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Hydrodynamic equations for the drift-wave instability containing the $\underset{\sim}{E} \times \underset{\sim}{B}$ convective nonlinearity are used to show that the three wave interactions lead to temporal chaos with broad-band frequency spectra in the saturated state.

In this Letter we investigate the dynamics of three interacting drift waves, one of which is linearly unstable, showing that the saturated state can be chaotic with a broad band frequency spectrum. The stochastization of the plasma oscillations described here is proposed as a possible explanation for the broad band frequency spectrum measured[1] by scattering microwaves from the electron density fluctuations with a well-defined $\underset{\sim}{k}$ vector and space volume.

For low pressure plasmas with $T_e > T_i$, as in ohmic tokamak discharges, the drift-wave instability may be described as quasineutral, electrostatic oscillations with cold hydrodynamic ions. The ion density evolves by convection $\underset{\sim}{v}_E = (cT_e/eB)\hat{b} \times \nabla\phi$ and the compressible polarization drift $\underset{\sim}{v}_p = \underset{\sim}{v}_p^{\ell} + \underset{\sim}{v}_p^{n\ell} = -\rho^2(\partial_t + \underset{\sim}{v}_E \cdot \nabla)\nabla_\perp\phi$, according to

$$\frac{\partial n_i}{\partial t} + \underset{\sim}{v}_E \cdot \nabla n_i + \nabla \cdot (n_i \underset{\sim}{v}_p) = 0 \quad . \tag{1}$$

The electron behavior is dissipative and approximately linear with the density fluctuation given by

$$\frac{\delta n_e(\underset{\sim}{x},t)}{n_e} = \phi(\underset{\sim}{x},t) + \mathcal{L}^a\phi(\underset{\sim}{x},t) \quad , \tag{2}$$

where $\mathcal{L}^a$ is the anti-Hermitian operator giving the electron-wave dissipation. The density fluctuations are eliminated through quasi-neutrality to give the nonlinear equation for the electrostatic potential. The potential equation is made dimensionless with unit strength for the nonlinearity by measuring space in units of $\rho = c(m_i T_e)^{1/2}/eB$, time in units of r_n/c_s where $r_n^{-1} = -\partial_r \ell n\, \bar{n}_e(r)$, $c_s = (T_e/m_i)^{1/2}$ and the potential in units of $(\rho/r_n)(T_e/e)$. In this work we consider the collisionless or plateau electron regime where $\mathcal{L}^a(\underset{\sim}{k}) = i\delta_0 k_y(k_\perp^2 - \frac{1}{2}\eta_e)$ with $\delta_o = (\pi/2)^{1/2}(m_e/m_i)^{1/2}(L_c/r_n)$ a constant of order unity. Other models for the linear electron dissipation yield similar conclusions.

In the Fourier representation $\phi(\underset{\sim}{x}t) = \sum \phi_{\underset{\sim}{k}}(t)\exp(i\underset{\sim}{k} \cdot \underset{\sim}{x})$ the nonlinearity is given by the sum over interactions of modes $\underset{\sim}{k}_1$ and $\underset{\sim}{k}_2$ such that $\underset{\sim}{k}_1 + \underset{\sim}{k}_2 = -\underset{\sim}{k}_3$. The lowest order set of self-consistent nonlinear interactions from mode coupling occurs for three modes $\phi_i(t) = \phi_{\underset{\sim}{k}_i}(t)$ with $i = 1,2,3$. Introducing the amplitude $a_j(t)$ and phase $\alpha_j(t)$ according to $(1 + k_j^2)^{1/2} \phi_j(t) = a_j(t)\exp[i\alpha_j(t)]$, the three complex equations for $\phi_j(t)$ reduce to four real equations for the amplitudes and the total phase $\alpha = \alpha_1 + \alpha_2 + \alpha_3$:

$$\frac{da_j}{dt} = \gamma_j a_j - A a_k a_\ell (F_j \cos\alpha + G_j \sin\alpha) \tag{3}$$

$$\frac{d\alpha}{dt} = -\Delta\omega + A \sum_{j,k,\ell} \frac{a_k a_\ell}{a_j} (F_j \sin\alpha - G_j \cos\alpha) \tag{4}$$

where j,k,ℓ are cyclic permutations of 1,2,3. In Eqs. (3) and (4) we have $\Delta\omega = \omega_1 + \omega_2 + \omega_3$ and γ_j is the linear growth of the $\underset{\sim}{k}_j$ mode. The linear frequency and growth rate are given by $\omega_j + i\gamma_j = k_{yj}/[1+k_j^2 + \mathcal{L}^a(\underset{\sim}{k}_j)]$. For $\delta_0 < 1$ the linear frequency is $\omega_j \simeq k_{yj}/(1+k_j^2)$ and the linear growth rate is $\gamma_j \simeq \delta_0 k_{yj}^2 (k_j^2 - \eta_e/2)/(1+k_j^2)^2$. The coupling strength A is proportional to the area of the triangle formed by the $\underset{\sim}{k}$ vectors of the interacting modes $A = 1/2(\underset{\sim}{k}_1 \times \underset{\sim}{k}_2 \cdot \hat{z})/[(1+k_1^2)(1+k_2^2)(1+k_3^2)]^{1/2}$. The coupling factors F_j and G_j are the real and imaginary parts of the symmetrized complex susceptibility arising from the E × B convection, $F_j - iG_j = \chi(\underset{\sim}{k}_k) - \chi(\underset{\sim}{k}_\ell)$ where $\chi(\underset{\sim}{k}) = \underset{\sim}{k}^2 - \mathcal{L}^a(\underset{\sim}{k}) = k_\perp^2 - i\delta_0 k_y (k_\perp^2 - \eta_e/2)$. This complex susceptibility combines the coupling models given earlier by Horton[2] with the nonlinear convection of density and dissipative electrons with that of Hasegawa[3] with the nonlinear polarization drift and adiabatic electrons. The complex susceptibilities in the three mode equations is generic to the drift wave problem and is an important generalization of earlier work[4-7] for the decay of a wave into its own subharmonic in which the three wave coupling coefficients are real.

The three amplitudes and phases evolve in a four-dimensional phase space defined by generalized polar coordinates with radii a_j and the angle α. The rate of change of a volume V element in this system is $dV/dt = 2\gamma_t$ where $\gamma_t = \gamma_2 + \gamma_2 + \gamma_3$. A necessary condition for bounded, stable, asymptotic behavior is that $\gamma_t < 0$, giving a volume contracting flow in the phase space.

In the limit of no dissipation ($\mathcal{L}^a \to 0$) the vectors $\underset{\sim}{\gamma}$ and $\underset{\sim}{G}$ vanish simultaneously and the system (3) and (4) is integrable. To show integrability we give the action-angle variables $J_j = a_j^2/F_j$ and α_j, and the Hamiltonian

$$H(J,\alpha) = -\sum_{i=1,2,3} \omega_i J_i + A(F_1F_2F_3)^{1/2}(J_1J_2J_3)^{1/2}\sin(\alpha_1 + \alpha_2 + \alpha_3) \qquad (5)$$

as the generator of the equations of motion $\dot{\alpha}_j = \partial H/\partial J_j$ and $\dot{J}_j = -\partial H/\partial \alpha_j$. The three constants of the motion are $J_1 - J_2 = m_{12}$, $J_1 - J_3 = m_{13}$ and $H(J,\alpha)$. From a physical point of view, the integrals may be expressed in terms of the wave energy $W = 1/2\Sigma(1+k_\perp^2)|\phi_{\underset{\sim}{k}}|^2 = \frac{1}{2}\Sigma a_j^2 = -1/2F_2(J_1-J_2) - 1/2F_3(J_1-J_3)$ and the potential enstrophy $U = 1/2\Sigma k^2(1+k_\perp^2)|\phi_{\underset{\sim}{k}}|^2 = 1/2F_2k_2^2(J_1-J_2) - 1/2F_3k_3^2(J_1-J_3)$. The plasma wave energy density W is the electron electrostatic energy density W_e =

$1/2e\delta n_e\phi$ in the electron distribution and the ion kinetic energy $W_i = 1/2 n_i m_i V_E^2$. The enstrophy U is related to the plasma wave-momentum density $P_y = \Sigma(k_y/\omega_{\underset{\sim}{k}})W(\underset{\sim}{k}) = 1/2\Sigma(1+k_\perp^2)^2|\phi_{\underset{\sim}{k}}|^2 = W+U$. From the integrals W, U, H the motion is one-dimensional.

When dissipation is retained in the system $\gamma, F, G \neq 0$, the existence of a quadrature is unknown. For $G \neq 0$ the enstrophy U is not conserved by the nonlinear coupling. We have performed many numerical integrations of the dissipative equations from which we conclude that there are perhaps isolating integrals for special values of γ, F, G but that generally the system is nonintegrable and chaotic. As has been done for simpler systems of equations[4-7], the behavior of this system is analyzed in terms of the stability of the fixed points and limit cycles. This stability analysis delineates regions of constant, periodic, aperiodic and stochastic behavior. The fixed points of the system are the roots of $\dot{a}_j = \dot{\alpha} = 0$ obtained from Eqs. (5),(6).

The stability $\exp(\lambda t)$ of the fixed point is determined by the eigenvalues $\Sigma c_n\lambda^n = 0$ of the 4×4 matrix of the linearized equations of motion. Four necessary and sufficient conditions for stability are given by $c_0 > 0$, $c_3 > 0$, $c_2c_3 > c_1^2$ and $c_1(c_2c_3 - c_1^2) > c_0c_3^2$, where $c_3 = -2\gamma_t$, $c_2 = \gamma_t^2 - \Delta\omega^2 + 2\Sigma[\gamma_j^2(F_j\sin\alpha - G_j\cos\alpha)^2/(F_j\cos\alpha + G_j\sin\alpha)^2]$, $c_1 = 4\gamma_1\gamma_2\gamma_3 + 4\gamma_1\gamma_2\gamma_3\Sigma(F_k\sin\alpha - G_k\cos\alpha)(F_\ell\sin\alpha - G_\ell\cos\alpha)/[(F_k\cos\alpha + G_k\sin\alpha)(F_\ell\cos\alpha + G_\ell\sin\alpha)]$ and $c_0 = 4\gamma_1\gamma_2\gamma_3\gamma_t - 4\gamma_1\gamma_2\gamma_3\Sigma\gamma_j(F_j\sin\alpha - G_j\cos\alpha)^2/(F_j\cos\alpha + G_j\sin\alpha)^2$.

There are numerous ways in which the parameters may be varied to make the system pass successively from the region of a stable fixed point to a region of chaotic behavior. We describe two methods and comment on their relevance to drift waves. In one, this succession is obtained by taking $|\underset{\sim}{F}| \ll |\underset{\sim}{G}|$ and increasing the magnitude of the damping of the stable modes relative to the growth rate of the unstable mode. In the other method, this succession occurs when the values of the growth and damping rates remain comparable in magnitude, but $|\underset{\sim}{F}|$ is increased relative to $|\underset{\sim}{G}|$. In this case stochastic behavior occurs for $|\underset{\sim}{F}| \gtrsim \frac{1}{3} |\underset{\sim}{G}|$. The first process is essentially the method applied in the studies of Refs. 4 through 7. The critical ratio Γ_c of the damping-to-growth for the onset of chaotic behavior is rather large ($\Gamma_c \sim 20$) and is atypical for drift waves due to the wavenumber constraint $\Sigma \underset{\sim}{k}_i = 0$ and the functional form of $\gamma(\underset{\sim}{k}_j)$. These constraints make growth rates with comparable amplitudes typical while the functional form of $\underset{\sim}{F}$ and $\underset{\sim}{G}$ make $|\underset{\sim}{F}| \sim |\underset{\sim}{G}|$. It is thus the complex susceptibilities which are instrumental in making the three drift wave interaction chaotic in contrast to the large Γ in earlier models.

The time evolution of the amplitudes and phase in the stochastic regime is depicted in Fig. 1 where the amplitudes and cosine of the phase are given for a typical drift wave simulation. The amplitude and the phase appear random with the phase covering the entire range $(0,2\pi)$ rather uniformly. After an initial transient there is saturation of the amplitudes and the total wave energy is constant in a time-average sense. The assertion that the motion is stochastic is supported by numerical evaluation of the maximal Lyapunov characteristic exponent and the autocorrelation of $\phi_j(t) = a_j \cos\alpha_j$. The Lyapunov exponent is positive and the autocorrelation function decays rapidly confirming two important aspects of stochastic signals.

In light of the stochasticity in the three wave interactions, we consider two important aspects of drift wave turbulence theory: the applicability of the random phase approximation and the frequency spectrum for fixed wavenumber $\underset{\sim}{k}$. Because of the random nature of the phase evolution in the three wave interactions, a random phase approximation seems plausible. We construct a set of reduced equations of motion based on the random values of $\alpha(t)$ and compare their solution with the exact amplitudes. Assumed in this procedure is the separation of time scales of the random fluctuations on the time scale τ_c from the long time scale for the evolution of the average amplitudes $I_j = \langle a_j^2 \rangle = (1+k_j^2)\langle|\phi_j|^2\rangle$. The random phase equations are

$$\frac{dI_j}{dt} = 2\pi\tau_{123}A^2\{(F_j^2 + G_j^2)I_k I_\ell + (F_j F_k + G_j G_k)I_\ell I_j + (F_j F_\ell + G_j G_\ell)I_k I_\ell\} + 2\gamma_j I_j \tag{6}$$

where $\tau_{123} = \nu/(\Delta\omega^2 + \nu^2)$ is the maximum interaction time. The undriven equilibrium of Eq. (6) is $I_j = I_0$ for $G \neq 0$ and $I_j = I_0/(1+\beta k_j^2)$ for $G = 0$, where I_0 and β are constants.

We have compared the numerical solutions of the exact Eqs. (3) and (4) for typical drift wave parameters with numerical solution of the random phase Eq. (6). In these comparisons the random phase equations are observed to faithfully reproduce the average behavior of the exact equations. This includes correctly giving the relative values of the mean amplitudes at saturation and the absolute values of these amplitudes when τ_{123} is correctly chosen. We conclude that the intrinsic stochasticity present in the interaction of the drift waves is a sufficient condition to justify the use of the random phase approximation for three wave interactions.

The frequency spectrum for the largest amplitude component $\phi_1(t)$ of the electric potential is shown in Fig. (2). The peak frequency is well above the maximum linear frequency $|\omega(\underset{\sim}{k})|$, and the width of the frequency spectrum $\nu(\underset{\sim}{k})$ is of the same order as the peak frequency.

In conclusion, we show that a broad band stochastic process occurs from the interaction of three electron drift waves for representative values of the system parameters. In the stochastic regime the random phase approximation is found to predict the average behavior of the wave amplitudes. Finally, the process given here appears to be a candidate for explaining the broad frequency spectra observed in the electromagnetic scattering experiments.

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FIGURE CAPTIONS

1. Time history of (a) W, U (b) a_1, a_2, a_3 and (c) $\cos\alpha$ for three drift waves with $\underset{\sim}{k}_1$ = (.296,-.716), k_2 = (.210,.506), k_3 = (-.506,.210) and (ω_1,γ_1) = (-.447,.160), (ω_2,γ_2) = (.389,-.197) and (ω_3,γ_3) = (.161,-.0338).

2. Frequency spectrum of the electrostatic potential $a_1(t)\cos\alpha_1(t)$ for the time signals shown in Fig. 1.

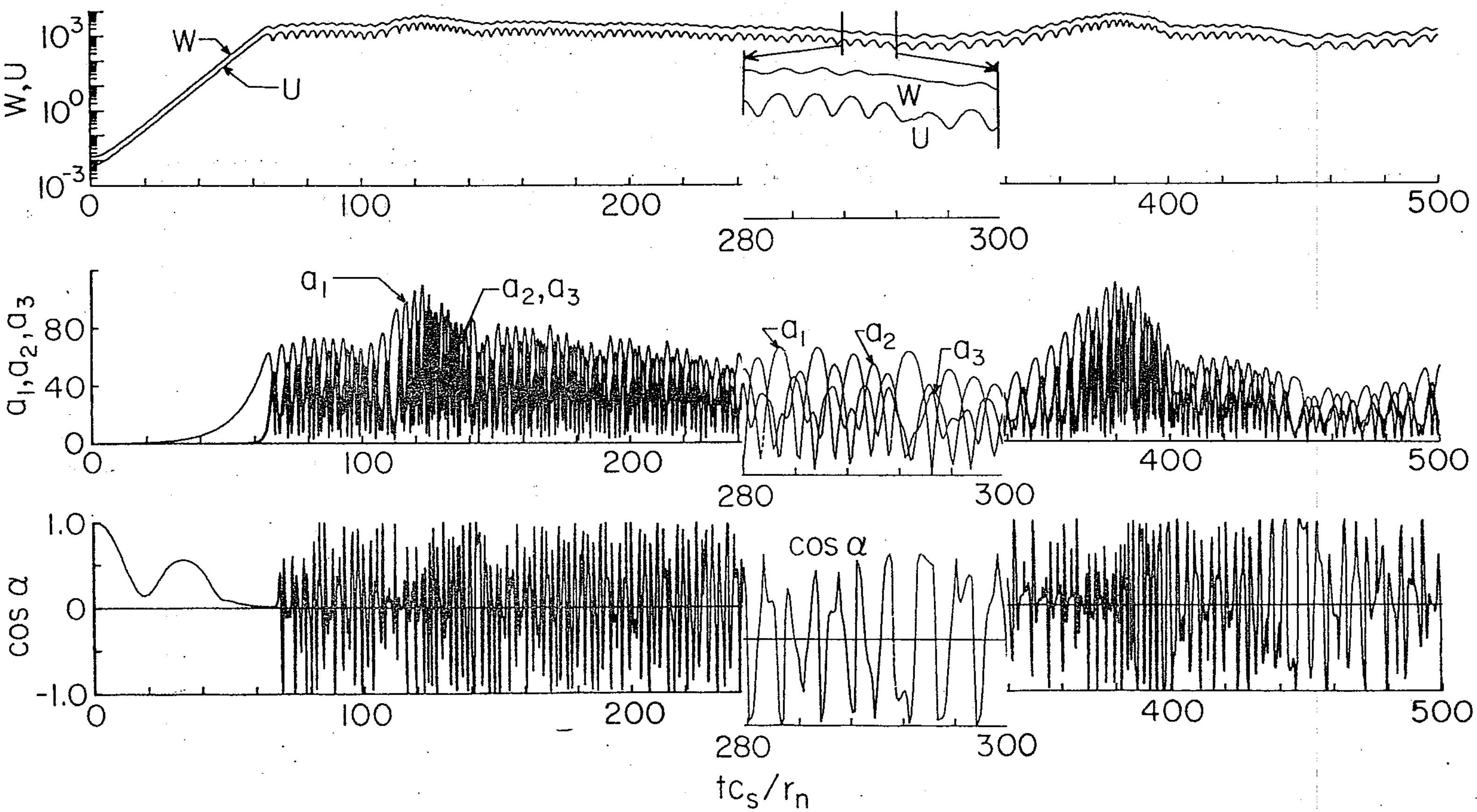
W,U
10^3
10^0
10^{-3}
W
U
a_1,a_2,a_3
a_1
a_2,a_3
a_3
a_2
80
40
0
cos α
1.0
-1.0
0
100
200
280
300
400
500
tc_s/r_n

Fig. 1

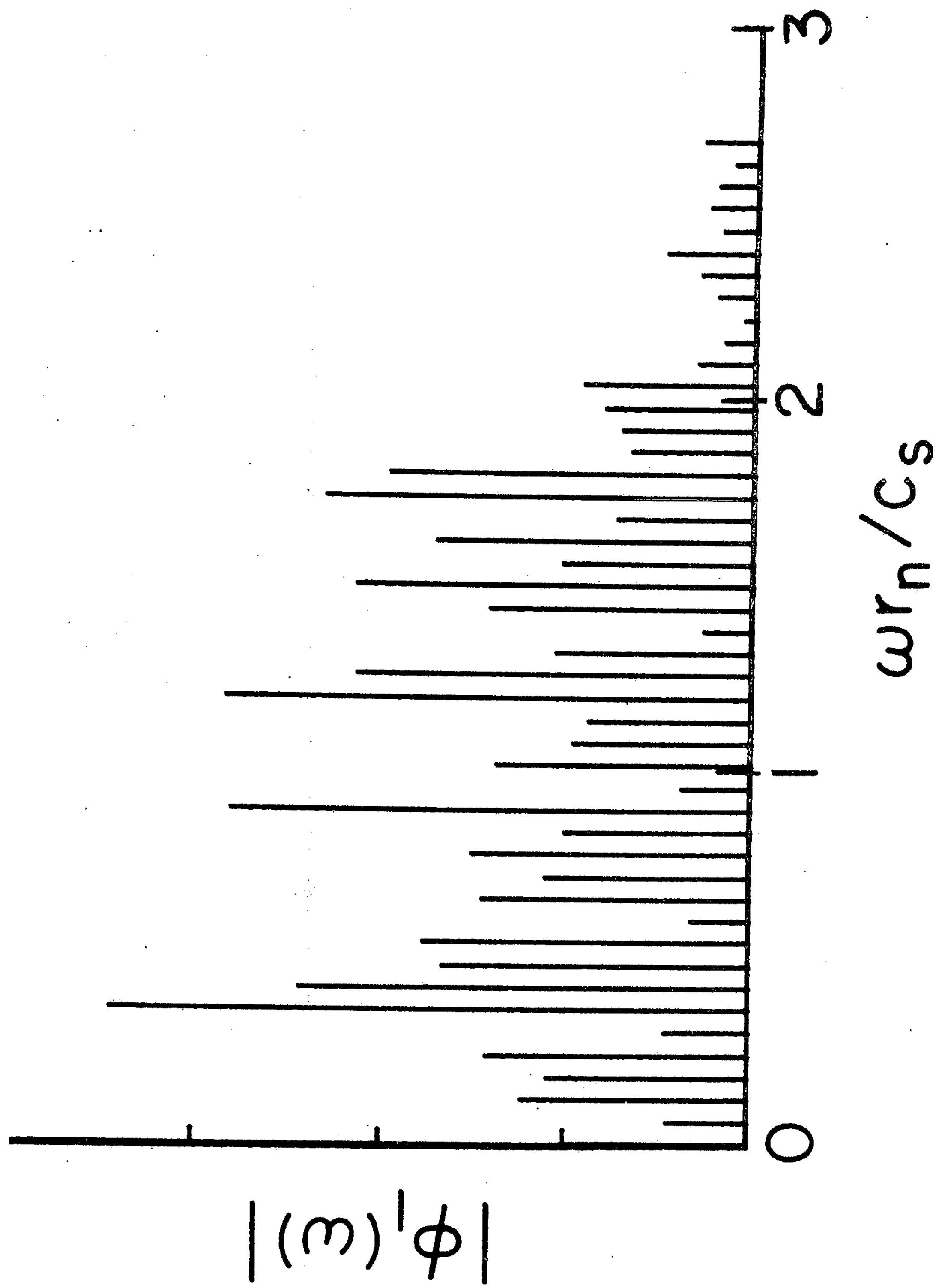

Fig. 2