DOE/ET/53088-54 IFSR #54

POISSON BRACKETS FOR FLUIDS AND PLASMAS

Philip J. Morrison

Institute for Fusion Studies

February 1982

POISSON BRACKETS FOR FLUIDS AND PLASMAS

Philip J. Morrison

Institute for Fusion Studies
Department of Physics
The University of Texas at Austin
Austin, Texas 78712
U. S. A.

To appear in "Mathematical Methods in Hydrodynamics and Integrability in Related Dynamical Systems", *AIP Conference Proceedings, La Jolla, December 1981, M. Tabor (ed.).*

1. Introduction

§ 1.1 Overview

The traditional method for obtaining a Hamiltonian system is by way of a Lagrangian, that is obtained by physical considerations. The system is then Legendre transformed (if possible) to obtain Hamilton's equations in canonical form, a form that is conveniently representable in terms of the Poisson bracket. Canonical transformations preserve the form of the Poisson bracket; the idea of canonical conjugacy is maintained. An arbitrary coordinate transformation does not preserve the form of the Poisson bracket and consequently the canonical form of Hamilton is obscured. Conjugate variables cannot be discerned and the Poisson bracket may depend explicity on the dynamical variables. In spite of the obscured form, certain algebraic properties of the Poisson bracket are maintained: bilinearity, antisymmetry, and the Jacobi condition (c.f., below). This motivates an alternate definition of Hamiltonian: A system is Hamiltonian if one can find a Poisson bracket, with these algebraic properties, and a Hamiltonian, such that together they generate the time evolution of the system. For the case of even-(nondegenerate) finite-dimensional systems, the theorem of Darboux[1,2] provides an algorithm for locally constructing canonical variables. Also, there exists an extension of Darboux's theorem[3] for the case of infinite dimensional systems. (The situation here is subtle--gauge conditions may be necessary for a canonical description.)

In this paper we present noncanonical yet Hamiltonian descriptions of many of the non-dissipative field equations that govern fluids and plasmas. The dynamical variables here are the usually encountered physical variables. These descriptions have the advantage that gauge conditions are absent, but at the expense of introducing peculiar Poisson brackets. Clebsch-like potential descriptions that reverse this situation are also introduced.

In the remainder of Sec. 1 the ideas sketched above are considered. The presentation here is admittedly non-rigorous. The reader who is interested in a more rigorous formulation of some of these ideas is directed to Refs. 4 - 11. Section 2 deals with the ideal three-dimensional compressible fluid. The noncanonical Poisson bracket for ideal magnetohydrodynamics[12] is presented. Various fluid descriptions are seen to be represented by portions of this bracket. The plasma equations of Chew, Goldberger and Low[13] are considered. The constants of motion for MHD are discussed and the bracket is shown to generate the infinitesimal transformations of the ten-parameter Galilean group. This section is concluded by presenting a canonical formalism. Various potential decompositions of the fluid velocity and the magnetic field are discussed. Section 3 deals with the Hamiltonization of the equations of two-dimensional vortex fluids and guiding center plasmas.[14] The sole noncanonical dynamical variable in this case is the scalar vorticity. The canonical description is given. Section 4 is concerned with the equations that govern fully nonlinear ion-acoustic waves in plasmas. This is the system from which the Korteweg-de Vries equation is obtained by approximation. Section 5 covers the Maxwell-Vlasov[15-18] equations. The noncanonical Poisson bracket is presented. The way to "canonize" this form[19] is indicated at the end of Sec. 6. The body of Sec. 6 deals with the Vlasov-Poisson equations.[15] It is observed that these equations possess the same non-canonical Poisson bracket as that for two-dimensional vortex fluids.[19] A Clebsch-like potential decomposition is seen to yield a canonical Hamiltonian description.[19]

§ 1.2 Generalized Hamiltonian Field Theory

Consider the following system of autonomous evolution equations:

$$u^i_t(t,\vec{x}) = F^i(\vec{u},\vec{x}) \qquad i = 1,2,\ldots,m \tag{1.1}$$

Here, each u^i is a function of time t and $\vec{x}$, where $\vec{x} \in V \subset R^n$ for some integer n. The F^i are general nonlinear partial differential or integral operators on $\vec{u}$. Specifically the F^i may be any functions (with a finite number of arguments) of the following:

i) $\vec{u}$ and $\vec{x}$

ii) $u^i_k \equiv \dfrac{\partial^k u^i}{\partial x_i^{k_1} \partial x_2^{k_2} \ldots \partial x_n^{k_n}}$

where the x_i's are the components of $\vec{x}$,

$k = |\vec{k}| = \sum_{i=1}^{n} k_i$

and $\vec{k}$ has components k_i which are positive integers.

iii) $\int_V K(\vec{x}|\vec{x}')f(\vec{u}_k)d\tau$ where

f is some function and the kernel K is independent of $\vec{u}$.

We denote this class of operators by $\mathcal{L}$. (I.e., $F^i \in \mathcal{L}$.)

We are not concerned with specific auxiliary conditions necessary for existence and uniqueness of solutions, but suppose solutions do exist and are elements of a vector space ω (over $\mathcal{R}$) that is equipped with the inner product

$$\langle f|g \rangle = \int_V fg \, d\tau \quad , \tag{1.2}$$

where $d\tau$ is the volume element for $V \subset R^n$.

Customarily in field theory certain integrals or functionals arise. For example, the integral of the Hamiltonian density is that particular functional that generates the evolution. Here the evolution will be generated via generalized Poisson brackets that operate on functionals. To this end we define a vector space Ω (over $\mathcal{R}$) of differentiable functionals that have the form

$$\mathcal{G}[\vec{u}] = \int_{V} G(\vec{u},\vec{x})\, d\tau \tag{1.3}$$

where $G \in \mathcal{L}$ is an operator on ω. We define differentiation of functionals in the usual way

$$\frac{d}{d\epsilon} \mathcal{F}\left[u^i + \epsilon w\right]\Bigg|_{\epsilon=0} = \left\langle \frac{\delta \mathcal{F}}{\delta u^i} \middle| w \right\rangle , \tag{1.4}$$

where the variation is taken with respect to functions w that vanish at the boundary of V. Equation (1.4) defines the functional derivative $\delta \mathcal{F}/\delta u^i$, which is in general a non-linear operator of the class $\mathcal{L}$ that operates on ω.

Before proceeding, consider the following examples of functional differentiation:

i) Suppose $\mathcal{F}[u] = \int_0^{2\pi} F(x,u,u_x,u_{xx},\ldots)dx$

where the function u is defined on $(0,2\pi)$ and F is C^∞ in all its (finite number) of arguments. By Eq. (1.4) we observe

$$\frac{\delta \mathcal{F}}{\delta u} = \frac{\partial F}{\partial u} - \frac{d}{dx}\frac{\partial F}{\partial u_x} + \frac{d^2}{dx^2}\frac{\partial F}{\partial u_{xx}} - \ldots$$

ii) Suppose $\mathcal{F}[\vec{u}] = u^i(\vec{x}\,')$, i.e., the functional composed of functions u^i evaluated at the point $\vec{x}\,'$. Using the Dirac delta function $\delta(\vec{x})$, we can represent this in the form of Eq. (1.3) as

$$F[\vec{u}] = \int_V u^i(\vec{x})\,\delta(\vec{x} - \vec{x}')\,d\tau ,$$

then from Eq. (1.4) we obtain

$$\frac{\delta u^i(\vec{x}')}{\delta u^j(\vec{x})} = \delta_{ij}\,\delta(\vec{x} - \vec{x}') ,$$

where

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} .$$

Continuing now, we recall that the usual Poisson brackets[20] of field theory have the form

$$[F,G] = \sum_k \int_V \left[\frac{\delta F}{\delta \eta_k}\frac{\delta G}{\delta \pi_k} - \frac{\delta G}{\delta \eta_k}\frac{\delta F}{\delta \pi_k}\right] d\tau \qquad (1.5)$$

where the dynamical equations generated by some Hamiltonian, H, are

$$\frac{\partial \eta_k}{\partial t} = \left[\eta_k, H\right] , \qquad \frac{\partial \pi_k}{\partial t} = \left[\pi_k, H\right] .$$

Clearly the operator of Eq. (1.5) is antisymmetric and it is well known that it satisfies the Jacobi condition (cf. below). We generalize this form by defining the following generic bilinear product on Ω :

$$[F,G] = \left\langle \frac{\delta F}{\delta u^i} \,\middle|\, O^{ij}\,\frac{\delta G}{\delta u^j} \right\rangle , \qquad (1.6)$$

where repeated index notation is used and $O^{ij} \in \mathcal{L}$. We desire our form, Eq. (1.6), to possess the same algebraic properties as Eq. (1.5), i.e.,

i) $[F,G] = -[G,F]$ for $F,\ G \in \Omega$

ii) the Jacobi condition

$$[E,[F,G]] + [F,[G,E]] + [G,[E,F]] = 0$$

for every $E,\ F,\ G \in \Omega$.

The first condition requires that the operator O^{ij} be anti-self-adjoint with respect to the inner product on ω . The second condition is more stringent and will be discussed in the next subsection. We note that a bracket of the form of Eq. (1.6), with properties i) and ii), defined on Ω defines a Lie Algebra of functionals. We now define what we mean by Hamiltonian.

<u>Definition.</u> A system of equations of the form (1.1) is Hamiltonian if there exists an operator $O^{ij} \in \mathcal{L}$ and a functional H such that Eq. (1.1) can be cast into the form

$$\frac{\partial u^i}{\partial t} = \left[u^i, H\right]$$

where $[\ ,]$ makes Ω a Lie Algebra.

§ 1.3 The Jacobi Condition

We now pinpoint what is required of an anti-self-adjoint O^{ij} in order for the Jacobi condition to be satisfied. Since the Jacobi condition involves nested Poisson brackets we require the functional derivative of a

Poisson bracket. To this end, we first obtain a property of second variation. We conclude this subsection by considering two general classes of $\mathcal{O}^{ij}$: those that are independent of $\vec{u}$ and those that are linear in $\vec{u}$ in a particular way.

Recall Eq. (1.4)

$$\frac{d}{d\epsilon} F\left[u^i + \epsilon w\right]\Bigg|_{\epsilon=0} = \left\langle \frac{\delta F}{\delta u^i} \,\middle|\, w \right\rangle \equiv G .$$

G as defined here is clearly an element of Ω . Differentiating again, we obtain

$$\frac{d}{d\eta} G\left[u^j + \eta z\right]\Bigg|_{\eta=0} = \left\langle \frac{\delta^2 F}{\delta u^j \delta u^i} z \,\middle|\, w \right\rangle . \qquad (1.7)$$

Equation (1.7) defines the operator $\delta^2 F/\delta u^j \delta u^i \in \mathcal{L}$ that operates on $\vec{u}$ as well as operating linearly on z . Since the order of differentiation is immaterial, we must have the following:

$$\left\langle \frac{\delta^2 F}{\delta u^i \delta u^j} w \,\middle|\, z \right\rangle = \left\langle w \,\middle|\, \frac{\delta^2 F}{\delta u^j \delta u^i} z \right\rangle . \qquad (1.8)$$

Since the Poisson bracket of any two functionals is also a functional, formally we have

$$\frac{d}{d\epsilon} [F,G]\left[u^k + \epsilon w\right]\Bigg|_{\epsilon=0} = \left\langle \frac{\delta [F,G]}{\delta u^k} \,\middle|\, w \right\rangle . \qquad (1.9)$$

By Eq. (1.6) we also have

$$\frac{d}{d\epsilon}[F,G][u^k + \epsilon w]\Big|_{\epsilon=0} = \frac{d}{d\epsilon}\left\langle \frac{\delta F}{\delta u^i} \;\middle|\; \mathcal{O}^{ij}\,\frac{\delta G}{\delta u^i} \right\rangle [u^k + \epsilon w]\Big|_{\epsilon=0}$$

$$= \left\langle \frac{\delta^2 F}{\delta u^k \delta u^i}\, w \;\middle|\; \mathcal{O}^{ij}\,\frac{\delta G}{\delta u^j} \right\rangle + \left\langle \frac{\delta F}{\delta u^i} \;\middle|\; \mathcal{O}^{ij}\,\frac{\delta^2 G}{\delta u^k \delta u^j}\, w \right\rangle$$

$$+ \left\langle \frac{\delta F}{\delta u^i} \;\middle|\; \frac{\delta \mathcal{O}^{ij}}{\delta u^k}(w)\,\frac{\delta G}{\delta u^j} \right\rangle . \qquad (1.10)$$

The first two terms of Eq. (1.10) come from the $d/d\epsilon$ acting on $\delta F/\delta u^i$ and $\delta G/\delta u^j$ respectively. The last term arises from the dependence of the operator $\mathcal{O}^{ij}$ on $\vec{u}$. This term is complicated in that the symbol $\delta \mathcal{O}^{ij}(w)/\delta u^k \in \mathcal{L}$ is used for an operator that acts on $\vec{u}$, linearly on w, and also on $\delta G/\delta u_j$ to its right. We require that this term be written as follows:

$$\left\langle K_k^{ij}\left(\frac{\delta F}{\delta u^i} \,;\, \frac{\delta G}{\delta u^j}\right) \;\middle|\; w \right\rangle \qquad (1.11)$$

where w is now isolated from the operator. (For the case when $\mathcal{O}^{ij}$ only involves partial differentiation, this is obtained by integration by parts.) Using Eqs. (1.8), (1.9), (1.10), and (1.11), the anti-self-adjointness of $\mathcal{O}^{ij}$, and the fact that these relations hold for arbitrary w within a wide class, we strip away the integration to obtain

$$\frac{\delta[F,G]}{\delta u^k} = \frac{\delta^2 F}{\delta u^i \delta u^k}\,\mathcal{O}^{ij}\,\frac{\delta G}{\delta u^j} - \frac{\delta^2 G}{\delta u^j \delta u^k}\,\mathcal{O}^{ij}\,\frac{\delta F}{\delta u^i} + K_k^{ij}\left(\frac{\delta F}{\delta u^i} \,;\, \frac{\delta G}{\delta u^j}\right) . \qquad (1.12)$$

Using Eq. (1.12) in the Jacobi condition yields

$$[E,[F,G]] + \text{cyc} = \left\langle \frac{\delta E}{\delta u^m} \,\middle|\, \mathcal{O}^{m\ell} K_{\ell}^{ij} \left(\frac{\delta F}{\delta u^i}, \frac{\delta G}{\delta u^j} \right) \right\rangle + \text{cyc} \quad . \tag{1.13}$$

Here cyc means cyclic permutation and we observe that the only surviving terms are those that involve the K_{ℓ}^{ij}. The terms that involve the second variation cancel by virtue of the anti-self-adjointness of $\mathcal{O}^{ij}$ and Eq. (1.8). The following theorem is apparent.

<u>Theorem 1.</u> If $\mathcal{O}^{ij}$ is independent of $\vec{u}$ (including any operator of class $\mathcal{L}$ on $\vec{u}$), then anti-self-adjointness is sufficient for the Jacobi condition.

Now we consider a special case where $\mathcal{O}^{ij}$ depends linearly on $\vec{u}$. We suppose $\mathcal{O}^{ij}$ has the manifestly anti-self-adjoint form

$$\mathcal{O}^{ij} = \sum_{k,r} \left[a_r^{ij,k}\, u_k \partial_r + a_r^{ji,k}\, \partial_r u_k \right] \tag{1.14}$$

where $k = 1,2,\ldots,m$; $r = 1,2,\ldots,n$; $\partial_r = \partial/\partial x_r$; and $a_r^{ij,k} \in \mathbb{R}$ for all i, j, k, and r. With this form, the quantity $\delta\mathcal{O}^{ij}(w)/\delta u^k$ of Eq. (1.10) is seen to be

$$\frac{\delta \mathcal{O}^{ij}}{\delta u^k}(w) = \sum_r \left(a_r^{ij,k}\, w \partial_r + a_r^{ji,k}\, \partial_r w \right) \quad .$$

From this we obtain the quantity K_k^{ij} by integration by parts

$$K_k^{ij}\left(\frac{\delta F}{\delta u^i}, \frac{\delta G}{\delta u^j}\right) = \sum_r \left(\frac{\delta F}{\delta u^i} a_r^{ij,k} \partial_r \frac{\delta G}{\delta u^j} - \frac{\delta G}{\delta u^j} a_r^{ji,k} \partial_r \frac{\delta F}{\delta u^i}\right). \tag{1.15}$$

Inserting Eq. (1.15) into Eq. (1.13) yields a complicated expression that vanishes if the coefficients $a_r^{ij,k}$ satisfy certain properties.

Theorem 2. Poisson brackets made up of operators $\mathcal{O}^{ij}$ of the form of Eq. (1.14) satisfy the Jacobi condition if

$$\text{i)} \quad \sum_k \left(a_r^{\ell k,m}\, a_t^{ij,k} - a_r^{ik,m}\, a_t^{\ell j,k}\right) = 0$$

and

$$\text{ii)} \quad \sum_k \Big(a_r^{\ell k,m}\, a_t^{ij,k} + a_r^{ki,m}\, a_t^{\ell j,k} - a_t^{\ell k,m}\, a_r^{ji,k}$$
$$- a_t^{kj,m}\, a_r^{\ell i,k}\Big) = 0$$

for all $r, t, \ell, m, i,$ and j.

We conclude this section by noting that many of the Poisson brackets presented in this paper are of the forms of Theorems 1 and 2. The Jacobi condition for the others can similarly be established by the procedures discussed here.

2. The Ideal Fluid and Magnetohydrodynamics (Double Adiabatic Equations)

The equations of ideal magnetohydrodynamics are

$$\vec{v}_t = -\nabla\left(\frac{v^2}{2}\right) + \vec{v} \times (\nabla \times \vec{v}) - \rho^{-1}\nabla(\rho^2 U_\rho) - \rho^{-1}\nabla\cdot\overleftrightarrow{T}_B \tag{2.1}$$

$$\rho_t = -\nabla\cdot(\rho\vec{v}) \tag{2.2}$$

$$s_t = -\vec{v}\cdot\nabla s \tag{2.3}$$

$$\vec{B}_t = -\vec{B}\nabla\cdot\vec{v} + \vec{B}\cdot\nabla\vec{v} - \vec{v}\cdot\nabla\vec{B} \quad . \tag{2.4}$$

The variables of Eqs. (2.1) - (2.4), ρ, $\vec{v}$, $\vec{B}$ and s, are functions of three spatial coordinates and one time coordinate. Equation (2.1) is the equation of motion for a fluid with density ρ and velocity $\vec{v}$. The magnetic body force term is represented in terms of a symmetric stress tensor $\overleftrightarrow{T}_B = (B^2/2)\overleftrightarrow{I} - \vec{B}\vec{B}$ where $\vec{B}$ is the magnetic field. The symmetry of $\overleftrightarrow{T}_B$ precludes the existence of internal torque densities; the equation obtains a symmetry without the use of the initial condition $\nabla\cdot\vec{B} = 0$. Also in Eq. (2.1) the internal energy per unit mass, $U(\rho,s)$, is a prescribed function of ρ and the entropy per unit mass, s. The intensive variable, pressure p and temperature T, are obtained from this function $p = \rho^2 U_\rho$, $T = U_s$. Equation (2.2) is mass conservation and Eq. (2.3) expresses entropy advection. Equation (2.4) is Faraday's law assuming $\vec{E} + \vec{v} \times \vec{B} = 0$. It is written in a form which is manifestly

Galilean invariant. Below we obtain the Poisson brackets for specific subsets of Eqs. (2.1) - (2.4). The equations of Chew, Goldberger and Low are also expressed in Poisson bracket form.

§ 2.1 Noncanonical Poisson Brackets

The MHD equations are known to possess several conservation laws. In addition to ρ, the momentum density $\rho\vec{v}$ and the energy density $\frac{1}{2}\rho v^2 + \rho U + (B^2/2)$ are densities of conservation laws. The symmetry of $\overleftrightarrow{T}_B$ assures that the angular momentum density $(\vec{x} \times \vec{v})\rho$, is conserved and also one can show that $\rho(\vec{x} - \vec{v}t)$ is conserved.[21] Similarly, the entropy per unit volume $\sigma \equiv s\rho$ is conserved (more generally $\rho f(s)$ for arbitrary function f). Also $\vec{B}$, $\vec{A}\cdot\vec{B}$ (where $\vec{B} = \nabla \times \vec{A}$) and $\vec{v}\cdot\vec{B}$ are conserved densities if $\nabla\cdot\vec{B} = 0$. Below we will discuss these constants in the context of our Poisson structure.

The natural choice for the Hamiltonian is the energy functional

$$H = \int_V \left(\frac{1}{2}\,\rho v^2 + \rho U(\rho, s) + \frac{B^2}{2} \right) d\tau \quad . \tag{2.5}$$

With this as Hamiltonian, the following Poisson bracket[12] produces the Eqs. (2.1) - (2.4):

$$[F,G] = \tag{2.6}$$

$$-\int_V \left\{ \left[\frac{\delta F}{\delta \rho} \nabla \cdot \frac{\delta G}{\delta \vec{v}} + \frac{\delta F}{\delta \vec{v}} \cdot \nabla \frac{\delta G}{\delta \rho} \right] \right. \tag{2.6.1}$$

$$+ \left[\rho^{-1} (\nabla \times \vec{v}) \cdot \left(\frac{\delta G}{\delta \vec{v}} \times \frac{\delta F}{\delta \vec{v}} \right) \right] \tag{2.6.2}$$

$$+ \left[\rho^{-1} \nabla s \cdot \left(\frac{\delta F}{\delta s} \frac{\delta G}{\delta \vec{v}} - \frac{\delta G}{\delta s} \frac{\delta F}{\delta \vec{v}} \right) \right] \tag{2.6.3}$$

$$+ \vec{B} \cdot \left[\frac{1}{\rho} \frac{\delta F}{\delta \vec{v}} \cdot \nabla \frac{\delta G}{\delta \vec{B}} - \frac{1}{\rho} \frac{\delta G}{\delta \vec{v}} \cdot \nabla \frac{\delta F}{\delta \vec{B}} \right]$$

$$\left. + \vec{B} \cdot \left[\left(\nabla \frac{1}{\rho} \frac{\delta F}{\delta \vec{v}} \right) \cdot \frac{\delta G}{\delta \vec{B}} - \left(\nabla \frac{1}{\rho} \frac{\delta G}{\delta \vec{v}} \right) \cdot \frac{\delta F}{\delta \vec{B}} \right] \right\} d\tau \tag{2.6.4}$$

The first term, Eq. (2.6.1), is a natural extension to higher dimension of the K-dV bracket obtained by Gardner.[4] Considered as a binary operation on functionals of ρ and $\vec{v}$, Eq. (2.6.1) satisfies the Jacobi condition. If Eq. (2.5) with $\vec{B}$ set to zero and the s dependence of U suppressed, is used as Hamiltonian, one obtains the ideal fluid equation of motion with $\nabla \times \vec{v} = 0$, and the continuity equation (2.2). The inclusion of Eq. (2.6.2) with the same Hamiltonian produces Eq. (2.1) with the $\nabla \times \vec{v}$ term. The sum of these terms satisfies the Jacobi condition. If Eq. (2.6.3) is added to the previous two, then the resulting bracket considered as an operator on functionals of ρ, $\vec{v}$, and s can be shown to satisfy the Jacobi condition. If Eq. (2.5), with the s dependence of U included and $\vec{B} = 0$, is used as Hamiltonian then Eqs. (2.1) (with $\vec{B} = 0$), (2.2) and (2.3) are produced.

The remaining term, Eq. (2.6.4), accounts for the magnetic field. The last two terms here are doubly contracted dyads, i.e.,

$$B\cdot\left(\nabla \frac{1}{\rho} \frac{\delta F}{\delta \vec{v}}\right) \cdot \frac{\delta G}{\delta \vec{B}} = \left(\nabla \frac{1}{\rho} \frac{\delta F}{\delta \vec{v}}\right) : \left(\frac{\delta G}{\delta \vec{B}} \vec{B}\right)$$

$$= \sum_{i,j=1}^{3} B_i \frac{\partial G}{\partial B_j} \partial_i \left(\frac{1}{\rho} \frac{\partial F}{\partial v_j}\right) .$$

If one considers a bracket composed of this term added to Eqs. (2.6.1) and (2.6.2), then Eqs. (2.1), (2.2) and (2.4) are produced with $H = \int(\rho v^2/2 + \rho U(\rho) + B^2/2)d\tau$. It can be shown that this satisfies the Jacobi requirement. (We note that the Jacobi condition in no way depends upon the initial condition $\nabla\cdot\vec{B} = 0$). Finally, the entire bracket, Eq. (2.6) satisfies the Jacobi requirement. If Eq. (2.5) is used as Hamiltonian then as noted Eqs. (2.1)-(2.4) are obtained. We summarize the above paragraph in Table (1).

Let us now return to the constants. We divide them into three groups, the first we call generators

$$H = \int_V \left(\frac{1}{2}\rho v^2 + \rho U(\rho,s) + \frac{B^2}{2}\right) d\tau \tag{2.7}$$

$$\vec{\mathcal{P}} = \int_V \rho\vec{v}\, d\tau \tag{2.8}$$

$$\vec{\mathcal{L}} = \int_V \vec{x} \times \rho\vec{v}\, d\tau \tag{2.9}$$

$$\vec{G} = \int_V \rho(\vec{x} - \vec{v}t)\,d\tau \quad . \tag{2.10}$$

These constants together with the Poisson bracket defined by Eq. (2.6) generate the infinitesimal transformations of the ten parameter Galilean group. H, of course, generates time translation, while $\vec{P}$ and $\vec{L}$ generate infinitesimal changes due to space translations and rotations respectively. For example, using the k^{th} component of $\vec{P}$ we obtain

$$[\rho, P_k] = -\partial_k \rho$$

$$[v_\ell, P_k] = -\partial_k v_\ell$$

$$[s, P_k] = -\partial_k s$$

$$[B_\ell, P_k] = -\partial_k B_\ell \quad .$$

The remaining constant, $\vec{G}$, physically corresponds to uniform motion of the center of mass of the fluid, i.e., $\vec{x}_{cm} = \vec{v}_{cm} t$ + const. It can be interpreted as an embodyment of Newton's third law; all internal forces occur in action-reaction pairs. The only forces that can be imparted to the center of mass occur through surface terms that here are assumed to vanish. This constant generates changes due to Galilean transformation. We obtain

$$[\rho, G_k] = t\partial_k \rho$$

$$[v_\ell, G_k] = t\partial_k v_\ell - \delta_{\ell k}$$

$$[s, G_k] = t\partial_k s$$

$$[B_\ell, G_k] = t\partial_k B_\ell \quad .$$

The Kronecker delta term of $[v_\ell, G_k]$ allows for the boost in velocity.

The second group of constants commute with any functional of the dynamical variables. That is, for

$$M = \int_V \rho d\tau \tag{2.11}$$

$$S = \int_V \rho s \, d\tau \tag{2.12}$$

we have

$$[\chi, M] = [\chi, S] = 0$$

for arbitrary χ .

The third group of constants is composed of the magnetic constants

$$\vec{\mathcal{B}} = \int_V \vec{B} \, d\tau \tag{2.13}$$

$$\vec{\mathcal{T}} = \int_V \vec{A} \cdot \vec{B} \tag{2.14}$$

$$\mathcal{W} = \int_V \vec{v} \cdot \vec{B} \, d\tau \quad . \tag{2.15}$$

These functionals commute with the Hamiltonian [Eq. (2.7)] only for the initial condition $\nabla \cdot \vec{B} = 0$. The constant $\mathcal{W}$ also requires constant entropy per unit mass.

The double adiabatic equations of Chew, Goldberger and Low can also be produced from the bracket Eq. (2.6). These equations account for the presence of a strong magnetic field through an anisotropic pressure tensor. The pressure parallel to the direction of the magnetic field $p_{\parallel}$ differs from that perpendicular, $p_{\perp}$. If the internal energy depends on B, the magnitude of the magnetic field, in addition to ρ and s, then our bracket produces the double adiabatic equations, if we make the following identifications:

$$p_{\parallel} = \rho^2 \frac{\partial U}{\partial \rho}$$

and

$$p_{\perp} = \rho^2 \frac{\partial U}{\partial \rho} + \rho B \frac{\partial U}{\partial B} \quad .$$

To conclude this subsection we present an alternate, more symmetric form of the bracket defined in Eq. (2.6). If we transform to the set of dynamical variables $\{\rho, \sigma, \vec{M}, \vec{B}\}$, where $\sigma = \rho s$ is the entropy per unit volume and $\vec{M} = \rho\vec{v}$ is the momentum density, then Eqs. (2.1)-(2.4) become eight conservation equations (if one adjoins $\nabla \cdot B = 0$). The pressure is now determined by $p = \rho^2 (\tilde{U}_\rho + \sigma\rho^{-1} \tilde{U}_\sigma)$ where $\tilde{U}(\rho,\sigma) = U(\rho,s)$. As a result of the transformation

$$\left.\frac{\delta}{\delta\rho}\right|_{\vec{v},s} = \left.\frac{\delta}{\delta\rho}\right|_{\vec{M},\sigma} + \rho^{-1}\,\vec{M}\cdot\frac{\delta}{\delta\vec{M}} + \sigma\rho^{-1}\,\frac{\delta}{\delta\sigma} \quad ,$$

together with similar transformations for the other variables, Eq. (2.6) becomes

$$[F,G] = -\int_V \Bigg\{ \rho \left[\frac{\delta F}{\delta \vec{M}} \cdot \nabla \frac{\delta G}{\delta \rho} - \frac{\delta G}{\delta \vec{M}} \cdot \nabla \frac{\delta F}{\delta \rho} \right]$$

$$+ \vec{M} \cdot \left[\frac{\delta F}{\delta \vec{M}} \cdot \nabla \frac{\delta G}{\delta \vec{M}} - \frac{\delta G}{\delta \vec{M}} \cdot \nabla \frac{\delta F}{\delta \vec{M}} \right]$$

$$+ \sigma \left[\frac{\delta F}{\delta \vec{M}} \cdot \nabla \frac{\delta G}{\delta \sigma} - \frac{\delta G}{\delta \vec{M}} \cdot \nabla \frac{\delta F}{\delta \sigma} \right] + \vec{B} \cdot \left[\frac{\delta F}{\delta \vec{M}} \cdot \nabla \frac{\delta G}{\delta \vec{B}} - \frac{\delta G}{\delta \vec{M}} \cdot \nabla \frac{\delta F}{\delta \vec{B}} \right]$$

$$+ \vec{B} \cdot \left[\left(\nabla \frac{\delta F}{\delta \vec{M}} \right) \cdot \frac{\delta G}{\delta \vec{B}} - \left(\nabla \frac{\delta G}{\delta \vec{M}} \right) \cdot \frac{\delta F}{\delta \vec{B}} \right] \Bigg\} \, d\tau \quad . \qquad (2.16)$$

Notice that each term of Eq. (2.16) is linear in one Eulerian variable and there are no terms, like those of Eq. (2.6), with the density ρ in the denominator. This feature facilitates evaluating the bracket when polynomial or Fourier representations are used for the dynamical variables. Also we observe that Eq. (2.16) is of the form discussed in Theorem 2 of § 1.3 .

§ 2.2 Potential Representations

The use of potentials to represent vector fields has a history that transcends the familiar potential decomposition of electricity and magnetism. In this subsection, we discuss potential representations that pertain to our Poisson bracket [Eq. (2.6)]. (We note that the historical account presented here should not be taken as complete. Such a task is hampered by a great deal of rediscovery in this area. The interested reader is directed to Refs. 22 - 29.) In particular, our main goal is to represent the fluid velocity field in a form that facilitates a canonical Hamiltonian description, and to show how this form transforms to Eq. (2.6). Various forms of potential representations "canonize" the subsets of the MHD equations discussed in §2.1. The magnetic field, of course, can also be subjected to potential decomposition. We conclude this subsection with a highly symmetric description where this decomposition, in addition to that for the velocity field, is done.

Euler (1769) in his investigation of fluids, represented the solenoidal vector field $\vec{v}$, where $\nabla \cdot \vec{v} = 0$, in the form

$$\vec{v} = \nabla F \times \nabla G \quad . \tag{2.17}$$

This decomposition in terms of the "Euler potentials" F and G can be shown to be locally general. This contravariant representation manifestly assures $\nabla \cdot \vec{v} = 0$. Locally, F and G must define independent surfaces. The intersection of these surfaces defines flow lines. (In plasma physics it is common, as we do below, to represent the magnetic field in this form; the intersection of these surfaces in this case defines field lines.) This representation is clearly not unique, since any function of G may be added to F (and vice versa) without changing $\vec{v}$. More generally any two functions $\alpha(F,G)$ and $\beta(F,G)$ can replace F and G provided the Jacobian $\partial(\alpha,\beta)/\partial(F,G) = 1$. [Note, one can add the gradient of an arbitrary harmonic function, ϕ, to Eq. (2.17) without destroying the solenoidal property. In the case where $\nabla\cdot\vec{v} \neq 0$ and ϕ is not harmonic, we have a form, in the same vein as the Helmholtz representation, which was presented by Monge (1784).]

We now present (as a stepping stone) a representation due to Clebsch (1859), which yields a variational description of the incompressible Eulerian fluid equations. If

$$\vec{v} = \alpha\nabla\beta + \nabla\phi \quad , \tag{2.18}$$

where ϕ is chosen such that $\nabla\cdot\vec{v} = 0$, then Euler's equations can be represented in Hamiltonian form. The potential α is seen to be canonically conjugate, in the usual sense, to the potential β.

A generalization of Eq. (2.18) that includes density variation is the following:

$$\rho\vec{v} = \lambda\nabla\mu + \rho\nabla\phi \quad . \tag{2.19}$$

This decomposition allows (at the expense of obtaining gauge conditions) a Hamiltonian description for a compressible fluid. The density ρ is seen to be conjugate to the potential ϕ and similarly, λ and μ are canonically conjugate. The Poisson bracket in terms of these potentials is

$$[F,G] = \int \left[\left(\frac{\delta F}{\delta\rho}\frac{\delta G}{\delta\phi} - \frac{\delta G}{\delta\rho}\frac{\delta F}{\delta\phi}\right) + \left(\frac{\delta F}{\delta\lambda}\frac{\delta G}{\delta\mu} - \frac{\delta G}{\delta\lambda}\frac{\delta F}{\delta\mu}\right)\right] d\tau \ , \tag{2.20}$$

where F and G are functionals of ρ, ϕ, λ and μ. If the Hamiltonian $H = \int[\frac{1}{2}\rho v^2 + \rho U(\rho)]d\tau$ is represented in terms of these variables by making use of Eq. (2.19), then the equations of motion are obtained in the usual manner (e.g., $\phi_t = [\phi, H]$). Now suppose

$$F[\rho,\phi,\lambda,\mu] = F'[\rho,\vec{v}] \quad ,$$

then the chain rule for functional differentiation yields

$$\frac{\delta F}{\delta\phi} = -\nabla \cdot \frac{\delta F'}{\delta\vec{v}} \quad , \qquad \frac{\delta F}{\delta\rho} = \frac{\delta F'}{\delta\rho} - \frac{\lambda}{\rho^2}\nabla\mu \cdot \frac{\delta F'}{\delta\vec{v}} \tag{2.21}$$

and similar expressions for λ and μ. Substitution of these expressions into Eq. (2.20) yields a portion of our Poisson bracket, Eq. (2.6.1) plus Eq. (2.6.2). [Note by Eq. (2.21), exclusion of λ and μ yields the irrotational portion of the bracket Eq. (2.6.1).]

Similarly, entropy advection is alloted for by the inclusion of an additional potential. Consider the following covariant form:

$$\rho\vec{v} = \lambda\nabla\mu + \rho\nabla\phi + \sigma\nabla\psi \quad . \tag{2.22}$$

Here ψ, the additional potential, is canonically conjugate to σ the entropy per unit volume. As above, the chain rule for Eq. (2.22) yields the Poisson bracket that is the sum of Eqs. (2.6.1), (2.6.2), and (2.6.3).

Consider now a form that includes the magnetic field

$$\rho\vec{v} = \vec{B} \times (\nabla \times \vec{T}) + \nabla\phi \quad . \tag{2.23}$$

Zakharov and Kuznetsov[29] (1971) presented a Hamiltonian description for MHD (with constant entropy/mass), where the vector potential $\vec{T}$ of Eq. (2.23) is seen to be conjugate to $\vec{B}$ in addition to maintaining the ρ,ϕ conjugacy. We emphasize that this form cannot be transformed into our bracket. The appropriate form, which respects the distinction between the initial condition $\nabla\cdot\vec{B} = 0$ and the dynamical symmetries of invariance under Galilean transformation and rotation, is

$$\rho\vec{v} = (\nabla\vec{T}) \cdot \vec{B} - \vec{B}\cdot\nabla\vec{T} - \vec{T}\nabla\cdot\vec{B} + \rho\nabla\phi \quad . \tag{2.24}$$

The following Poisson bracket:

$$[F,G] = \int_V \left\{ \left(\frac{\delta G}{\delta\phi}\frac{\delta F}{\delta\rho} - \frac{\delta F}{\delta\phi}\frac{\delta G}{\delta\rho} \right) + \left(\frac{\delta F}{\delta\vec{B}}\cdot\frac{\delta G}{\delta\vec{T}} - \frac{\delta G}{\delta\vec{B}}\cdot\frac{\delta F}{\delta\vec{T}} \right) \right\} d\tau$$

yields with Eq. (2.24) and the chain rule, the Poisson bracket Eq. (2.6) with the exception of the entropy term [Eq. (2.6.3)]. The entire bracket is obtained by adding $\sigma\nabla\psi$ to Eq. (2.24) and considering the canonical structure that includes σ conjugate to ψ.

To conclude this subsection, we present a formulation that entails a decomposition of $\vec{B}$ as well as $\vec{v}$. If we expand $\vec{B}$ in terms of Euler potentials as in Eq. (2.17)

$$\vec{B} = \nabla\alpha \times \nabla\beta \quad ,$$

then the appropriate expression for $\vec{v}$ is

$$\rho\vec{v} = a\nabla\alpha + b\nabla\beta + \rho\nabla\phi \quad . \tag{2.25}$$

In this representation the advected field labels α and β are seen to be conjugate to the potentials a and b. The initial condition $\nabla \cdot \vec{B} = 0$ is now inherent to the dynamics. The connection to the formulation of Eq. (2.23) is easily seen to be made through the following:

$$a = -\nabla\beta \cdot (\nabla \times \vec{T}) \quad , \qquad b = \nabla\alpha \cdot (\nabla \times \vec{T}) \quad .$$

We note that the entire canonical formulation is obtained by appending $\sigma\nabla\psi$ to Eq. (2.25). These results are summarized in Table 2.

3. Two-Dimensional Vortex Fluids and Guiding Center Plasmas

The equations for vortex advection in two spatial dimensions are used to model large scale motions that occur in atmospheres and oceans. The same equations have arisen in the study of plasma transport perpendicular to a uniform magnetic field. If we assume the usual euclidean coordinate system with uniformity in the $\hat{z}$ direction then the scalar vorticity is $\omega(\vec{x},t) = \vec{z}\cdot\nabla \times \vec{v}(\vec{x},t)$, where $\vec{v}$ is the fluid velocity such that $\vec{v}\cdot\vec{z} = 0$. For the guiding center plasma, ω corresponds to the charge density and $\vec{v}$ to the $\vec{E} \times \vec{B}$ drift velocity. The equations under consideration are the following:

$$\omega_t = -\vec{v}\cdot\nabla\omega \tag{3.1}$$

$$\nabla\cdot\vec{v} = 0 \quad . \tag{3.2}$$

For an unbounded fluid v can be eliminated from Eq. (3.1) by

$$\vec{v} = \int \omega(\vec{x}\,')\ \vec{M}(\vec{x}\ |\ \vec{x}\,')\ d\tau' \quad , \tag{3.3}$$

where we display only the arguments necessary to avoid confusion. Here $\vec{M} = \hat{z} \times \nabla\, k(\vec{x}|\vec{x}\,')$ and $k(\vec{x}|\vec{x}\,')$ is the Green function for Laplace's equation in two dimensions

$$k(\vec{x}|\vec{x}\,') = \frac{1}{4\pi}\ \ell n[(x-x')^2+(y-y')^2] \quad .$$

The integration in Eq. (3.3) is over the entire x-y plane; $d\tau \equiv dxdy$. Observe Eq. (3.2) is manifestly satisfied by Eq. (3.3). Eq. (3.1) becomes

$$\omega_t = -\int \omega(\vec{x}\,') \, \vec{M}(\vec{x}|\vec{x}\,') d\tau' \cdot \nabla\omega(\vec{x}) \quad . \tag{3.4}$$

Equations (3.1) and (3.2) are known to possess conserved densities, e.g. any function of ω is conserved. In addition, the kinetic energy, which is the natural choice for the Hamiltonian, is conserved. With the density set to unity we have

$$H[\omega] = \int \frac{v^2}{2} d\tau = -\frac{1}{2} \int k(\vec{x}|\vec{x}\,') \; \omega(\vec{x}\,') \; \omega(\vec{x}) \; d\tau \; d\tau' . \tag{3.5}$$

The functional derivative of Eq. (3.5) is

$$\frac{\delta H}{\delta\omega} = -\int k(\vec{x}|\vec{x}\,') \; \omega(\vec{x}\,') \; d\tau' \quad .$$

The Poisson bracket[14] that produces Eq. (3.4) is the following:

$$[F,G] = \int \omega(\vec{x}) \left\{ \frac{\delta F}{\delta\omega} , \frac{\delta G}{\delta\omega} \right\} d\tau \tag{3.6}$$

where $\{f,g\} = (\partial f/\partial x)(\partial g/\partial y) - (\partial f/\partial y)(\partial g/\partial x)$. We note that the bracket defined by Eq. (3.6) is precisely that for the one-dimensional Vlasov-Poisson equations[15,19] (see Sec. 6) if one replaces the vorticity by the phase space density and and the phase space (x,y) by (x,v). Also observe that any two functionals composed of functions of ω alone are in involution with respect to Eq. (3.6).

We conclude this section by transforming Eq. (3.6) to canonical form. The discussion of potentials in §2.2 indicates the following representation of the vorticity

$$\omega = \{\alpha, \beta\} \quad . \tag{3.7}$$

The chain rule for functional differentiation yields

$$\frac{\delta F}{\delta \alpha} = \left\{ \beta, \frac{\delta F}{\delta \omega} \right\} , \quad \frac{\delta F}{\delta \beta} = \left\{ \frac{\delta F}{\delta \omega}, \alpha \right\} \tag{3.8}$$

where on the left F is now regarded as a functional of α and β. The canonical Poisson bracket for α and β is

$$[F,G] = \int \left(\frac{\delta F}{\delta \alpha} \frac{\delta G}{\delta \beta} - \frac{\delta F}{\delta \beta} \frac{\delta G}{\delta \alpha} \right) d\tau \tag{3.9}$$

which upon substitution of Eqs. (3.8) yields the bracket Eq. (3.6). (This is easily accomplished by making use of the relation $\int f\{g,h\}\, d\tau = \int g\{h,f\}\, d\tau$ and the Jacobi requirement).

4. Fully Nonlinear Ion-Acoustic Waves

In this section we present the Poisson bracket for a particular approximation of the two-fluid equations of plasma physics that models nonlinear ion-acoustic waves.[30] In the limit that the electron temperature greatly exceeds the ion temperature, the ion dynamics are governed by the cold fluid momentum transport and continuity equations,

$$v_t = -vv_x - \phi_x \tag{4.1}$$

$$N_t = -(Nv)_x \quad . \tag{4.2}$$

Here, v is the ion fluid velocity, normalized to the ion sound speed $c_s = \sqrt{T_e/m_i}$ where T_e is the electron temperature and m_i the ion mass, N is the ion density that is normalized to n_o the quasi-neutral electron or ion density, x and t are expressed in units of the electron Debye length $\lambda_D = \sqrt{T_e/4\pi n_o e^2}$ and ion plasma frequency $\omega_{pi} = \sqrt{4\pi n_o e^2/m_i}$ respectively. The electrostatic potential ϕ couples the ion dynamics to the electrons through Poisson's equation

$$\phi_{xx} = n(\phi) - N \quad . \tag{4.3}$$

Here, ϕ is normalized to e/T_e and the electron density, $n(\phi)$, is assumed to be a function of ϕ. Typically, since the electron mass is greatly exceeded by the ion mass, electron inertial terms are neglected and the approximation of isothermal electrons is justifiable. In this case $n(\phi) = e^{\phi}$. The structure that we present makes no restrictions on n except that it be a function of ϕ.

In the case $n(\phi) = e^{\phi}$, since $\phi_x = n_x/n$, it is customary to envision the electrons as supplying the ion pressure. Alternatively with $n(\phi)$ specified the constraint equation (4.3) can be interpreted as supplying a non-local equation of state for the ion pressure. It is through this non-local equation of state that dispersion is introduced into the dynamics. It is well-known that in addition to shock wave solutions these equations possess solitary wave solutions. Equations (4.1) - (4.3) are the starting point for the reductive perturbation procedure which yields the K-dV equation for ion-acoustic solitons.[31]

The three known integral constants for Eqs. (4.1) - (4.3) are

$$\mathcal{N} = \int_{\mathcal{R}} N \, dx \tag{4.4}$$

$$\mathcal{P} = \int_{\mathcal{R}} Nv \, dx \tag{4.5}$$

$$\mathcal{H} = \int_{\mathcal{R}} \left(\frac{Nv^2}{2} + \mathcal{L}N \right) dx \tag{4.6}$$

where in Eq. (4.6) $\mathcal{L}$ is a nonlocal operator determined by Eq. (4.3) such that

$$\mathcal{L}N = \frac{\phi_x^2}{2} + \int^{\phi} \phi' \frac{\partial n(\phi')}{\partial \phi'} d\phi' \quad . \tag{4.7}$$

Equation (4.7) represents a nonlocal internal energy function. The obvious choice for the Hamiltonian is, of course, the energy, Eq. (4.7). (We note that in terms of the Poisson bracket presented below Eq. (4.9), P and N are in involution.) Observe

$$\frac{\delta H}{\delta v} = Nv$$

and subsequently we will show

$$\frac{\delta H}{\delta N} = \frac{v^2}{2} + \phi \quad . \tag{4.8}$$

The following bracket, which is the one-dimensional restriction of the first term of Eq. (2.6), yields the equations of motion:

$$[F,G] = \int_R \left(\frac{\delta G}{\delta v} \partial \frac{\delta F}{\delta N} - \frac{\delta F}{\delta v} \partial \frac{\delta G}{\delta N} \right) dx \tag{4.9}$$

where $\partial \equiv d/dx$. Clearly

$$N_t = [N,H] = -(Nv)_x$$

and assuming Eq. (4.8),

$$v_t = [v,H] = -(v^2/2 + \phi)_x \ .$$

To justify Eq. (4.8), suppose $\mathcal{P}[\phi]$ is some functional of ϕ, i.e.,

$$\mathcal{P}[\phi] = \int_{\mathcal{R}} P(\phi)\,dx \quad .$$

Varying this we obtain

$$\delta\mathcal{P}(\phi;\delta\phi) = \int_{\mathcal{R}} \frac{\delta\mathcal{P}}{\delta\phi}\,\delta\phi\,dx \quad . \tag{4.10}$$

To see how a variation in ϕ is related to a variation in N we linearize Eq. (4.3) and obtain

$$\left(\partial^2 - \frac{\partial n}{\partial\phi}(\phi)\right)\delta\phi = -\delta N \quad ,$$

which upon formally inverting yields

$$\delta\phi(x) = -\int_{\mathcal{R}} K(\phi,x,x')\,\delta N(x')\,dx' \tag{4.11}$$

where K satisfies

$$\left(\partial^2 - \frac{\partial n}{\partial\phi}\right)K = \delta(x-x') \quad . \tag{4.12}$$

Here, $\delta(x)$ is the Dirac delta function and we seek solutions with asymptotic charge neutrality and vanishing electric field. Substituting Eq. (4.11) into Eq. (4.10) yields

$$\int_{\mathcal{R}}\left[\frac{\delta\mathcal{P}}{\delta N} + \int_{\mathcal{R}}\frac{\delta\mathcal{P}}{\delta\phi}\,K\,dx'\right]\delta N\,dx = 0 \quad .$$

For our special case where

$$\mathcal{P}[\phi] = \int_{\mathcal{R}} \left[\frac{\phi_x^2}{2} + \int^{\phi} \phi' \frac{\partial n}{\partial \phi'} \, d\phi' \right] dx ,$$

we obtain

$$\frac{\delta \mathcal{P}}{\delta N} = \int_{\mathcal{R}} \left(\partial'^2 \phi + \phi(x') \frac{\partial n}{\partial \phi} \right) K \, dx' ,$$

which with Eq, (4.12) implies

$$\frac{\delta \mathcal{P}}{\delta N} = \phi .$$

To conclude this section we obtain a canonical form for the bracket Eq. (4.9). With the substitution

$$v = \psi_x ,$$

where ψ now replaces v as a dynamical variable, and the chain rule for functional differentiation

$$\frac{\delta \mathcal{F}}{\delta \psi} = - \partial \frac{\delta \mathcal{F}}{\delta v} ,$$

Eq. (4.9) becomes

$$[\mathcal{F}, \mathcal{G}] = \int_{\mathcal{R}} \left(\frac{\delta \mathcal{F}}{\delta N} \frac{\delta \mathcal{G}}{\delta \psi} - \frac{\delta \mathcal{F}}{\delta \psi} \frac{\delta \mathcal{G}}{\delta N} \right) dx .$$

(Observe that the subtitution $N = \psi_x$ will also achieve the same end). Clearly the substitution (4.13) makes Eqs. (4.1) - (4.3) variational in the sense that we can construct the action

$$\mathcal{J} = \int_{\mathcal{R}} \int_{\mathcal{T}} N_t \, \psi \, dx \, dt - \int_{\mathcal{T}} \mathcal{H}[N,\psi] \, dt$$

which upon variation with respect to N and ψ produces the dynamical equations.

5. The Vlasov-Maxwell Equations

If a plasma is sufficiently hot and tenuous, then collisions become unimportant. When this is the case, fast time scale plasma phenomena is decribed by the following set of equations:

$$f_{\alpha t}\,(\vec{x},\vec{v},t) = -\vec{v}\cdot \frac{\partial f_\alpha}{\partial \vec{x}} - \frac{e_\alpha}{m_\alpha}\,[\vec{E}+\vec{v}\times\vec{B}]\cdot \frac{\partial f_\alpha}{\partial \vec{v}} \tag{5.1}$$

$$\vec{B}_t(\vec{x},t) = -\nabla\times\vec{E} \tag{5.2}$$

$$\vec{E}_t(\vec{x},t) = \nabla\times\vec{B} - \sum_\alpha e_\alpha \int_R \vec{v}\, f_\alpha\, d^3v \quad . \tag{5.3}$$

Equation (5.1) is the evolution equation for the single particle distribution function, f_α , which is a function of the six phase-space coordinates together with time. Here α designates species and e_α and m_α are the signed charge and mass respectively. Equation (5.2) is Faraday's law relating the magnetic field $\vec{B}$ and the electric field $\vec{E}$. Equation (5.3) is Ampere's law with the inclusion of the displacement current. (We use rationalized Gaussian units with the speed of light set to unity.)

It is well known that this system, Eqs. (5.1)-(5.3), conserves energy. The natural choice for the Hamiltonian functional is the following:

$$\mathcal{H}[f_\alpha,\vec{E},\vec{B}] = \sum_\alpha \int_{\mathcal{D}} \frac{1}{2}\, m_\alpha\, v^2 f_\alpha\; d^3x d^3v + \frac{1}{2}\int_R (E^2+B^2)\, d^3x \quad . \tag{5.4}$$

For this Hamiltonian observe

$$\frac{\delta H}{\delta f_\alpha} = \frac{1}{2} m_\alpha v^2 \, , \qquad \frac{\delta H}{\delta \vec{E}} = \vec{E} \, , \qquad \frac{\delta H}{\delta \vec{B}} = \vec{B} \quad .$$

With Eq. (5.4) as Hamiltonian it is not difficult to show that the following Poisson bracket[15-17] produces Eqs. (5.1)-(5.3):

$$[F,G] = \tag{5.5}$$

$$\sum_\alpha \int_{\mathcal{D}} \frac{f_\alpha}{m_\alpha} \left\{ \frac{\delta F}{\delta f_\alpha}, \frac{\delta G}{\delta f_\alpha} \right\} d^3x \; d^3v \tag{5.5.1}$$

$$+ \int_{R} \left(\frac{\delta F}{\delta \vec{E}} \cdot \nabla \times \frac{\delta G}{\delta \vec{B}} - \frac{\delta G}{\delta \vec{B}} \cdot \nabla \times \frac{\delta F}{\delta \vec{E}} \right) d^3x \tag{5.5.2}$$

$$+ \sum_\alpha \frac{e_\alpha}{m_\alpha} \int_{\mathcal{D}} \frac{\partial f_\alpha}{\partial \vec{v}} \cdot \left[\frac{\delta F}{\delta \vec{E}} \frac{\delta G}{\delta f_\alpha} - \frac{\delta G}{\delta \vec{E}} \frac{\delta F}{\delta f_\alpha} \right] d^3x \quad d^3v \tag{5.5.3}$$

$$+ \sum_\alpha \frac{e_\alpha}{m_\alpha^{\,2}} \int_{\mathcal{D}} f_\alpha \, \vec{B} \cdot \left[\frac{\partial}{\partial \vec{v}} \frac{\delta F}{\delta f_\alpha} \times \frac{\partial}{\partial \vec{v}} \frac{\delta G}{\delta f_\alpha} \right] d^3x \quad d^3v \quad . \tag{5.5.4}$$

In the first term, Eq. (5.5.1), the curly brackets are used to indicate the usual particle Poisson bracket of two phase functions $\{g,h\} = \partial g/\partial \vec{x} \cdot \partial h/\partial \vec{v} - \partial g/\partial \vec{v} \cdot \partial h/\partial \vec{x}$. This term with Eq. (5.4) produces Eq. (5.1) without the terms which couple in the electric and magnetic fields. It can be shown to satisfy the Jacobi requirement. (In Section 6 we present a construction where this bracket is the entire bracket for the Vlasov-Poisson system.) The second term, Eq. (5.5.2), produces Maxwell's equations in vacuum. This term was apparently first written down by Born and Infeld.[32] It satisfies the Jacobi condition. The next two terms, Eqs. (5.5.3) and (5.5.4) supply the coupling between Eqs. (5.1) and (5.3). Observe the e_α/m_α multiplying each. The first of these yields the electric field coupling term. The last term, Eq. (5.5.4), completes the coupling. This term is due to J. Marsden and A. Weinstein[16], who obtained it through consideration of the underlying Lie group. The Jacobi condition is satisfied for this term only if the space of functionals, on which the bracket acts, is restricted to vector fields $\vec{B}$ that satisfy $\nabla\cdot\vec{B} = 0$. For arbitrary functionals E, F, and G we obtain

$$[E,[F,G]] + \text{cyc} = \int f\, \nabla\cdot\vec{B}\, \epsilon_{s\ell t} \frac{\delta E}{\delta v_\ell} \frac{\delta F}{\delta v_s} \frac{\delta G}{\delta v_t}\, d^3v\, d^3x \quad ,$$

where $\epsilon_{s\ell t}$ is the Levi-Civita tensor. If initially $\nabla\cdot\vec{B} = 0$ then the Jacobi condition is satisfied for all time.

We conclude this section by pointing out a recent motivation[33] of the bracket, Eq. (5.5), (see Ref. 18). Here, the relativistic generalization is made and the generators of the full Poincaré group are pointed out. Table 3 summarizes the above.

6. The Vlasov-Poisson Equations

In this section we write the Vlasov-Poisson equations in a form[15,19] very similar to that of the two-dimensional vortex fluid equations of Sec. 3. We will observe that these sets of equations possess the same noncanonical and canonical formulations. The Vlasov-Poisson equations are

$$\frac{\partial f_\alpha}{\partial t}(\vec{x},\vec{v},t) = -\vec{v}\cdot\frac{\partial f_\alpha}{\partial \vec{x}} + \frac{e_\alpha}{m_\alpha}\frac{\partial \phi}{\partial \vec{x}}\cdot\frac{\partial f_\alpha}{\partial \vec{v}} \tag{6.1}$$

$$\Delta\phi(\vec{x},t) = -\sum_\alpha e_\alpha \int f_\alpha d^3v \quad . \tag{6.2}$$

Here the only symbol not defined in Sec. 5 is ϕ, the electrostatic potential. If we seek solutions where ϕ is defined on $\mathcal{R}$, and if we assume asymptotic charge neutrality and vanishing electric field, then the Laplacian operator Δ can be inverted. Equations (6.1) and (6.2) can be written compactly as follows:

$$\frac{\partial f_\alpha}{\partial t} = -\vec{\omega}_\alpha \cdot \nabla_p f_\alpha \quad . \tag{6.3}$$

Here ∇_p is the six-dimensional phase-space gradient $(\partial/\partial\vec{x}\ ,\ \partial/\partial\vec{v})$ and $\vec{\omega}_\alpha$ is defined by

$$\vec{\omega}_\alpha = \left(v, \frac{e_\alpha}{m_\alpha}\frac{\partial}{\partial\vec{x}} \sum_\beta e_\beta \int K(\vec{x}|\vec{x}')f_\beta d^3x'\right). \tag{6.4}$$

Observe $\nabla_p\cdot\vec{\omega}_\alpha = 0$. In Eq. (6.4) $K(\vec{x}|\vec{x}')$ is the kernel for the inverse Laplacian; e.g., in one-dimension $K(\vec{x}|\vec{x}') = \frac{1}{2}|\vec{x} - \vec{x}'|$. The Hamiltonian for this system is

$$\mathcal{H} = \sum_{\alpha} \frac{1}{2} m_{\alpha} \int v^2 f_{\alpha} d^3 z - \frac{1}{2} \sum_{\alpha\beta} e_{\alpha} e_{\beta} \int K(\vec{x}|\vec{x}') f_{\alpha}(z) f_{\beta}(z') d^3 z d^3 z' \quad . \tag{6.5}$$

Here we have used $z \equiv (\vec{x}, \vec{v})$. The Poisson bracket is the first term of Eq. (5.5)

$$[F, G] = \sum_{\alpha} \int \frac{f_{\alpha}(z)}{m_{\alpha}} \left\{ \frac{\delta F}{\delta f_{\alpha}}, \frac{\delta G}{\delta f_{\alpha}} \right\} d^3 z \quad , \tag{6.6}$$

where the braces are as defined in Sec. 5. It is not difficult to see that

$$\frac{\partial f_{\alpha}}{\partial t} = [f_{\alpha}, \mathcal{H}] = -\vec{\omega}_{\alpha} \cdot \nabla_p f_{\alpha} \quad .$$

To obtain canonical form we consider the three-dimensional generalization of the potential representation of Sec. 3,

$$f_{\alpha} = \frac{1}{m_{\alpha}} \{\psi_{\alpha}, \chi_{\alpha}\} \quad . \tag{6.7}$$

With this substitution, ψ_{α} and χ_{α} become canonically conjugate variables. We note, in conclusion, that the entire bracket of Sec. 5, Eq. (5.5), can be put into canonical form by the substitution of Eq. (6.8) together with the usual canonical description of the fields in terms of the vector potential $\vec{A}$ and its conjugate $\vec{E}$.

Acknowlegments

This work began with the consideration of ideal MHD. My collaborator in this effort was John M. Greene. Our principle motivation was the work of Robert Littlejohn on the use of noncanonical variables for perturbation theory (Ref. 2) and the work of C. S. Gardner on the KdV equation (Ref. 4). It is with pleasure that I thank Allan Kaufman for his help and encouragement. Section 3 was motivated by several conversations with Harvey Segur. I have learned much about 2-D turbulence from Segur, Guido Sandri, and Rick Salmon. The work of Sec. 4 was prompted by many discussions with Predhiman Kaw. I have also benefited from conversations and/or correspondences with the following: D. Barnes, F. Henyey, D. Holm, J. Hubbard, M. Kruskal, R. Kulsrud, B. Kupershmidt, J. Marsden, C. Oberman, J. B. Taylor, W. B. Thompson, and A. Weinstein.

References

1. V. I. Arnold, Mathematical Methods of Classical Mechanics, (Springer-Verlag, New York, 1978).

2. R. G. Littlejohn, J. Math. Phys. 20, 2445 (1979).

3. J. E. Marsden, Lectures on Geometric Methods in Mathematical Physics, (Siam, Philadelphia, 1981).

4. C. S. Gardner, J. Math. Phys. 12, 1548 (1971).

5. P. D. Lax, Comm. Pure Appl. Math. 28, 141 (1975).

6. D. R. Lebedev and Y. I. Manin, Funk. Anal. Appl. 13, 40 (1979).

7. I. M. Gel'fand and I. Y. Dorfman, Funk. Anal. Appl. 13, 248 (1979).

8. B. A. Kupershmidt, Rutgers 1981 Proceedings, Lecture Notes in Mathematics (to appear).

9. Y. I. Manin, J. Sov. Math. 11, 1 (1979).

10. P. R. Chernoff and J. E. Marsden, Properties of Infinite Dimensional Hamiltonian Systems, Lect. Notes in Math. 425, (Springer-Verlag, Berlin, 1974).

11. R. Abraham, J. E. Marsden, Foundations of Mechanics, (Benjamin-Cummings, London, 1978).

12. P. J. Morrison and J. M. Greene, Phys. Rev. Lett. 45, 790 (1980). Errata to appear.

13. G. F. Chew, M. L. Goldberger, and F. E. Low, Proc. Roy. Soc. A236, 112 (1956).

14. P. J. Morrison, Princeton University Plasma Physics Laboratory Report, PPPL-1783 (1981).

15. P. J. Morrison, Phys. Lett. 80A, 383 (1980).

16. J. E. Marsden and A. Weinstein, Physica D. (to appear).

17. A. Weinstein and P. J. Morrison, Phys. Lett. 86A, 235 (1981).

18. I. Bialynicki-Birula and J. C. Hubbard, preprint.

19. P. J. Morrison, Princeton University Plasma Physics Laboratory Report, PPPL-1788 (1981).

20. H. Goldstein, Classical Mechanics, (Addison-Wesley, Reading, 1950).

21. Constants of this general form for dynamical systems are treated by I. L. Caldas and H. Tasso, Lett. Nuovo Cimento 24, 500 (1979).

22. D. P. Stern, Am. J. Phys. 38, 494 (1970).

23. H. Lamb, Hydrodynamics, (Dover, New York, 1945).

24. J. Serrin, Handbuch der Physik 8, 169 (1959).

25. P. A. M. Dirac, Proc. Roy. Soc. A212, 330 (1952).

26. A. Clebsch, J. Reine Angeev, Math. 54 293 (1857); 56, 1 (1859).

27. R. L. Seliger and G. B. Whitham, Proc. Roy. Soc. A305, 1 (1968).

28. M. G. Calkin, Can. J. Phys. 41, 2241 (1963).

29. V. E. Zakharov and E. A. Kuznetzov, Sov. Phys. Dokl. 15, 913 (1971).

30. N. A. Krall and A. W. Trivelpiece, Principles of Plasma Physics, (McGraw Hill, New York, 1973).

31. C. H. Su and C. S. Gardner, J. Math. Phys. 10, 536 (1969).

32. M. Born and L. Infeld, Proc. Roy. Soc. 150, 141 (1935).

33. This motivation is based on the early work of I. Bialynicki-Birula. See I. Bialynicki-Birula and Z. Bialynicka-Birula, Quantum Electrodynamics, (Pergamon, Oxford, 1975).

SATISFIES JACOBI	COMMENTS
Eq. (2.61)	Defined on functionals of ρ & $\vec{v}$. With $H = \int[\rho v^2/2 + \rho U(\rho)]d\tau$ produces Eq. (2,1) with $\nabla\times\vec{v} = 0$ and $\vec{B} = 0$, and Eq. (2.2).
Eq. (2.61) + Eq. (2.62)	Defined on functionals of ρ & $\vec{v}$. With $H = \int[\rho v^2/2 + \rho U(\rho)]d\tau$ produces Eq. (2.1) with $\vec{B} = 0$ and Eq. (2.2).
Eq. (2.61) + Eq. (2.62) + Eq. (2.63)	Defined on functionals of $\rho, \vec{v}$ and s. With $H = \int[\rho v^2/2 + \rho U(\rho,s)]\, d\tau$ produces Eq. (2.1) with $\vec{B} = 0$, Eq. (2.2) and Eq. (2.3).
Eq. (2.61) + Eq. (2.62) + Eq. (2.64)	Defined on functionals of $\rho, \vec{v}$ and $\vec{B}$. With $H = \int[\rho v^2/2 + \rho U(\rho) + B^2/2]\, d\tau$ produces Eq. (2.1), Eq. (2.2) and Eq. (2.4).
Eq. (2.61) + Eq. (2.62) + Eq. (2.63) + Eq. (2.64)	Defined on functionals of $\rho, \vec{v}, \vec{B}$ and s. With $H = \int[\rho v^2/2 + \rho U(\rho,s) + B^2/2]\, d\tau$ produces Eqs. (2.1) - (2.4)

TABLE 1

Kind of Fluid	Noncanonical Variables	Canonical Variables	Velocity Representation
Ideal, Irrotational, and Isentropic	$\vec{v}$ and ρ	ϕ and ρ	$\rho\vec{v} = \rho\nabla\phi$
Ideal and Isentropic	$\vec{v}$ and ρ	λ, μ, ϕ and ρ	$\rho\vec{v} = \lambda\nabla\mu + \rho\nabla\phi$
Ideal	$\vec{v}$, ρ and s	λ, μ, ϕ, ψ, σ and ρ	$\rho\vec{v} = \lambda\nabla\mu + \rho\nabla\phi + \sigma\nabla\psi$
Ideal Isentropic MHD	$\vec{v}$, ρ and $\vec{B}$	$\vec{T}$, $\vec{B}$, ϕ and ρ	$\rho\vec{v} = (\nabla\vec{T})\cdot\vec{B} - \vec{B}\cdot\nabla\vec{T} - \vec{T}\nabla\cdot\vec{B} + \rho\nabla\phi$
Ideal Isentropic MHD, $\vec{B} = \nabla\alpha\times\nabla\beta$	$\vec{v}$, ρ and $\vec{B}$	α, β, a, b, ϕ and ρ	$\rho\vec{v} = a\nabla\alpha + b\nabla\beta + \rho\nabla\phi$
Ideal MHD	$\vec{v}$, ρ, $\vec{B}$ and s	$\vec{T}$, $\vec{B}$, σ, ψ, ϕ and ρ	$\rho\vec{v} = (\nabla\vec{T})\cdot\vec{B} - \vec{B}\cdot\nabla\vec{T} - \vec{T}\nabla\cdot\vec{B} + \rho\nabla\phi + \sigma\nabla\psi$
Ideal MHD, $\vec{B} = \nabla\alpha\times\nabla\beta$	$\vec{v}$, ρ, $\vec{B}$ and s	α, β, a, b, σ, ψ, ϕ and ρ	$\rho\vec{v} = a\nabla\alpha + b\nabla\beta + \rho\nabla\phi + \sigma\nabla\psi$

Table 2

SATIFIES JACOBI	COMMENTS
Eq.(5.51)	with $H = \sum_{\alpha} \int \frac{1}{2} m_\alpha v^2 f_\alpha d^3x\, d^3v$ produces Eq. (5.1) with $\vec{E} = \vec{B} = 0$.
Eq.(5.52)	with $H = \int \frac{1}{2} (E^2+B^2) d^3x$ produces Maxwell's equations in vacuum.
Eq.(5.51) + Eq.(5.52) + Eq.(5.53) + Eq.(5.54)	with Eq. (5.4) as Hamiltonian produces the Vlasov-Maxwell equations. Requires the constraint $\nabla \cdot \vec{B} = 0$.

TABLE 3