) = U_0(|x|)$, also $F := |x| \times v^2$ are constant along the characteristics of the stationary Vlasov equation, it is natural to consider $f_0(x,v) = \phi(E)$ or $f_0(x,v) = \phi(E,F)$ with some function ϕ. The present work is restricted to the first case.

There exists a large number of investigations of both linear and nonlinear stability, cf. [1, 2, 11, 12, 13, 16, 18, 19, 21, 33, 34, 35]. The general and long standing opinion seems to be that — both in the plasma physics and in the stellar dynamics cases — a steady state is stable if ϕ is a decreasing function of the energy. Although these results are physically appealing and plausible, a distinction must be made between these results and rigorous mathematics. (Note, too, that certain conclusions drawn for anisotropic spherical systems are admittedly contradictory [13, p. 308].) Concerning nonlinear stability we mention the following rigorous results: In [9] it is shown, for the plasma physics case with spatial periodicity, that spatially homogeneous steady states are nonlinearly stable if ϕ is decreasing; the analogous result for the relativistic Vlasov-Maxwell system is shown in [23]. Both results are based on using the total energy of the system as a Lyapunov function, cf. also [26]. As for linear stability, we are not aware of any rigorous results. It is important to point out that in infinite dimensional dynamical systems such as the Vlasov-Poisson system, the relationship between the two concepts of stability is in general not clear.

The present investigation is intended to fill part of the gap between what is done in more physically motivated papers and what is mathematically established. We proceed as follows: In the next section the basic assumptions on the steady states under consideration are collected. In the third section we prove a global existence and uniqueness result for the linearized Vlasov-Poisson system. The problem here is two-fold. First, the solution concept has to be strong enough to yield the existence of a conserved quantity, the free energy of the system, which is investigated in the fourth section. Second, we have to be able to linearize around steady states where ϕ is discontinuous, since in the stellar dynamics case we obtain linear stability if $\phi' > 0$ on its support, and since the steady state has to have finite total mass this necessitates a jump discontinuity of ϕ. For the plasma physics case we obtain linear stability if $\phi' < 0$ on its support. This stability analysis is carried out in the fifth section. Finally, we show that there actually exist steady states that satisfy our assumptions. This is necessary since in the plasma physics case the existence of steady states in the above situation has not yet been demonstrated; however, we refer to [6, 14, 15, 31] for related results. In the stellar dynamics case the polytropes, which are investigated in [7] and [8], are not examples of stationary solutions that satisfy our assumptions.

2 Assumptions on the stationary solutions

We consider stationary solutions (f_0, U_0) of the Vlasov-Poisson system such that

$$f_0(x,v) = \phi\left(\frac{1}{2}v^2 + U_0(x)\right),\ x, v \in \mathbb{R}^3 ,$$

where ϕ satisfies the assumptions

(ϕ1) $\phi \in L^\infty_{\mathrm{loc}}(\mathbb{R})$, $\phi \geq 0$,

(ϕ2) $E_0 := \inf \{E \in \mathbb{R} \mid \phi(E') = 0$ a.e. for $E' > E\} \in]-\infty, \infty[$,

(ϕ3) $\phi \in C^1(]-\infty, E_0[)$ with $\phi' \in L^1_{\mathrm{loc}}(]-\infty, E_0])$,

and U_0 satisfies the assumptions

(U1) $U_0 \in C^2(\mathbb{R}^3)$,

(U2) U_0 is bounded; $U_{\min} := \inf_{x\in\mathbb{R}^3} U_0(x) < E_0$,

(U3) the set $B := \left\{(x,v) \in \mathbb{R}^6 \mid \frac{1}{2}v^2 + U_0(x) \leq E_0\right\}$ is bounded, and ∂B has measure zero.

Here $\phi \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ means that $\phi_{|K} \in L^\infty(K)$ for every compact interval $K \subset \mathbb{R}$, and $L^1_{\mathrm{loc}}(]-\infty, E_0])$ is defined analogously.

We write $(f_0, U_0) \in \mathcal{S}$ if (f_0, U_0) is a stationary solution of the Vlasov-Poisson system satisfying the above assumptions. In Sec. 6 we show that stationary solutions of this type exist both in the plasma physics and in the stellar dynamics cases.

Obviously, for $(f_0, U_0) \in \mathcal{S}$ there exists a radius $R_0 > 0$ such that $f_0(x,v) = 0$ for $|x| > R_0$, and

$$\int f_0(x,v)\, dv\, dx \leq \mathrm{vol}\,(B) \sup_{E\in[U_{\min}, E_0]} \phi(E) < \infty ,$$

i.e. the steady states under consideration always have finite radius and finite total mass or charge. Also

$$\int_{\mathbb{R}^3} |\phi'(E(x,v))|\,dv = 4\pi \int_{U_0(x)}^{E_0} |\phi'(E)|\sqrt{2(E-U_0(x))}dE$$

$$\leq 4\pi\sqrt{2(E_0-U_{\min})}\int_{U_{\min}}^{E_0} |\phi'(E)|dE$$

$$< \infty,\ x \in \mathbb{R}^3,$$

and

$$\int_{\mathbb{R}^6} |\phi'(E(z))|\,dz < (4\pi)^2\sqrt{2(E_0-U_{\min})}\int_{-U_{\min}}^{E_0} |\phi'(E)|dE \int_0^{R_0} r^2 dr < \infty\ ;$$

$z := (x,v) \in \mathbb{R}^6$. Throughout the paper constants which depend only on the steady state under consideration — such as the above integrals — and which may change from line to line, are denoted by C.

3 Global existence for the linearized Vlasov-Poisson system

Let $(f_0, U_0) \in \mathcal{S}$ and let $t \mapsto f_0 + f(t)$ be a solution of the Vlasov-Poisson system with initial condition $f_0 + \mathring{f}$. If the term which is quadratic in $f(t)$ in the Vlasov equation is neglected, we arrive at the linearized Vlasov-Poisson system for the (small) perturbation $f(t)$

$$\partial_t f + v \cdot \partial_x f - \partial_x U_0 \cdot \partial_v f = \partial_x U_f \cdot \partial_v f_0\ , \tag{3.1}$$

$$\triangle\, U_f = 4\pi\gamma\rho_f\ , \tag{3.2}$$

$$\rho_f(t,x) := \int f(t,x,v)\,dv\ , \tag{3.3}$$

together with the initial condition $f(0) = \mathring{f}$. Assuming that U_f vanishes at infinity, we obtain

$$U_f(t,x) = -\gamma \int \frac{\rho_f(t,y)}{|x-y|}\,dy,\ x \in \mathbb{R}^3\ . \tag{3.4}$$

Consider the system of characteristics corresponding to (3.1)

$$\dot{x} = v,\ \ \dot{v} = -\partial_x U_0(x)\ . \tag{3.5}$$

Due to the regularity of U_0 there exists for every $t \in \mathbb{R}$ and $z = (x, v) \in \mathbb{R}^6$ a unique global solution $Z(\cdot, t, z) = (X, V)(\cdot, t, x, v)$ of (3.5) with $Z(t, t, z) = z$. The mapping Z is continuously differentiable in all variables and $Z(s, t, \cdot) : \mathbb{R}^6 \to \mathbb{R}^6$ is a measure preserving diffeomorphism for all $s, t \in \mathbb{R}$. Using the flow Z, Eqn. (3.1) can be written in the form

$$\frac{d}{ds} f(s, Z(s,t,z)) = \Big(\partial_x U_f(s) \cdot \partial_v f_0\Big)(Z(s,t,z))\ , s, t \in \mathbb{R},\ z \in \mathbb{R}^6\ ,$$

which upon integration yields

$$f(t,z) = \mathring{f}(Z(0,t,z)) + \int_0^t \Big(\partial_x U_f(s) \cdot \partial_v f_0\Big)(Z(s,t,z))ds,\ t \geq 0,\ z \in \mathbb{R}^6\ .$$

Since for steady states of class $\mathcal{S}$ we have $f_0(x,v) = \phi\left(\frac{1}{2}v^2 + U_0(x)\right)$ and since the energy $E(x,v) = \frac{1}{2}v^2 + U_0(x)$ is invariant under the characteristic flow, the above relation becomes

$$f(t,z) = \mathring{f}(Z(0,t,z)) + \phi'(E(z)) \int_0^t \Big(\partial_x U_f(s) \cdot v\Big)(Z(s,t,z))ds,\ t \geq 0,\ z \in \mathbb{R}^6\ . \tag{3.6}$$

This motivates the following definition:

Definition 3.1 *Let $\mathring{f} \in L^1(\mathbb{R}^6)$. A function $f : [0, \infty[\times\mathbb{R}^6 \to \mathbb{R}$ is a solution of the linearized Vlasov-Poisson system with initial value $\mathring{f}$ iff*

(i) $f \in C([0,\infty[, L^1(\mathbb{R}^6))$,

(ii) $\rho_f \in C([0,\infty[, L^\infty(\mathbb{R}^3))$,

(iii) $U_f \in C([0,\infty[, C^1_b(\mathbb{R}^3))$,

(iv) *f satisfies (3.6) for $t \geq 0$ and $z \notin \partial B$,*

(v) $f(0) = \mathring{f}$.

Here ρ_f and U_f are defined by Eqn. (3.3) and (3.4) respectively, and $C_b^1(\mathbb{R}^3)$ denotes the space of continuously differentiable functions which are bounded together with their first derivatives.

Note that a solution in the above sense satisfies the linearized Vlasov-Poisson system classically if f and U_f are sufficiently regular.

Theorem 3.2 *Let $(f_0, U_0) \in \mathcal{S}$ and let $\mathring{f} \in L^1(\mathbb{R}^6)$ be a pointwise defined, $\mathbb{R}$-valued function such that $\mathring{f} \circ Z(0,\cdot)$ satisfies conditions (i) and (ii) of Definition 3.1. Then there exists a unique solution of the linearized Vlasov-Poisson system with initial value $\mathring{f}$.*

Proof : We shall construct a converging sequence of iterates in the set

$$M := \Big\{ g : [0,\infty[\times\mathbb{R}^6 \to \mathbb{R} \mid g \text{ satisfies (i), (ii), (v) of Definition 3.1} \Big\} .$$

Obviously, $f_1 := \mathring{f} \circ Z(0,\cdot) \in M$. Let $g \in M$. Then (i) implies that

$$\rho_g \in C([0,\infty[, L^1(\mathbb{R}^3)) ,$$

and the well-known estimates, cf. [4],

$$\|U_g(t)\|_\infty \le 2(2\pi)^{1/3} \|\rho_g(t)\|_\infty^{1/3} \|\rho_g(t)\|_1^{2/3} , \tag{3.7}$$

$$\|\partial_x U_g(t)\|_\infty \le 3(2\pi)^{2/3} \|\rho_g(t)\|_\infty^{2/3} \|\rho_g(t)\|_1^{1/3} ,$$

yield (iii) for U_g. Define

$$(Tg)(t,z) := f_1(t,z) + \phi'(E(z)) \int_0^t \Big(\partial_x U_g(s) \cdot v\Big)(Z(s,t,z))ds,\ t \ge 0 \ ,\ z \notin \partial B$$

and zero else. The estimate

$$\int_{\mathbb{R}^6} \left| \phi'(E(z)) \int_0^t \Big(\partial_x U_g(s) \cdot v\Big)(Z(s,t,z))ds \right| dz$$

$$\le \int_B |\phi'(E(z))| dz \ \sup_{z\in B} |v| \ \int_0^t \|\partial_x U_g(s)\|_\infty ds$$

$$\le C \int_0^t \|\rho_g(s)\|_\infty^{2/3} \|\rho_g(s)\|_1^{1/3} ds$$

shows that $(Tg)(t) \in L^1(\mathbb{R}^6)$ — recall that constants denoted by C may depend on the steady state under consideration. Let $0 \le \tau < t$, then

$$\|(Tg)(t) - (Tg)(\tau)\|_1 \le \|f_1(t) - f_1(\tau)\|_1 + C \int_\tau^t \|\partial_x U_g(s)\|_\infty ds$$

$$+ \int_0^\tau \int_B |\phi'(E(z))| \left| \left(\partial_x U_g(s) \cdot v\right)(Z(s,t,z)) - \left(\partial_x U_g(s) \cdot v\right)(Z(s,\tau,z)) \right| dz\, ds$$

$$\to 0 \text{ for } \tau \to t \ ,$$

since f_1 satisfies condition (i) and $(\partial_x U_g(s) \cdot v)(Z(s,\tau,z))$ is uniformly continuous on $[0,t]^2 \times B$. The case $\tau > t$ is analogous, and thus Tg satisfies condition (i). Next observe that

$$|\rho_g(t,x)| \le |\rho_{f_1}(t,x)| + C \int |\phi'(E(z))| dv \int_0^t \|\partial_x U_g(s)\|_\infty ds$$

$$\le |\rho_{f_1}(t,x)| + C \int_0^t \|\partial_x U_g(s)\|_\infty ds \ ,$$

which implies that $\rho_g(t) \in L^\infty(\mathbb{R}^3)$ for $t \ge 0$. Furthermore, for $\tau < t$,

$$\|\rho_g(t) - \rho_g(\tau)\|_\infty \le \|\rho_{f_1}(t) - \rho_{f_1}(\tau)\|_\infty + C \int_\tau^t \|\partial_x U_g(s)\|_\infty ds$$

$$+ C \int_0^\tau \sup_{z \in B} \left| \left(\partial_x U_g(s) \cdot v\right)(Z(s,t,z)) - \left(\partial_x U_g(s) \cdot v\right)(Z(s,\tau,z)) \right| ds$$

$$\to 0 \text{ for } \tau \to t \ ,$$

where the first term converges by the assumption on $\mathring{f}$, the second converges by (iii) and the last by the same argument as above. Since the case $\tau > t$ is analogous, we have (ii), and (v) being obvious we have shown that $Tg \in M$, i.e. T maps the set M into itself. Now let $g_1, g_2 \in M$, then the above estimates show that

$$\|(Tg_1)(t) - (Tg_2)(t)\|_1 \le C \int_0^t \|\rho_{g_1}(s) - \rho_{g_2}(s)\|_\infty^{2/3} \|g_1(s) - g_2(s)\|_1^{1/3} ds \ ,$$

and

$$\|\rho_{Tg_1}(t) - \rho_{Tg_2}(t)\|_\infty \le C \int_0^t \|\rho_{g_1}(s) - \rho_{g_2}(s)\|_\infty^{2/3} \|g_1(s) - g_2(s)\|_1^{1/3} ds \ .$$

Hence, if we define $f_{n+1} := Tf_n$, $n \geq 1$, it follows that there exist functions $f \in C([0,\infty[, L^1(\mathbb{R}^6))$ and $\rho \in C([0,\infty[, L^\infty(\mathbb{R}^3))$ such that $f_n(t) \to f(t)$ in $L^1(\mathbb{R}^6)$, t-locally uniformly on $[0,\infty[$ and $\rho_n(t) \to \rho(t)$ in $L^\infty(\mathbb{R}^3)$, t-locally uniformly on $[0,\infty[$. Since $\rho_{f_n}(t) \to \rho_f(t)$ in $L^1(\mathbb{R}^3)$ we have $\rho = \rho_f$ and $\rho_{f_n}(t) \to \rho_f(t)$ in $L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, t-locally uniformly on $[0,\infty[$. This implies that $U_{f_n}(t) \to U_f(t)$ in $C_b^1(\mathbb{R}^3)$, t-locally uniformly on $[0,\infty[$. Passing to the limit in the relation

$$f_{n+1}(t,z) = f_1(t,z) + \phi'(E(z)) \int_0^t \Big(\partial_x U_{f_n}(s) \cdot v\Big)(Z(s,t,z))ds, \ t \geq 0, \ z \notin \partial B \ ,$$

we obtain

$$f(t,z) = \mathring{f}(Z(0,t,z)) + \phi'(E(z)) \int_0^t \Big(\partial_x U_f(s) \cdot v\Big)(Z(s,t,z))ds, \ t \geq 0, \ z \notin \partial B$$

after redefining f on a set of measure zero. Since condition (v) is clear, f is a solution in the sense of Def. 3.1. Uniqueness of the solution is obvious. □

Corollary 3.3 *The solution f obtained in Thm. 3.2 has the following properties for $t \geq 0$:*

(a) $f(t,z) = \mathring{f}(Z(0,t,z))$ *for $z \notin B$, in particular, if $\mathring{f}$ vanishes outside B then so does $f(t)$.*

(b) $\frac{d}{dt} f(t, Z(t,0,z)) = \phi'(E(z))\Big(\partial_x U_f(t) \cdot v\Big)(Z(t,0,z)), \ z \notin \partial B.$

(c) $\int \mathring{f}(z)\, dz = \int f(t,z)\, dz.$

(d) *If $\mathring{f}$ has compact support or vanishes sufficiently rapidly at infinity then*

$$\int U_f(t,x) f(t,z)\, dz = -\frac{\gamma}{4\pi} \int |\partial_x U_f(t,x)|^2 dx \ .$$

Proof : (a) is obvious, (b) follows from replacing z in (3.6) by $Z(t,0,z)$ and differentiating the resulting equation, (c) follows by integrating (3.6) with respect to z and using the fact that the flow Z preserves measure and that the term $\partial_x U_f(s,x) \cdot v$ is odd in v; note that the set B is invariant with respect to $(x,v) \mapsto (x,-v)$.

(d) If $\rho_f(t)$ is, in addition, Hölder-continuous, then we have

$$\begin{aligned}
&\int U_f(t,x) f(t,z)\,dz = \int U_f(t,x)\rho_f(t,x)\,dx \\
&= \lim_{r\to\infty}\int_{|x|\le r} U_f(t,x)\rho_f(t,x)\,dx = \frac{1}{4\pi\gamma}\lim_{r\to\infty}\int_{|x|\le r} U_f(t,x)\,\triangle\, U_f(t,x)\,dx \\
&= \frac{\gamma}{4\pi}\lim_{r\to\infty}\left[\int_{|x|=r} U_f(t,x)\partial_x U_f(t,x)\cdot n(x)\,d\omega(x) - \int_{|x|\le r} |\partial_x U_f(t,x)|^2\,dx\right] \\
&= -\frac{\gamma}{4\pi}\int |\partial_x U_f(t,x)|^2\,dx
\end{aligned}$$

if the decay of $U_f(t,x)\partial_x U_f(t,x)$ at spatial infinity is such that the boundary term vanishes; if $\mathring{f}$ has compact support then $U_f(t,x)\partial_x U_f(t,x) = O(|x|^{-3})$. In case $\rho_f(t)$ is not Hölder-continuous we can use a mollification of $\rho_f(t)$ to get the result. □

4 Conservation of free energy

Theorem 4.1 *Let $(f_0, U_0) \in \mathcal{S}$ and let $\mathring{f} \in L^1(\mathrm{I\!R}^6)$ be as in Thm. 3.2 with $\mathring{f}(z) = 0$ for $z \notin B$ and*

$$\int_B \frac{\mathring{f}^2(z)}{|\phi'(E(z))|}\,dz < \infty\ . \tag{4.1}$$

Then

$$\int_B \frac{f^2(t,z)}{|\phi'(E(z))|}\,dz < \infty,\ t \ge 0\ ,$$

and

$$\mathcal{F}(t) := -\int_B \frac{f^2(t,z)}{\phi'(E(z))}\,dz + \int U_f(t,x) f(t,z)\,dz = \mathcal{F}(0),\ t \ge 0\ . \tag{4.2}$$

Here the quotient is defined for every $z \notin \partial B$ by

$$\frac{g^2(z)}{\phi'(E(z))} = \begin{cases} \infty & \text{for } \phi'(E(z)) = 0 \text{ and } g(z) \neq 0 \\ 0 & \text{for } \phi'(E(z)) \text{ arbitrary and } g(z) = 0\ . \end{cases}$$

Proof : Assume that

$$\int_B \frac{f^2(t,z)}{|\phi'(E(z))|}\,dz < \infty$$

for some $t \geq 0$, then with Cor. 3.3,

$$\begin{aligned}
\mathcal{F}(t) &= -\int_B \frac{f^2(t,Z(t,0,z))}{\phi'(E(z))}\,dz - \gamma \iint \frac{f(t,Z(t,0,z))f(t,Z(t,0,z'))}{|X(t,0,z)-X(t,0,z')|}\,dz\,dz' \\
&= -\int_B \frac{\mathring{f}^2(z)}{\phi'(E(z))}\,dz - \int_B\int_0^t \frac{\frac{d}{ds} f^2(s,Z(s,0,z))}{\phi'(E(z))}\,ds\,dz - \gamma \iint \frac{\mathring{f}(z)\mathring{f}(z')}{|x-x'|} dz\,dz' \\
&\quad -\gamma \iint \int_0^t \frac{d}{ds} \frac{f(s,Z(s,0,z))f(s,Z(s,0,z'))}{|X(s,0,z)-X(s,0,z')|}\,ds\,dz\,dz' \\
&= \mathcal{F}(0) - 2\int_B\int_0^t f(s,Z(s,0,z))\Big(\partial_x U_f(s)\cdot v\Big)(Z(s,0,z))\,ds\,dz \\
&\quad +\gamma \iint \int_0^t \frac{X(s,0,z)-X(s,0,z')}{|X(s,0,z)-X(s,0,z')|^3}(V(s,0,z)-V(s,0,z')) \\
&\qquad\qquad f(s,Z(s,0,z))f(s,Z(s,0,z'))ds\,dz\,dz' \\
&\quad -2\gamma \int_B\int_0^t \phi'(E(z))\Big(\partial_x U_f(s)\cdot v\Big)(Z(s,0,z)) \\
&\qquad\qquad \int \frac{f(s,Z(s,0,z'))}{|X(s,0,z)-X(s,0,z')|}\,dz'\,ds\,dz \\
&= \mathcal{F}(0) - 2\int_0^t \int_B f(s,z)\partial_x U_f(s,x)\cdot v\,dz\,ds \\
&\quad +\gamma \int_0^t \iint \frac{x-x'}{|x-x'|^3}(v-v')f(s,z)f(s,z')\,dz\,dz'\,ds \\
&\quad +2\int_0^t \int_B \phi'(E(z))\partial_x U_f(s,x)\cdot v\,U_f(s,x)\,dz\,ds \\
&= \mathcal{F}(0) - 2\int_0^t \int_B f(s,z)\partial_x U_f(s,x)\cdot v\,dz\,ds \\
&\quad +2\int_0^t \int_B f(s,z)\partial_x U_f(s,x)\cdot v\,dz\,ds \\
&\quad +2\int_0^t \int_B \phi'(E(z))\,\partial_x U_f(s,x)\cdot v\,U_f(s,x)\,dz\,ds
\end{aligned}$$

$= \mathcal{F}(0)$;

the last integral vanishes because the integrand is odd in v. Retracing all the steps of the argument, we observe that all the integrals exist by the boundedness of the term $\partial_x U_f(s,x)\cdot v$ on B and the integrability of $\phi' \circ E$, and Fubini's Theorem applies. $\square$

Remark: The energy expression of (4.2), restricted to monotonically decreasing (nonvanishing) stationary phase space densities, was first obtained in [24] in a more general plasma physics model than that of the Vlasov-Poisson system. Imposing condition (4.1) allows one to consider stationary solutions of compact support by restricting the class of initial conditions. If one supposes *dynamically accessible* [3, 17, 22, 27, 28, 29] perturbations of the stationary state, i.e., that $\mathring{f}$ arises from perturbations of the underlying characteristics (particle orbits), where the perturbations are caused by electrostatic or gravitational forces, then $\mathring{f} = [g, f_0]$ for some function g, where $[\cdot,\cdot]$ denotes the usual Poisson bracket. In this case condition (4.1) turns into the following condition on g

$$\int [g,E]^2(z)|\phi'(E(z))|dz < \infty ,$$

and the singularity in (4.1) due to the vanishing of ϕ' outside B disappears.

5 Linear stability

Theorem 5.1 *Let $(f_0, U_0) \in \mathcal{S}$, assume that $\gamma\phi'(E) > 0$ for $U_{\min} \le E < E_0$, and define the weighted L^2-norm*

$$||g||^2_{2,\phi} := \gamma \int_B \frac{g^2(z)}{\phi'(E(z))}\, dz .$$

Then (f_0, U_0) is linearly stable in the following sense: For every $\mathring{f}$ as in Thm. 4.1 with $||\mathring{f}||_1^{2/3} ||\rho_{\mathring{f}}||_\infty^{1/3} \le 1$ the corresponding solution f of the linearized Vlasov-Poisson system satisfies the estimate

$$||f(t)||^2_{2,\phi} \le c_0 ||\mathring{f}||_{2,\phi} + ||\mathring{f}||^2_{2,\phi},\ t \ge 0 ,$$

where the constant c_0 depends only on the stationary solution (f_0, U_0).

Proof :

$$
\begin{aligned}
\|f(t)\|_{2,\phi}^2 &= \gamma \int_B \frac{f^2(t,z)}{\phi'(E(z))}\, dz \\
&= -\gamma \mathcal{F}(t) - \frac{\gamma^2}{4\pi} \int |\partial_x U_f(t,x)|^2 dx \\
&\leq -\gamma \mathcal{F}(0) = \gamma \int_B \frac{\mathring{f}^2(z)}{\phi'(E(z))}\, dz - \gamma \int_B U_{\mathring{f}}(x) \mathring{f}(z)\, dz \\
&\leq \|\mathring{f}\|_{2,\phi}^2 - \gamma \int_B \frac{\mathring{f}(z)}{\sqrt{\gamma \phi'(E(z))}} \sqrt{\gamma \phi'(E(z))}\, U_{\mathring{f}}(x) dz \\
&\leq \|\mathring{f}\|_{2,\phi}^2 + \left(\int_B |\phi'(E(z))| U_{\mathring{f}}^2(x) dz \right)^{1/2} \|\mathring{f}\|_{2,\phi} \\
&\leq \|\mathring{f}\|_{2,\phi}^2 + 2(2\pi)^{1/3} \|\mathring{f}\|_1^{2/3} \|\rho_{\mathring{f}}\|_\infty^{1/3} \left(\int_B |\phi'(E(z))|\, dz \right)^{1/2} \|\mathring{f}\|_{2,\phi} ,
\end{aligned}
$$

where the last estimate follows from (3.7). Thus, the proof is complete, with

$$
c_0 := 2(2\pi)^{1/3} \left(\int_B |\phi'(E(z))|\, dz \right)^{1/2} .
$$

□

Remarks:

1. Using [35, Lemma 2] to estimate the potential energy corresponding to $\mathring{f}$ in the above proof we obtain the alternative stability estimate
$$
\|f(t)\|_{2,\phi}^2 \leq c_1 \|\mathring{f}\|_2^2 + \|\mathring{f}\|_{2,\phi}^2,\ t \geq 0 ,
$$
for all initial data as in Thm. 4.1, where c_1 again depends on the stationary solution (f_0, U_0).

2. If $0 < c_- \leq |\phi'(E)| \leq c_+ < \infty$ on $]-\infty, E_0[$ then the norm $\|\cdot\|_{2,\phi}$ is equivalent to the usual L^2 norm, and we obtain the stability estimate
$$
\|f(t)\|_2 \leq c_3 \|\mathring{f}\|_2,\ t \geq 0
$$
for all initial data as in Thm. 4.1.

3. The stellar dynamics case where $\phi'(E) > 0$ is of particular interest. This result requires the jump discontinuity in ϕ and the restricted class of initial conditions, $\mathring{f}$, described in Sec. 4. It is natural to question the physical relevance of and the sensitivity to these assumptions. One would expect collisions, i.e. the effect of short range interactions, to smooth out the jump discontinuity in ϕ and produce a transition region where $\phi'(E) > 0$ (and large). In this way collisions can provide a mechanism for the onset of instability. The assumption that $\mathring{f}$ vanish outside the set B is of physical importance, since, as noted in Sec. 4, perturbations caused by electrostatic or gravitational forces acting on the point mass orbits are naturally of this form [3, 28].

6 Examples

In this section we establish the existence of a large class of stationary solutions $(f_0, U_0) \in \mathcal{S}$. Among these there are steady states satisfying the stability condition of Thm. 5.1, i.e. $\phi'(E) < 0$ in the plasma physics case and $\phi'(E) > 0$ in the stellar dynamics case.

Any f_0 of the form

$$f_0(x, v) = \phi(E) = \phi\left(\frac{1}{2}v^2 + U_0(x)\right) \tag{6.1}$$

automatically satisfies Vlasov's equation, since the energy E is constant along characteristics. Therefore, the stationary Vlasov-Poisson system is reduced to the semilinear Poisson equation

$$\triangle U_0(x) = 4\pi(\rho^+(x) + \gamma h_\phi(U_0(x)), \ x \in \mathbb{R}^3 \ ,$$

where

$$h_\phi(u) := \int \phi\left(\frac{1}{2}v^2 + u\right) \, dv \ .$$

Here we investigate spherically symmetric solutions of this problem, i.e. solutions of

$$\frac{1}{r^2}\left(r^2 U_0'(r)\right)' = 4\pi(\rho^+(r) + \gamma h_\phi(U_0(r)) \ , r > 0 \ , \tag{6.2}$$

where

$$h_\phi(u) := \frac{2\pi}{r^2} \int_0^\infty \int_0^\infty \phi\left(\frac{1}{2}w^2 + \frac{F}{2r^2} + u\right) dF\, dw\ ; \tag{6.3}$$

$r := |x|$, $w := \frac{x \cdot v}{|x|}$, $F := |x \times v|^2 = x^2 v^2 - (x \cdot v)^2$. The distribution function f_0 is then a function of r, w, F, and $\rho_0(r) = h_\phi(U_0(r))$ is a function of r, i.e. the whole steady state is spherically symmetric.

For the rest of this section let ϕ satisfy the conditions (ϕ1) and (ϕ2) from Sec. 2 and assume in addition

(ϕ4) *Case (S)* (stellar dynamics case): $\phi(E_0) := \lim_{E \nearrow E_0} \phi(E)$ exists and $\phi(E_0) > 0$,

Case (P) (plasma physics case): $\rho^+ \in C([0,\infty[)$, $\rho^+ \geq 0$, $r^2\rho^+ \in L^1([0,\infty[)$, and there exist constants $r_0 > 0$ and $\rho_0^+ > 0$ such that $\rho^+(r) \geq \rho_0^+$, $r \in [0, r_0]$.

Theorem 6.1 *Let the assumptions (ϕ1), (ϕ2), and (ϕ4) be satisfied. Then there exists a constant $\alpha_0 < E_0$ such that for $\alpha \in]\alpha_0, E_0[$ the problem (6.2) has a unique solution $U_0 \in C^2([0,\infty[)$ with $U_0(0) = \alpha$, where h_ϕ is defined by (6.3). U_0 is strictly increasing, $U_0'(0) = 0$, $E_0 < \lim_{r\to\infty} U_0(r) < \infty$, and there exists $R_0 > 0$ such that $U_0(R_0) = E_0$ and $U_0'(R_0) > 0$. Consequently, if ϕ in addition satisfies (ϕ3) and if we define f_0 by Eq. (6.1) and $\rho_0 := h_\phi \circ U_0$, then $(f_0, U_0) \in \mathcal{S}$, $\rho_0 \in C^1([0,\infty[)$, and $\rho_0(r) = 0$ for $r \geq R_0$, $\rho_0(r) > 0$ for $r < R_0$. Under the further assumption that $\gamma\phi'(E) > 0$ for $E < E_0$, the steady state (f_0, U_0) is stable in the sense of Thm. 5.1.*

For the proof of this result the following lemma is useful:

Lemma 6.2 Let ϕ satisfy the assumptions (ϕ1) and (ϕ2). Then (6.3) defines a function $h_\phi \in C^1(\mathbb{R})$, $h_\phi(u) = 0$ for $u \geq E_0$, and

$$h_\phi(u) = 4\pi\sqrt{2} \int_u^\infty \phi(E)\sqrt{E-u}\, dE,$$

$$h_\phi'(u) = -\frac{4\pi}{\sqrt{2}} \int_u^\infty \phi(E) \frac{dE}{\sqrt{E-u}}\ , \quad u \in \mathbb{R}.$$

The proof of this lemma is an easy application of Lebesgue's theorem on dominated convergence and therefore omitted.

Proof of Thm. 6.1: Local existence and uniqueness of the solution for arbitrary $\alpha \in \mathbb{R}$ follow by the contraction mapping principle, applied to the following reformulation of the problem:

$$U_0'(r) = \frac{4\pi}{r^2} \int_0^r s^2 \left(\rho^+(s) + \gamma h_\phi\Big(\alpha + \int_0^s U_0'(\tau)d\tau\Big)\right) ds \ .$$

Let $U_0 \in C^1([0, R[)$ be the solution, extended to its maximal interval of existence $[0, R[$, and $\rho_0(r) := h_\phi(U_0(r))$. Then $U_0 \in C^2(]0, R[)$, $U_0''(r)$ has a limit for $r \to 0$, $U_0'(0) = 0$, and this implies the regularity assertions for U_0 and ρ_0. For the rest of the proof, we have to treat the two cases (P) and (S) separately.

Case (P): Take $\alpha_0 < E_0$ such that

(i) $h_\phi(u) < \rho_0^+/2$ for $u \in]\alpha_0, E_0[$,

(ii) $E_0 - \alpha_0 < \frac{\pi}{3}\rho_0^+ r_0^2$,

and let $\alpha \in]\alpha_0, E_0[$. Then by (i) $U_0'(r) > 0$, and U_0 is strictly increasing on $[0, r_0] \cap [0, R[$, in fact, because h_ϕ is decreasing,

$$\begin{aligned} U_0'(r) &= \frac{4\pi}{r^2} \int_0^r s^2(\rho^+(s) - h_\phi(U_0(s))ds \\ &> \frac{4\pi}{r^2} \int_0^r s^2 \left(\rho_0^+ - \frac{\rho_0^+}{2}\right) ds \\ &= \frac{2\pi}{3}\rho_0^+ r \ . \end{aligned}$$

Since $h_\phi(u) = 0$ for $u \geq E_0$ this implies that $R > r_0$, and by condition (ii)

$$U_0(r_0) > \alpha + \frac{\pi}{3}\rho_0^+ r_0^2 > E_0 \ .$$

Thus there exists $R_0 \in]0, r_0[$ with $U_0(R_0) = E_0$ and $U_0(r) > E_0\,(< E_0)$ for $r > R_0\,(< R_0)$ which implies the assertions on ρ_0.

Case (S): First of all we note that for any $\alpha < E_0$ the potential U_0 is strictly increasing on $[0, R[$. Either $U_0(r) < E_0$ for $r \in [0, R[$ in which case U_0 is bounded and thus exists globally, or $U_0(r) \geq E_0$ for $r \geq R_0$ and some R_0 in which case ρ_0 vanishes for $r \geq R_0$, and U_0 again exists globally. To prove that actually the latter case holds, we rely on the analysis in [5]. The existence of R_0 follows if we show that the (possibly infinite) limit

$$L := \lim_{r\to\infty} U_0(r) > E_0 \ .$$

Assume $L < E_0$. Then the monotonicity of U_0 and h_ϕ implies that $h_\phi(U_0(r)) \geq h_\phi(L)$ for $r \geq 0$, and thus,

$$U_0'(r) \geq \frac{4\pi}{r^2} \int_0^r s^2 h_\phi(L)\, ds = \frac{4\pi}{3}\, h_\phi(L)\, r,\ r \geq 0\ .$$

But this means that $U_0(r) \to \infty$ for $r \to \infty$, a contradiction. Thus it remains to show that the assumption $L = E_0$ leads to a contradiction as well. To this end, define $y(r) := E_0 - U_0(r),\ r \geq 0$, then

(i) $y(r) > 0$ and $y'(r) < 0$ for $r \geq 0$,

(ii) $\lim_{r\to\infty} y(r) = 0$, $\lim_{r\to\infty} r\, y(r) > 0$, and $\lim_{r\to 0} \dfrac{y'(r)}{r} < 0$,

(iii) $(r^2 y'(r))' = -H(r, y(r)),\ r > 0$, where $H(r, y) := 4\pi r^2 h_\phi(E_0 - y)$.

Here the assertions in (i) follow from the strict monotonicity of U_0, and the first assertion in (ii) is our assumption $L = E_0$. The second assertion in (ii) follows by l'Hospital's rule:

$$\begin{aligned}\lim_{r\to\infty} r(L - U_0(r)) &= \lim_{r\to\infty} \frac{L - U_0(r)}{1/r} = \lim_{r\to\infty} \frac{U_0'(r)}{1/r^2}\\ &= \lim_{r\to\infty} 4\pi \int_0^r s^2 \rho_0(s)\, ds = 4\pi \int_0^\infty s^2 \rho_0(s)\, ds > 0\ .\end{aligned}$$

Finally,

$$\frac{y'(r)}{r} = -\frac{U_0'(r)}{r} = -\frac{4\pi}{r^3} \int_0^r s^2 \rho_0(s) ds \to \frac{4\pi}{3}\, \rho_0(0) = -\frac{4\pi}{3}\, h_\phi(\alpha) < 0$$

for $r \to 0$. Condition (iii) is Eq. (6.2), rewritten for y. Obviously, H satisfies the relation

(iv) $r\partial_r H(r,y) = 2H(r,y),\ r \geq 0,\ y \in \mathbb{R}$.

The assumption ($\phi 4$) in case (S) implies the existence of constants $\alpha_0 < E_0$ and $0 < c_1 < c_2$ such that

$$c_1 \leq \phi(E) \leq c_2 \text{ for } E \in [\alpha_0, E_0[\ .$$

Take $\alpha \in]\alpha_0, E_0[$, then $0 < y(r) \leq E_0 - \alpha < E_0 - \alpha_0$ or $\alpha_0 < E_0 - y(r) < E_0$ for $r \geq 0$. Thus,

$$\begin{aligned}
\partial_y H(r, y(r)) &= -4\pi r^2 h'_\phi(E_0 - y(r)) \\
&= \frac{1}{\sqrt{2}}(4\pi)^2 r^2 \int_{E_0 - y(r)}^{E_0} \phi(E) \frac{dE}{\sqrt{E - E_0 + y(r)}} \\
&\leq \frac{(4\pi)^2}{\sqrt{2}} r^2\, c_2\, 2\sqrt{y(r)} \\
&= (4\pi)^2 \sqrt{2}\, c_2\, r^2 \sqrt{y(r)}\ ,
\end{aligned}$$

and

$$\begin{aligned}
H(r, y(r)) &\geq (4\pi)^2 \sqrt{2} c_1 r^2 \int_{E_0 - y(r)}^{E_0} \sqrt{E - E_0 + y(r)}\, dE \\
&= (4\pi)^2 \sqrt{2}\, \frac{2}{3}\, c_1\, r^2\, y(r)^{3/2}\ .
\end{aligned}$$

For α_0 sufficiently close to E_0 we can assume that

$$\frac{c_2}{c_1} < \frac{10}{3}\ .$$

Then there exists a constant $m \in]1, 5[$ such that $c_2/c_1 \leq 2m/3$ i.e. $c_2 \leq \frac{2}{3} c_1 m$, which in view of the above estimates for $H(r, y(r))$ and $\partial_y H(r, y(r))$ implies that

(v) $y(r)\partial_y H(r, y(r)) \leq m H(r, y(r)),\ r \geq 0$.

Conditions (i) to (v) now lead to the desired contradiction in the following way; cf. also [5]: Define

$$q(r) := \frac{r y^{\frac{m+1}{2}}(r)}{y'(r)}\ ,\ r \geq 0$$

and

$$Q(r) := r^2 y(r)^{\frac{1-m}{2}} y'(r)^2 q'(r)$$

$$= 3r^2 y(r) y'(r) + \frac{m+1}{2} r^3 y'(r)^2 + r y(r) H(r, y(r)) \ , \ r \geq 0 \ ,$$

then

$$Q'(r) = \frac{5-m}{2} r^2 y'(r)^2 + (r\partial_r H - 2H) y(r) + (y\partial_y H - mH) r y'(r)$$

$$\geq \frac{5-m}{2} r^2 y'(r)^2 \ , \ r \geq 0$$

by (iv) and (v), and

$$\lim_{r\to 0} Q(r) = 0 \ .$$

Thus

$$r^2 y(r)^{\frac{1-m}{2}} y'(r)^2 q'(r) = Q(r) = \int_0^r Q'(s) ds > 0 \ ,$$

which implies that $q'(r) > 0$ for $r > 0$, and we have shown that q is strictly increasing. The third assertion in (ii) yields

$$\lim_{r\to 0} q(r) = -A, \ A > 0 \ ,$$

and we conclude that

$$-A < q(r) = \frac{r y^{\frac{m+1}{2}}(r)}{y'(r)} \ , \ r > 0 \ .$$

Therefore,

$$(-A) y(r)^{-\frac{m-1}{2}} < (-A) \left(y(r)^{-\frac{m-1}{2}} - y(0)^{-\frac{m-1}{2}} \right)$$

$$= (-A) \int_0^r \left(-\frac{m-1}{2} \right) y(s)^{-\frac{m+1}{2}} y'(s) \, ds$$

$$< \frac{1-m}{2} \int_0^r s \, ds = \frac{1-m}{4} r^2, \ r > 0 \ ,$$

which implies that

$$ry(r) < Cr^{\frac{m-5}{m-1}} \ , \ r > 0$$

for some constant $C > 0$. But this contradicts the second assertion in (ii). Thus, the only remaining possibility is $L > E_0$ which implies that $U_0(R_0) = E_0$ for some $R_0 > 0$ also in the stellar dynamics case. In both cases U_0 is strictly increasing and $U_0'(r) \sim r^{-2}$ for $r > R_0$ so that $E_0 < \lim_{r\to\infty} U_0(r) < \infty$. Furthermore, $U'(R_0) > 0$ so that $\partial_z E(z) = (\frac{x}{r} U_0'(r), v) \neq 0$ for $z = (x, v) \in \partial B$. This shows that ∂B is a C^1-submanifold of $\mathbb{R}^6$ and is of measure zero. Since $U_0 \in C^2([0,\infty[)$ and $U_0'(0) = 0$, U_0 is C^2 when interpreted as a function on $\mathbb{R}^3$, and the proof is complete. $\square$

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