IFS #593

Hamiltonian Dynamics in Tokamak Configuration

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Abstract

The exact Hamiltonian with guiding-center variables is obtained through canonical transformation for the particles in tokamaks. The canonical variables are simply related to cylindrical coordinates, R, φ, Z. The drift Hamiltonian is calculated to the second order. The gyrokinetic equation is derived in a very simple way.

The simplicity and accuracy of the canonical transformation make the formalism much easier in the kinetic and stochastic theories in tokamaks.

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I Introduction

The derivation of the canonical variables for a given magnetic field has been given by Taylor,[1] Boozer,[2] and Littlejohn.[3] However, for a general magnetic field the canonical variables are algebraically complicated. They are further developed by Catto and Tang,[4] White,[5] Meiss and Hazeltine.[6]

In this paper, very simple canonical variables are discovered by Hamiltonian formalism through canonical transformation for the tokamak configuration.

There are many advantages to the canonical variables. They conserve phase space volume element, Jacobian equal to unity. The equations of motion are obtained easily from the Hamiltonian with canonical variables. The perturbation theory becomes easy in canonical variables.

A gyrokinetic formalism is derived in a very simple way, which is more accurate than the one developed by Lee, Myra, and Catto, because our canonical variables are accurate.

Applications of the Hamiltonian dynamics to the kinetic theory,[7] to the stochastic theory, and to other nonlinear theory are easy. Revolutionary change of the formalism in plasma physics in tokamak may take place.

In Sec. II, using canonical transformation, we get an exact Hamiltonian with guiding-center variables; drift Hamiltonian is given to the first order of ρ/r; equations of motion follow immediately. In Sec. III, the drift Hamiltonian is derived to the second order. The linear gyrokinetic equation is derived in the last section.

II Canonical Transformation

The Lagrangian equation is derived from a variational principle,[8]

$$\delta \int_{t_1}^{t_2} L(q, \dot{q}, t) dt = 0 \ , \tag{1}$$

where

$$L(q, \dot{q}, t) = \sum_i p_i \, \dot{q}_i - H(p, q, t) \ . \tag{2}$$

The integrated form of Eq. (1) in the two sets of coordinates can differ by a complete differential,

$$\sum_i p_i \, \dot{q} - H(p, q, t) = \sum_i \overline{p}_i \, \dot{\overline{q}}_i - \overline{H}(\overline{p}, \overline{q}, t) + \frac{d}{dt} F_1(q, \overline{q}, t) \ . \tag{3}$$

Expanding the total derivative of F_1, we obtain

$$\frac{d \, F_1(q, \overline{q}, t)}{dt} = \sum_i \frac{\partial F_1}{\partial q_i} \, \dot{q}_i + \sum_i \frac{\partial F_1}{\partial \overline{q}_i} \, \dot{\overline{q}}_i + \frac{\partial F_1}{\partial t} \ . \tag{4}$$

Taking the variables in Eq. (4) to be independent, we find, on comparing terms in Eq. (3) and requiring the $\dot{q}_i$ and $\dot{\overline{q}}_i$ terms to vanish separately, that

$$p_i = \frac{\partial F_1}{\partial q_i} \ , \tag{5}$$

$$\overline{p}_i = -\frac{\partial F_1}{\partial \overline{q}_i} \ , \tag{6}$$

$$\overline{H}(\overline{p}, \overline{q}, t) = H(p, q, t) + \frac{\partial}{\partial t} F_1(q, \overline{q}, t) \ . \tag{7}$$

F_1 is called generating functions. If F_2 is put in the place of F_1, we get, through the same procedure,

$$p_i = \frac{\partial F_2}{\partial q_i} \ , \tag{8}$$

$$\overline{q}_i = \frac{\partial F_2}{\partial \overline{p}_i} \ , \tag{9}$$

$$\overline{H}(\overline{p},\overline{q},t) = H(p,q,t) + \frac{\partial}{\partial t} F_2(q,\overline{p},t) \ , \tag{10}$$

where

$$F_2(q,\overline{p},t) = F_1(q,\overline{q},t) + \sum_i \overline{q}_i \, \overline{p}_i \ . \tag{11}$$

In tokamak configuration, the Hamiltonian of a charged particle can be expressed in (R, φ, Z) coordinate system,

$$H = \frac{1}{2M} \left[(P_R - eA_R)^2 + (P_z - eA_z)^2 + (P_\varphi - eA_\varphi)^2 / R^2 \right] + e\Phi \ , \tag{12}$$

where A_R, A_z, and A_φ are the vector potential components of the magnetic field, Φ the electric potential which is assumed a function of Ψ only, M the mass of the charged particle which we set equal to unity for simplicity, and e the charge. P_R, P_φ, and P_z are the canonical momenta conjugate to R, φ, and Z respectively,

$$P_R = V_R + eA_R \ , \tag{13}$$

$$P_\varphi = R\,V_\varphi + eR\,A_\varphi \ , \tag{14}$$

$$P_z = V_z + eA_z \ . \tag{15}$$

The magnetic field can be expressed as

$$\mathbf{B} = \nabla\varphi \times \nabla\Psi + I\,\nabla\varphi \ , \tag{16}$$

where Ψ is related to the potential flux of the magnetic field, I is related to the poloidal current, R_0 is the major radius. Then,

$$A_R = 0 \ , \quad A_z = -I\,\ell n \frac{R}{R_0} \ , \quad A_\varphi = -\frac{\Psi}{R} \ . \tag{17}$$

We introduce a generating function

$$F_1 = -\frac{\Omega_0 R_0^2}{2} \exp\left(\frac{X}{\Omega_0 R_0}\right) \left(\ell n \frac{R}{R_0} - \frac{X}{\Omega_0 R_0}\right)^2 \tan\alpha - ZX \ , \tag{18}$$

$$X = \Omega_0 R_0 \,\ell n \left(\frac{R_c}{R_0}\right) \ , \qquad R = R_c \exp\left(-\frac{\rho \cos\alpha}{R_c}\right) \ , \tag{19}$$

where Ω is the toroidal gyrofrequency, ρ the Larmor radius, α the gyrophase, subscripts 0 and c refer respectively to the values at the magnetic axis and the guiding center, X and α are the new coordinates conjugate to the momenta, from Eq. (6),

$$P_x = Z + \rho \sin\alpha + \frac{\rho^2}{4R_c} \sin 2\alpha \ , \tag{20}$$

$$P_\alpha = \frac{1}{2}\Omega_c \rho^2 \ , \tag{21}$$

where P_x is actually the guiding center of z coordinate. The Hamiltonian is rewritten as

$$H = \frac{1}{2}(\Omega_c \rho)^2 \left[\left(\frac{R_c}{R}\right)^2 \sin^2\alpha + \cos^2\alpha\right] + \frac{1}{2R^2}[P_\varphi + e\Psi]^2 + e\Phi \ . \tag{22}$$

This is an exact Hamiltonian for particles in tokamaks. After averaging over a period of α, one finds that P_α is a constant of motion for all order of ρ/r, since P_α is conjugate to α. To the first order, the Hamiltonian is obtained in a form

$$H = \Omega_c P_\alpha + \frac{1}{2R_c^2}[P_\varphi + e\Psi(X, P_x)]^2 + e\Phi \ , \tag{23}$$

where Ω_c and R_c are functions of X through Eq. (19).

Along with conservation of the canonical momentum in toroidal direction, we get a set of equations of motion for the guiding center,

$$\frac{dR}{d\tau} = \frac{B_R}{B_\varphi}\left(\frac{J}{R} - \frac{E_r}{B_p}\right) \ , \tag{24}$$

$$\frac{dZ}{d\tau} = \frac{B_z}{B_\varphi}\left(\frac{J}{R} - \frac{E_r}{B_p}\right) + \frac{P_\alpha}{R} + \left(\frac{J}{R}\right)^2 \ , \tag{25}$$

$$\frac{dJ}{d\tau} = \frac{B_R}{B_\varphi}\left(\frac{P_\alpha}{R} + \frac{J^2}{R^2}\right) + e\,E_\varphi R \ , \tag{26}$$

where $J = R\,V_\varphi$, all lengths are normalized by R_0, velocity by $\Omega_0 R_0$, time by Ω_0^{-1}. If the toroidal electric field can be neglected, the Hamiltonian is conservative. The subscripts are omitted for simplicity. The velocities in R and Z directions are easy to change to the r

and θ directions through rotating the coordinates. For a circular configuration, the particle guiding-center equations are reduced in (r, θ, φ) coordinates,

$$\frac{dr}{d\tau} = \frac{B_R}{B_\varphi}\left(\frac{P_\alpha}{R} + \frac{J^2}{R^2}\right) , \tag{27}$$

$$\frac{d\theta}{d\tau} = \frac{B_p}{r\,B_\varphi}\left(\frac{J}{R} - \frac{E_r}{B_p}\right) + \frac{B_z}{r\,B_p}\left(\frac{P_\alpha}{R} + \frac{J^2}{R^2}\right) , \tag{28}$$

where $B_R/B_p = -\sin\theta$, $B_z/B_p = -\cos\theta$, and $B_p/r\,B_\varphi = q\,R_0$, q the safety factor.

III The Second Order Gyroaveraged Hamiltonian

Using low β ordering

$$\varepsilon = \frac{r}{R_0} \sim \frac{\rho}{r} \sim \frac{B_p}{B_\varphi} , \tag{29}$$

we find that ρ/R_0 is the second order. R and Z in Eqs. (19) and (20) are coordinates in real space. Putting them into Eq. (22) and expanding with the ordering in Eq. (29) to the second order, after gyroaveraging along a line on which $P_\alpha, P_x, P_\varphi, x$, and φ are constant, we get

$$\overline{H} = \Omega_c\,P_\alpha\left(1 + \frac{B_p^2}{2B_\varphi^2}\right) + \frac{1}{2R_c^2}\left[P_\varphi + e\Psi + \iota\,P_\alpha\right]^2 + e\Phi + \frac{P_\alpha}{2B}\nabla^2\,\Phi , \tag{30}$$

where ι is the rotation transform equal to $\mu_0\,Rj_\varphi/2B_\varphi$. If the average does not change the Hamiltonian nature too much, we get the gyrofrequency,

$$\omega_\alpha = \frac{d\alpha}{dt} = \frac{\partial\overline{H}}{\partial p_\alpha} = \Omega_c\left(1 + \frac{B_p^2}{2B_p^2}\right) + \frac{V_\varphi}{qR} + \frac{\nabla^2\phi}{2B} , \tag{31}$$

which is shifted because of the motion in φ-direction and the shear of the radial electric field. And the gyrofrequency is related to B instead of B_φ. Not only electric potential Φ and flux function Ψ, but also the charge density and current density affect the particle dynamics.

IV Gyrokinetics

In this section the linear gyrokinetic equation is derived. We take the equilibrium distribution function F_m to be $F_m(H, P_\varphi)$, which is a Maxwellian form with H in the place of energy and P_φ in the place of Ψ_0. It is easy to see that F_m satisfies the unperturbed Vlasov equation. When plasma is perturbed, we have

$$F = F_m(H, P_\varphi) + f_1 \ , \tag{32}$$

$$\Phi = \Phi_0 + \Phi_1 \ , \tag{33}$$

$$A_\varphi = A_{\varphi 0} + A_{\varphi 1} \ , \tag{34}$$

where f_1, Φ_1, and $A_{\varphi 1}$ are the perturbations, for one harmonic

$$[f_1, \Phi_1, A_{\varphi 1}] = \left[\tilde{f}, \tilde{\phi}, \tilde{A}_\varphi\right] e^{i(k\rho\, \cos\alpha + m\,\delta \sin\alpha + kr + m\theta - n\varphi - \omega t)} \ , \tag{35}$$

$$\delta = \frac{\rho}{r} \ . \tag{36}$$

The Hamiltonian can be expanded,

$$H = H_0 + H_1 \ , \tag{37}$$

$$H_1 = e\Phi_1 - e\, v_\varphi\, A_{\varphi 1} \ . \tag{38}$$

And, we have

$$P_\varphi = P_{\varphi 0} + P_{\varphi 1} \ . \tag{39}$$

The distribution function can be expressed from Eqs. (32), (37), and (39),

$$F = F_m(H_0, P_{\varphi 0}) + H_1 \frac{\partial F_m}{\rho H_0} + P_{\varphi 1} \frac{\partial F_m}{\partial P_{\varphi 0}} + f_1 \ . \tag{40}$$

We put F into the Vlasov equation,

$$\frac{df_1}{dt} = -\frac{dH_1}{dt}\frac{\partial F_m}{\partial H_0} - \frac{dP_{\varphi 1}}{dt}\frac{\partial F}{\partial P_{\varphi 0}} = -\frac{dH_1}{dt}\frac{\partial F_m}{\partial H_0} + \frac{\partial H_1}{\partial \varphi}\frac{\partial F}{\partial P_{\varphi 0}}$$

$$= -\frac{ie}{T}(\omega - \omega^*)(\Phi_1 - V_\varphi\, A_{\varphi 1}) F_m \ , \tag{41}$$

where

$$\omega^* = \frac{nT}{\Omega_p R_0 r_n M} , \tag{42}$$

$$r_n = \frac{\partial}{\partial r} \ell n(F_m) . \tag{43}$$

We decompose f_1 into harmonics of the gyroangle,

$$f_1 = \sum_{n=-\infty}^{\infty} f_\ell e^{i\ell\alpha} . \tag{44}$$

Since f_ℓ is uncoupled, it follows that

$$(\omega - k\dot{r} - m\dot{\theta} + n\dot{\varphi} + \ell\Omega_c) f_\ell = \frac{e}{T}(\omega - \omega^*) F_m \oint \frac{d\alpha}{2\pi} (\tilde{\Phi} - v_\varphi \tilde{A}_\varphi) e^{i(k\rho\cos\alpha + m\delta\sin\alpha + kr + m\theta - n\varphi - \omega t - \ell\alpha)}$$

$$= \frac{e}{T}(\omega - \omega^*) F_m (\widehat{\Phi} - v_\varphi \tilde{A}_\varphi) i^\ell J_\ell(k\rho) e^{i(kr - m\theta - n\varphi - \omega t)} . \tag{45}$$

In Eq. (45), we have considered that δ is a small quantity. The resonance condition is θ-dependent not only for cyclotron frequency, but also for drift frequency.

Acknowledgments

We would like to thank Dr. R.D. Hazeltine and Dr. P. Morrison for checking the canonical transformation procedure and many helpful discussions. We are also grateful to Dr. J.B. Taylor for his comments on the constant motion to the second order.

Portions of this work were done while visiting the Institute for Fusion Studies, The University of Texas at Austin, Austin, Texas, and supported by the U.S. Department of Energy contract #DE-FG05-80ET-53088.

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