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Vortex Filament Evolution in Electron Magnetohydrodynamics and Three-Dimensional Model of Pinning

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Abstract

Three-dimensional vortex filament motion in the framework of local approximation is reduced to the 1D nonlinear Schrödinger equation. The essential feature of electron MHD model is that the skinning of the magnetic field of the vortex at the London scale leads to an *algebraic* accuracy of the local approximation, contrary to logarithmic one as in the well-known Hasimoto vortex in ideal liquid. Filament behavior in an inhomogeneous medium provides a new model of the pinning attraction of the filament to the minima of density.

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Vortex filaments are universal hydrodynamic objects arising in many problems such as atmospheric whirlwinds [1], vortex in superconductor [2], filaments in liquid Helium [3, 4], vortices in neutron stars, etc. Under "vortex filament" we understand a 1D line curling in 3D fluid, whose vorticity is localized inside the line. In resent years the model of localized vortex line was introduced in plasma physics, cf. [5, 6, 7]. Although plasma MHD was the motivation of this work, the authors believe that results to follow are more general. The existence of vortex filaments as well as vortices follows from the freezing-in of the generalized vorticity $\mathbf{\Omega}$. This means that the curl of the Lagrange particles' generalized momentum is frozen in the fluid,

$$\mathbf{\Omega}_t = \mathbf{\nabla} \times (\mathbf{v} \times \mathbf{\Omega}), \tag{1}$$

where the frozen in quantity (the generalized vorticity) is

$$\mathbf{\Omega} \equiv \mathbf{\nabla} \times \mathbf{P}. \tag{2}$$

Here $\mathbf{P} = m\mathbf{v} + e\mathbf{A}/c$ is the generalized momentum, $\mathbf{v}$ the local fluid velocity, $\mathbf{A}$ the vector potential. For many-component plasmas the explicit generalized vorticity was considered in [8, 9]. The freezing-in of the generalized vorticity is based on the Hamiltonianity of the fluid particles dynamics and coincides with the conservation of Poincare's relative integral invariant $\oint \mathbf{p} d\mathbf{q}$ [10]. Physically, the freezing-in of the vorticity means that the field lines of $\mathbf{\Omega}$ move with the fluid, the fluid particles are tied to the $\mathbf{\Omega}$ lines and carry them in the course of their motion. The vortex filament is a limiting case for thin vortices. Here $\mathbf{\Omega}$ is localized in an infinitely thin filament and is equal to zero elsewhere. The whirlpool and whirlwind are common examples of phenomena that may be modelled by vortex filaments. It is known that the vortex filament motion is approximately "local" in the ideal liquid [11, 12]: the logarithmic divergence of the self-action of the filament at small scale leads to a "local" equation for its dynamics:

$$\mathbf{v} = \frac{I\Lambda}{4\pi}\mathbf{r}_s \times \mathbf{r}_{ss}, \quad \Lambda \equiv \ln\frac{r_{max}}{r_{min}}. \tag{3}$$

Here $\mathbf{v}$ is the local velocity of the filament region, s the length along the filament, $I = \oint \mathbf{v}d\mathbf{r}$ the velocity circulation around the filament, which is an invariant of motion for the filament. The logarithmic factor is related to the divergence mentioned above, r_{max} and r_{min} are the characteristic filament curvature radius and the self-radius (thickness), respectively. The unusual and wonderful feature of Eq. (3) is that the Hasimoto transformation [13],

$$\psi = \sigma \exp i \int \kappa\, ds, \tag{4}$$

where σ is the filament curvature and κ the torsion reduces the filament dynamics to the nonlinear Schrödinger equation,

$$i\psi_t + I\Lambda/4\pi(\psi_{ss} + 1/2|\psi|^2\psi) = 0. \tag{5}$$

This leads, in particular, to the integrability [14] of the filament motion described by Eq. (3). Note that the form of this equation is the same as in the Heisenberg model of magnetics [15, 16]

$$\boldsymbol{\tau}_t = (I\Lambda/4\pi)\boldsymbol{\tau} \times \boldsymbol{\tau}_{ss}, \tag{6}$$

where $\boldsymbol{\tau} \equiv \mathbf{r}_s$.

Unfortunately, these beautiful equations possess only a logarithmic accuracy in the framework of hydrodynamics of ideal fluid. The omitted terms are not small and not local. We show that, unlike the case of Euler fluid, the local model of vortex filament is much more accurate in the framework of electron magnetohydrodynamics. Inspired by this observation, we apply this model to the description of pinning.

The electron magnetohydrodynamics (EMHD) approximation [17] was developed in order to describe the fast electron evolution in plasmas where the heavy ion background was regarded as motionless. The plasma density is then assumed specified and time-independent. For usual plasmas the principal criterion of validity of this approximation is the smallness of the parameter

$$\Pi_i = \omega_{pi}^2 a^2/c^2,$$

proportional to the line ion density, where $\omega_{pi} = (4\pi n e^2/M_i)^{1/2}$ is the ion plasma frequency and a the characteristic spatial scale. For more details about EMHD see review [17]. Let us prove the freezing-in of the generalized vorticity in the framework of EMHD. We start with the equation

$$\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v} = (e/m)(\mathbf{E} + \mathbf{v}\times\mathbf{B})$$

where e and m are the electron charge and mass, respectively. Upon taking curl and substituting $\nabla\times\mathbf{E} = c^{-1}\mathbf{B}$ one gets the equation of the freezing-in of vorticity (1), where

$$\boldsymbol{\Omega} = \nabla\times\mathbf{P} = \nabla\times[m\mathbf{v} - (e/c)\mathbf{A}].$$

In the EMHD approximation, $\mathbf{v} = c\nabla\times\mathbf{B}/(4\pi e n)$. Despite the presence of the field component (vector potential $\mathbf{A}$) in the generalized momentum $\mathbf{P}$, the vortex filament motion at $r_{min} \le c/\omega_{pe}$ is described by the same equation (3), but the accuracy of this equation is drastically improved. Namely, at $r_{max} \gg c/\omega_{pe}$ the logarithmic divergence at long scales disappears because of the screening effect at the London scale c/ω_{pe}. To prove this we note that outside the filament

$$\boldsymbol{\Omega} = \nabla\times\mathbf{P} = \nabla\times(m\mathbf{v} - e/c\mathbf{A}) = 0. \qquad (7)$$

Together with the relationships

$$\nabla\times\mathbf{B} = -4\pi e n\mathbf{v}e/c, \quad \mathbf{B} = \nabla\times\mathbf{A}, \quad n_e = n_i = n(\mathbf{r}),$$

Eq. (7) yields

$$\mathbf{B} + \nabla\times([c^2/\omega_{pe}]^2\nabla\times\mathbf{B}) = 0, \quad \omega_{pe}^2 = 4\pi n e^2/m. \qquad (8)$$

In the case of uniform background, $c^2/\omega_{pe}^2 = const$, Eq. (8) is reduced to the usual screening equation,

$$c^2/\omega_{pe}^2 \nabla^2 \mathbf{B} = \mathbf{B},$$

with the characteristic scale c/ω_{pe}. This means that the magnetic field is directed along the filament and falls off exponentially outside the filament. The screening eliminates the self-action of the filament at scale more than c/ω_{pe}. In addition, the screening leads to a more definite value of the logarithmic factor. Instead of the ambiguous scale r_{max}, the London screening scale c/ω_{pe} enters formula (3) for Λ.

It is very remarkable that the motion of an EMHD filament is much more integrable than that of an Euler filament [13, 14]. The additional term to the nonlinear Schrödinger equation is small as a power of $c/(\omega_{pe} r_{max})$. The authors have grounds to believe that this term is linear in ψ,

$$i\psi_t + I\Lambda/4\pi(\psi_{ss} + 1/2|\psi|^2\psi + \epsilon[c/\omega_{pe}]^2\psi_{ssss}) = 0, \quad \epsilon \simeq 1, \tag{9}$$

which will not be proved here. The Hamilton perturbations to the nonlinear Schrödinger equation (5) make solitons statistical attractors [18, 19], and the filament tends to form small localized loops from an arbitrary initial perturbation. (It is known that a soliton solution corresponds a loop on the vortex filament through the Hasimoto transformation [13].)

Generally speaking, there is no reason to assume the background density n in the EMHD model to be homogeneous. Therefore the problem arises of the filament dynamics in an inhomogeneous medium. We shall show that, for $\nabla n \neq 0$ and $c/\omega_{p_e}|\nabla \ln n| \ll 1$ (the characteristic spatial scale of n is much larger than the screening scale c/ω_{pe}), there arises an additional term in Eq. (8) proportional to $\nabla \ln n$:

$$\mathrm{v} = I\Lambda/4\pi \mathbf{r}_s \times (\mathbf{r}_{ss} - \nabla \ln n), \quad \Lambda = \ln[c/(\omega_{pe} r_{min})]. \tag{10}$$

In fact, this equation is written simply as the sum of the well known velocity in a uniform medium and the vortex drift in a nonuniform medium, such as atmosphere, ocean, or plasma

[17]. Below we provide a derivation. Choose the cylindrical coordinates r, ϕ, z so that the origin $r = 0,\ z = 0$ is situated at the local curvature center of the filament and the filament tangent $\boldsymbol{\tau} = \mathbf{r}_s$ is directed along $\widehat{\boldsymbol{\phi}}$. Neglecting higher derivatives we may regard the filament as an arc of a circle. This symmetry leads to that Eq. (8) does not depend on ϕ if rewritten in the r, ϕ, z coordinates:

$$c^2/\omega_{pe}^2\,(B_{rr} + B_{zz} - [\ln(nr)]_r B_r - [\ln(n)]_z B_z) = B. \tag{11}$$

Assuming the curvature and the inhomogeneity scales to be large compared with the screening scale,

$$c/\omega_{pe} \ll \min(r,\ |\boldsymbol{\nabla} \ln n|^{-1}),$$

one may seek the solution of Eq. (11) in the form

$$B = (1 + \mathbf{a} \cdot \delta\mathbf{r}) B_0(|\delta\mathbf{r}|), \tag{12}$$

where $\delta\mathbf{r} = (r - r_0, z)$, r_0 is the curvature radius, and B_0 the solution for the homogeneous $(r_0 = \infty, \boldsymbol{\nabla} n = 0)$ limit of Eq. (11):

$$c^2/\omega_{pe}^2 (B_{0rr} + B_{0zz}) = B_0.$$

Upon substituting (12) into (11) one finds

$$2(\mathbf{a} \cdot \boldsymbol{\nabla}) B_0 - \boldsymbol{\nabla} \ln(nr) \cdot \boldsymbol{\nabla} B_0 - [\mathbf{a} \cdot \boldsymbol{\nabla} \ln(nr)] B_0 = 0.$$

The third term on the LHS is negligible due to the smallness of the screening scale, hence we find

$$\mathbf{a} = \boldsymbol{\nabla} \ln(nr)/2.$$

The filament velocity is given by expression

$$\mathbf{v} = -\mathbf{j}/(en) = -c/(4\pi en) \boldsymbol{\nabla} \times \mathbf{B},$$

where the RHS is calculated at an appropriate point of the filament. Calculations result in

$$\mathbf{v} = c/(4\pi en)(\nabla B_0 \times \hat{\phi} + B_0 \mathbf{a} \times \hat{\phi}).$$

The first term on the RHS corresponds to the velocity circulation around the filament and should be omitted. The second term corresponds to the filament self-action. As B_0 is related to the circulation $I = \oint \mathbf{v} \cdot d\mathbf{r}$ at $|\delta \mathbf{r}| \ll c/\omega_{pe}$ by the relationship $B_0 = I/2\pi \ln[c/(\omega_{pe} r_{min})]$, one arrives at the expression (3). Formula (3) implies that the inhomogeneity of the medium leads to the filament motion perpendicular to the density gradient. Vortex pinning can be considered as the filament capture by local density minima. This phenomenon is observed in superconductors and may be relevant to some phenomena in the atmosphere.

The filament motion described by expression (3) provides a simple and transparent model of pinning. One can find that filament motion in the framework of Eq. (3) leads to conservation law

$$(n\, ds)_t = 0.$$

This implies that the infinitesimal value of $nds, where\ ds$ is the filament length element, is conserved during the motion. Calculations show that nds is proportional to the filament element energy:

$$dE = n\, ds\, \Lambda I^2 m_e / 4\pi, \quad \Lambda = \ln[c/(\omega_{pe} r_{min})],$$

where it is assumed that $r_{min} \ll c/\omega_{pe}$. This allows us to propose a new physical mechanism of filament capture by local density minima. Figure 1 exhibits this process. When passing by a density minimum region, the filament starts rotating around it. The rotation is accompanied by waves propagating along the filament from the rotation region. The waves carry away the energy and allow the filament to enter into the density minimum and to decrease its energy $dE \propto n\, ds$. The filament attraction to the density minima can also be seen from the evident trend of magnetic field to expand together with frozen-in electrons.

A variational principle can be used to describe the filament equilibrium in an inhomogeneous medium ($\nabla n \neq 0$). Minimization of the filament energy $E \propto \int n\, ds$ leads to the condition

$$\delta \int n\, ds = 0,$$

which exactly coincides with the optical variational principle defining the light ray geometry in a medium with inhomogeneous refractional coefficient $n = n(\mathbf{r})$. Using this analogy one can find the critical angle θ_c of filament confinement by a density well,

$$\theta_c = \arcsin(n/n_0),$$

where n_0 is the density of the surrounding medium. Filament attraction to density minima persist for $r_{min} \geq |\nabla \ln n|^{-1}$ (small scale of the density variation) but is no longer described by this simple geometrical-optics analogy.

We consider the filament motion in the standard approximation of a thin filament. It is essential that the stability analysis for arbitrary perturbations done by Ivonin [20] has shown the stability towards particular core structures. This is a strong argument for the real existence of vortex lines. A vortex filament in classical (non-quantum) fluids is not typical, but makes a serious impression when observed (tornado etc.). Our theory is non-quantum, but the authors believe that it may be applicable to suprconductors. A quantum theory renders only the meaning to parameters in Eq. (10). This equations can be used, for example, for the investigation of the propagation of electromagnetic perturbations through superconductors in the form of vortex waves. The principal results of this work are (1) unusually high accuracy of local approximation in electron MHD and (2) the simple model of filament attraction to density minima. The equation describing the filament motion in inhomogeneous media can be also used to study the creep effect (creep is a release of captured filaments observed in superconductors). The processes of filament reconnection, in analogy with the well known MHD [21, 22] process and EMHD [23] reconnection, appear to be important in this context.

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References

[1] W. Thompson, Phil. Mag. **10**, 155 (1880).

[2] P. Minnhagen, Rev. Mod. Phys. **59**, 1001 (1987).

[3] E. B. Sonin, Rev. Mod. Phys. **59**, 87 (1987).

[4] M. M. Salomaa and G. E. Volovik, Rev. Mod. Phys. **59**, 533 (1987).

[5] S. F. Edwards and J. B. Taylor, Proc. R. Soc. Lond. **A 336**, 257 (1974).

[6] Y. H. Ichikawa, private communication (unpublished).

[7] R. Kinney, T. Tajima, N. Petviashvili, and J. C. Williams, to be submitted to Phys. Rev. A (unpublished).

[8] S. I. Braginskii, in *Reviews of plasma physics*, edited by M. A. Leontovich (Consultants Bureau, New York, 1965), Vol. 1, p. 205.

[9] V. E. Zakharov, Zh. Eksp. Teor. Fiz. **60**, 1714 (1971), [Sov. Phys. JETP **33**(5), 927 (1971)].

[10] V. I. Arnold, *Mathematical methods in classical mechanics* (Springer, Heidelberg, 1978).

[11] L. S. D. Rios, Rend. Ctrc. Mat. Palermo **22**, 117 (1906), [in Italian].

[12] R. Betchov, J. Fluid Mech. **22**, 471 (1965).

[13] H. Hasimoto, J. Fluid Mech. **51**, 477 (1972).

[14] V. E. Zakharov and A. B. Shabat, JETP **37**, 823 (1973).

[15] M. Laksmann, Phys. Lett. **61A**, 53 (1977).

[16] L. A. Takhtajan, Phys. Lett **61A**, 235 (1977).

[17] A. S. Kingsep, K. V. Chukbar, and V. V. Yankov, in *Reviews of Plasma Physics*, edited by B. B. Kadomtsev (Consultants Bureau, New York, 1990), Vol. 16, p. 243, [Russian edition 1987].

[18] V. I. Petviashvili and V. V. Yankov, in *Reviews of Plasma Physics*, edited by B. B. Kadomtsev (Consultants Bureau, New York, 1989), Vol. 14, p. 1, [Russian edition 1985].

[19] V. E. Zakharov, A. N. Pushkarev, V. F. Shvets, and V. V. Yankov, JETP Lett. **48**, 83 (1988).

[20] I. A. Ivonin, Sov. J. Plasma Phys. **18**, 302 (1992), [Graduate work, Moscow Institute for Physics and Technology, 1988].

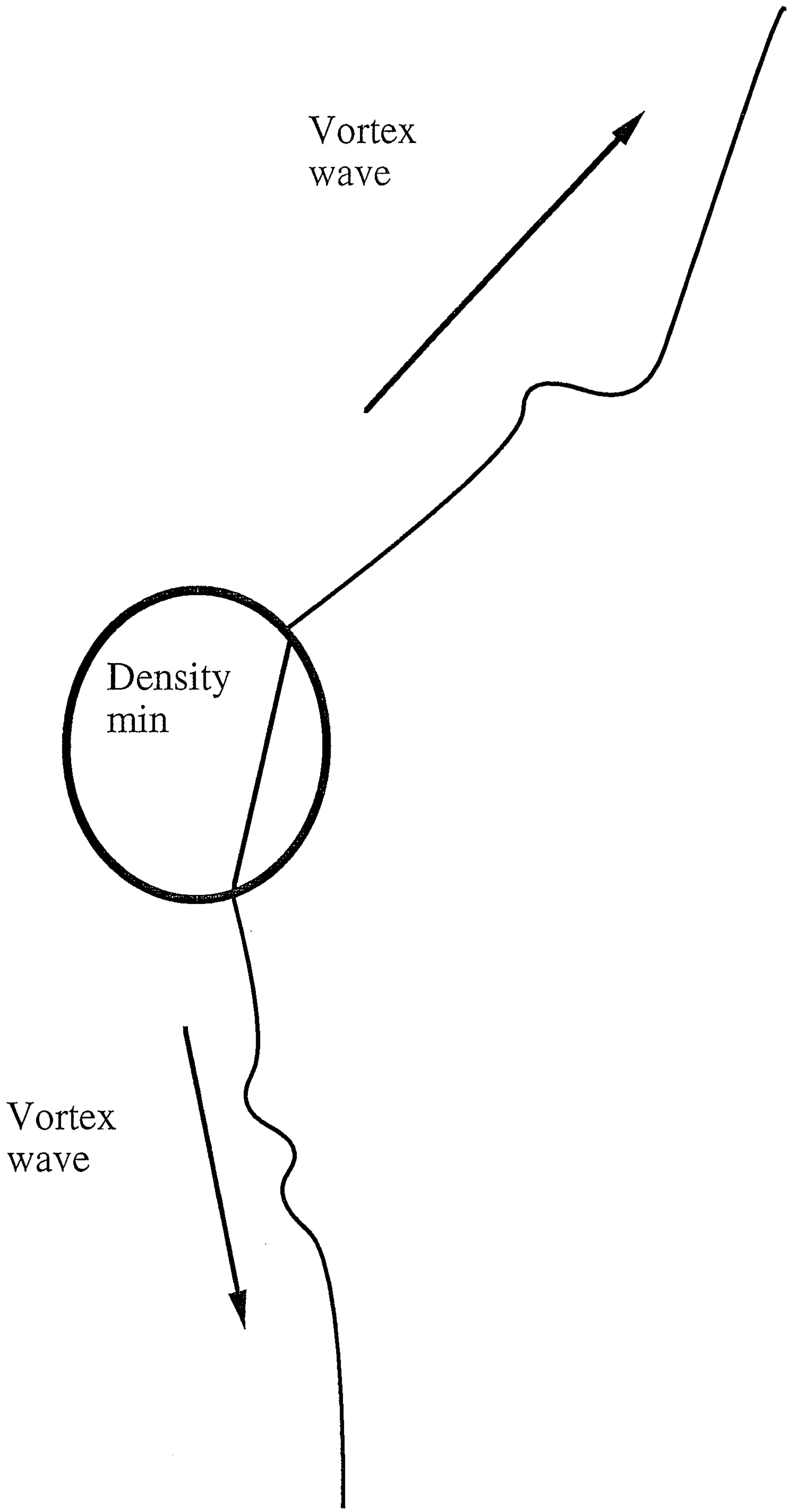
[21] E. N. Parker, Astrophys. J. Suppl. **77**, 310 (1963).

[22] J. B. Taylor, Rev. Mod. Phys. **58**, 741 (1986).

[23] S. V. Bulanov, F. Pegoraro, and A. S. Sakharov, Phys. Fluids B **4**, 2499 (1992).

Figure captions

Pinning of the vortex filament by a density minimum.


Vortex wave
Density min
Vortex wave