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Dielectric Energy versus Plasma Energy, and Hamiltonian Action-Angle Variables for the Vlasov Equation

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Abstract

Expressions for the energy content of one-dimensional electrostatic perturbations about homogeneous equilibria are revisited. The well-known dielectric energy, $\mathcal{E}_D$, is compared with the exact plasma free energy expression, $\delta^2 F$, that is conserved by the Vlasov-Poisson system [Phys. Rev. A **40**, 3898 (1989) and Phys. Fluids B **2**, 1105 (1990)]. The former is an expression in terms of the perturbed electric field amplitude, while the latter is determined by a generating function, which describes perturbations of the distribution function that respect the important constraint of *dynamical accessibility* of the system. Thus the comparison requires solving the Vlasov equation for such a perturbation of the distribution function in terms of the electric field. This is done for neutral modes of oscillation that occur for equilibria with stationary inflection points, and it is seen that for these special modes $\delta^2 F = \mathcal{E}_D$. In the case of unstable and

corresponding damped modes it is seen that $\delta^2 F \neq \mathcal{E}_D$; in fact $\delta^2 F \equiv 0$. This failure of the dielectric energy expression persists even for arbitrarily small growth and damping rates since $\mathcal{E}_D$ is nonzero in this limit, whereas $\delta^2 F$ remains zero. In the case of general perturbations about stable equilibria, the two expressions are not equivalent; the exact energy density is given by an expression proportional to $\frac{\omega|E(k,\omega)|^2|\epsilon(k,\omega)|^2}{\epsilon_I(k,\omega)}$, where $E(k,\omega)$ is the Fourier transform in space and time of the perturbed electric field (or equivalently the electric field associated with a single Van Kampen mode) and $\epsilon(k,\omega)$ is the dielectric function with ω and k real and independent. The connection between the new exact energy expression and the at-best approximate $\mathcal{E}_D$ is described. The new expression motivates natural definitions of Hamiltonian action variables and signature. A general linear integral transform (or equivalently a coordinate transformation) is introduced that maps the linear version of the noncanonical Hamiltonian structure, which describes the Vlasov equation, to action-angle (diagonal) form.

I. Introduction

Expressions for the energy contained in the perturbation away from equilibria are important for, among other things, ascertaining stability. If such an energy is positive definite then the system is stable, while if the energy is indefinite then either the system is unstable or there exist negative energy modes. Negative energy modes are of importance since their presence can lead to nonlinear (finite or infinitesimal amplitude) instability and if dissipation is added they can become linearly unstable. Therefore, a precise understanding of the energy in a perturbation is important to have.

Early work on such electrostatic instabilities in homogeneous plasmas [1, 2, 3, 4, 5] and their relation to energy were based on the well-known expression for the energy of a dispersive dielectric medium [6, 7, 8] and generalizations thereof [9],

$$\mathcal{E}_D = \frac{V}{16\pi} \frac{\partial(\omega\epsilon_R)}{\partial\omega} \left|\mathbf{E}(\mathbf{k},\omega)\right|^2 , \tag{1}$$

where ϵ_R is the real part of the dielectric function, V is the volume of a periodicity box, and $\mathbf{E}(\mathbf{k},\omega)$ is perturbed electric field amplitude for a mode with wave vector $\mathbf{k}$ and frequency $\omega(\mathbf{k})$. This expression is derived for general media described by Maxwell's equations and the dielectric function. It is often believed that relation (1) is valid if the imaginary part of the dielectric function, ϵ_I, is negligible, but it will be seen below that this is not sufficient.

Another expression was discovered by Kruskal and Oberman [10] for the perturbed energy, which in the case of the one-dimensional Vlasov-Poisson systems with homogeneous monotonic equilibria, is given by

$$\mathcal{E}_{KO} = -\sum_\nu \frac{m_\nu}{2} \int_V \int \frac{v\,\delta f_\nu^2}{\partial f_\nu^0/\partial v}\, dv d^3x + \frac{1}{8\pi} \int_V \delta E^2\, d^3x \ , \tag{2}$$

where ν is the species label, δE is the perturbed electric field, f_ν^0 is the equilibrium distribution function that is assumed to be a monotonic decreasing function of the square of the velocity, and δf_ν is the perturbation of the distribution function.

In previous work [11, 12, 13, 14] we derived a general expression for the energy of arbitrary perturbations of arbitrary three-dimensional Vlasov-Maxwell equilibria, an expression that does not suffer from having a singularity at extremal points of the equilibrium distribution function. (The origin of this singularity is discussed in Sec. VIII.) The derivation proceeds from the general nonlinear Vlasov-Maxwell energy expression, which is expanded up to second order in the perturbations in such a way as to preserve the constraint of *dynamical accessibility* of the system. Since it preserves this constraint we have called it the free energy, and since it is a second order quantity we have denoted it in previous work by $\delta^2 F$.

One purpose of the present paper is to compare $\delta^2 F$, specialized to one-dimensional electrostatic perturbations of homogeneous magnetic field free plasmas, with the dielectric energy expression, $\mathcal{E}_D$. Since the derivation of $\mathcal{E}_D$ is not general, accordingly, it will be seen that this quantity is *not* in general correct. The correct expression is given by Eq. (98) below.

Expression (98) for $\delta^2 F$ suggests a transformation to action-angle variables. A second purpose of this paper is to introduce a general linear integral transform pair that accomplishes this feat. This transform is a coordinate transformation that maps the linearization of the noncanonical Hamiltonian structure (Poisson bracket) [15] that describes the Vlasov (and other) systems to the action-angle variables, thereby solving the spectral problem for stable equilibria.

In Sec. II a derivation of the dielectric energy is given, a derivation that is more complete than usual. Section III contains a simple derivation of the exact Vlasov free energy expression for the one-dimensional case, which is similar to Eq. (2), but is valid for arbitrary equilibrium distribution functions; it does not become singular at velocities for which $\partial f_\nu^{(0)}/\partial v$ vanishes. The crucial point is, as mentioned above, to impose the constraint of dynamical accessibility. In Sec. IV we solve the linearized Vlasov equation for the distribution function in terms of the electrostatic potential and the initial value of the perturbation of the distribution function.

Two choices of initial conditions and forms for the electric field are considered in Secs. V and VI. First we consider a special kind of neutral (undamped) mode [17] that occurs at stationary inflection points of stable equilibrium distribution functions. It is seen that the energy of these modes is identical to the dielectric energy of Eq. (1). This is followed by showing that for "real" damped and growing modes, as distinct from Landau modes, the energy is identically zero. In Sec. VII we consider the energy of arbitrary perturbations about stable equilibria by expanding initial conditions of the linear problem in terms of Van Kampen modes. After reviewing the Van Kampen decomposition, in VII.A, we calculate the energy for such a general perturbation and obtain the new energy expression given by Eq. (98) of VII.B. In VII.C we show the relationship between the new energy and the commonly used, although at-best only approximate, dielectric energy. In Sec. VIII we discuss dynamical accessibility; in particular, we show the consequences that arise if this condition is not imposed. As noted above the new energy (of VII.B) leads to a natural definition of Hamiltonian type action variables, which is described in IX.A along with a discussion of signature and bifurcations. The transform pair is introduced in IX.B and used in IX.C. We conclude with Sec. X.

II. $\mathcal{E}_D$ for One-Dimensional Electrostatic Perturbations of Homogeneous Plasmas

In this section the energy is derived in a model where the electric field is described by the appropriate Maxwell equation, while the "plasma" is described by a phenomenological dielectric function. Comparison to a plasma described by the Vlasov equation is made.

Consider a gedanken experiment in which a current and field free plasma is perturbed by an electric field in the x-direction. This field is assumed to result from a current δj_e in the x-direction that flows in an artificial medium that spatially coexists with the plasma. The current does not arise from an electric field but is imposed by an external agent. The only interaction between the artificial medium and the plasma is by means of the electric field. The Maxwell equation that describes this situation is

$$\frac{\partial \delta E}{\partial t} + 4\pi(\delta j + \delta j_e) = 0 \ . \tag{3}$$

Here δE and δj are, respectively, the electric field and plasma current density of the perturbation. Assuming

$$\delta j_e \sim e^{-i\omega t + ikx} \ , \tag{4}$$

where $\omega = \omega_R + i\mu$ and, for now, $\mu > 0$ and $-\infty < t \leq 0$. It is assumed that δE and δj are generated solely by δj_e; thus, their space and time dependencies are identical to those of δj_e, and Eq. (3) becomes

$$-i\omega\delta E + 4\pi(\delta j + \delta j_e) = 0 \ . \tag{5}$$

According to usual response theory, the plasma is assumed to be adequately described by a dielectric function $\epsilon(k,\omega)$,

$$\delta E + i\frac{4\pi}{\omega}\delta j = \epsilon(k,\omega)\,\delta E \ , \tag{6}$$

and hence,

$$\delta j_e = \frac{i\omega}{4\pi}\epsilon(k,\omega)\,\delta E \ . \tag{7}$$

Now the energy absorbed by the plasma, $\mathcal{E}_P$, due to δj_e, is calculated from the power absorbed by the plasma. The latter quantity, which is equal to that liberated by the artificial medium, is given by

$$P = -\frac{1}{4}\int_V (\delta j_e + \delta j_e^*)(\delta E + \delta E^*)\, d^3x = -\frac{V}{4}(\delta j_e^* \delta E + \delta j_e \delta E^*) \ . \tag{8}$$

Here, and henceforth, real quadratic expressions like the power are evaluated by inserting real quantities, e.g. in this case

$$\frac{1}{2}(\delta j_e + \delta j_e^*) \ , \qquad \frac{1}{2}(\delta E + \delta E^*) \ .$$

Upon making use of (7), (8) becomes

$$P = i\frac{V}{4}\frac{|\delta E|^2}{4\pi}\left(\omega^* \epsilon^*(k,\omega) - \omega\epsilon(k,\omega)\right) \ . \tag{9}$$

Assuming $\epsilon(k,\omega_R)$ possesses real and imaginary parts; i.e.

$$\epsilon(k,\omega_R) = \epsilon_R(k,\omega_R) + i\epsilon_I(k,\omega_R) \ , \tag{10}$$

Eq. (9) can be written as

$$P = i\frac{V}{16\pi}|\delta E|^2\left(\omega^*\epsilon_R(k,\omega^*) + i\omega^*\epsilon_I(k,\omega) - \omega\epsilon_R(k,\omega) - i\omega\epsilon_I(k,\omega)\right) \ . \tag{11}$$

Also, assuming $\epsilon(k,\omega_R + i\mu)$ can be approximated by

$$\epsilon(k,\omega_R + i\mu) \approx \epsilon(k,\omega_R) + i\mu\frac{\partial\epsilon(k,\omega_R)}{\partial\omega_R} \ , \tag{12}$$

yields for (11)

$$P = i\frac{V}{16\pi}|\delta E|^2\left(2\mu\frac{\partial}{\partial\omega_R}\left(\omega_R\epsilon_R(k,\omega_R)\right) + 2\omega_R\epsilon_I(k,\omega_R)\right) \ . \tag{13}$$

Since the power is related to the plasma energy by $P = 2\mu\mathcal{E}_P$,

$$\mathcal{E}_P = \frac{V}{16\pi}|\delta E|^2\left(\frac{\partial}{\partial\omega_R}\left(\omega_R\epsilon_R(k,\omega_R)\right) + \frac{\omega_R}{\mu}\epsilon_I(k,\omega_R)\right) \ . \tag{14}$$

In Eq. (14) no connection between k and ω_R was assumed; however, now such a connection is established. Assuming $\epsilon_I \neq 0$ the dispersion relation $\epsilon(k,\omega_R + i\gamma) = 0$ can be approximately solved, in the so-called small growth rate expansion, as follows:

$$\epsilon_R(k,\omega_R) = 0 \ , \qquad \gamma = -\frac{\epsilon_I(k,\omega_R)}{\partial\epsilon_R/\partial\omega_R} \ . \tag{15}$$

Recall μ is a property of the current δj_e, while in light of the above γ arises from the dispersion relation. Because of the expansions used, both quantities must be small. With (15), (14) becomes

$$\mathcal{E}_P = \frac{V}{16\pi}|\delta E|^2\omega_R\frac{\partial}{\partial\omega_R}\epsilon_R(k,\omega_R)\left(1 - \frac{\gamma}{\mu}\right) \ . \tag{16}$$

For unstable plasmas one can take $\mu = \gamma$ and obtain the result $\mathcal{E}_P = 0$, a result that is in fact correct for a Vlasov plasma, as will be seen below in Sec. V. In this unstable case $\delta E \neq 0$ at $t = 0$ is obtained with $\delta j_e = 0$; i.e. only the self-consistent δE and δj contribute. This case could be called self-consistent "adiabatic" turn-on. For a mode with $\gamma < 0$ one can choose $\mu < 0$ and in this case the time interval $0 \leq t < \infty$ is considered. The energy at $t = 0$ is given by the energy that has been transferred to the artificial medium during this time interval. If $\gamma = \mu$ then again $\delta j_e = 0$ and $\mathcal{E}_P = 0$, again a valid result for a Vlasov plasma, as will be seen in Sec. V below. This case could be called self-consistent "adiabatic" turn-off.

It is important to point out that the validity of the above results, for both the growing and damped modes, depends upon γ being the imaginary part of a root of the dielectric function. In the case of a Vlasov plasma such modes may exist, but these must be distinguished from solutions of the Landau problem where the contour of integration is deformed. In the latter case the above analysis is invalid. For a stable Vlasov plasma a dielectric function $\epsilon(k, \omega)$ strictly speaking *does not* exist. The expression with the deformed contour used for obtaining Landau damping is only asymptotically valid in the limit of large time where the electric field decays exponentially, and one cannot self-consistently turn-off, as in the above case of a stable mode, a perturbed electric field that is only asymptotically of the form $\delta E \sim e^{-\gamma t}$.

Several authors [9] have attempted to obtain energy expressions by solving the linearized Vlasov equation with the adiabatic turn-on assumption. Generally these expressions are deficient in two respects. Firstly, they are not constants of motion so their use in energy arguments must be viewed with caution. Secondly, the presence of resonant particles leads to singularities. This is because a finite amount of energy is deposited in the plasma in each wave period over an infinite interval of time. This behavior is recovered from Eq. (16) by keeping γ fixed and taking the limit $\mu \to 0$.

The limit where $\mu >> |\gamma|$, but still small, is also of interest, since in this case Eq. (16)

reduces to $\mathcal{E}_D$ of Eq. (1). Although this limit can be appropriated for dielectric media, it is *only* valid for a Vlasov plasma when there exist the neutral modes described in Sec. IV where $\epsilon_I = 0$. In the case of weakly Landau damped modes a self-consistent exponential adiabatic turn-on (or turn-off) is not possible. In Sec. VII.C we will discuss this point further.

We conclude this section by remarking that $\mathcal{E}_D$ as given by (1) corresponds to the first term of (14). This quantity is sometimes referred to as the wave energy, while the second term, the one involving ϵ_I, is sometimes identified with the energy of the resonant particles. Such a distinction might be useful, but makes sense only for $\mu = \gamma$; i.e. in the self-consistent case. With Landau damping this is not possible for the reasons given above.

III. $\delta^2 F$ for One-Dimensional Electrostatic Perturbations of Homogeneous Plasmas

Here we present a simple derivation of the Vlasov energy expression for homogeneous current and field free equilibria. The unperturbed distribution function $f_\nu^{(0)}(\mathbf{v})$ is general except for the requirement that it allow purely electrostatic perturbations with the electric field vector $\delta\mathbf{E}$ in the x-direction. Here the Maxwell equation that describes these perturbations is

$$\frac{\partial \delta E}{\partial t} + 4\pi \delta j = 0 \, , \tag{17}$$

where δj, as in Sec. II, is the current density of the perturbation in the x-direction. From Eq. (17) one obtains

$$\frac{\partial}{\partial t}\frac{(\delta E)^2}{8\pi} + \delta j \, \delta E = 0 \, . \tag{18}$$

This relation is now integrated over the periodicity box of volume V, where the limit $V \to \infty$ can be taken. The second term of the resulting equation can be expressed as

$$\int_V \delta j \, \delta E \, d^3x = \sum_\nu e_\nu \int_V d^3x \int d^3v f_\nu^{(1)} \delta E \tag{19}$$

with

$$\mathbf{v} = (v_x \equiv v, v_y, v_z)$$

where $f_\nu^{(1)}(\mathbf{x},\mathbf{v},t)$ is the perturbation of the distribution function. Introduction of

$$\delta f_\nu = \int f_\nu^{(1)} dv_y dv_z \tag{20}$$

leads to

$$\int_V \delta j\,\delta E\, d^3x = \sum_\nu e_\nu \int_V d^3x \int dv\,\delta f_\nu\,\delta E \ . \tag{21}$$

The quantity $f_\nu^{(1)}$ obeys the first-order Vlasov equation

$$\frac{\partial f_\nu^{(1)}}{\partial t} + v\frac{\partial f_\nu^{(1)}}{\partial x} = -\frac{e_\nu}{m_\nu}\delta E\frac{\partial f_\nu^{(0)}}{\partial v} \ , \tag{22}$$

from which it follows that

$$\frac{\partial \delta f_\nu}{\partial t} + v\frac{\partial \delta f_\nu}{\partial x} = -\frac{e_\nu}{m_\nu}\delta E\frac{\partial f_\nu^{0}}{\partial v} \ . \tag{23}$$

In this equation we have introduced the definition

$$f_\nu^0 \equiv \int f_\nu^{(0)} dv_y dv_z \ . \tag{24}$$

Now the important condition of dynamical accessibility is imposed. This condition stipulates that the initial δf_ν denoted by $\delta\hat{f}_\nu$ must be producible by regular forces δK_ν that must be derivable from a Hamiltonian. The generation of $\delta\hat{f}_\nu$ by δK_ν also requires an initial condition, which by definition must be taken as $\delta\hat{f}_\nu = 0$. It follows then from the first-order Vlasov (or Liouville) equation with $e_\nu\delta E$ replaced by δK_ν that

$$\delta f_\nu(x,v,t=0) = \delta\hat{f}_\nu(x,v,s) = \delta\hat{f}_\nu(x,v,s=0) - \frac{\partial f_\nu^0}{\partial v}\int_0^s \delta K_\nu(x+v(\tau-s),\tau)\,d\tau \ , \tag{25}$$

where we have used a mock time s to generate the real initial condition, $\delta f_\nu(x,v,t=0)$, for the dynamics under the self-consistent force $e_\nu\delta E$.

In light of the initial condition $\delta\hat{f}_\nu(x,v,s=0)=0$, Eq. (25) has the form

$$\delta f_\nu = q_\nu(x,v,t)\frac{\partial f_\nu^0}{\partial v} \ , \tag{26}$$

where $q_\nu(x,v,t)$ is regular at the zeros of $\partial f_\nu^0/\partial v$. This quantity q_ν obeys the equation

$$\frac{\partial q_\nu}{\partial t} + v\frac{\partial q_\nu}{\partial x} = -\frac{e_\nu}{m_\nu}\delta E \ , \tag{27}$$

which does *not* contain $\partial f_\nu^0/\partial v$. Relation (26) is a special case of the form for $f_\nu^{(1)}$ found in Ref. [14] for general three-dimensional equilibria and general three-dimensional perturbations:

$$f_\nu^{(1)} = [g_\nu, f_\nu^{(0)}] \ , \tag{28}$$

where the bracket on the right-hand side means the Poisson bracket,

$$[a,b] \equiv \frac{\partial a}{\partial \mathbf{x}} \cdot \frac{\partial b}{\partial \mathbf{p}} - \frac{\partial a}{\partial \mathbf{p}} \cdot \frac{\partial b}{\partial \mathbf{x}} \ . \tag{29}$$

The functions $g_\nu(\mathbf{x},\mathbf{v},t)$ are first-order generating functions for canonical transformations. For $g_\nu = g_\nu(x,v,t)$ one also finds that

$$\delta f_\nu = [g_\nu, f_\nu^0] = \frac{1}{m_\nu}\frac{\partial g_\nu}{\partial x}\frac{\partial f_\nu^0}{\partial v} \ , \tag{30}$$

and therefore

$$q_\nu(x,v,t) = \frac{1}{m_\nu}\frac{\partial g_\nu}{\partial x} \ , \tag{31}$$

which also follows directly from (25); since δK_ν is a Hamiltonian force it must be derivable from a potential. The generating function g_ν obeys the $\partial f_\nu^0/\partial v$-independent equation

$$\frac{\partial g_\nu}{\partial t} + v\frac{\partial g_\nu}{\partial x} = e_\nu \delta\phi \ , \tag{32}$$

where $\delta\phi$ is the electrostatic potential associated with δE. Equation (32) is a special case of the following general equation of [14]:

$$\frac{\partial g_\nu}{\partial t} + [g_\nu, H_\nu^{(0)}] = \delta H_\nu \ , \tag{33}$$

where $H_\nu^{(0)}$ is the unperturbed Hamiltonian and δH_ν is its perturbation. The derivation in Ref. [14] makes use of Lie-type canonical transformations, that guarantee dynamic accessibility in general. This method allows one to obtain the perturbations of the distribution functions to arbitrary order. Especially, the second-order perturbation is given by

$$f_\nu^{(2)} = [g_\nu^{(2)}, f_\nu^{(0)}] + \frac{1}{2}\left[g_\nu, [g_\nu, f_\nu^{(0)}]\right] \ , \tag{34}$$

where $g_\nu^{(2)}$ is a second-order quantity while g_ν, as mentioned before, is a first-order quantity. This representation was used to obtain the following second-order energy from the exact nonlinear energy expression, for arbitrary systems and arbitrary perturbations:

$$\delta^2 F = \sum_\nu \frac{1}{2} \int d^3x\, d^3p \left[H_\nu^{(0)}, g_\nu\right] \left[g_\nu, f_\nu^{(0)}\right] + \frac{1}{8\pi} \int \delta E^2\, d^3x \ . \tag{35}$$

Expression (35) is a free energy since by its derivation the perturbed quantities are forced to satisfy the dynamical accessibility constraint. One can show explicitly that the perturbations given by Eqs. (30) and (34) automatically preserve all the well-known invariants,

$$C_\nu[f_\nu] = \int_V \int \mathcal{C}_\nu(f_\nu)\, d^3x d^3v \ , \tag{36}$$

to first and second order, respectively. Therefore, writing the perturbations in the form of (30) and (34) is a mathematical way of stating the Gardner [16] restacking principle to first and second order, respectively.

For the present derivation, which is not possible in general, the second-order distribution function is not needed. Also, the representation (30) need not be used explicitly, but knowledge of this representation allows us to solve the first-order Vlasov equation (23) for δE instead of for δf_ν:

$$\delta E = -\frac{m_\nu}{e_\nu}\left(\frac{\partial \delta f_\nu}{\partial t} + v\frac{\partial \delta f_\nu}{\partial x}\right)\frac{1}{\partial f_\nu^0/\partial v} \ . \tag{37}$$

From (30) it is evident that there are no problems where $\partial f_\nu^0/\partial v$ vanishes. Insertion of Eq. (37) in Eq. (21) yields

$$\int_V \delta j\, \delta E\, d^3x = -\sum_\nu m_\nu \int_V d^3x \int_{-\infty}^{\infty} dv\, v\, \delta f_\nu \left(\frac{\delta f_\nu}{\partial t} + v\frac{\partial \delta f_\nu}{\partial x}\right)\frac{1}{\partial f_\nu^0/\partial v} \ . \tag{38}$$

Since

$$\delta f_\nu \frac{\partial \delta f_\nu}{\partial x} = \frac{1}{2}\frac{\partial(\delta f_\nu)^2}{\partial x} \ , \tag{39}$$

the integration over x makes the second term of (38) vanish. In a similar way, the $\partial\delta f_\nu/\partial t$ term leads to

$$\int_V \delta j\,\delta E d^3x = -\frac{\partial}{\partial t}\sum_\nu \frac{m_\nu}{2}\int_V d^3x\int_{-\infty}^{\infty} dv\frac{v(\delta f_\nu)^2}{\partial f_\nu^0/\partial v}\,. \tag{40}$$

Combination of this result with the electric field contribution of Eq. (18) yields

$$\frac{\partial}{\partial t}\int_V d^3x\left(-\sum_\nu \frac{m_\nu}{2}\int_{-\infty}^{\infty} dv\,\frac{v(\delta f_\nu)^2}{\partial f_\nu^0/\partial v} + \frac{(\delta E)^2}{8\pi}\right) = 0\ . \tag{41}$$

Therefore, we have obtained the following constant of motion:

$$\delta^2 F = \int_V d^3x\left(-\sum_\nu \frac{m_\nu}{2}\int_{-\infty}^{\infty} dv\,\frac{v(\delta f_\nu)^2}{\partial f_\nu^0/\partial v} + \frac{(\delta E)^2}{8\pi}\right)\ . \tag{42}$$

According to its derivation this quantity is the energy of the perturbation. It is formally the same expression as that of Eq. (2), but here δf_ν is restricted by the condition of dynamical accessibility. Thus, $\partial f_\nu^0/\partial v$ is allowed to vanish at various velocities. Also, f_ν^0 can be any function of v and is not restricted to depend on v^2 alone.

In Sec. VIII we discuss more generally the implications of dynamical accessibility.

IV. $\delta^2 F$ in Terms of $\delta E(x,t)$ and $\delta f(x,v,t=0)$; Energy Transfer During Landau Damping

Now we express the energy relation (42) in terms of the initial value solution of Eq. (23). This is a first step in the task of writing the energy in terms of the field amplitudes and will be of special interest for explaining the energy transport caused by Landau damping. Henceforth, a single species and a constant neutralizing background are assumed.

The general solution of Eq. (23), in light of Eq. (26), can be written as

$$\delta f = \frac{\partial f^0}{\partial v}\left[\hat{q}(x-vt,v) - \frac{e}{m}\int_0^t \delta E(x+v(\tau-t),\tau)\,d\tau\right]\ , \tag{43}$$

where $\hat{q}(x,v) = q(x,v,0)$. (The hat notation is used throughout the paper to denote initial values.) Inserting (43) into (42) yields

$$\delta^2 F = -\frac{1}{2}\int_V d^3x\int_{-\infty}^{\infty} dv\,v\,\frac{\partial f^0}{\partial v}\left\{\hat{q}^2 + e^2\left(\int_0^t \delta E\,d\tau\right)^2\right.$$

$$- 2e\,\hat{q}\int_0^t \delta E\,d\tau\Big\} + \frac{1}{8\pi}\int_V d^3x\,\delta E^2(x,t)\ , \tag{44}$$

where the unspecified arguments of $\hat{q}$ and δE are given by

$$\hat{q} = \hat{q}(x - vt, v)$$

$$\delta E = \delta E(x + v(\tau - t), \tau)\ . \tag{45}$$

Observe that Eq. (44) is only partially written in terms of δE, since it contains terms involving the initial perturbed distribution function through $\hat{q}$. Later we will fulfill the mentioned task of writing $\delta^2 F$ *entirely* in terms of $\delta E(x,t)$. That this is possible is somewhat surprising, since there does not exist a unique perturbed distribution function corresponding to a given initial perturbed electric field (as evidenced by Landau damping). This "paradox" will be discussed further in VII.B.

Now we restrict to the case of a single plane wave perturbation:

$$\hat{q}(x,v) = \frac{1}{2}\left(\frac{e}{m}\hat{Q}_k(v)e^{ikx} + c.c\right)\ ; \quad \delta E(x,t) = \frac{1}{2}\left(E_k(t)e^{ikx} + c.c\right)\ . \tag{46}$$

More general perturbations can be represented in terms of Fourier integrals by simply summing over these plane waves and taking the limit $V \to \infty$. The energy for plane wave perturbations is obtained from Eq. (44) upon insertion of Eq. (46),

$$\delta^2 F = \frac{V}{16\pi}|E_k|^2 - \frac{V}{16\pi}\omega_p^2\int_{-\infty}^{\infty} dv\, v\,\frac{\partial f_0}{\partial v}$$
$$\times\left(|\hat{Q}_k|^2 - \frac{1}{2}(\hat{Q}^*\int_0^t E_k(\tau)e^{ikv\tau}d\tau + c.c.) + |\int_0^t E_k(\tau)e^{ikv\tau}d\tau|^2\right)\ , \tag{47}$$

where we have defined

$$f^0 \equiv n_0 f_0\ . \tag{48}$$

The form of the energy given by (47) sheds light on the energy transfer for Landau-damped waves. The electric field perturbation $E_k(t)$ approaches zero as $t \to \infty$. Therefore the first term, which represents the electric field energy, vanishes asymptotically. This energy then shows up in the terms containing time integrals that are zero at $t = 0$ but do not necessarily vanish as $t \to \infty$. These time integrals are part of the particle contribution to the energy; i.e., their kinetic energy. Kinetic energy is also contained in the constant term involving $|Q_k|^2$, that represents the initial perturbation of the particle energy. Therefore, the energy expression (47) provides a clear description of the energy flow, a description that cannot be obtained from the dielectric energy $\mathcal{E}_D$. In Section VII we continue this discussion, although in a different way.

In Appendix A we develop an expression analogous to Eq. (47) for general electrostatic perturbations about general three-dimensional equilibria that possess action-angle variables for the equilibrium trajectories. Similar arguments about energy transfer apply.

V. Energy of Neutral Oscillations

With this section we begin the investigation of special types of perturbations. First, the neutral oscillations with real frequency ω discussed in Ref. [17] are considered. These neutral modes can occur for marginally stable equilibrium distribution functions that have a point v_c such that $\partial f_0(v_c)/\partial v = \partial^2 f_0(v_c)/\partial v^2 = 0$. We note here that there are other kinds of marginally stable equilibria for which neutral modes do not exist. For these equilibria the limit of vanishing damping is only obtained for $k = 0$, but ω/k nonzero and finite. Since $k = 0$ there exists no perturbation. Nonvanishing perturbations of these equilibria are Landau-damped modes, which will be treated later in Sec. VII.

The frequency ω and wave number k of the neutral modes are determined from $v_c = \omega/k \equiv u$ and

$$\epsilon = 1 - \frac{\omega_p^2}{k^2} \int_{-\infty}^{\infty} \frac{\partial f_0}{\partial v} \frac{dv}{v - u} = 0\,, \tag{49}$$

where the integral is along the real v-axis (as is the case for all v-integrals in this paper). For complex perturbations proportional to $e^{ikx - i\omega t}$ the quantities δf and δE are uniquely related by

$$\delta f_k = i\frac{en_0}{mk}\,\delta E_k\,\frac{\partial f_0/\partial v}{v-u}\ . \tag{50}$$

The energy expressions require real quantities, i.e.

$$\frac{\delta f_k + \delta f_k^*}{2}\ , \qquad \frac{\delta E_k + \delta E_k^*}{2}\ .$$

The dielectric energy then becomes

$$\mathcal{E}_D = \omega\frac{\partial\epsilon}{\partial\omega}\frac{V}{16\pi}|\delta E_k|^2 = -\frac{\omega_p^2\omega}{k^3}\int_{-\infty}^{\infty}\frac{\partial f_0/\partial v}{(v-u)^2}dv\frac{V}{16\pi}|\delta E_k|^2\ . \tag{51}$$

The first term in the energy expression (42) gives the following contribution to $\delta^2 F$:

$$-\frac{m}{4n_0}\int_{-\infty}^{\infty}\frac{v|\delta f_k|^2}{\partial f_0/\partial v}dv = -\frac{\omega_p^2}{k^2}\int_{-\infty}^{\infty}\frac{\partial f_0/\partial v}{(v-u)^2}dv\frac{1}{16\pi}|\delta E_k|^2\ , \tag{52}$$

where E_k has been introduced according to Eq. (46). Upon writing v as

$$v = (v-u) + u$$

the $v-u$ contribution is seen to cancel the electric field energy in Eq. (42) because $\epsilon = 0$, while the u contribution yields *exactly* $\mathcal{E}_D$. Thus complete agreement exists between the two kinds of energy formulae for these neutral modes. This is possible because there exists a unique relation between δf and δE at any time. Since $\epsilon_I = 0$ according to Eq.(14) $\mathcal{E}_P$ is exactly $\mathcal{E}_D$.

The energy of these neutral modes was previously obtained in Ref. [17].

VI. Energy of Growing and Damped Modes

The next type of special perturbations are growing and corresponding damped modes in an unstable system. The perturbations δf and δE are related to each other in the same way as for the neutral modes of the preceding section. The relation is given by Eq. (50), but u is complex since the dielectric function $\epsilon(k,\omega)$ now has complex roots $\omega/k \equiv u$, where

$\omega = \omega_R + i\gamma$. Since ϵ is a real function of ω/k and k, there are always pairs of complex conjugate roots u and u^* and hence, corresponding growing and damped modes. It is worth noting that such damped modes must exist for all unstable equilibria of Hamiltonian systems like the Vlasov equation (see e. g. [13]), since for these systems discrete eigenvalues occur in pairs or quartets; i.e. as $\pm\omega_R \pm i\gamma$ (see e. g. [18]) . We should like to emphasize that these damped modes are normal modes in the strict sense and not Landau damped modes. Normal modes are solutions that are valid for all times, while Landau modes are only approximate solutions, valid in the limit $t \to +\infty$.

Equation (44) with (50) now yields the energy

$$\delta^2 F = \frac{V}{16\pi}|E_k|^2 \left(1 - \frac{\omega_p^2}{k^2}\int_{-\infty}^{\infty} \frac{\partial f_0}{\partial v}\frac{v}{|v-u|^2}dv\right) . \tag{53}$$

With

$$\frac{v}{|v-u|^2} = \frac{1}{u-u^*}\left(\frac{u}{v-u} - \frac{u^*}{v-u^*}\right) ,$$

Eq. (53) becomes

$$\delta^2 F = \frac{V}{16\pi}|E_k|^2 \left(1 - \frac{u}{u-u^*}\frac{\omega_p^2}{k^2}\int_{-\infty}^{\infty} \frac{\partial f_0/\partial v}{v-u}dv + \frac{u^*}{u-u^*}\frac{\omega_p^2}{k^2}\int_{-\infty}^{\infty} \frac{\partial f_0/\partial v}{v-u^*}dv\right) . \tag{54}$$

Since both u and u^* satisfy the dispersion relation $\epsilon = 0$, it holds that

$$\frac{\omega_p^2}{k^2}\int_{-\infty}^{\infty} \frac{\partial f_0/\partial v}{v-u}dv = \frac{\omega_p^2}{k^2}\int_{-\infty}^{\infty} \frac{\partial f_0/\partial v}{v-u^*}dv = 1 . \tag{55}$$

Hence, $\delta^2 F = 0$. Although this result was presaged in Sec. II, it could have easily been obtained without the foregoing calculations: for unstable and corresponding damped waves $\delta^2 F$ must be at once proportional to $e^{\pm\gamma t}$, and time independent; in order for both properties to be fulfilled simultaneously, $\delta^2 F = 0$.

To evaluate the dielectric energy the real part of ϵ,

$$\epsilon_R = 1 - \frac{\omega_p^2}{k^2}\int_{-\infty}^{\infty} \frac{v-u_R}{|v-u|^2}\frac{\partial f_0}{\partial v}dv = 0 , \tag{56}$$

is needed, whence it follows that

$$\frac{d}{dk}\epsilon_R = \frac{\partial \epsilon_R}{\partial u_R} \cdot \frac{\partial u_R}{\partial k} + \frac{\partial \epsilon_R}{\partial k} = 0 \; . \tag{57}$$

Using (56) for ϵ_R and $\epsilon_R = 0$ one derives

$$\frac{\partial \epsilon_R}{\partial k} = \frac{2}{k} \; , \tag{58}$$

and it also holds that

$$\frac{\partial u_R}{\partial k} = \frac{1}{k}(v_g - v_p) \; ,$$

where v_g is the group velocity and $v_p = u_R$ is the phase velocity. Combining the above yields the known result [19]

$$\omega_R \frac{\partial \epsilon_R}{\partial \omega_R} = 2\frac{v_p}{v_p - v_g} \; . \tag{59}$$

Expression (59) is also valid for the neutral modes of the previous section, where $\epsilon_R = \epsilon$ and $\omega_R = \omega$. It is non-zero and therefore $\mathcal{E}_D$ is non-zero (except in a frame of reference in which $v_p = 0$). That $\mathcal{E}_D$ is non-zero is true even for $\gamma \to 0$, while the exact energy $\delta^2 F$ remains zero in this limit. As outlined in [13], the relevant frame of reference for homogeneous unperturbed systems is the center-of-mass rest frame in which v_p usually does not vanish.

The discrepancy between the exact and the dielectric energy has its origin in the fact that $\epsilon(k,\omega)$ is not analytic at $\gamma = 0$. This is easily seen by shifting the path of integration in the ϵ-expression (49) such that

$$\epsilon = 1 - \frac{\omega_p^2}{k^2} P \int \frac{1}{v} \frac{\partial f_0(v+u)}{\partial v} dv \mp i\pi \frac{\omega_p^2}{k^2} \frac{\partial f_0(u)}{\partial u} \; . \tag{60}$$

The upper sign holds when $u_I > 0$, while the lower sign holds when $u_I < 0$. The presence of the two different signs displays the non-analytic character of ϵ as $u_I \to 0$.

The usual procedure for obtaining the small growth or damping rate, γ, relies on the nonanalyticity of ϵ. Expanding ϵ , as in Sec. II, in the smallness of γ,

$$\epsilon \approx \epsilon_R(k,\omega_R) + i\epsilon_I(k,\omega_R) + i\gamma \frac{\partial \epsilon_R(k,\omega_R)}{\partial \omega_R} \approx 0 \; , \tag{61}$$

results

$$\epsilon_R(k,\omega_R)=0, \quad \gamma=-\frac{\epsilon_I(k,\omega_R)}{\partial\epsilon_R/\partial\omega_R}\ . \tag{62}$$

(Although ϵ is not analytic at $\gamma=0$ one can Taylor expand on each side of γ.) We emphasize that $\epsilon_I(k,\omega_R)$ would be zero if ϵ were analytic, as is the case, e.g. , for fluid theories or theories that exclude resonant particles. When ϵ is analytic the above procedure is not sufficient to determine γ; one must expand to second order:

$$\epsilon\approx\epsilon(k,\omega_R)+i\gamma\frac{\partial\epsilon_R}{\partial\omega_R}-\frac{\gamma^2}{2}\frac{\partial^2\epsilon_R}{\partial\omega_R^2}\approx 0\ , \tag{63}$$

which in contrast to the above would yield the result

$$\begin{aligned} i\gamma\frac{\partial\epsilon_R}{\partial\omega_R}&=0\ ,\\ \epsilon(k,\omega_R)-\frac{\gamma^2}{2}\frac{\partial^2\epsilon_R}{\partial\omega_R^2}&=0\ . \end{aligned} \tag{64}$$

Note, here $\partial\epsilon_R/\partial\omega_R=0$ determines ω_R, while the second equation determines γ. Thus if ϵ were analytic and unstable modes existed, $\partial\epsilon_R/\partial\omega_R=0$, and this would imply $\mathcal{E}_D=0$, even for arbitrarily small γ, as occurs for δ^2F. Also, note the discrepancy is evident from Eq. (14) where it is seen that $\mathcal{E}_P\neq\mathcal{E}_D$ when $\epsilon_I(k,\omega_R)\neq 0$.

This singular limit between growing and damped modes and neutral oscillations is not peculiar to the Vlasov equation. It arises in general Hamiltonian systems; e.g. the energy in a simple harmonic oscillator is a positive quantity proportional to the spring constant and the amplitude squared. If the spring constant changes sign, purely growing and damped modes with zero energy occur. The behavior of the energy in this transition, like that discussed above for the Vlasov equation, is nonanalytic, although unlike the above the energy is continuous.

VII. Energy of General Perturbations in Stable Plasmas

This section contains the derivation of the general energy expression for *stable* equilibria, an expression that is written in terms of the electric field associated with a single Van Kampen mode. We emphasize that although the equilibrium distribution function is stable it *need not be monotonic.* In VII.A the Van Kampen decomposition [20] is reviewed and interpreted in light of the dynamical accessibility condition. In VII.B the new energy expression is obtained. This section is concluded in VII.C where a comparison of the new energy expression and the dielectric energy is made.

A. Van Kampen Mode Review

Consider again the linearized Vlasov equation

$$\frac{\partial \delta f}{\partial t} + v\,\frac{\partial \delta f}{\partial x} + \frac{e}{m}\delta E\,\frac{\partial f^0}{\partial v} = 0 \tag{65}$$

and Poisson's equation

$$\frac{\partial \delta E}{\partial x} = 4\pi e \int_{-\infty}^{\infty} \delta f\, dv\ . \tag{66}$$

Van Kampen's procedure begins by deriving a two parameter family of solutions, labelled by real quantities k and $u \equiv \omega/k$, of the form

$$\delta f(x,v,t;u) = e^{ikx-ikut}\, h(k,u,v)\,, \tag{67}$$

where the function $h(k,u,v)$ remains to be determined. Inserting (67) into (66) produces the electric field associated with a single member of the family of incipient solutions

$$\delta E(x,t) = \tilde{E}(k,u)e^{ikx-ikut}\,. \tag{68}$$

Insertion of (67) and (68) in (65) yields

$$ik(v-u)h(k,u,v) = -\frac{e}{m}\,\tilde{E}(k,u)\,\frac{\partial f^0}{\partial v}\,. \tag{69}$$

Since u and v are real there exists a singularity at $u = v$ that is handled by solving (69) in a distributional sense according to

$$h(k,u,v) = \frac{ie}{mk}\,\tilde{E}\,\frac{\partial f^0}{\partial v}\,P\,\frac{1}{v-u} + C(k,u)\delta(v-u)\,, \tag{70}$$

where P denotes the principal value, $\delta(v-u)$ is the Dirac delta distribution, and C is yet to be determined. Another singularity exists at $k = 0$, but this is resolved by simply requiring $\tilde{E}(0,u) = 0$, a condition that removes a homogeneous perturbed electric field. The unknown C is obtained by inserting (67), (70) and (68) into Poisson's equation, giving

$$C(k,u) = \frac{ik}{4\pi e}\,\tilde{E} - \frac{ie}{mk}\,\tilde{E}\,P\int_{-\infty}^{\infty}\frac{\partial f^0/\partial v}{v-u}\,dv\,. \tag{71}$$

Now, substitution of this value for C in (70) yields

$$\begin{aligned} h(k,u,v) &= \frac{ik\tilde{E}}{4\pi e}\left\{\frac{\omega_p^2}{k^2}\,\frac{\partial f_0}{\partial v}\,P\,\frac{1}{v-u} + \delta(v-u)\left[1 - \frac{\omega_p^2}{k^2}\,P\int_{-\infty}^{\infty}\frac{\partial f_0/\partial v'}{v'-u}\,dv'\right]\right\} \\ &\equiv \frac{ik}{4\pi e}\,\tilde{E}(k,u)\,\mathcal{G}(k,u,v)\ , \end{aligned} \tag{72}$$

where the Van Kampen mode is denoted by $\mathcal{G}(k,u,v)$.

It remains to show that $\mathcal{G}(k,u,v)$ forms a complete basis for expanding the general solution as

$$\delta f = \frac{1}{2}\sum_k \int_{-\infty}^{\infty}\frac{ik}{4\pi e}\,E(k,u)\mathcal{G}(k,u,v)e^{ikx-ikut}\,du\,, \tag{73}$$

which requires that the $\mathcal{G}$'s be capable of expanding an (essentially) arbitrary initial condition. [Note, the factor of 1/2 is consistent with our convention for representing real quantities and the original field amplitudes $\tilde{E}(k,u)$ have been replaced by $E(k,u)\,du$.] Therefore, it must be possible to satisfy the following equation for the Fourier coefficients of an arbitrary initial perturbed distribution function $\hat{f}(k,v)$:

$$\hat{f}(k,v) = \frac{ik}{4\pi e}\int_{-\infty}^{\infty} du\,E(k,u)\mathcal{G}(k,u,v)\,. \tag{74}$$

Given the initial value $\hat{f}(k,v)$ the expansion requires finding the amplitudes $E(k,u)$. Since the basis $\mathcal{G}$ is given by

$$\mathcal{G}(k,u,v) = \epsilon_I(k,kv)\,\frac{1}{\pi}P\,\frac{1}{u-v} + \epsilon_R(k,kv)\,\delta(v-u)\ , \tag{75}$$

insertion of (75) in (74) leads to the following expression that must be solved for E:

$$\hat{f}(k,v) = \frac{ik}{4\pi e}\frac{\omega_p^2}{k^2}\frac{\partial f_0}{\partial v}P\int_{-\infty}^{\infty}\frac{E(k,u)}{v-u}\,du + \frac{ik}{4\pi e}E(k,v)\left(1-\frac{1}{k^2}P\int_{-\infty}^{\infty}\frac{\partial f_0/\partial v'}{v'-v}\,dv'\right)\ . \tag{76}$$

The task of solving for E is accomplished by splitting the functions $\partial f_0/\partial v$ and E, and the principal value expressions into parts that are analytic in the upper and lower half complex v-plane, respectively. Appealing to (B-3) and (B-4) of Appendix B yields

$$\begin{aligned}
\hat{f}(k,v) &= -\frac{k\,\omega_p^2}{4k^2e}\left[\left(\frac{\partial f_0}{\partial v}\right)_+ + \left(\frac{\partial f_0}{\partial v}\right)_-\right](E_- - E_+) + \frac{ik}{4\pi e}(E_+ + E_-) \\
&\quad + \frac{k\,\omega_p^2}{4k^2e}\left[\left(\frac{\partial f_0}{\partial v}\right)_+ - \left(\frac{\partial f_0}{\partial v}\right)_-\right](E_+ + E_-) \\
&= \frac{ik}{4\pi e}E_+\left[1-2\pi i\frac{\omega_p^2}{k^2}\left(\frac{\partial f_0}{\partial v}\right)_+\right] + \frac{ik}{4\pi e}E_-\left[1+2\pi i\frac{\omega_p^2}{k^2}\left(\frac{\partial f_0}{\partial v}\right)_-\right]\ .
\end{aligned} \tag{77}$$

Observe that $\pm$ products do not occur, so that $\hat{f}$ is the sum of terms analytic in the upper and lower half planes, respectively. Defining

$$\begin{aligned}
\epsilon(k,kv) &\equiv 1 - 2\pi i\,\frac{\omega_p^2}{k^2}\left(\frac{\partial f_0}{\partial v}\right)_+\ , \\
\epsilon^*(k,kv) &\equiv 1 + 2\pi i\,\frac{\omega_p^2}{k^2}\left(\frac{\partial f_0}{\partial v}\right)_-\ ,
\end{aligned} \tag{78}$$

and assuming $\hat{f}$ has the splitting described in Appendix B, yields

$$\hat{f} = \hat{f}_+ + \hat{f}_- = \frac{ik}{4\pi e}E_+\,\epsilon + \frac{ik}{4\pi e}E_-\,\epsilon^*\ . \tag{79}$$

Since the splitting is unique (c.f. [20])

$$\hat{f}_+ = \frac{ik}{4\pi e}E_+\epsilon; \qquad \hat{f}_- = \frac{ik}{4\pi e}E_-\epsilon^*\ , \tag{80}$$

and since the plasma is assumed to be stable,

$$\epsilon(k, kv) \neq 0 \qquad \text{Im } kv > 0 \ ,$$

$$\epsilon^{*}(k, kv) \neq 0 \qquad \text{Im } kv < 0 \ ; \tag{81}$$

therefore

$$E_{+} = \frac{4\pi e}{ik}\frac{\hat{f}_{+}}{\epsilon} \qquad E_{-} = \frac{4\pi e}{ik}\frac{\hat{f}_{-}}{\epsilon^{*}} \ . \tag{82}$$

The quantities E_{+} and E_{-} are used to construct $E(k, u)$:

$$\begin{aligned} E(k,u) = E_{+} + E_{-} &= \frac{4\pi e}{ik}\left(\frac{\hat{f}_{+}}{\epsilon} + \frac{\hat{f}_{-}}{\epsilon^{*}}\right) \\ &= \frac{4\pi e}{ik}\frac{1}{|\epsilon|^2}\left(\epsilon_R(\hat{f}_{+} + \hat{f}_{-}) - i\epsilon_I(\hat{f}_{+} - \hat{f}_{-})\right) \\ &= \frac{4\pi e}{ik}\frac{1}{|\epsilon|^2}\left(\epsilon_R \hat{f}(k,u) - \epsilon_I \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\hat{f}(k,v')}{v'-u}\, dv'\right) \\ &= \frac{4\pi e}{ik}\frac{1}{|\epsilon|^2}\left(\epsilon_R \hat{f}(k,u) + \frac{\omega_p^2}{k^2}\frac{\partial f_0(u)}{\partial u} P \int_{-\infty}^{\infty} \frac{\hat{f}(k,v')}{v'-u}\, dv'\right) \ . \end{aligned} \tag{83}$$

Equation (83) determines $E(k, u)$ in terms of the initial condition $\hat{f}$. Substituting this result together with (75) into (73), one obtains the solution $\delta f(x, v, t)$.

We note here that if the initial condition $\hat{f}$ is chosen to be proportional to $\partial f_0/\partial v$, then $\delta f(x, v, t)$ is also proportional to $\partial f_0/\partial v$ and hence fulfills the condition of dynamical accessibility for all time.

B. Calculation of the Energy of a General Perturbation

Now we are equipped to obtain the energy of a general perturbation that is expanded in terms of Van Kampen modes. The expansion of the perturbed distribution function, as given by Eq. (73), is inserted into the energy expression of Eq. (42). The difficulty occurs in the second term where there are several integrations that need to be performed. First consider the x-integration of this second term. Since $\partial f_0/\partial v$ is independent of x,

$$-\frac{m}{2}\int_V\int_{-\infty}^{\infty}\frac{v\,(\delta f)^2}{\partial f^0/\partial v}\;dvd^3x=-\frac{mV}{8}\sum_k\int_{-\infty}^{\infty}dv\,du_1\,du_2\,\frac{v}{\partial f^0/\partial v}\left(\frac{k}{4\pi e}\right)^2$$

$$\times E^*(k,u_1)E(k,u_2)\mathcal{G}(k,u_1,v)\mathcal{G}(k,u_2,v)e^{-ik(u_1-u_2)t}\;, \tag{84}$$

where we have used the reality condition $E^*(k,u)=E(-k,u)$, which follows from (73) since $\mathcal{G}(k,u,v)=\mathcal{G}(-k,u,v)$. The v-integration amounts to evaluating the quantity,

$$\int_{-\infty}^{\infty}\frac{\mathcal{G}(k,u_1,v)\mathcal{G}(k,u_2,v)v}{\partial f_0/\partial v}\,dv\,, \tag{85}$$

where $\mathcal{G}$ is given by (75). This integral is complicated because $\mathcal{G}$ possesses singular as well as regular parts. However, the energy is time independent and one can do the evaluation for $t\to\infty$. Terms that possesses bounded, integrable, absolute values must phase mix to zero by the Riemann-Lebesgue lemma, while singular terms can give rise to nonvanishing contributions. We denote the δ-function portion of $\mathcal{G}$ by δ and consider the three types of terms separately. For convenience let

$$\epsilon_p(u)\equiv 1-\frac{\omega_p^2}{k^2}P\int_{-\infty}^{\infty}\frac{1}{v-u}\frac{\partial f_0}{\partial v}\,dv\;, \tag{86}$$

where the k-dependence is suppressed.

Consider first the term that scales as the product of the two δ-functions, the $\delta-\delta$ contribution; it behaves as follows:

$$\int_{-\infty}^{\infty}\delta(v-u_1)\delta(v-u_2)\epsilon_p(u_1)\epsilon_p(u_2)\,\frac{v}{\partial f_0/\partial v}\,dv=\delta(u_1-u_2)\frac{[\epsilon_p(u_1)]^2u_1}{\partial f_0/\partial u_1}\,, \tag{87}$$

which clearly gives rise to a nonvanishing term.

Next consider the two δ-nonδ contributions. The first is given by

$$-\int_{-\infty}^{\infty}\delta(v-u_1)\epsilon_p(u_1)\frac{\omega_p^2}{k^2}P\,\frac{1}{u_2-v}\,\frac{\partial f_0}{\partial v}\,\frac{v}{\partial f_0/\partial v}\,dv=\lim_{\nu\to 0}\,\epsilon_p(u_1)\frac{\omega_p^2}{k^2}\,\frac{u_1(u_1-u_2)}{(u_2-u_1)^2+\nu^2}\,, \tag{88}$$

where the equality follows from the definition of principal value given by Eq. (B-4) of Appendix B, which amounts to

$$P\frac{1}{u_2-v}=\frac{1}{2}\left[\frac{1}{u_2-v-i\nu}+\frac{1}{u_2-v+i\nu}\right]\,.$$

The second contribution is simply obtained by interchanging the subscripts 1 and 2,

$$\lim_{\nu\to 0} \epsilon_p(u_2)\frac{\omega_p^2}{k^2}\frac{u_2(u_2-u_1)}{(u_1-u_2)^2+\nu^2}\,. \tag{89}$$

Summing (88) and (89) yields

$$\lim_{\nu\to 0}\frac{\omega_p^2}{k^2}\frac{[u_1\epsilon_p(u_1)-u_2\epsilon_p(u_2)](u_1-u_2)}{(u_1-u_2)^2+\nu^2}\,, \tag{90}$$

a quantity that is regular in u_1 and u_2. Thus, when (90) is multiplied by the E's, which are assumed to be obtained from reasonable initial conditions and are thus integrable, and multiplied by the factor $\exp(-ik(u_1-u_2)t)$ and then integrated, it vanishes in the limit $t\to\infty$.

Lastly, consider the nonδ-nonδ contribution:

$$\int dv\,\frac{v}{\partial f_0/\partial v}\frac{\omega_p^2}{k^2}P\frac{1}{u_2-v}\frac{\partial f_0}{\partial v}\frac{\omega_p^2}{k^2}P\frac{1}{u_1-v}\frac{\partial f_0}{\partial v}\,. \tag{91}$$

Treatment of the principal values in this expression is somewhat delicate so several steps are included:

$$P\,\frac{1}{u_1-v}P\frac{1}{u_2-v}=\frac{1}{4}\left[\frac{1}{u_1-v-i\nu}+\frac{1}{u_1-v+i\nu}\right]\left[\frac{1}{u_2-v-i\nu}+\frac{1}{u_2-v+i\nu}\right]$$

$$=\frac{1}{4}\left[\frac{1}{u_1-v-i\nu}\cdot\frac{1}{u_2-v-i\nu}+\frac{1}{u_1-v+i\nu}\cdot\frac{1}{u_2-v+i\nu}\right.$$

$$\left.+\frac{1}{u_1-v-i\nu}\cdot\frac{1}{u_2-v+i\nu}+\frac{1}{u_1-v+i\nu}\cdot\frac{1}{u_2-v-i\nu}\right]$$

$$=\frac{1}{4}\left[\frac{1}{u_1-v-i\nu}-\frac{1}{u_2-v-i\nu}\right]\frac{1}{u_2-u_1}+\frac{1}{4}\left[\frac{1}{u_1-v+i\nu}-\frac{1}{u_2-v+i\nu}\right]\frac{1}{u_2-u_1}$$

$$+\frac{1}{4}\left[\frac{1}{u_1-v-i\nu}-\frac{1}{u_2-v+i\nu}\right]\frac{1}{u_2-u_1+2i\nu}$$

$$+\frac{1}{4}\left[\frac{1}{u_1 - v + i\nu} - \frac{1}{u_2 - v - i\nu}\right]\frac{1}{u_2 - u_1 - 2i\nu}\,. \tag{92}$$

After performing the necessary v-integration, the first two terms above are seen to be regular at $u_1 - u_2 = 0$ and therefore yield vanishing contributions when $t \to \infty$. When the last two expressions are decomposed by the well-known formula,

$$\lim_{\nu\to 0}\frac{1}{x \pm i\nu} = P\frac{1}{x} \mp i\pi\delta(x)\ , \tag{93}$$

the δ-functions in these brackets lead to

$$\frac{i\pi}{4}\left(\delta(u_1 - v) + \delta(u_2 - v)\right)\frac{1}{u_2 - u_1 + 2i\nu} - \frac{i\pi}{4}\left[\delta(u_1 - v) + \delta(u_2 - v)\right]\frac{1}{u_2 - u_1 - 2i\nu}$$

$$= \frac{\pi^2}{2}\left[\delta(u_1 - v) + \delta(u_2 - v)\right]\delta(u_2 - u_1)\ ; \tag{94}$$

whence we obtain

$$\pi^2\frac{\omega_p^4}{k^4}\,u_1\,\frac{\partial f_0}{\partial u_1}\,\delta(u_1 - u_2)\ . \tag{95}$$

Finally, because of Landau damping the electrostatic field energy vanishes when $t \to \infty$. Therefore the only surviving contributions are Eqs. (87) and (96), which yield

$$\delta^2 F = -V\frac{m}{8n_0}\sum_k\int du_1\left(\frac{k}{4\pi e}\right)^2|E(k,u_1)|^2$$

$$\times\left[\frac{u_1\,\epsilon_p(u_1)^2}{\partial f_0/\partial u_1} + \pi^2\frac{\omega_p^4}{k^4}\,u_1\,\frac{\partial f_0}{\partial u_1}\right]\ . \tag{96}$$

Using

$$ku_1 = \omega, \qquad \epsilon_R(k,\omega) = \epsilon_p(u_1), \qquad \epsilon_I(k,\omega) = -\pi\frac{\omega_p^2}{k^2}\frac{\partial f_0(\omega/k)}{\partial v}\ , \tag{97}$$

we obtain the following remarkable formula for the free energy:

$$\delta^2 F = \frac{V}{32\pi}\sum_k\int du\;u\,\pi\frac{|\epsilon(k,ku)|^2}{\epsilon_I(k,ku)}\,|E(k,u)|^2$$

$$= \frac{V}{32\pi}\sum_k\int d\omega\;\omega\,\pi\frac{|\epsilon(k,\omega)|^2}{\epsilon_I(k,\omega)}\,|E(k,\omega)|^2\ . \tag{98}$$

The latter equality follows upon introducing the usual Fourier transformed electric field as

$$E(k,\omega) \equiv \frac{1}{|k|} E(k,u) \; .$$

Although this formula for $\delta^2 F$ may seem similar to $\mathcal{E}_D$ it is not. We emphasize once again that unlike $\mathcal{E}_D$, $\delta^2 F$ is exact. The quantities k and ω that appear in ϵ are real and independent; they are not tied together by a dispersion relation, and the quantity $E(k,\omega)$ is the Fourier (not Laplace) transform of the electric field as described below, where k and ω are independent.

An interesting feature that is brought out by the above derivation is that the initial perturbed distribution function determines uniquely the perturbed electric field for all times, and conversely. Recall that $E(k,\omega)$ is given in (82) by the initial perturbation of the distribution function. From $E(k,\omega)$ one can clearly obtain $\delta E(x,t)$ for $t \geq 0$; however, in addition one obtains $\delta E(x,t)$ for $t < 0$. This artificial past history corresponds to solving the linearized Vlasov equation backwards in time, which leads to Landau damping backwards in time. Conversely, it is possible to arbitrarily prescribe $E(k,\omega)$ and then obtain the corresponding $\hat{f}$ by using Eq. (76). However, for dynamical accessibility $E(k,\omega)$ must be chosen proportional to $\partial f_0/\partial u$. Although the state of the system is completely determined by δf at a single time, its determination requires δE for all time; δE at a single time is incomplete.

Equation (98) looks as if it would diverge or would not be well-defined at places where $\epsilon_I(k,\omega)$ vanishes for some ω/k, i.e. at the zeros of $\partial f_0(\omega/k)/\partial v$. But this is not so if the condition of dynamical accessibility is fulfilled, since in this case $E(k,\omega) \propto \partial f_0(\omega/k)/\partial v$ [see Eq. (83)] and the integrand of $\delta^2 F$ is proportional to $\partial f_0(\omega/k)/\partial v$. Also, it is evident that the energy must be finite since (98) is numerically equal to $\delta^2 F$ as given by (42), an expression that is well-defined for dynamically accessible perturbations.

C. Deficiency of the Dielectric Energy - The Problem of Landau Damping

In Sec. II it was argued that the dielectric energy in the case of Landau damping does not relate in a well-defined way to the exact energy, or to what is commonly called the wave energy. In this subsection we demonstrate this explicitly by comparing $\mathcal{E}_D$ with the exact energy of Eq. (98). This is done by considering various time dependencies for the self-consistent electric field.

According to Eq. (83), $E(k,\omega) \propto \epsilon_I/|\epsilon|^2$ if the initial perturbation of the distribution function $\hat{f}$ fulfills the condition of dynamical accessibility. Therefore $E(k,\omega)$ has poles in the complex ω-plane at the solutions $\omega = \omega_0 - i\gamma$ of the Landau dispersion relation; i.e. $\epsilon = 0$ where now the contour is deformed into the lower half plane. This means that the main contribution to the ω-integral of Eq. (126) comes from the interval

$$\omega_0 - \gamma < \omega < \omega_0 + \gamma \; . \tag{99}$$

Assuming now the small "damping" rate ordering, we expand the dielectric function as before [c.f. (61) and (62)] as

$$\epsilon(k, \omega_0 - i\gamma) \approx \epsilon(k, \omega_0) - i\gamma \frac{\partial \epsilon_R}{\partial \omega_0} = 0 \; ; \tag{100}$$

whence it follows that

$$\epsilon_R(k,\omega_0) = 0, \qquad \epsilon_I(k,\omega_0) = \gamma \frac{\partial \epsilon_R}{\partial \omega_0} \; . \tag{101}$$

Taylor expanding $\epsilon_R(k,\omega)$ about ω_0 ,

$$\epsilon_R(k,\omega) \approx (\omega - \omega_0) \frac{\partial \epsilon_R}{\partial \omega_0} + \frac{1}{2} (\omega - \omega_0)^2 \frac{\partial^2 \epsilon_R}{\partial \omega_0^2} + \cdots \; , \tag{102}$$

yields

$$\frac{|\epsilon|^2}{\epsilon_I} \approx \left[\frac{(\omega - \omega_0)^2 + \gamma^2]}{\gamma} \right] \frac{\partial \epsilon_R}{\partial \omega_0} \; . \tag{103}$$

It is then natural to choose an approximate form for $E(k,\omega)$ that is in agreement with its aforesaid property; i.e.

$$E(k,\omega) = \frac{1}{\pi}\frac{\gamma}{(\omega-\omega_0)^2+\gamma^2}\,\delta E(k,\omega_0)\ . \tag{104}$$

This corresponds to the following time dependent field amplitude:

$$E(k,t) = \delta E(k,\omega_0)e^{-i\omega_0 t}e^{-\gamma|t|}\ , \tag{105}$$

which reflects the time behavior of Landau damping, including the backward Landau damping mentioned above. This could be viewed as a sort of self-consistent turn-on followed by a self-consistent turn-off. Of course, for physically reasonable $\hat{f}$ there is a smooth transition from negative to positive time, since then $E(k,\omega)$ vanishes for $\omega \to \pm\infty$ at a rate faster than any power of ω^{-1}. Substituting (103) and (104) into the energy expression (98) produces

$$\delta^2 F = \frac{V}{32\pi}\omega_0\frac{\partial\epsilon_R}{\partial\omega_0}|\delta E(k,\omega_0)|^2\ , \tag{106}$$

an expression that agrees to within a factor of one-half with the dielectric energy for $t=0$. However, we note here two caveats. First, to obtain this formula a number of approximations were required. If instead of (104) a form for $E(k,\omega)$ that possesses a more realistic asymptotic behavior at $t=0$ is used, then (106) is only reproduced to within a numerical factor. Such cases are considered below and in Appendix C. Second, the approximate expressions so obtained for $\delta^2 F$ are, like the exact expression (98), constant in time, whereas the dielectric energy has a damping factor $e^{-2\gamma t}$ on the right hand side. This damping factor must be present in the dielectric energy expression in order for this quantity to describe Landau damped perturbations. The presence of a damping factor pin points the *principal deficiency* of $\mathcal{E}_D$, for if $\mathcal{E}_D$ were an energy it would be a constant of the motion. Also, we emphasize that the discussion of $\mathcal{E}_D$ of Sec. II shows that the derivation of this quantity is generally defined only in terms of the initial value of the electric field.

A phenomenon that displays the deficiency of $\mathcal{E}_D$ in a striking way is the plasma echo. Although the echo phenomenon is nonlinear, linear waves launched with a time interval are of central importance. The two linear waves are Landau damped in turn, until the dielectric energy of both of them is nearly zero and there is no "apparent" disturbance in the plasma. If the dielectric energy were really the energy of the perturbations, then nothing further could happen. But, since $\delta^2 F$ is not zero, the two waves can still interact nonlinearly and make a phenomenon like the echo, whereby it is possible to recover a significant portion of the original dielectric energy.

To illustrate that the factor of one-half between $\delta^2 F$ and $\mathcal{E}_D$ seen above is incidental, $E(k,\omega)$ is chosen instead of (105) to be

$$E(k,\omega) = \frac{1}{4\gamma \cosh[\pi(\omega - \omega_0)/2\gamma]} \delta E(k,\omega_0) \; ; \tag{107}$$

hence,

$$E(k,t) = \frac{e^{-i\omega_0 t}}{2\cosh\gamma t} \delta E(k,\omega_0) \; . \tag{108}$$

The time dependence again shows forward and backward Landau damping for large $|t|$, but now there is a smooth transition from negative to positive times. Because of the proportionality between E and $|\epsilon|^2/\epsilon_I$, Eq. (103) should now be replaced by

$$\frac{|\epsilon|^2}{\epsilon_I} = \gamma \cosh[\pi(\omega - \omega_0)/2\gamma] \frac{\partial \epsilon_R}{\partial \omega_0} \; . \tag{109}$$

Using (108) and (109), one obtains

$$\delta^2 F = \frac{\pi}{8} \frac{V}{32\pi} \omega_0 \frac{\partial \epsilon_R}{\partial \omega_0} |\delta E(k,\omega_0)|^2 \; , \tag{110}$$

which is smaller by a factor of $\pi/8$ than (106). The proportionality constant in this case is positive, but in general it is determined by the time profile $E(k,t)$. Appendix C contains an example where this constant can even have either sign. This is an important result since it means that *the sign of the wave energy is not necessarily correctly given by the dielectric* *energy, even when the damping rate γ is sufficiently small.* Hence the distinction between positive and negative energy perturbations as well as the magnitude of the energy may not be correctly given by the dielectric energy in the small growth rate limit. Also, if γ is not small the exact energy expression must be used in all cases. We emphasize, in addition, that the dielectric energy is usually, especially for three-dimensional equilibria and three-dimensional electromagnetic perturbations, much more complicated to use than $\delta^2 F$.

VIII. Dynamical Accessibility

It was emphasized in several places in the text that the condition of dynamical accessibility is a crucial concept. First we describe here the ramification if this condition is not imposed. The derivation of relation (42) for $\delta^2 F$ indicates that the free energy may not be uniquely defined. Now this ambiguity is discussed. We restrict to equilibria with simple zeros v_{i_ν} of $\frac{1}{v}\frac{\partial f_0}{\partial v}$, although a similar treatment exists for higher order zeros.

The ambiguity in the derivation of relation (42) can be made explicit by writing

$$\frac{v}{\partial f_0/\partial v} = P\,\frac{v}{\partial f_0/\partial v} + c_\nu \delta\left(\frac{v}{\partial f_0/\partial v}\right) \,, \tag{111}$$

where the constants c_ν are arbitrary. Insertion of (111) in Eq. (42) yields

$$\begin{aligned}\delta^2 F = &\int_V d^3x \left(\frac{(\delta E)^2}{8\pi} - \sum_\nu \frac{m_\nu}{2} P \int_{-\infty}^{\infty} dv\, \frac{v(\delta f_\nu)^2}{\partial f_\nu^0/\partial v}\right) \\ &+ \sum_k c_\nu \frac{m_\nu}{2} \int_V d^3x \int_{-\infty}^{\infty} dv\, v(\delta f_\nu)^2\, \delta\left(\frac{v}{\partial f_0/\partial v}\right) \;;\end{aligned} \tag{112}$$

the integration over v in the last term yields

$$\sum_k c_\nu \frac{m_\nu}{2} \sum_{i_\nu} \frac{v_{i_\nu}}{\frac{\partial}{\partial v}\left(\frac{1}{v}\frac{\partial f_\nu^0}{\partial v}\right)_{v=v_{i_\nu}}} \int_V d^3x\, (\delta f_\nu(x, v_{i_\nu}, t))^2 \,. \tag{113}$$

For any differentiable function $G_{i_\nu}(\delta f_\nu(x, v_{i_\nu}, t))$ it holds that

$$\frac{d}{dt}\int_V d^3x\, G_{i_\nu}(\delta f_\nu(x, v_{i_\nu}, t)) = \int_V \frac{\partial G_{i_\nu}}{\partial \delta f_\nu}\frac{\partial \delta f_\nu}{\partial t} = \int_V d^3x\, \frac{\partial G_{i_\nu}}{\partial \delta f_\nu}\left(\frac{\partial \delta f_\nu}{\partial t} + v_{i_\nu}\frac{\partial \delta f_\nu}{\partial x}\right) = 0 \,, \tag{114}$$

which means that the total number of particles at v_{i_ν} stays constant. The contribution (113) to $\delta^2 F$ is therefore also a constant of motion; of course, otherwise the constants c_ν would not be arbitrary.

The "energy" expression (112) is therefore a mixture consisting of a genuine energy and other constants of motion. The question is, whether one can separate out the genuine energy. The principal value integral in (112) excludes particles with $v = v_{i_\nu}$. Since these particles as a whole, i.e. integrated over x, do not take part in the dynamics they should not be part of the energy. Therefore, the energy should only contain the principal value contribution and not that due to the δ-function, and the genuine energy $\delta^2 F$ is given by setting all the constants $c_\nu = 0$.

Although there is then an energy even without imposing the condition of dynamical accessibility, violation of this condition would nevertheless mean that one has left the framework of Vlasov theory.

As noted above initial perturbations that arise from Hamiltonian forces, such as the electromagnetic force are dynamically accessible. This includes, for example, self-consistent fluctuations in the plasma. However, initial perturbations of the distribution function that violate the condition of dynamical accessibility are possible with particle sources and sinks, such as, ionization, recombination, injection and losses through the plasma boundary. Dynamical accessibility is also violated if phenomenological friction (forces $\propto -v$) and diffusion in phase space are allowed. If such influences lead to an initial distribution $f^0_{\nu,new}$ and if one can distinguish between new unperturbed forces and exact forces (see the example below), then the time evolution of this function can be described by (see Ref.[14])

$$f_{\nu,new}(\mathrm{x}, \mathrm{v}, t) = e^{[g_{\nu,new},\cdot]} f^0_{\nu,new}(\mathrm{x}^{(0)}_0(\mathrm{x}, \mathrm{v}, t), \mathrm{v}^{(0)}_0(\mathrm{x}, \mathrm{v}, t)) \ , \tag{115}$$

where

$$\mathrm{x}^{(0)}_0(\mathrm{x}, \mathrm{v}, t) = const \ , \ \mathrm{v}^{(0)}_0(\mathrm{x}, \mathrm{v}, t) = const \ , \tag{116}$$

describe the "undisturbed" orbits of the "new system" and $g_{\nu,new}$ is a generating function for Lie type canonical transformations from exact new orbits to new unperturbed orbits with $g_{\nu,new}(\mathbf{x},\mathbf{v},0) = 0$. An undisturbed new system can, for instance, be defined by spatially averaging. To first order the distribution functions $f_{\nu,new}$ are

$$f_{\nu,new}(\mathbf{x},\mathbf{v},t) = f^0_{\nu,new}(\mathbf{x}_0^{(0)}(\mathbf{x},\mathbf{v},t),\mathbf{v}_0^{(0)}(\mathbf{x},\mathbf{v},t)) + [g_{\nu,new}, f^0_{\nu,new}] \ . \tag{117}$$

Now restrict to the case of an homogeneous undisturbed new system, for which the new unperturbed forces vanish, introduce the spatial average of $f^0_{\nu,new}$,

$$F^0_\nu(\mathbf{v}) \equiv < f^0_{\nu,new} >_x \ , \tag{118}$$

and define the perturbation δf_ν by

$$\begin{aligned}\delta f_\nu \equiv f_{\nu,new} - F^0_\nu = f^0_{\nu,new} - F^0_\nu + [g_{\nu,new}, f^0_{\nu,new}] \\ \equiv f_{\nu h}(\mathbf{x}_0^{(0)}(\mathbf{x},\mathbf{v},t),\mathbf{v}_0^{(0)}(\mathbf{x},\mathbf{v},t)) + [g_{\nu,new}, F^0_\nu] \ .\end{aligned} \tag{119}$$

Note that F^0_ν is allowed to deviate considerably from the original unperturbed distribution f^0_ν, but the new δf_ν must be small again.

The function $f_{\nu h}$ solves the unperturbed Vlasov equation, i.e.,

$$\frac{\partial f_{\nu h}}{\partial t} + v\frac{\partial f_{\nu h}}{\partial x} = 0 \ . \tag{120}$$

Therefore, $f_{\nu h}$ drops out of the linearized Vlasov equation

$$\frac{\partial \delta f_\nu}{\partial t} + v\frac{\partial \delta f_\nu}{\partial x} = -\frac{e_\nu}{m_\nu}\delta E\frac{\partial \delta F^0_\nu}{\partial v} \ , \tag{121}$$

and one can solve this equation in the form

$$\delta E = \frac{m_\nu}{e_\nu}\frac{1}{\partial F^0_\nu/\partial v}\left(\frac{\partial[g_{\nu,new}, F^0_\nu]}{\partial t} + v\frac{\partial[g_{\nu,new}, F^0_\nu]}{\partial x}\right) \ . \tag{122}$$

The right-hand side has no singularities at the zeros of $\partial F^0_\nu/\partial v$. Insertion of (122) and (119) yields the new relation

$$\int_V \delta j\, \delta E\, d^3x =$$

$$-\sum_\nu m_\nu \int_v d^3x \int_{-\infty}^{\infty} dv \, \frac{v}{\frac{\partial F_\nu^0}{\partial v}} (f_{\nu h} + [g_{\nu,new}, F_\nu^0]) \left(\frac{\partial [g_{\nu,new}, F_\nu^0]}{\partial t} + v \frac{\partial [g_{\nu,new}, F_\nu^0]}{\partial x} \right) \; . \tag{123}$$

The Poisson bracket in the first parenthesis yields again (40) with δf_ν replaced by $[g_{\nu,new}, F_\nu^0]$ and f_ν^0 by F_ν^0. Integration by parts of the $f_{\nu h} v \frac{\partial [g_{\nu,new}, F_\nu^0][g_{\nu,new}, F_\nu^0]}{\partial x}$ contribution with respect to x transforms this expression into $-[g_{\nu,new}, F_\nu^0] v \frac{\partial f_{\nu h}}{\partial x}$ and because of (120) further into $[g_{\nu,new}, F_\nu^0] v \frac{\partial f_{\nu h}}{\partial t}$. Hence, the total $f_{\nu h}$ contribution has $\frac{\partial}{\partial t}(f_{\nu h}[g_{\nu,new}, F_\nu^0])$ in the integrand and therefore

$$\int_V \delta j \, \delta E \, d^3x =$$

$$-\frac{\partial}{\partial t} \sum_\nu \frac{m_\nu}{2} \int_v d^3x \int_{-\infty}^{\infty} dv \, \frac{v}{\frac{\partial F_\nu^0}{\partial v}} \left([g_{\nu,new}, F_\nu^0]^2 + 2[g_{\nu,new}, F_\nu^0] f_{\nu h} \right) \; . \tag{124}$$

Combination with the electric field energy term yields the constant of the motion

$$\int_V d^3x \left\{ -\sum_\nu \frac{m_\nu}{2} \int_v d^3x \int_{-\infty}^{\infty} dv \, \frac{v}{\frac{\partial F_\nu^0}{\partial v}} \left([g_{\nu,new}, F_\nu^0]^2 + 2[g_{\nu,new}, F_\nu^0] f_{\nu h} \right) + \frac{(\delta E)^2}{8\pi} \right\} = const \; . \tag{125}$$

This expression differs from (42) in that its initial value is given by the electric field energy only because of $g_{\nu,new} = 0$ initially. If $f_{\nu h} = [g_{\nu h}, F_\nu^0]$ then the missing initial particle energy perturbation is just given by a term $[g_{\nu h}, F_\nu^0]^2$ in addition to the term $[g_{\nu,new}, F_\nu^0]^2$ in (125), leading there to a replacement of the quantity in parentheses by $[g_{\nu,new} + g_{\nu h}, F_\nu^0]^2$. This is then again of the form (42) and a constant of the motion. If $f_{\nu h} \neq [g_{\nu h}, F_\nu^0]$ one is confronted again with the problem described above. The dynamically relevant energy is given by (125) with $f_{\nu h}^2$ added in the parentheses and the integral defined as a principal value integral.

We point out that for general equilibria, where the equilibrium distribution function is a function of the energy, analysis similar to that described in this section is possible. In general the ambiguity occurs at critical energy surfaces.

IX. Canonical Hamiltonian Description - Action-Angle Variables and Signature

In this section we obtain action-angle variables for the linear Vlasov problem. This is done in IX.A by appealing to the form of $\delta^2 F$ of Eq. (98). Also, in IX.A the notion of signature is introduced and a general discussion of the importance of action-angle variables and signature is made by comparison with the case for finite degree-of-freedom Hamiltonian systems. In IX.B a general linear integral transform is introduced, for which the Van Kampen mode development of Sec. VII is a special case. New completeness and orthogonality type relations that are needed in IX.C are proven. The transform transcends the application of this paper and is of general utility for solving linear fluid and plasma problems in terms of singular eigenfunctions. This is seen in IX.C where the transform is used to map the linearized version of the noncanonical bracket structure of [15] to canonical action-angle variables. Since noncanonical bracket structures of this form describe a plethora of continuum models, the transform in effect applies with remarkable generality to a variety of linear problems. The general treatment with additional examples will be presented in a future publication.

A. Discussion of Action-Angle Variables and Signature

It is of interest to compare the energy expression of Eq. (98) with the Hamiltonian for a stable, nondegenerate, N degree-of-freedom system written in terms of action-angle variables:

$$H = \sum_{\alpha}^{N} \omega_\alpha J_\alpha \ , \tag{126}$$

where by choice the action variable J_α is positive and the sign of ω_α can be positive or negative. Upon defining $\alpha \equiv (k, u)$,

$$\omega_\alpha \equiv |ku| \,\mathrm{sgn}\,(ku\epsilon_I) \tag{127}$$

and

$$J_\alpha \equiv \frac{V}{16}\frac{1}{k}\frac{|\epsilon(k,u)|^2}{|\epsilon_I(k,ku)|}\,|E(k,u)|^2 \ , \tag{128}$$

Eq. (98) takes the form of Eq. (126); i.e.

$$\delta^2 F = \sum_{k=1}^{\infty} \int du\ \omega_\alpha\, J_\alpha\ . \tag{129}$$

By its derivation this expression is valid for stable equilibria even though ϵ_I can have either sign. If the neutral modes of Sec. IV are present then a discrete sum over these modes must be added to (129). The definitions of (127) and (128) are somewhat arbitrary. The definition of the action variable J_α is incomplete since there is no guarantee that this quantity is a canonical variable. Also, we have quite arbitrarily attached the signature to ω_α [for finite systems it can also be determined from the Lagrange bracket of the linear eigenfunctions (c.f. [18])], but this is not important so long as there is a unique sign attached to each mode.

Signature is known to be important for *finite* systems because it can determine the kinds of bifurcations that are possible. Stable nondegenerate systems can always be written in the form of Eq. (126), but if a parameter of the system is varied so that the frequencies move along the real axis of the complex ω-plane then a transition to instability is possible when frequencies collide, at which point the Hamiltonian can no longer be written in this form. Since for Hamiltonian systems real frequencies must occur in pairs $\pm\omega$, two types of collisions are possible. A mode ω can collide with its mate $-\omega$ at the origin giving birth to a damped and a growing mode. The signature of the colliding pair is not important for this bifurcation, since the signature of the Hamiltonian can change as the frequencies go through zero. If the Hamiltonian was originally positive definite, and thus stable, it can change signature and become unstable. However, signature is important for the second kind of bifurcation, which occurs when pairs collide simultaneously on the positive and negative parts of the real axis. The possible outcomes are described by a theorem due to Krein [21], which states that in order for a transition to instability the colliding modes must have opposite signature. If the Hamiltonian were initially positive definite then a collision of positive signature modes away from the origin cannot change the Hamiltonian to indefinite, and therefore instability is not possible.

In infinite dimensional Hamiltonian systems such as the Vlasov equation [15, 22, 13, 23, 25], the situation is more complicated since in addition to discrete eigenmodes there is continuous spectrum. Equation (127) assigns a signature to a Van Kampen mode. This opens the possibility for interesting bifurcations. For example, two positive signature neutral modes (whose presence requires a slight generalization of the formalism in this paper) imbedded in a negative continuum could collide and become unstable, or perhaps unstable modes could be born at the boundaries where negative and positive continua meet. Positive definiteness of the energy does not rule out these possibilities. Classification of bifurcations with continua will be the subject of future work.

In the remainder of this section the heuristic identification of the action variables above is made precise.

B. A General Integral Transform Pair; Orthogonality, Completeness and Other Relations

We introduce the following general linear integral transform:

$$f_k(v,t) = \frac{ik}{4\pi e}\int_{-\infty}^{\infty} E_k(u,t)\,\mathcal{G}_k^{\varepsilon}(u,v)\,du \ , \tag{130}$$

where

$$\mathcal{G}_k^{\varepsilon}(u,v) \equiv \varepsilon_I(k,v)\,\frac{1}{\pi}P\,\frac{1}{u-v} + \varepsilon_R(k,v)\,\delta(v-u) \ . \tag{131}$$

This differs from the Van Kampen decomposition since *(130) is to be viewed as a coordinate transformation between arbitrary time dependent Fourier amplitudes $E_k(u,t)$ and $f_k(v,t)$.* Here the notation has been slightly modified for convenience and to emphasize the generality of the transform. The symbol $\epsilon = \epsilon_R + i\epsilon_I$ has been changed to $\varepsilon = \varepsilon_R + i\varepsilon_I$ to emphasize that the identities we are to derive are valid for general complex valued functions that do not possesses zeros for real v. The choice for ε depends upon the problem at hand, since one desires a $\mathcal{G}_k^{\varepsilon}$ that diagonalizes the Hamiltonian for the linear dynamics. The prefactor $(ik)/(4\pi e)$ has been retained for cultural heritage. From the development of VII.A [c.f. in particular (83)] and Appendix B it is evident that the inverse of this transform exists and is given by

$$E_k(u,t) = \frac{4\pi e}{ik} \int_{-\infty}^{\infty} f_k(v,t)\, \tilde{\mathcal{G}}_k^{\varepsilon}(u,v)\, dv \ , \tag{132}$$

where

$$\tilde{\mathcal{G}}_k^{\varepsilon}(u,v) \equiv \frac{\varepsilon_I(k,u)}{|\varepsilon(k,u)|^2} \frac{1}{\pi} P \frac{1}{u-v} + \frac{\varepsilon_R(k,u)}{|\varepsilon(k,u)|^2} \delta(v-u) \ . \tag{133}$$

In (131) and (133) symmetry in k is assumed.

The above transform is a general physical decomposition that may apply to a variety of systems. The delta function represents free streaming or free particle propagation, while the principal part represents the effect of interaction.

Insertion of (132) in (130) yields

$$f_k(v,t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{\mathcal{G}}_k^{\varepsilon}(u,v')\mathcal{G}_k^{\varepsilon}(u,v)\, f_k(v',t)\, du\, dv' \ . \tag{134}$$

Since the basis $\mathcal{G}_k^{\varepsilon}$ is complete and since $\tilde{\mathcal{G}}_k^{\varepsilon}$ defines the inverse transform, the completeness relation follows from (134),

$$\int_{-\infty}^{\infty} \tilde{\mathcal{G}}_k^{\varepsilon}(u,v')\mathcal{G}_k^{\varepsilon}(u,v)\, du = \delta(v-v') \ . \tag{135}$$

Similarly, insertion of (130) in (132) yields,

$$\int_{-\infty}^{\infty} \tilde{\mathcal{G}}_k^{\varepsilon}(u,v)\mathcal{G}_k^{\varepsilon}(u',v)\, dv = \delta(u-u') \ . \tag{136}$$

Orthogonality relations similar to (135) and (136) appear in the works of Case [24].

In the remainder of this subsection two additional nontrivial orthogonality type relations are obtained. These relations, given by (144) and (146) below, are needed in IX.C, to where the reader can turn without loss of continuity.

Consider the integral

$$\int_{-\infty}^{\infty} Q(u)\, \mathcal{G}_k^{\varepsilon}(u,v')\mathcal{G}_k^{\varepsilon}(u,v)\, du = Q(v)\varepsilon_R^2 \delta(v-v')$$

$$-\frac{1}{\pi}\frac{P}{v'-v}\Big(Q(v)\varepsilon_R(k,v)\varepsilon_I(k,v')-Q(v')\varepsilon_R(k,v')\varepsilon_I(k,v)\Big)$$

$$+\frac{1}{\pi^2}\varepsilon_I(k,v)\varepsilon_I(k,v')\int_{-\infty}^{\infty}Q(u)\,\frac{P}{v-u}\frac{P}{v'-u}\,du\ , \qquad (137)$$

where the equality follows simply upon inserting (131) and performing the δ-function integrals. The integral of the third term on the right-hand side of (137) is evaluated by again decomposing the principal value product as in (92). This results in

$$\int_{-\infty}^{\infty}Q(u)\,\frac{P}{v-u}\frac{P}{v'-u}\,du=\pi^2\,Q(v)\,\delta(v-v')$$

$$+\frac{1}{v'-v}\Big(P\int_{-\infty}^{\infty}\frac{Q(u)}{v-u}\,du-P\int_{-\infty}^{\infty}\frac{Q(u)}{v'-u}\,du\Big)\ . \qquad (138)$$

Combining (137) with (138) yields the following identity:

$$\int_{-\infty}^{\infty}Q(u)\ \mathcal{G}_k^{\varepsilon}(u,v')\mathcal{G}_k^{\varepsilon}(u,v)\,du=Q(v)|\varepsilon|^2\delta(v-v')$$

$$+\frac{1}{\pi^2}\frac{\varepsilon_I(k,v)\varepsilon_I(k,v')}{v'-v}\left(P\int_{-\infty}^{\infty}\frac{Q(u)}{v-u}\,du-P\int_{-\infty}^{\infty}\frac{Q(u)}{v'-u}\,du\right)$$

$$-\frac{1}{\pi}\frac{P}{v'-v}\Big(Q(v)\varepsilon_R(k,v)\varepsilon_I(k,v')-Q(v')\varepsilon_R(k,v')\varepsilon_I(k,v)\Big)\ . \qquad (139)$$

Now consider the well-known Kramers-Kronig relation for causal complex-valued functions,

$$\Phi_R(v)=1+\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\Phi_I(u)}{u-v}du\ . \qquad (140)$$

This relation is valid for functions $\Phi(u)$ that are analytic in the upper half complex u-plane and obtain there the value unity for $|u|\to\infty$. A class of functions fulfilling these conditions is given by

$$\Phi(v)=1+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{R(u)}{u-v-i\nu}du\ ,\ \nu>0\ , \qquad (141)$$

where $R(u)$ is continuous and satisfies condition (B-2). If $R(u)$ is real and $\nu\to+0$, the Kramers-Kronig relations (140) for Φ are the same relations as obtained directly from the definition of Φ, which yields

$$\Phi_R(v) - 1 = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{R(u)}{u-v} du \ , \ \ \Phi_I(v) = R(v) \ .$$

Another Kramers-Kronig type relation exists for the function $1/\Phi(v)$. This function fulfills the above mentioned conditions if $\Phi(v)$ has no zeros in the upper half v-plan, including the real axis. Since

$$Re\frac{1}{\Phi} = \frac{\Phi_R}{|\Phi|^2} \ , \ Im\frac{1}{\Phi} = -\frac{\Phi_I}{|\Phi|^2} \ , \tag{142}$$

the Kramers-Kronig relations yield

$$\frac{\Phi_R}{|\Phi|^2} - 1 = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\Phi_I(u)/|\Phi|^2}{u-v} du \ . \tag{143}$$

For completeness we mention that any function of $\Phi(v)$ that is analytic in the upper half v-plane and that is unity for $\Phi = 1$ can be used in the Kramers-Kronig relations.

Now choosing $\Phi = \varepsilon$, $Q(u) = \varepsilon_I/|\varepsilon|^2$ and making use of (143), Eq. (139) produces

$$\int_{-\infty}^{\infty} \frac{\varepsilon_I(k,u)}{|\varepsilon(k,u)|^2} \mathcal{G}_k^\varepsilon(u,v') \mathcal{G}_k^\varepsilon(u,v)\, du = \varepsilon_I(k,v)\delta(v-v') \ . \tag{144}$$

Another relation like (139), but involving $\tilde{\mathcal{G}}_k^\varepsilon$, exists.

$$\int_{-\infty}^{\infty} Q(v) \ \ \tilde{\mathcal{G}}_k^\varepsilon(u,v)\tilde{\mathcal{G}}_k^\varepsilon(u',v)\, dv = \frac{1}{|\varepsilon(k,u)|^2}\frac{1}{|\varepsilon(k,u')|^2}\Big[Q(u)|\varepsilon(k,u)|^2\delta(u-u')$$

$$+\frac{1}{\pi^2}\frac{\varepsilon_I(k,u)\varepsilon_I(k,u')}{u'-u}\left(P\int_{-\infty}^{\infty}\frac{Q(v)}{u-v}\,dv - P\int_{-\infty}^{\infty}\frac{Q(v)}{u'-v}\,dv\right)$$

$$+\frac{1}{\pi}\frac{P}{u'-u}\Big(Q(u)\varepsilon_R(k,u)\varepsilon_I(k,u') - Q(u')\varepsilon_R(k,u')\varepsilon_I(k,u)\Big)\Big] \ . \tag{145}$$

Upon setting $Q = \varepsilon_I$, $\Phi = \varepsilon$, and making use of (140), (145) yields

$$\int_{-\infty}^{\infty} \varepsilon_I(k,v)\, \tilde{\mathcal{G}}_k^\varepsilon(u,v)\tilde{\mathcal{G}}_k^\varepsilon(u',v)\, dv = \frac{\varepsilon_I(k,u)}{|\varepsilon(k,u)|^2}\delta(u-u') \ . \tag{146}$$

This last expression is used in a transformation below; the inverse of this transformation requires (144).

C. Transformation to Action-Angle Variables

The Vlasov-Poisson equation is an infinite dimensional Hamiltonian system or field theory, but because the distribution function does not constitute canonically conjugate variables the Poisson bracket is of the following noncanonical form [15]:

$$\{F, G\} = \int f \left[\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right] d^3x\, d^3v \ , \tag{147}$$

where F and G are arbitrary functionals, $[,]$ is the ordinary Poisson bracket of (29), and $\delta F/\delta f$ is the functional derivative. In terms of (147) the Vlasov equation is compactly written as

$$\frac{\partial f}{\partial t} = \{f, H\} \ , \tag{148}$$

where the Hamiltonian H is the total energy functional,

$$H = \int \frac{1}{2} m v^2 f d^3x d^3v + \frac{1}{8\pi} \int E^2 d^3x \ . \tag{149}$$

Two features of the bracket of Eq. (147) warrant mention: first, the form is obviously not canonical (note e.g. it is an explicit function of f) and second, the bracket is degenerate in the sense that

$$\{C, F\} = 0 \ , \tag{150}$$

for all functionals F, where C, the so-called Casimir invariants, are given by (36). Because of the degeneracy the bracket $\{,\}$ can only generate dynamics in constraint "surfaces" (sometimes called symplectic leaves) determined by the constants C; hence the degeneracy is tantamount to dynamical accessibility. For further details we refer the reader to [26, 11, 22, 13, 23, 25].

The Hamiltonian description of the linearized dynamics of interest here is obtained by expanding both the above noncanonical Poisson bracket and the Hamiltonian. Assuming $f = f^0(v) + \delta f$ and expanding yields the linearized bracket

$$\{F, G\}_L = \int f^0 \left[\frac{\delta F}{\delta \delta f}, \frac{\delta G}{\delta \delta f}\right] d^3x\, dv \ , \tag{151}$$

in terms of which the linearized Vlasov-Poisson equation can be compactly written as follows:

$$\frac{\partial \delta f}{\partial t} = \{f, \delta^2 F\}_L . \tag{152}$$

Representing δf as a Fourier series,

$$\delta f(x,v,t) = \frac{1}{2} \sum_{k=-\infty}^{\infty} f_k(v,t)\, e^{ikx} , \tag{153}$$

the functional derivative has the Fourier series (c.f. [27]),

$$\begin{aligned} \frac{\delta F}{\delta \delta f} &= \sum_{k=-\infty}^{\infty} \left(\frac{\delta F}{\delta \delta f}\right)_k e^{ikx} \\ &= \frac{2}{V} \sum_{k=-\infty}^{\infty} \frac{\delta F}{\delta f_{-k}}\, e^{ikx} , \end{aligned} \tag{154}$$

and upon insertion of (154) and the corresponding expression for G in (151), the bracket becomes

$$\{F, G\}_L = \frac{4i}{mV} \sum_{k=1}^{\infty} k \int_{-\infty}^{\infty} dv\, \frac{\partial f^0}{\partial v} \left(\frac{\delta F}{\delta f_k} \frac{\delta G}{\delta f_{-k}} - \frac{\delta G}{\delta f_k} \frac{\delta F}{\delta f_{-k}} \right) dv . \tag{155}$$

Note the $k = 0$ component vanishes; this is part of the degeneracy associated with dynamical accessibility, which arises when one assumes the existence of the Fourier transform of the quantities above.

In order to transform from the independent coordinates f_k and f_{-k} to E_k, and E_{-k} as defined by (132), the chain rule for functional differentiation must be obtained. Varying an arbitrary functional $F[f_k] = F[f_k[E_k]]$ yields,

$$\delta F = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} dv\, \frac{\delta F}{\delta f_k}\, \delta f_k = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} du\, \frac{\delta F}{\delta E_k}\, \delta E_k . \tag{156}$$

From Eq. (132)

$$\delta E_k(u,t) = \frac{4\pi e}{ik} \int_{-\infty}^{\infty} \delta f_k(v,t)\, \tilde{\mathcal{G}}_k^{\varepsilon}(u,v)\, dv , \tag{157}$$

which, upon substitution into (156) and noting that δf_k is an arbitrary variation, results in the following connection between the functional derivatives:

$$\frac{\delta F}{\delta f_k(v,t)} = \frac{4\pi e}{ik} \int_{-\infty}^{\infty} \tilde{\mathcal{G}}_k^{\varepsilon}(u,v)\, \frac{\delta F}{\delta E_k(u,t)}\, du . \tag{158}$$

Inserting (158) and a similar formula for $\delta G/\delta E_k$ into (155) yields,

$$\{F,G\}_L = \frac{4i}{V}\frac{(4\pi e)^2}{m}\sum_{k=1}^{\infty}\frac{1}{k}\int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} du' \,\frac{\partial f^0}{\partial v}\,\tilde{\mathcal{G}}_k^{\varepsilon}(u,v)\tilde{\mathcal{G}}_{-k}^{\varepsilon}(u',v)$$
$$\times\left(\frac{\delta F}{\delta E_k(u)}\frac{\delta G}{\delta E_{-k}(u')} - \frac{\delta G}{\delta E_k(u')}\frac{\delta F}{\delta E_{-k}(u)}\right) . \tag{159}$$

Upon requiring

$$\varepsilon_I(k,u) = \epsilon_I(k,ku) = -\pi\frac{\omega_p^2}{k^2}\frac{\partial f_0}{\partial u} , \tag{160}$$

reverting to the notation $E(k,u) = E_k(u)$, and making use of Eq. (146), (159) becomes

$$\{F,G\}_L = -\frac{16i}{V}\sum_{k=1}^{\infty} k\int_{-\infty}^{\infty} du\,\frac{\epsilon_I(k,ku)}{|\epsilon(k,ku)|^2}\left(\frac{\delta F}{\delta E(k,u)}\frac{\delta G}{\delta E(-k,u)} - \frac{\delta G}{\delta E(k,u)}\frac{\delta F}{\delta E(-k,u)}\right) . \tag{161}$$

It is clear from (161) that dynamics with $\partial f_k/\partial t \propto \partial f^0/\partial v$ is impossible. This is the remaining part of the degeneracy associated with dynamical accessibility. It is now possible to eliminate the degeneracy altogether and define the dynamics in terms of canonical coordinates that lie within and span the constraint surfaces (that are sometimes called symplectic leaves).

From (161) the time dependence of $E(k,ku,t)$ is generated:

$$\frac{\partial E}{\partial t} = \{E,\delta^2 F\}_L = -ikuE , \tag{162}$$

where the last equality follows from $\delta^2 F$ of Eq. (98) rewritten as

$$\delta^2 F = \frac{V}{16}\sum_{k=1}^{\infty}\int_{-\infty}^{\infty} du\,u\,\frac{|\epsilon(k,ku)|^2}{\epsilon_I(k,ku)}\,|E(k,u)|^2 . \tag{163}$$

Observe that the time dependence derived in this way is precisely that *assumed* for the Van Kampen mode decomposition.

The action variables given in IX.A, with their corresponding angles, follow directly from the transformation

$$E(-k,u) = \sqrt{\frac{16|\epsilon_I|}{kV\,|\epsilon|^2}\,J_\alpha}\;\; e^{i\theta_\alpha}$$

$$E(k,u) = \sqrt{\frac{16|\epsilon_I|}{kV\,|\epsilon|^2}\, J_\alpha}\;\; e^{-i\theta_\alpha}\;, \tag{164}$$

when $\epsilon_I > 0$, and the following when $\epsilon_I < 0$:

$$E(k,u) = \sqrt{\frac{16|\epsilon_I|}{kV\,|\epsilon|^2}\, J_\alpha}\;\; e^{i\theta_\alpha}\;,$$

$$E(-k,u) = \sqrt{\frac{16|\epsilon_I|}{kV\,|\epsilon|^2}\, J_\alpha}\;\; e^{-i\theta_\alpha}\;. \tag{165}$$

With the chain rule the Poisson bracket becomes

$$\{F,G\}_L = \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} du \left(\frac{\delta F}{\delta\theta_\alpha}\frac{\delta G}{\delta J_\alpha} - \frac{\delta G}{\delta\theta_\alpha}\frac{\delta F}{\delta J_\alpha} \right) \;. \tag{166}$$

The content of this section amounts to solving the complete spectral problem for a large class of operators; the method can be interpreted either as a diagonalizing coordinate transformation to action-angle variables or as an integral transform technique.

X. Conclusions

One goal of this paper energy was to comprehensively consider energy expressions for perturbations of homogeneous Vlasov-Poisson equilibria. To this end the dielectric energy and the exact plasma free energy, which respects the important constraint of dynamical accessibility, were compared in special cases. For the case of stable equilibria we were led to the general energy expression of (98). Another goal of this paper was to obtain in an unambiguous way the Hamiltonian action variables for stable linear Vlasov equilibria. This required the introduction of the integral transform pair, which maps the linearized noncanonical Poisson bracket for the Vlasov equation to the desired canonical action-angle form.

Several avenues for future work naturally come to mind. One is to obtain the generalization of (98) for electromagnetic perturbations about homogeneous equilibria and inhomogeneous equilibria with various field configurations. Also, the action-angle variables are natural variables to use for perturbation theory. In the case of unstable equilibria Hamiltonian systems no longer possess action-angle variables, but other normal forms exist and the techniques of [24] can be used to define a transformation to these coordinates. Since the Poisson bracket of IX.C applies to a variety of fluid and plasma systems, the integral transform pair is a most general transformation. One can transform a plethora of systems to these variables for diagonalization. This will be the subject of a future paper.

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Appendix A: Electrostatic Energy for Three-Dimensional Equilibria

Assuming $f_\nu^{(0)} = f_\nu^{(0)}(H_\nu^{(0)})$ alone, where $H_\nu^{(0)} = \frac{m_\nu v^2}{z} + e_\nu \phi^{(0)}$, Eq. (35) can be written as

$$\delta^2 F = -\sum_\nu \int d^3x\, d^3p\, \frac{1}{2} \left[g_\nu, H_\nu^{(0)}\right]^2 \frac{\partial f_\nu^{(0)}}{\partial H_\nu^{(0)}} + \frac{1}{8\pi} \int \delta E^2\, d^2x\ . \tag{A-1}$$

For convenience, below we omit the species label ν and assume a volume of size 2π.

Enroute to eliminating g in terms of $\delta\mathbf{E}$ we write $g = g(\mathbf{J}, \boldsymbol{\theta})$ where $(\mathbf{J}, \boldsymbol{\theta})$ are equilibrium action-angle variables *for the particles*, which are assumed to exist. Recall

$$J_i = \frac{1}{2\pi} \oint_{\gamma_i} \mathbf{p} \cdot d\mathbf{x}$$

where the individual components J_i are defined in terms of the closed contours γ_i. In terms of action-angle variables

$$H^{(0)} = H^{(0)}(\mathbf{J})\ ;$$

i.e. all the angles conjugate to $\mathbf{J}$ are ignorable. The main reason for using these variables is that Eq. (33) simplifies to

$$\frac{\partial g}{\partial t} + \boldsymbol{\Omega}(\mathbf{J}) \cdot \frac{\partial g}{\partial \boldsymbol{\theta}} = \delta H \tag{A-2}$$

where $\delta H = e\delta\phi(\mathbf{x}(\theta, \mathbf{J}), \mathrm{t})$ and $\boldsymbol{\Omega} \equiv \frac{\partial H^{(0)}}{\partial \mathbf{J}}$. The general solutions of Eq. (A-2) is given by

$$g(\mathbf{J}, \boldsymbol{\theta}, t) = \hat{g}(\boldsymbol{\theta} - \boldsymbol{\Omega} t, J) + \int_0^t \delta H\left(\mathbf{x}(\boldsymbol{\theta} + \boldsymbol{\Omega}(\tau - t), \mathbf{J}), \tau\right) d\tau\ . \tag{A-3}$$

The perturbed distribution function is obtained from (28) as follows:

$$f^{(1)} = [g, f_0] = \frac{\partial f_0}{\partial H^{(0)}} \left[g, H^{(0)}\right] \tag{A-4}$$

where

$$\left[g, H^{(0)}\right] = \boldsymbol{\Omega} \cdot \frac{\partial \hat{g}}{\partial \boldsymbol{\theta}} - e \int_0^t \left[\left(\boldsymbol{\Omega} \cdot \frac{\partial}{\partial \boldsymbol{\theta}}\right) \mathbf{x}\right] \cdot \delta\mathbf{E}(\mathbf{x}, t)\, d\tau\ , \tag{A-5}$$

where $\mathbf{x}$ is a shorthand for $\mathbf{x} = \mathbf{x}(\boldsymbol{\theta} + (\tau - t)\boldsymbol{\Omega}, \mathbf{J})$. Inserting (A-5) into (A-1) yields

$$\delta^2 F = -\frac{1}{2}\int d^3\phi\, d^3J\, \frac{\partial f_0}{\partial H^{(0)}}\left\{ \left(\boldsymbol{\Omega}\cdot\frac{\partial \hat{g}}{\partial \boldsymbol{\theta}}\right)^2 + e^2\left(\int_0^t \delta\mathbf{E}\cdot\frac{\partial \mathbf{x}}{\partial \boldsymbol{\theta}}\cdot\boldsymbol{\Omega}\, d\tau\right)^2 \right.$$

$$\left. - 2e\,\boldsymbol{\Omega}\cdot\frac{\partial \hat{g}}{\partial \boldsymbol{\theta}}\int_0^t \delta\mathbf{E}\cdot\frac{\partial \mathbf{x}}{\partial \boldsymbol{\theta}}\cdot\boldsymbol{\Omega}\, d\tau \right\} + \frac{1}{8\pi}\int \delta E^2\, d^3x\ , \tag{A-6}$$

where the volume element transforms as $d^3x\, d^3p = d^3\theta\, d^3J$.

The existence of action-angle variables requires periodicity, and so we can expand as follows:

$$\hat{g}(\boldsymbol{\theta}, \mathbf{J}) = \sum_{\mathbf{m}} g_{\mathbf{m}}(\mathbf{J}) e^{i\mathbf{m}\cdot\boldsymbol{\theta}}\ ,$$

$$\mathbf{x}(\boldsymbol{\theta}, \mathbf{J}) = \sum_{\mathbf{m}} \mathbf{x}_m(\mathbf{J}) e^{i\mathbf{m}\cdot\boldsymbol{\theta}}\ , \tag{A-7}$$

$$\delta\mathbf{E}(\mathbf{x}(\boldsymbol{\theta}, \mathbf{J}), t) \equiv \delta\tilde{\mathbf{E}}(\boldsymbol{\theta}, \mathbf{J}, t) = \sum_{\mathbf{m}} E_{\mathbf{m}}(\mathbf{J}, t) e^{i\mathbf{m}\cdot\boldsymbol{\theta}}\ .$$

Inserting Eqs. (A-7) into (A-6), and integrating over θ results in the following energy expression:

$$\delta^2 F = -4\pi^3 \sum_{\mathbf{m}} \int d^3J\, \frac{\partial f_0}{\partial H^{(0)}} (\mathbf{m}\cdot\boldsymbol{\Omega})^2 |g_{\mathbf{m}}|^2$$

$$- 8\pi^3 e \sum_{\boldsymbol{\ell},\mathbf{m},\mathbf{p}} \int d^3J\, \frac{\partial f_0}{\partial H^{(0)}} (\boldsymbol{\Omega}\cdot\mathbf{m})(\boldsymbol{\Omega}\cdot\mathbf{p}) g_{\mathbf{m}} \delta(\mathbf{m}+\boldsymbol{\ell}+\mathbf{p}) e^{-i\boldsymbol{\Omega}\cdot(\mathbf{m}+\boldsymbol{\ell}+\mathbf{p})t}$$

$$\times \left[\int_0^t \mathbf{x}_{\mathbf{p}}\cdot\mathbf{E}_{\boldsymbol{\ell}}(\tau) e^{i\boldsymbol{\Omega}\cdot(\boldsymbol{\ell}+\mathbf{p})\tau}\, d\tau\right] + 4\pi^3 e^2 \sum_{\mathbf{q},\mathbf{p},\boldsymbol{\ell},\mathbf{m}} \int d^3J\, \frac{\partial f_0}{\partial H^{(0)}} (\boldsymbol{\Omega}\cdot\mathbf{p})$$

$$\times (\boldsymbol{\Omega}\cdot\mathbf{m}) \delta(\boldsymbol{\ell}+\mathbf{p}+\mathbf{m}+\mathbf{q}) e^{-i(\boldsymbol{\ell}+\mathbf{p}+\mathbf{m}+g)t} \left[\int_0^t \mathbf{x}_{\mathbf{q}}\cdot\mathbf{E}_{\mathbf{m}}(\tau) e^{i\boldsymbol{\Omega}\cdot(\mathbf{m}+\mathbf{q})\tau}\, d\tau\right]$$

$$\times \left[\int_0^t \mathbf{x}_{\mathbf{p}}\cdot\mathbf{E}_{\boldsymbol{\ell}}(\tau') e^{i\boldsymbol{\Omega}\cdot(\boldsymbol{\ell}+\mathbf{p})\tau'}\, d\tau'\right] + \pi^3 \sum_{\boldsymbol{\ell},\mathbf{m},\mathbf{p}} \mathbf{E}_m\cdot\mathbf{E}_{\boldsymbol{\ell}} \left|\frac{\partial \mathbf{x}}{\partial \theta}\right|_{\mathbf{p}} \delta(\mathbf{m}+\boldsymbol{\ell}+\mathbf{p})\ , \tag{A-8}$$

where $\delta(\mathbf{m}) = 0$ unless $\mathbf{m} = 0$ in which case it is unity.

Appendix B: The $f = f_+ + f_-$ Splitting

The method used by Van Kampen for solving the initial value problem requires a splitting of a function $f(x)$ of a single real variable into the sum of two functions with complex extension that are analytic in the upper and lower half planes, respectively. In order to obtain this splitting we represent $f(x)$ as follows:

$$f(x) = \lim_{\nu \to 0^+} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\nu}{(x - x')^2 + \nu^2} f(x')\, dx' \; . \tag{B-1}$$

The validity of this formula requires $f(x)$ to be continuous and $\lim_{|x| \to \infty} |f(x)| \leq M|x|^\alpha$ with $\alpha < 1$ and M constant. Jump discontinuities can be allowed, but the right-hand side produces the arithmetic mean of the values of f on each side.

Assuming further that

$$\lim_{|x| \to \infty} f(x) = 0 \; , \tag{B-2}$$

Eq. (B-1) can be decomposed as

$$\begin{aligned} f(x) &= \lim_{\nu \to 0^+} \frac{1}{2\pi i} \left\{ \int_{-\infty}^{\infty} \frac{f(x')dx'}{x' - x - i\nu} - \int_{-\infty}^{\infty} \frac{f(x')dx'}{x' - x + i\nu} \right\} \\ &\equiv f_+(x) + f_-(x) \; . \end{aligned} \tag{B-3}$$

This defines the splitting needed in the text.

The extension into the complex plane of the function $f_{+(-)}(z)$ has several important properties: (i) it is analytic in the upper (lower) half plane, (ii) it approaches $f_{+(-)}(x)$ as $y \to 0$, (iii) it approaches 0 as $y \to +(-)\infty$. Also, most importantly, the splitting is unique [20].

Similarly the principal value integral can be represented as

$$\begin{aligned} P \int_{-\infty}^{\infty} \frac{f(x')\, dx'}{x' - x} &= \lim_{\nu \to 0^+} \int_{-\infty}^{\infty} \frac{x' - x}{(x - x')^2 + \nu^2} f(x')\, dx' \\ &= \lim_{\nu \to 0^+} \frac{1}{2} \left\{ \int_{-\infty}^{\infty} \frac{f(x')dx'}{x' - x - i\nu} + \int_{-\infty}^{\infty} \frac{f(x')dx'}{x' - x + i\nu} \right\} \\ &\equiv i\pi(f_+(x) - f_-(x)) \; . \end{aligned} \tag{B-4}$$

For this formula to be valid, the conditions on $f(x)$ are the same as those required for Eq. (B-3).

The above decomposition is introduced somewhat differently than that originally given by Van Kampen, which begins with the Fourier transform of $f(x)$. The comparison can be made by writing f_+ and f_- in terms of Fourier integrals, making use of the following identities:

$$\frac{1}{x'-x-i\nu} = i\int_0^\infty e^{i(x-x'+i\nu)p}\,dp$$

$$\frac{-1}{x'-x+i\nu} = i\int_{-\infty}^0 e^{i(x-x'-i\nu)p}\,dp\,.$$

Substituting these into (C-3), and then interchanging the p and x' integrations yields

$$f(x) = \frac{1}{2\pi}\int_0^\infty dp\,e^{+ipx}\left\{\int_{-\infty}^\infty f(x')e^{i(-x'+i\nu)p}\,dx'\right\}$$
$$+\frac{1}{2\pi}\int_{-\infty}^0 dp\,e^{+ipx}\left\{\int_{-\infty}^\infty f(x')e^{i(-x'-i\nu)p}\,dx'\right\}\,.$$

Upon defining

$$F(p) = \left\{\begin{array}{ll} \frac{1}{2\pi}\int_{-\infty}^\infty f(x')e^{(-ix'-\nu)p}\,dx' \equiv F_+(p) & p>0 \\ \frac{1}{2\pi}\int_{-\infty}^\infty f(x')e^{-(ix'+\nu)p}\,dx' \equiv F_-(p) & p<0 \end{array}\right\}\,,$$

we obtain

$$f(x) = f_+(x) + f_-(x) = \int_0^\infty dpe^{+ipx}\,F_+(p) + \int_{-\infty}^0 dpe^{+ipx}\,F_-(p) = \int_{-\infty}^\infty dpe^{+ipx}\,F(p)\,,$$

where it is seen that f_+ and f_- are the positive and negative frequency parts of the Fourier transform, respectively.

Appendix C: Proportionality Constant of VII.C

In this appendix an example is given where the proportionality constant between $\delta^2 F$, in the small γ approximation, and $\mathcal{E}_D$ can have either sign. To this end we choose

$$E(k,\omega) = \left(\frac{1}{\pi}\frac{\gamma}{(\omega-\omega_0)^2+\gamma^2} + \beta\frac{1}{\sqrt{2\pi}\gamma}e^{-\frac{1}{2}(\omega-\omega_0)^2/\gamma^2}\right)\delta E(k,\omega_0)\ . \tag{C-1}$$

The first term on the right-hand side could also be replaced by any other function leading to an asymptotic behavior

$$E(k,t) \to e^{-i\omega_0 t}e^{-\gamma|t|}\delta E(k,\omega_0)\ . \tag{C-2}$$

From (C-1)

$$E(k,t) = e^{-i\omega_0 t}\left(e^{-\gamma|t|} + \beta e^{-\frac{1}{2}\gamma^2 t^2}\right)\delta E(k,\omega_0)\ , \tag{C-3}$$

which also possesses the correct asymptotic behavior.

The requirements for $|\epsilon|^2/\epsilon_I$ are

$$\left.\frac{|\epsilon|^2}{\epsilon_I}\right|_{\omega=\omega_0} = \gamma\frac{\partial\epsilon_R}{\partial\omega}\ ,\ \frac{|\epsilon|^2}{\epsilon_I} \propto \frac{1}{E}\ . \tag{C-4}$$

From (C-1) one obtains

$$E(k,\omega_0) = \left(\frac{1}{\pi\gamma} + \frac{\beta}{\sqrt{2\pi}\gamma}\right)\delta E(k,\omega_0)\ . \tag{C-5}$$

One should therefore represent $|\epsilon|^2/\epsilon_I$ as

$$\frac{|\epsilon|^2}{\epsilon_I} = \frac{1}{E}\left(\frac{1}{\pi} + \frac{\beta}{\sqrt{2\pi}}\right)\delta E(k,\omega_0)\frac{\partial\epsilon_R}{\partial\omega_0}\ . \tag{C-6}$$

Insertion in (98) yields

$$\delta^2 F = \frac{V}{16\pi}|\delta E(k,\omega_0)|^2(1+\sqrt{\frac{\pi}{2}}\beta)(1+\beta)\omega_0\frac{\partial\epsilon_R}{\partial\omega_0}\ . \tag{C-7}$$

The factor $(1+\sqrt{\frac{\pi}{2}}\beta)(1+\beta)$ obtains its minimum for $\beta = \beta_{min} = \frac{1}{2}(\sqrt{\frac{2}{\pi}}+1)$. Its value there is $-\frac{1}{4}\sqrt{\frac{\pi}{2}}(1-\sqrt{\frac{2}{\pi}})^2$. Thus the factor ranges in value from some finite negative number to $+\infty$.

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