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Two Dimensional Aspects of Toroidal Drift Waves in the Ballooning Representation

Y.Z. Zhang, S.M. Mahajan
and
X.D. Zhang
Institute for Fusion Studies
The University of Texas at Austin
Austin, Texas 78712

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Institute for Fusion Studies
The University of Texas at Austin
Austin, Texas 78712

Abstract

By systematically doing the higher order theory, the predictions of the conventional ballooning theory (CBT) are examined for non-ideal systems. For the complex solvability condition to be satisfied, radial variation of the lowest order mode amplitude needs to be invoked. It turns out, however, that even this procedure with its concomitant modifications of eigenvalues and eigenstructures, is not sufficient to justify the predictions of many CBT solutions; only a small set of the CBT solutions could be put on a firm footing. To demonstrate our general conclusions, theoretical and numerical results are presented for system of fluid drift waves with non-adiabatic electron response.

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In an axisymmetric toroidal pinch, like a tokamak, the turbulence due to high toroidal number (n) modes is generally considered to be responsible for anomalous plasma transport. In order to understand the nature of this turbulence, one begins with investigating linear instabilities of the high n modes. In toroidal geometry, a proper understanding of these modes necessitates solving a two dimensional (2-D) eigenvalue problem. The first reasonably successful analytical method consisted in devising the so-called ballooning transform,[1-3] a consequence of the translational invariance of the lowest order system with $1/\sqrt{n}$ as an expansion parameter. It is this translational invariance (the ballooning symmetry) that reduces the intrinsic 2-D equation to the one-dimensional (1-D) ballooning equation.

For the solutions of the ballooning equation to be meaningful, the perturbative techniques of this kind requires that a well defined solvability condition must be satisfied.[4-6] This constraint should normally determine the radial stationary point r_0, at which the plasma parameters occurring in the ballooning equation are to be evaluated. For a non-ideal system, however, the solvability condition is complex (two real equations),[6] implying the need for one more free parameter for a possible solution. The fact that this additional freedom can indeed be found in the framework of the ballooning theory, has been appreciated for some time.[7] The sought after parameter is the symmetry breaking, or amplification factor λ_I, [$\mathrm{Im}\,\theta_\kappa$ of Ref. 7], which allows for a zeroth order variation of the poloidal Fourier mode amplitudes. To clarify our notions, we go back to the 2-D ballooning transform[6]

$$\phi(x,l) = \oint d\lambda dk \exp[ik(x-l) - i\lambda l]\widehat{\varphi}(k,\lambda) \ , \tag{1}$$

where $\phi(x,l)$ is the Fourier coefficient in the expansion of the physical mode $\Phi(x,\theta,\zeta) = \exp(in\zeta - im\theta)\sum_l \exp(-il\theta)\phi(x,l)$ with $x = n[q(r)-q(r_0)]$, $q(r)$ as the safety factor, and $m = nq(r_0)$. One can see that the variable λ may have a parametric imaginary part λ_I, which can be naturally introduced through the symmetry breaking ansatz: $\phi(x,l) = \exp(\lambda_I l)\widetilde{\phi}(x,l)$. Notice that λ_I is yet to be determined, and the 2-D ballooning transform [Eq. (1)] for $\widetilde{\phi}(x,l)$ is on the real variable λ. For non-zero λ_I (henceforth, the analytical continuation $\lambda \to \lambda_r + i\lambda_I$ with a parametric λ_I is understood) the translational invariance of the lowest order mode amplitude in the radial direction is immediately destroyed. Taking $\widehat{\varphi}(k,\lambda) \to \widehat{\varphi}_0(k,\lambda) \sim \delta(\lambda - \lambda^*)$ in the lowest order,[6] where λ^* stands for the localization in λ space, we readily find that for an arbitrary $\overline{m}, \Phi_0(x+\overline{m},\theta,\zeta) = \exp[-i\overline{m}(\lambda^* + \theta)]\Phi_0(x,\theta,\zeta)$, resulting in radial amplification of the (lowest order) mode amplitude by $\exp(\overline{m}\lambda_I)$. Notice that $\lambda_I (\equiv \operatorname{Im}\lambda^*)$ as well as $\operatorname{Re}\lambda^*$ must be determined by the higher order equations of ballooning theory.

To explore the effects induced by λ_I, let us consider a 2-D eigenmode equation which resembles the fluid drift wave equation (in a circular flux surface equilibrium) described in Ref. 6. The non-ideal feature of the system is mainly characterized by the non-adiabatic electron response. The relevant equation is

$$\rho_s^2 \nabla_\perp^2 \Phi - \left(1 - i\delta_e - \frac{\widehat{\omega}_e^*}{\omega}\right)\Phi - \frac{c_s^2}{\omega^2}\nabla_\parallel^2 \Phi - 2\frac{\widehat{\omega}_{de}}{\omega}\Phi = 0 \ , \tag{2}$$

where $\nabla_\parallel = [q(r)\partial/\partial\zeta + \partial/\partial\theta]/qR$, $\nabla_\perp^2 = (1/r)(\partial/\partial r)r(\partial/\partial r) + (1/r^2)(\partial^2/\partial\theta^2)$, $\widehat{\omega}_e^* = i(T_e c/Ben_0 r)(dn_0/dr)(\partial/\partial\theta)$, $c_s^2 = T_e/m_i$, $\widehat{\omega}_{de} = -i(T_e c/BeRr)(\sin\theta r\partial/\partial r + \cos\theta\partial/\partial\theta)$, $\rho_s^2 = T_e c/eB\omega_{ci}$ with ω the mode frequency, T_e the electron temperature, n_0 the plasma density, B the magnetic field, $e(>0)$ the electron charge, R the major radius of the torus, r the radial position, m_i the ion mass, c the speed of light, $\omega_{ci} = eB/cm_i$ the ion cyclotron frequency, and δ_e stands for the non-adiabaticity of electron response. For simplicity B and R are assumed constant throughout the note. This model is appropriate for the present purpose, because the drift wave has essentially the characteristics of a ballooning mode.[8] The 2-D ballooning transform (defined by Eq. (1)) of Eq. (2) in the $k-\lambda$ representation,[6] is

$$\left[L^{(0)} + L^{(1)}\frac{\partial}{\partial\lambda} + L^{(2)}\frac{\partial^2}{\partial\lambda^2} + L^{(\overline{1})} + \text{higher orders}\right]\widehat{\varphi}(k,\lambda) = 0 \ , \tag{3}$$

where $L^{(i)} = \Pi_1^{(i)}\partial^2/\partial k^2 + \Pi_2^{(i)}k^2 + \Pi_3^{(i)} + \cos(k+\lambda)\Pi_4^{(i)} + \sin(k+\lambda)\Pi_5^{(i)}k (i = 0,1,2,\overline{1})$, and $\Pi_j^{(0)} \sim O(1)$, $\Pi_j^{(1)} \sim O(1/n)$, $\Pi_j^{(2)} \sim O(1/n^2)$ are independent of k, λ, determined completely by the local parameters at r_0, and $\Pi_j^{(1)} = f(k) + g(k)\partial/\partial k \sim O(1/n)$. Expressions for all Π's can be derived in a straightforward manner for a given equilibrium.[6] The existence of non-zero $L^{(1)}$, $L^{(2)} \cdots$ reflects the fact that the translational invariance of the operator holds only approximately.

The lowest order of Eq. (3) with all $[(1/n)\partial/\partial\lambda]^i$'s neglected, is the ballooning equation

$$\widehat{L}^{(0)}(\lambda)\chi \equiv \Pi_1^{(0)}\frac{\partial^2\chi}{\partial k^2} + \left[\Pi_2^{(0)}k^2 + \cos(k+\lambda)\Pi_4^{(0)} + \sin(k+\lambda)\Pi_5^{(0)}k\right]\chi(k,\lambda) = -\Pi(\lambda)\chi(k,\lambda) \;, \tag{4}$$

where $\Pi(\lambda)$ is the 'ballooning' eigenvalue parametrically dependent on λ via $\cos\lambda$ and $\sin\lambda$. The reflection symmetry of Eq. (4) in $k \to -k$, $\lambda \to -\lambda$ indicates that a cosine Fourier series $\Pi_0 + \Pi_1\cos\lambda + \Pi_2\cos 2\lambda + \cdots$ can serve as a good representation for $\Pi(\lambda)$. The next step in the process is to express the 2-D wave function as $\widehat{\varphi}(k,\lambda) = \Psi(\lambda)\chi(k,\lambda) +$ higher orders, where $\Psi(\lambda)$ represents the fast variation in λ. It is now clear that the conventional ballooning theory (CBT) corresponds to the most localized $\Psi(\lambda)$ with $\lambda_I = 0$.

The solvability condition with a finite λ_I can be obtained by solving Eq. (3) perturbatively.[6] Provided $\Pi_2, \Pi_3 \ldots$ are small, the solvability condition for the most localized $\Psi \sim \exp(-n\sqrt{p}\lambda_r^2/2)$ becomes

$$F \equiv n\left\langle\chi L^{(1)}\chi\right\rangle - i\Pi_1 \sinh\lambda_I/\sqrt{p} = 0 \tag{5}$$

with $p \equiv -\Pi_1\cosh\lambda_I/2n^2[\left\langle\chi L^{(2)}\chi\right\rangle + \left\langle\chi L^{(1)}\overline{\varphi}_1\right\rangle]$, where $\langle\ldots\rangle \equiv \int dk\ldots/\int dk\chi^2$, and $\overline{\varphi}_1$ is the inhomogeneous solution of the equation $L^{(0)}\overline{\varphi}_1 + (L^{(1)} - \left\langle\chi L^{(1)}\chi\right\rangle)\chi = 0$. For an ideal system with *a priori* $\lambda_I = 0$, the solvability condition Eq. (5) reduces to $\left\langle\chi L^{(1)}\chi\right\rangle = 0$ [Eq. (8) of Ref. 6], and is likely to be satisfied by merely adjusting r_0. This is the situation investigated in Ref. 7, which concludes that the CBT[9] applies for circular flux surface equilibria. This conclusion should be accepted with caution, because Ref. 7 deals only with the marginal stability situation ($\delta_e = 0$), which is operationally equivalent to an ideal system. In addition, it does not explore the consequences of the full solvability condition. If dissipation is essential to plasma instability, it is almost impossible to have a purely real system. Then, one has to solve Eq. (5) with Eq. (4) to determine r_0 and λ_I simultaneously. It follows then that the eigenvalues of the ballooning equation with finite λ_I may have an $\mathcal{O}(1)$ correction to those from CBT. Notice that a complete determination of the 2-D eigenvalue will require solving the equation for $\Psi(\lambda)$ [Eq. (9) of Ref. 6]. But this will merely yield an $\mathcal{O}(1/n)$ correction to the eigenvalues of the ballooning equation with λ_I. When this small correction is neglected, the above manipulation correct to $\mathcal{O}(1/\sqrt{n})$ is sufficient for determination of the eigenvalues.

In this brief communication, we examine in detail the fluid drift wave [Eq. (2)] problem within the framework of the theoretical model [broken ballooning symmetry] just discussed. The equilibrium is characterized by a constant density scale length L_n, a constant electron dissipation δ_e, a T_e-profile $T_e(\rho) = T_e(0)(1-\rho^2)^2$, and a q-profile $q(\rho) = q_0+(q_a-2q_0)\rho^2+q_0\rho^4$, where $\rho = r_0/a$ is the radial position normalized to the plasma minor radius a. In some limits the problem can be solved analytically by using the explicit perturbative solutions of Eq. (4). The procedure is straightforward, but is very cumbersome, and will be presented elsewhere. Here, we present only some typical numerical results. The numerical procedure consists in solving [using a shooting code] Eq. (4) for $\mathcal{X}(k,\lambda)$ and $\Pi(\lambda)$, from which $\overline{\varphi}_1$ and the related integrals [such as $\left\langle \mathcal{X}L^{(1)}\mathcal{X}\right\rangle$, $\left\langle \mathcal{X}L^{(1)}\overline{\varphi}_1\right\rangle$] are calculated numerically. The present investigation is limited to the case where $\Pi(\lambda)$ is very well approximated by $\Pi_0 + \Pi_1 \cos\lambda$, which may not be always true. For some values of r_0 and certain branches, $\Pi(\lambda)$ can have very different λ-dependence. Our results indicate that the solvability condition Eq. (5) can be satisfied mostly at small r_0 ($\rho < 0.35$), and then also with $\rho_s^2 k_\theta^2 \sim O(1)$! A typical example leads to the stationary point $\rho = 0.19$ for the following parameters: $b_{\theta,0} \equiv T_e(0)cn^2/eB\omega_{ci}a^2 = 0.04$, $\delta_e = 0.15$, $L_n/R = 0.2$, $q_0 = 1.0$, $q_a = 3.0$ with $\lambda_r = 0.$, $\lambda_I = 1.3$, yielding the eigenmode frequency $\omega/\omega_e^* = 0.344$, and the growth rate $\gamma/\omega_e^* = 0.0228$. We point out that even though $\lambda_I > 1$ for this solution, there is very little modification to the eigenvalues ($\lambda_I = 0$); the small shear at $\rho = 0.19$ makes the eigenvalues of Eq. (4) quite insensitive to λ. A different set of parameters: $b_{\theta,0} = 0.2$, $\delta_e = 0.8$, $L_n/R = 0.2$, $q_0 = 1.0$, $q_a = 3.0$ result in the solution $\rho = 0.3$, $\lambda_r = 0$, $\lambda_I = 1.1$, the eigenmode frequency $\omega/\omega_e^* = 0.258$, and the growth rate $\gamma/\omega_e^* = 0.0608$. The corresponding values for no λ_I are $\omega/\omega_e^* = 0.243$, and the growth rate $\gamma/\omega_e^* = 0.0518$.

In the ballooning k-space, the mode structures pertaining to the above two examples are shown in Figs. 1a, 1b and 1c, 1d for two different sets of parameters. It is readily seen that a finite λ_I has significant effects on the mode structures exhibiting a strong asymmetry in the ballooning space. The radial envelope of the physical mode $\Phi(x, \theta, \zeta)$ is also modified by a finite λ_I; the peak of the envelope is shifted from r_0 to a radial position $r_0^{\lambda_I}$ estimated as $q(r_0^{\lambda_I}) \sim q(r_0) + \mathrm{Re}(\sqrt{p})\lambda_I$ for the localized $\Psi \sim \exp(-n\sqrt{p}\lambda_r^2/2)$. The width of the envelope is still order $r_0/\sqrt{n}$ necessary for the validity of the ordering. If $q(r_0^{\lambda_I})$ is well beyond the range of the safety factor within the plasma, the corresponding mode is not physically interesting. The finite difference between r_0 and $r_0^{\lambda_I}$ seems to reflect a global feature of the localized modes. This observation would be significant for understanding both the numerical simulations and the experimental measurements.

For $\rho > 0.5$ we found that the solvability condition Eq. (5) is generally not satisfied for the above equilibrium model, even if a finite λ_I is allowed. A typical search is shown in Fig. 2, where the function $F(\rho, \lambda_I)$ defined by Eq. (5) is plotted for $b_{\theta,0} = 0.04$ with $\rho = 0.5 - 0.9$ and various λ_I. It seems, therefore, that there does not exist a one to one correspondence between stability predicted by CBT and stability pertinent to the physical 2-D system. Among others, some of the most unstable ones (e.g. $\rho_s^2 k_\theta^2 \ll 1$, which also are the physically most interesting modes) are excluded. The preceding discussion, by no means, implies that there are no unstable modes with $\rho_s k_\theta \ll 1$, it merely points to the limitations of the present ballooning theory that is based on the assumption of the most localized $\Psi(\lambda)$. The higher harmonic solutions in λ-space, which are less localized, are not explored in this study, and are beyond the scope of this brief communication. At this stage we would like to emphasize that the true expansion "parameter" of the (generalized) ballooning theory is $(1/n)(\partial/\partial\lambda)$. As a result, the eigenmode structure in λ space is crucial in determining the effective expansion parameter. It is only for the most localized modes ($\partial/\partial\lambda \sim \sqrt{n}$), that the expansion parameter becomes $1/\sqrt{n}$. On the other hand, the localization of wave function in λ-space, is not necessarily a general feature of the high-n ballooning type of modes. For example, $\Psi(\lambda)$ for the high-n toroidal Alfvén eigenmode[10,11] does have a λ spread, revealing another aspect of departure from the CBT.

In summary, we have shown that the complex solvability condition [Eq. (5)] for a non-ideal system can in principle be satisfied by introducing a symmetry breaking factor, λ_I, i.e., the ballooning solution may describe a class of solutions for the non-ideal 2-D eigenvalue problems. In practice, we find that for a typical equilibrium (used in this brief communication), it is impossible [even with $\lambda_I \neq 0$] to satisfy the solvability condition unless $n \gg 1$. Thus, we must be careful in accepting the stability predicted by a ballooning analysis for moderate n, i.e., the range of n characteristic of the experimentally measured fluctuations.

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Figure Captions

1. (a) Eigenmode structure in the ballooning k-space for $\lambda_I = 0$, and $\rho = 0.19, b_{\theta,0} = 0.04, \delta_e = 0.15$.

   (b) Eigenmode structure in the ballooning k-space with the symmetry breaking factor $\lambda_I = 1.3$ for parameters of Fig. 1a.

   (c) Eigenmode structure in the ballooning k-space for $\lambda_I = 0$, and $\rho = 0.3, b_{\theta,0} = 0.2, \delta_e = 0.8$.

   (d) Eigenmode structure in the ballooning k-space with the symmetry breaking parameter $\lambda_I = 1.1$ for the parameters of Fig. 1c.

2. The complex solvability condition $F(\rho, \lambda_I)$ for scanning ρ and λ_I at $b_{\theta,0} = 0.04$, $\delta_e = 0.8$, $L_n/R = 0.2$, $\lambda_r = \pi$, $q_0 = 1.0$, and $q_a = 3.0$. The curve a, b, c, d, e, f represents $\rho = 0.5, 0.6, 0.7, 0.8, 0.9$ respectively. The symbols □, ■, ○, ●, △, ▲ represent $\lambda_I = 0., -0.2, -0.4, -0.6, -0.8, -1.0$ respectively.

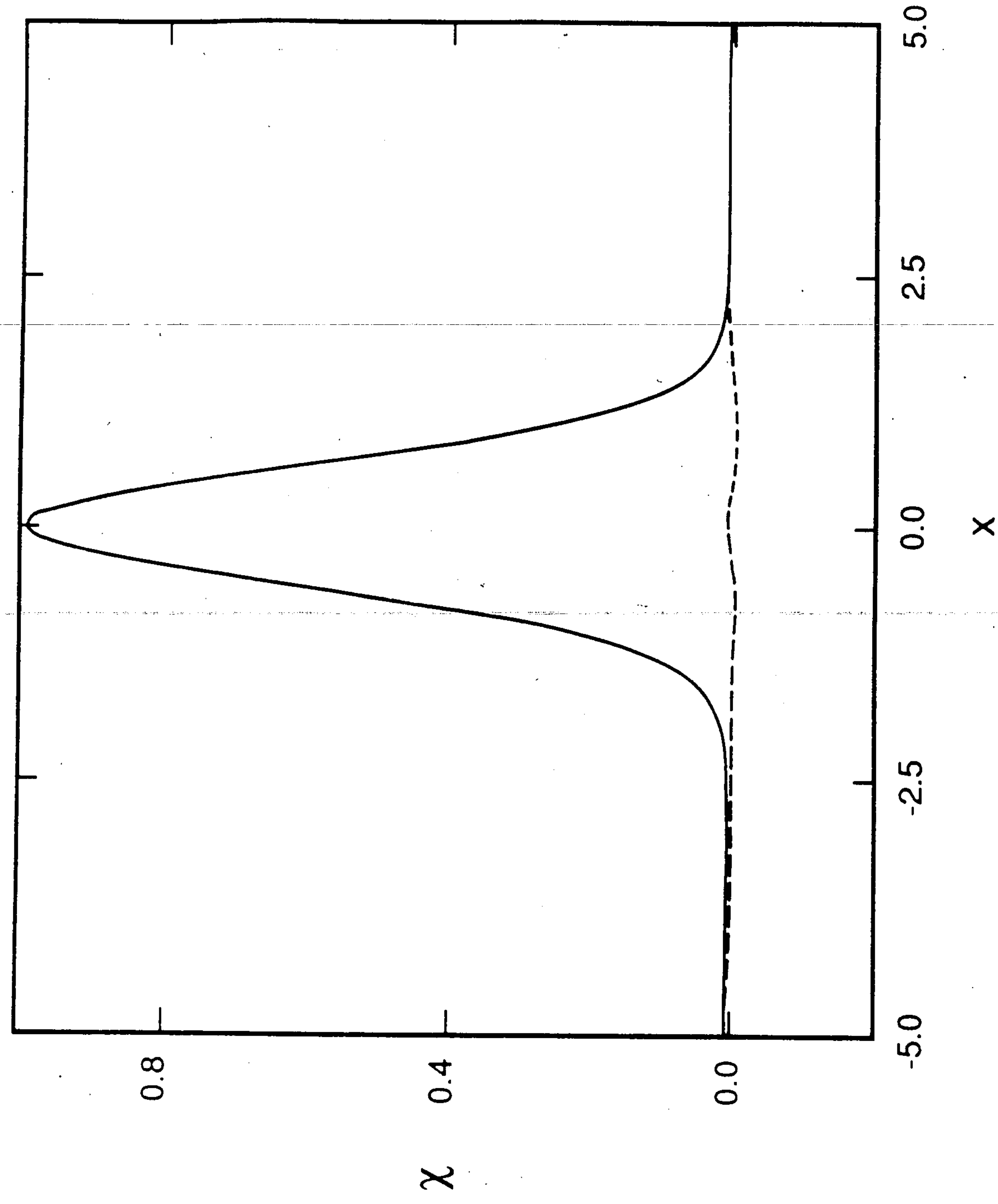

Fig. 1a

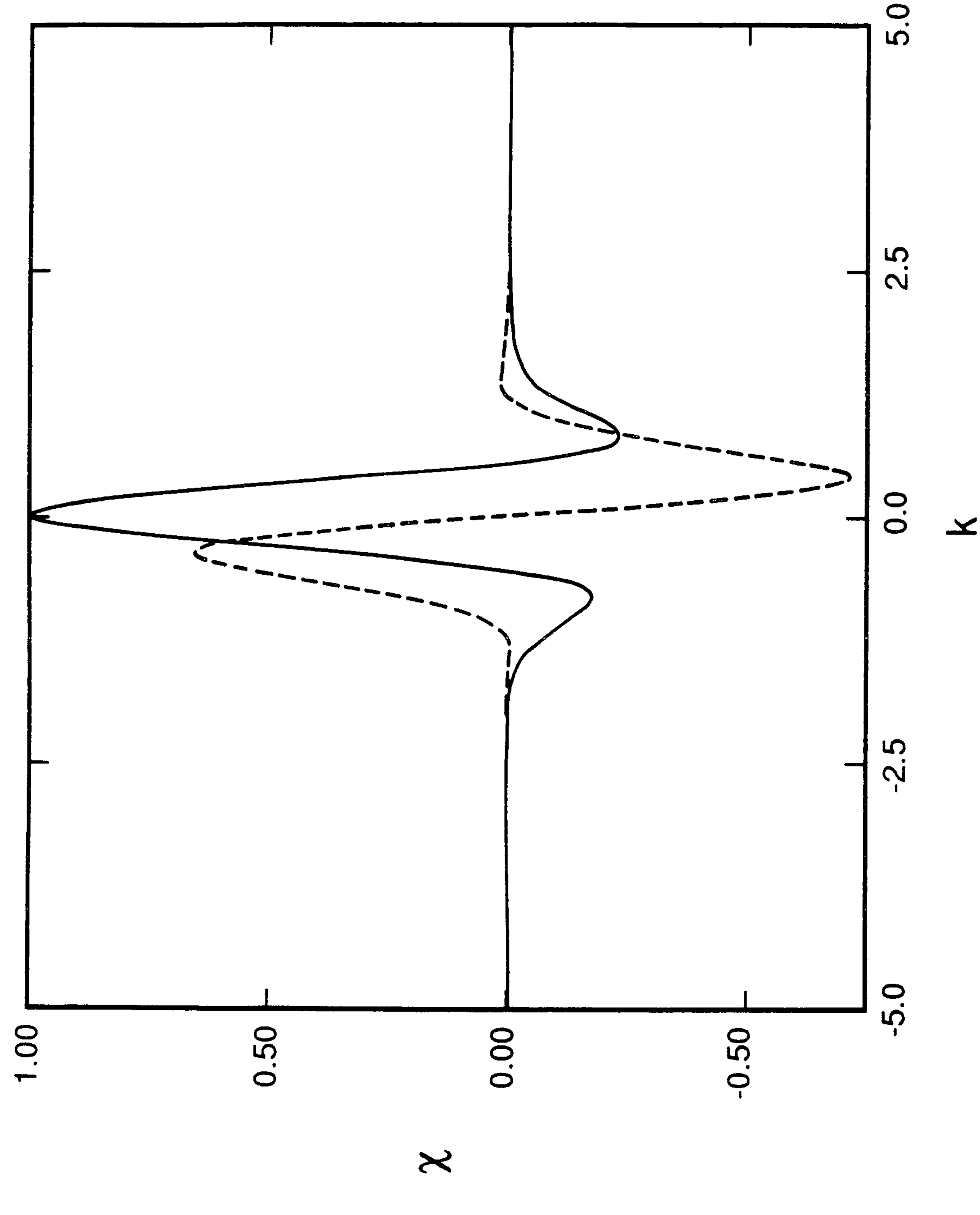

Fig. 1b

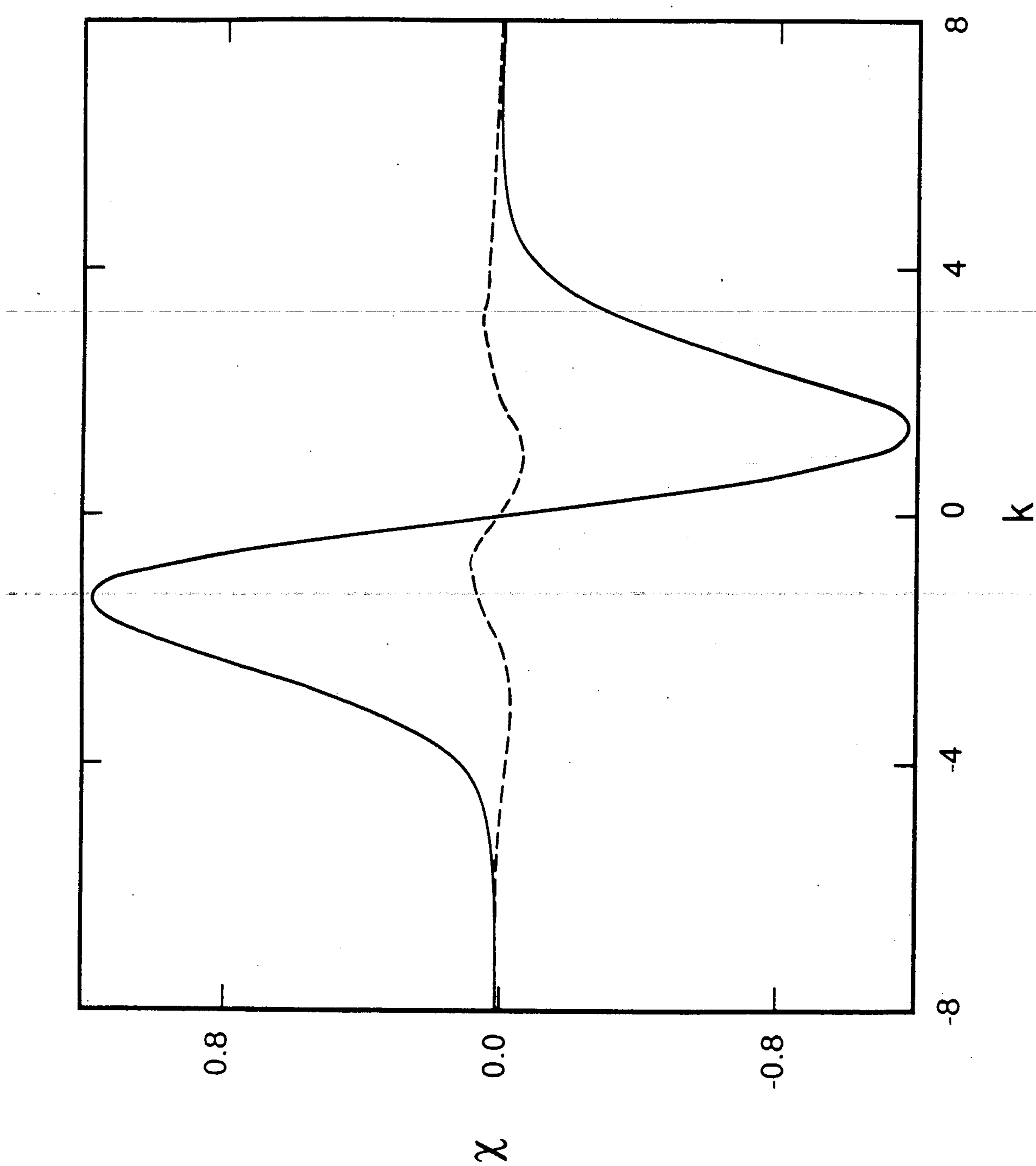

Fig. 1c

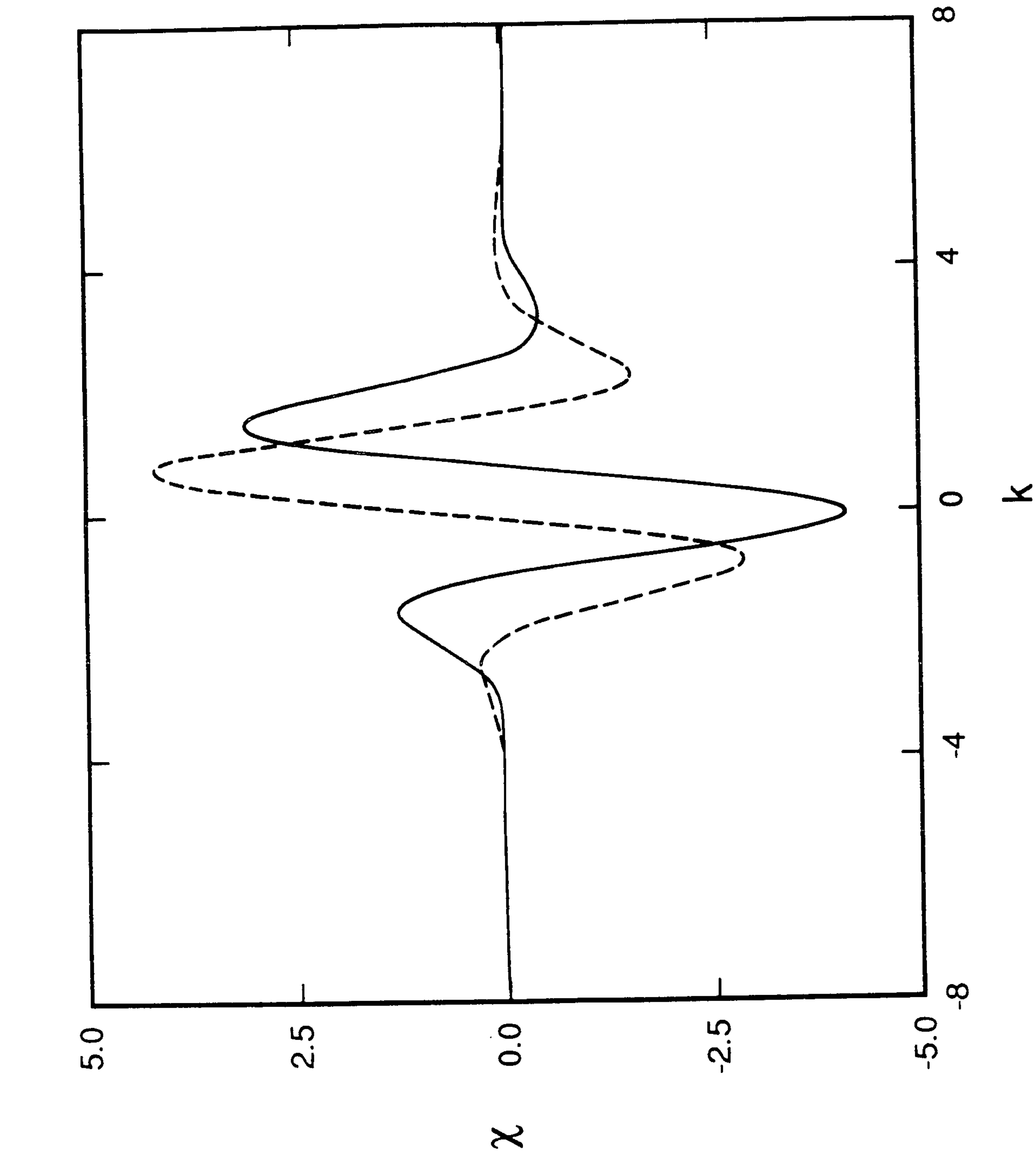

Fig. 1d

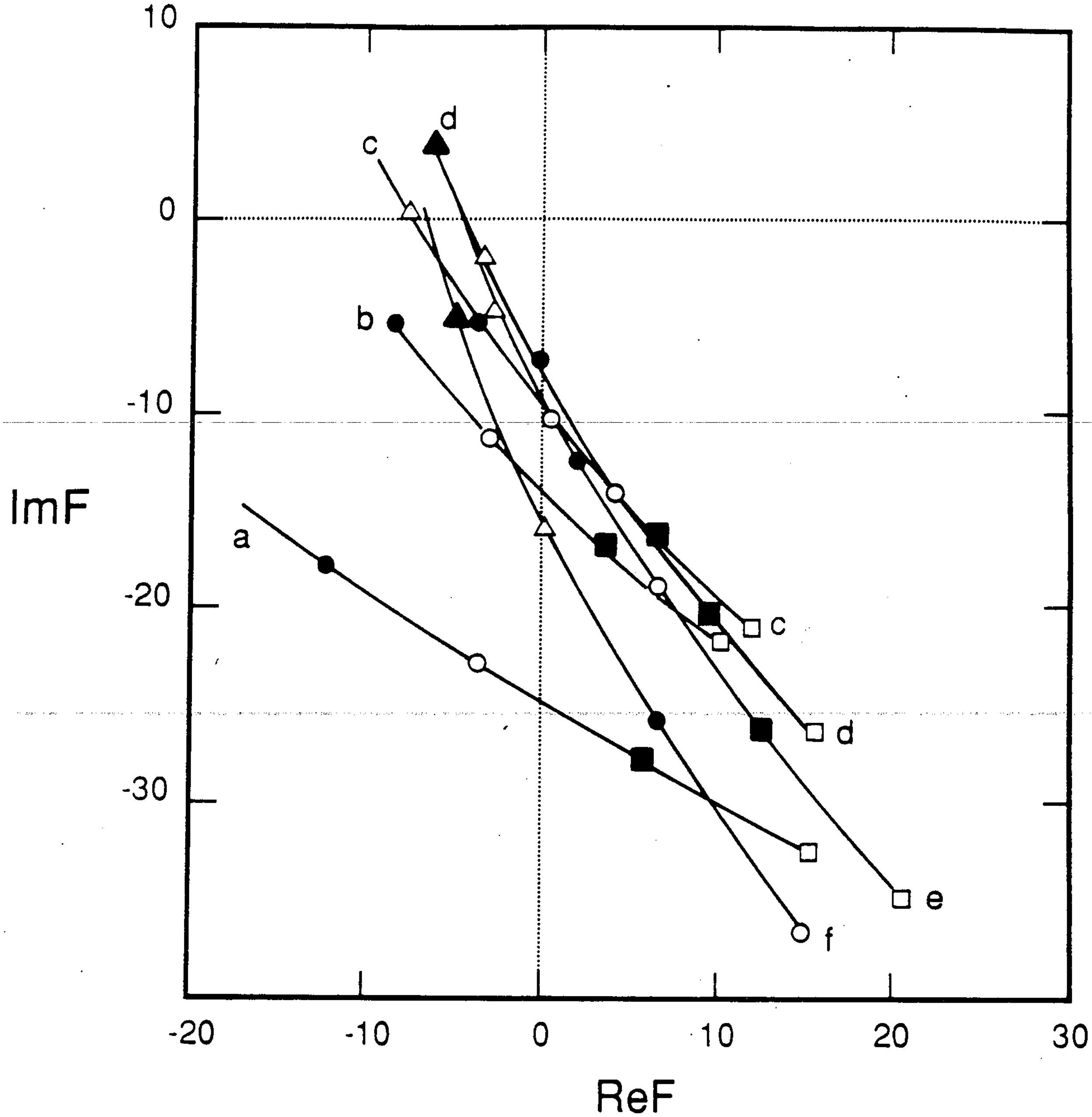

Fig. 2