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VACUUM MAGNETIC FIELDS WITH DENSE FLUX SURFACES

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ABSTRACT

A procedure is given for eliminating resonances and stochasticity in nonaxisymmetric vacuum toroidal magnetic field. The results of this procedure are tested by the surface of section method. It is found that one can obtain magnetic fields with increased rotational transform and decreased island structure while retaining basically the same winding law.

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The flow of magnetic field lines through three-dimensional space has an exact analogy[1] with the evolution of trajectories in a Hamiltonian system of one and a half degrees of freedom. Field lines may be closed, they may ergodically cover a two-dimensional flux surface (which must be topologically equivalent to a torus[2]), or they may wander ergodically throughout a volume. It is of practical and intrinsic interest to know the conditions for a magnetic field to have a dense set of flux furfaces. In analogy with Hamiltonian systems, such fields are termed integrable. Without any restrictions other than $\nabla \cdot B = 0$, it is easy to construct such magnetic fields. However, if the vacuum condition, $\nabla \times B = 0$, or the magnetohydrodynamic equilibrium condition[2-4] $(\nabla \times B) \times B = \nabla P$ is imposed, then magnetic fields with a dense set of periodic, ergodically covered flux surfaces are known only for the special cases of translational, rotational, or helical symmetry.

Whether there exist such magnetic fields in the absence of those special symmetries has been an open question. The purpose of this Letter is to address a part of this question: Do there exist vacuum magnetic fields with a dense set of bounded flux surfaces, rotational transform, no magnetic islands, and no currents inside the flux surfaces? The requirement of bounded flux surfaces eliminates fields with helical or translational symmetry; and the requirement of no internal currents eliminates the azimuthally symmetric systems such as the Levitron.[5] Thus, the state of the art is that such systems are not known. Here it will be shown that such systems can be found. A perturbation procedure is developed to find the magnetic field, and the final approximate result is checked numerically by a surface of section analysis and found to have significantly reduced island structure and stochasticity.

The ability to obtain a magnetic field with these properties is important to the present effort to magnetically confine plasma for the purpose of creating controlled fusion power. Having a dense set of bounded flux surfaces ensures particle confinement to lowest order in the ratio of Larmor radius to scale length. Rotational transform reduces or eliminates the first order drift of particles away from a flux surface. Lack of an internal conductor eliminates the problems associated with the interaction of the plasma with the conductor support. Absence of magnetic islands reduces the collisional transport of low-energy particles.

From the known examples of systems with good surfaces it is obvious that symmetry plays an important role. To determine the specific symmetry needed, it is useful to consider a variational description of field line flow, $\delta \int \underset{\sim}{A}(\underset{\sim}{r})\cdot(d\underset{\sim}{r}/d\lambda)d\lambda = 0$, in which the path is varied with certain endpoint restrictions. The Euler-Lagrange equations yield $\underset{\sim}{B} \times d\underset{\sim}{r}/d\lambda = 0$. The parameter λ is arbitrary. It may, for example, be chosen to be one of the coordinates. This variational principle can be transformed to any coordinates. In alternate coordinates, z^i, $i=1,2,3$, the Lagrangian is given by $L = \underset{\sim}{A} \cdot (\partial\underset{\sim}{r}/\partial z^i)\, dz^i/d\lambda = A_i(z)\, dz^i/d\lambda$, in which A_i is the covariant representation of the vector potential, and summation over repeated indices is assumed. Application of Noether's theorem to this Lagrangian implies that $A_1(z)$ is a flow invariant, if each component A_i of the vector potential is independent of z^1. Thus, the symmetry needed to prove the existance of flux-surfaces is a symmetry of the covariant representation of the vector potential. This is illustrated by the straight stellarator[6], for which the covariant components of the vector potential depend only on the radius and the helical angle.

To discuss the toroidal case it is convenient to introduce the toroidal coordinates ξ, η, φ, in terms of which the cartesian coordinates are $z = \xi \sin\eta/(1-\xi\cos\eta)$, $x = (1-\xi^2)^{1/2}\cos\varphi/(1-\xi\cos\eta)$, and $y = (1-\xi^2)^{1/2}\sin\varphi/(1-\xi\cos\eta)$. In these coordinates, Laplace's equation separates (Ref. 7, p.1301). The vacuum magnetic field is found in terms of the scalar magnetic potential $B = \nabla\Phi(\nabla^2\Phi=0)$. The result is[7]

$$\Phi = B_0\varphi + (1-\xi\cos\eta)^{1/2} \sum_{\ell,m} b_{\ell m} T_{\ell m}(\xi)\cos(\ell\eta + m\varphi + \phi_{\ell m}), \qquad (1)$$

in which B_0, $b_{\ell m}$, and $\phi_{\ell m}$ are constants. The functions $T_{\ell m}(\xi)$ are related to the modified Legendre functions of half-integer order according to $T_{\ell m}(x) = (-1)^m(2/\pi)^{1/2}(2\ell)!!\, x^{-1/2} Q^m_{\ell-1/2}(x^{-1})/\Gamma(\ell+m+1/2)$.

To look for symmetries, we must calculate the covariant components of the vector potential. In the gauge where $A = 0$, which can be chosen for the $m \neq 0$ terms, the remaining components of the vector potential are given by

$$A_\xi = \sum_{\substack{\ell \\ m\neq 0}} b_{\ell m} \frac{T_{\ell m}(\xi)}{m\xi(1-\xi\cos\eta)} \frac{\partial}{\partial\eta}\left[(1-\xi\cos\eta)^{1/2}\sin(\ell\eta + m\varphi + \phi_{\ell m})\right], \qquad (2a)$$

and

$$A_\eta = \frac{1}{2} B_0 \left(\frac{\ln\left|\frac{1-\xi\cos\eta}{1-\xi}\right|}{1-\cos\eta} + \frac{\ln\left|\frac{1-\xi\cos\eta}{1+\xi}\right|}{1+\cos\eta} \right)$$

$$+ \sum_{\substack{\ell \\ m\neq 0}} b_{\ell m} \frac{\xi(\xi^2-1)}{m(1-\xi\cos\eta)} \frac{\partial}{\partial\xi}\left[(1-\xi\cos\eta)^{1/2} T_{\ell m}(\xi)\right] \sin(\ell\eta + m\varphi + \phi_{\ell m}) . \qquad (2b)$$

As one can see, any single solution depends on all three variables independently because of coupling terms like $\xi \cos \eta$. In fact, one can show that no choice of the $b_{\ell m}$'s will make the components A_i a function of only one helical combination of η and φ. This does not mean that no systems with invariant surfaces exist, since there may be other coordinates or another gauge in which the symmetry would be apparent.

The surface of section technique can be used to analyze these systems lacking apparent symmetry. In this technique on numerically integrates the field line equations, $d\eta/d\varphi = B^\eta/B^\varphi$ and $d\xi/d\varphi = B^\xi/B^\varphi$, and plots the intersections of the field line with the $\varphi = 0$ plane. If the points lie on a curve, the field lines lie on a surface. Fig. 1 is a surface of section for the vacuum field of Eq. (1) corresponding to a single toroidal harmonic function: $B_0=1$, $b_{3,7}=4$, $b_{\ell,m}=0$ for $(\ell,m) \neq (3,7)$, and $\phi_{\ell,m}=0$ for all (ℓ,m). The field line generally lie on surfaces out to some final surface. The outermost confined line was determined to an accuracy of 10^{-4} of the minor radius. This line was found to have a rotational transform of .8232.

This surface of section shows the loss of the outer flux surfaces by stochastic effects. In an integrable system, such as the straight stellarator, the outermost confined line would be on the separatrix. The x-point of the separatrix is fixed in helical angle, $\ell\Delta\eta + m\Delta\varphi = 0$. This implies that the rotational transform of the last field lie of an integrable (ℓ=3, m=7) system would be $|\iota_s| = |\Delta\eta/\Delta\varphi| = 2\frac{1}{3}$. Yet in Fig. 1 the observed final rotational transform is ι_f = .8232. This indicates that all surfaces with rotational transform between .8232 and 2.333 have been destroyed by toroidal effects.

To analyze this surface destruction we introduce an averaging method analogous to that of Hamiltonian mechanics. We define the helical angle, $h \equiv \ell_0\eta+m_0\varphi$, and then we divide the covariant components of the vector potential into two pieces, $A_i(\xi,\eta,h) = \overline{A}_i(\xi,h) + \tilde{A}_i(\xi,\eta,h)$, one of which is the average over the poloidal angle at constant helical angle of the vector potential components:

$$\overline{A}_i(\xi,h) \equiv \frac{1}{2\pi} \int_0^{2\pi} d\eta \; A_i(\xi,\eta,h).$$

The averaged piece $\overline{A}_i$ has good surfaces because it is independent of η, while $\tilde{A}_i$ is a perturbation. Both fields are toroidal, but only their sum yields a curl free magnetic field.

The effect of the perturbation $\tilde{A}$ is to destroy the exact surfaces of the averaged vector potential $\overline{A}$. The cause of the surface destruction is the formation of resonances (magnetic islands). The calculation of these resonances proceeds by writing the perturbation field in terms of the flux variables of the averaged field $\overline{A}$, and then finding the fourier amplitudes in the angle variables of the magnetic field normal to the unperturbed flux surfaces. The resonance amplitude, the fourier amplitude evaluated at the resonant layer, determines the island width. Finally, the island widths are used in the Chirikov overlap criterion[8] to estimate the surface destruction. The details of this calculation are presented in Ref. 1.

This indicates that one could reduce the stochasticiy by selecting the perturbation to make the resonance amplitudes ~~vanish. This is accomplished by adding additional vacuum~~ fields to obtain a total field,

$$A = \bar{A} + \tilde{A} + \sum_{(\ell,m)\neq(\ell_0,m_0)} b_{\ell,m} A_{\ell,m}.$$

The condition of vanishing resonance amplitude results in a set of equations for the amplitudes $b_{\ell m}$ and the phases $\phi_{\ell m}$ of the additional fields. These equations are easily solved, since they are linear in the parameters $b_{\ell m}\cos\phi_{\ell m}$ and $b_{\ell m}\sin\phi_{\ell m}$. In principle one could eliminate all of the first-order resonances by this method; there is one-to-one correspondence between the additional fields and the resonance amplitudes. At this point there would remain smaller, higher-order resonances which could be eliminated by higher-order theory.

This method was applied to the system of Fig. 1. It was decided to add in additional vacuum fields with (ℓ,m) = (4,7),(5,7),..., (10,7). This allows the elimination of the primary resonances at rotational transform values of 7/10, 7/9, 7/8, ..., 7/4, i.e. all of the primary resonances in the stochastic region. Table I shows the actual amplitudes. As one can see, the amplitudes decrease rapidly with increasing ℓ.

Figure 2 shows the last magnetic field line for this particular vacuum toroidal magnetic field. The final rotational transform is found to be 1.205, which is significantly greater than the original .8232. Thus, a partial restoration of surfaces has been effected. This is also seen in that the final surface of the improved case is significantly more pointed , and, hence, much closer to a separatrix.

One might expect the volume contained by the last good surface to be significantly increased by this method. A comparison of Figs. 1 and 2 shows that this is not the case. The reason is that the rate of change of rotational transform with respect to volume is very large near the separatrix.

this does not mean that small-aspect-ratio stellarator fields with good surfaces cannot be found. To find such fields one simply begins with an unimproved field of smaller aspect ratio, which can be obtained by decreasing the helical amplitude b_{ℓ_0,m_0}.

The main tangible result of this work is a method for finding nonaxisymmetric vacuum magnetic fields with significantly increased rotational transform and decreased area of stochasticity and resonances. The problem of calculating the actual coils remains. This calculation is doable in principle by superimposing coils of various helicities. However, the determination of practical (e.g. modular[9]) coil configurations is a nontrivial problem. The results of Table I are encouraging in this respect. The rapid decrease of the harmonic amplitude with poloidal mode number ℓ indicates that the distortions of present coil designs will not be too rich in harmonic structure.

In addition, this work strongly indicates that there do exist nonaxisymmetric vacuum magnetic fields with a dense set of ergodically covered magnetic surfaces. No proof has been given, but since first-order theory significantly reduces the stochasticity, it is reasonable to believe that infinite-order theory would produce a completely nonstochastic system.

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Figure Captions

Fig. 1 Surface of section for an $\ell=3$, $m=7$ stellarator. This corresponds to the vacuum solution of Eq. (1) for the choice $B_0 = 1$, $b_{3,7} = 4$, $w_{\ell,m} = 0$ for $(\ell,m) \neq (3,7)$, and $\phi_{\ell,m} = 0$ for all (ℓ,m).

Fig. 2 Surface of section for an improved $\ell=3$, $m=7$ stellarator. Vacuum field amplitudes are given in Table I.

Table Caption

Table I Vacuum field amplitudes for the improved $\ell=3,m=7$ stellarator of Fig. 2.

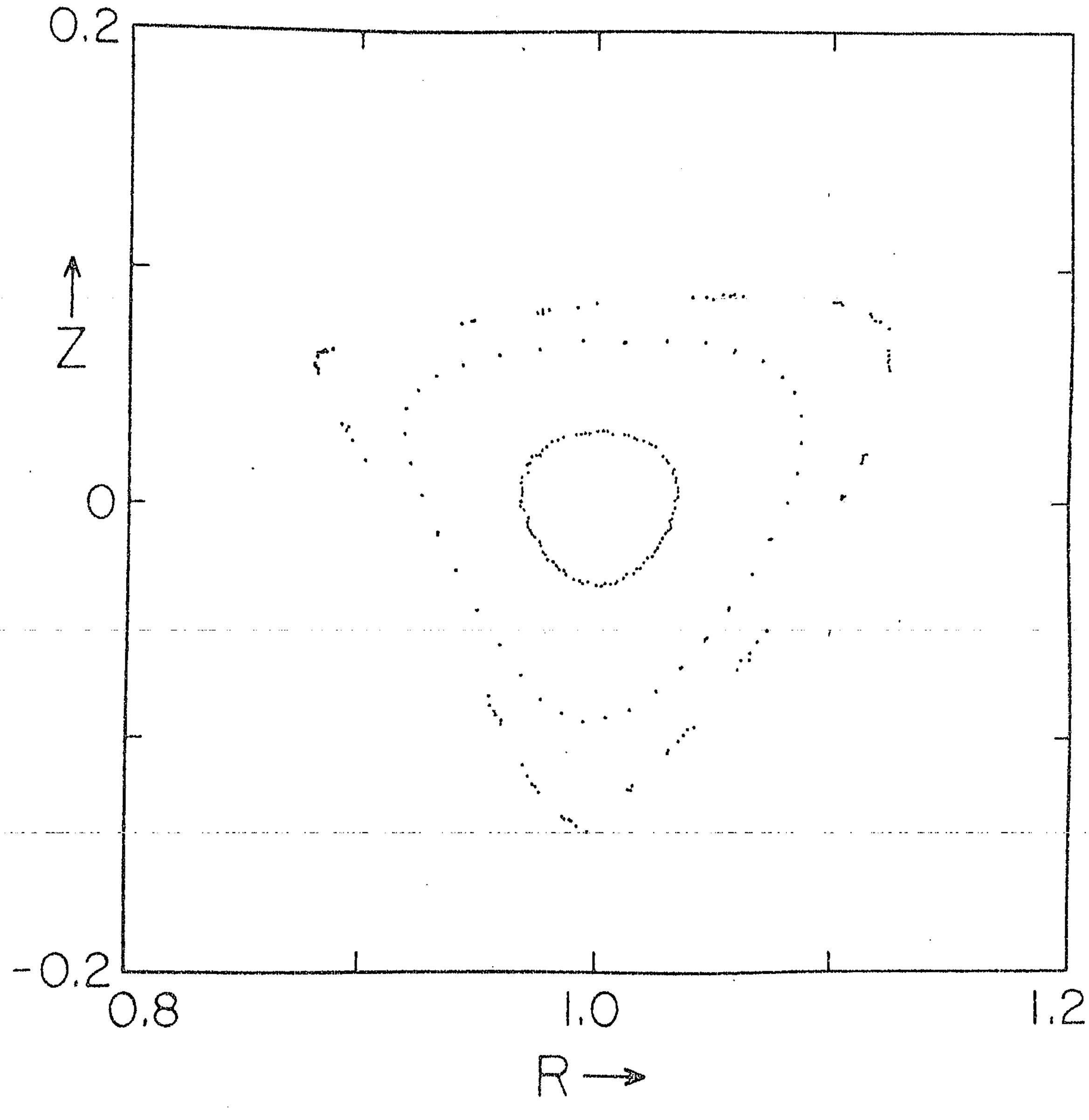

FIGURE 1

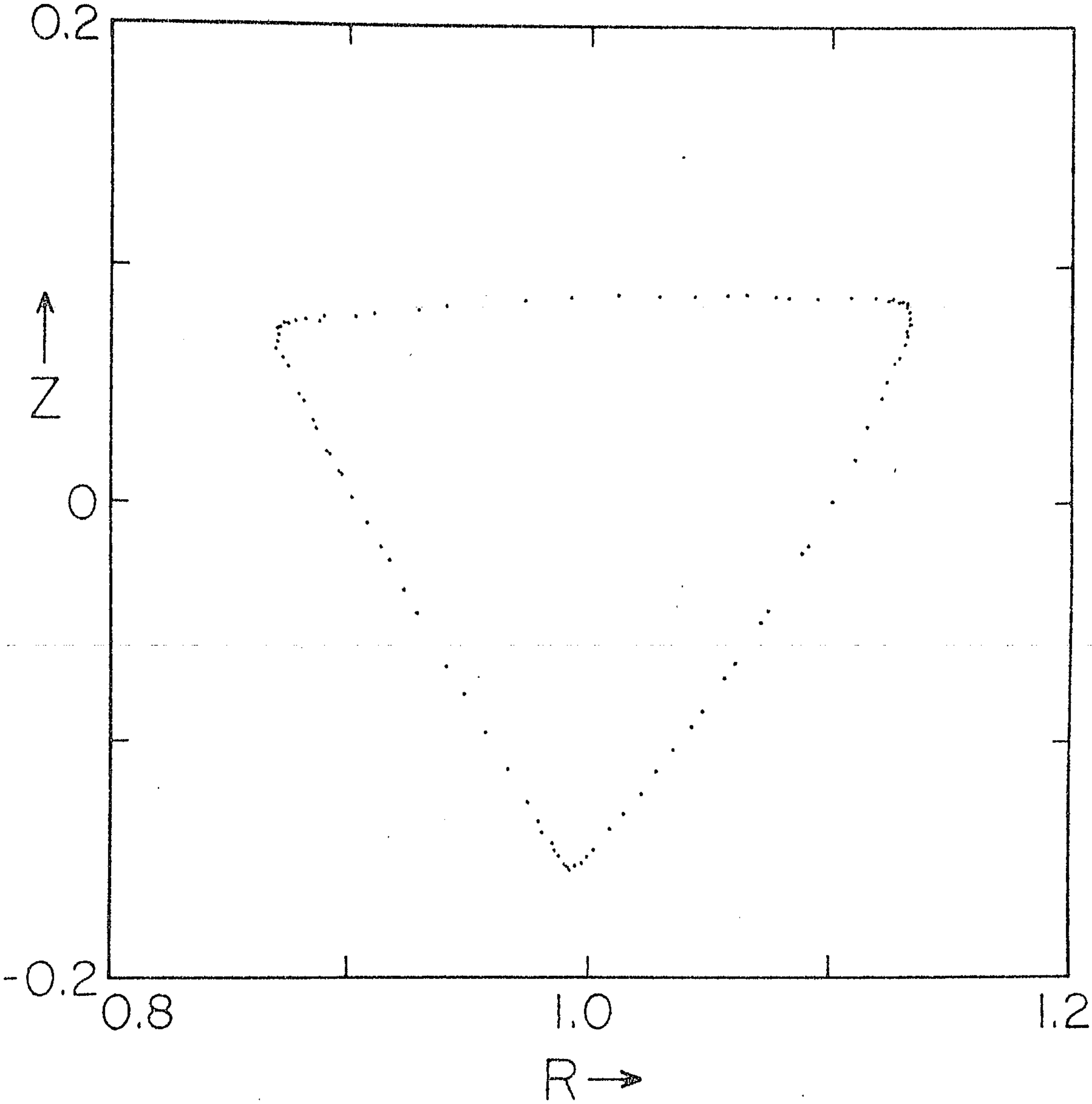

FIGURE 2

ℓ	m	$b_{\ell m}$	$\phi_{\ell m}$
3	7	4.0	0.0
4	7	2.5	5.6×10^{-4}
5	7	0.5	6.6×10^{-4}
6	7	0.2	5.5×10^{-4}
7	7	0.09	4.7×10^{-4}
8	7	0.04	4.3×10^{-4}
9	7	0.018	5.0×10^{-4}
10	7	0.008	8.5×10^{-7}

TABLE 1