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The Rayleigh-Taylor Instability in an Expanding Plasma

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Abstract

In a number of fluid systems, overall fluid expansion has a retarding effect on the growth of Rayleigh-Taylor (RT) instabilities: the growth of RT instabilities relative to the expansion of the fluid is slowed and is often sub-exponential. We give two new analytical examples of this phenomenon of reduced growth or stabilization: one with incompressible fluids and one with an adiabatic fluid. Confirmation of this phenomenon is also obtained from a new MHD code constructed specifically for modelling fluids undergoing nearly homogeneous (but not necessarily isotropic) expansion or contraction. In the code, expansion is included by making each point of the computational grid co-moving with a predetermined overall expansion, which is equivalent to using an expanding metric.

I. Introduction

Many physical fluid systems of interest undergo overall expansion or contraction. Examples include inertial confinement fusion targets [1,2,3], supernovae [4,5], the plasma of the early Universe [6], Z-pinch plasmas undergoing Felber oscillations [7], and, probably, D-T ice crystals in muon-catalyzed fusion reactors [8]. It is to be expected that such global motion will affect the character of many fluid processes such as the propagation of waves and the growth of instabilities.

In this paper, we concern ourselves with the effect of homogeneous expansion on the Rayleigh-Taylor (RT) instability. This topic has been addressed for several particular systems. A perusal of the literature will show that expansion has a slowing effect on the instability: the instability growth, relative to the (growing) size of the fluid system, is sub-exponential. This is to be expected since the quantities driving the instability, namely the gradients in pressure and in the magnetic field strength, are depleted by the expansion.

We present two new analytical studies of expanding fluid systems undergoing RT unstable motion. In both of these systems, we find sub-exponential relative growth of the instability. We also present a numerical algorithm for simulating expanding (and contracting) fluid systems. Simulation results obtained with this algorithm show a marked retardation of the growth of a Rayleigh-Taylor instability in an expanding fluid.

II. Analytical Studies of RT Instabilities in Expanding Fluids

A. Incompressible Flow

We look first at expansion in an R-T unstable, two-fluid, incompressible system. We take gravity to be in the y direction. The fluid interface lies in the x-z plane ($y = 0$). Zero-order motion in the z direction is given by

$$v_{z0} = \frac{\dot{a}(t)}{a(t)}\, z \ .$$

Of course, we cannot have purely expanding motion in an incompressible fluid. To satisfy incompressibility, v_{x0} and v_{y0} can be taken to be

$$v_{x0} = -\alpha\, \frac{\dot{a}(t)}{a(t)}\, x \quad ; \quad v_{y0} = (\alpha - 1)\, \frac{\dot{a}(t)}{a(t)}\, y$$

where α is a constant.

We assume that $\rho_0 = \rho_0(y)$. Solution of the zero-order momentum equation shows

$$\begin{aligned} p_0(x,y,z,t) = &- \int_0^x \rho_0(y)(-\alpha x) \left(\partial_t \left(\frac{\dot{a}}{a}\right) - \alpha \left(\frac{\dot{a}}{a}\right)^2 \right) dx \\ &- \int_0^y \left[\rho_0(y)(\alpha - 1) y \left(\partial_t \left(\frac{\dot{a}}{a}\right) + (\alpha - 1) \left(\frac{\dot{a}}{a}\right)^2 \right) + \rho_0(y) g \right] dy \\ &- \int_0^z \rho_0(y) \left(\partial_t \left(\frac{\dot{a}}{a}\right) + \left(\frac{\dot{a}}{a}\right)^2 \right) dz + p_0(0,0,0,t) \ . \end{aligned}$$

With this in mind, we perturb the fluid velocity field with $\delta \mathbf{v} = \delta v_x \widehat{x} + \delta v_y \widehat{y}$ and linearize the equations of motion:

$$\partial_t \delta\rho + (\alpha - 1)\, \frac{\dot{a}}{a}\, y\, \delta\rho - \alpha\, \frac{\dot{a}}{a}\, x \partial_x\, \delta\rho + \delta v_y\, \partial_y \rho_0 = 0 \tag{1}$$

$$\begin{aligned} \rho_0 &\left(\partial_t\, \delta v_x - \alpha\, \frac{\dot{a}}{a}\, x\, \partial_x\, \delta v_x + (\alpha - 1)\, \frac{\dot{a}}{a}\, y\, \partial_y\, \delta v_x - \alpha\, \frac{\dot{a}}{a}\, \delta v_x \right) \\ &+ \left(\alpha^2 \left(\frac{\dot{a}}{a}\right)^2 - \alpha\, \partial_t \left(\frac{\dot{a}}{a}\right) \right) x\, \delta\rho + \partial_x\, \delta p = 0 \end{aligned} \tag{2}$$

$$\begin{aligned} \rho_0 &\left(\partial_t\, \delta v_y - \alpha\, \frac{\dot{a}}{a}\, x\, \partial_x\, \delta v_y + (\alpha - 1)\, \frac{\dot{a}}{a}\, y\, \partial_y\, \delta v_y + (\alpha - 1)\, \frac{\dot{a}}{a}\, \delta v_y \right) \\ &+ \left((\alpha - 1)^2 \left(\frac{\dot{a}}{a}\right)^2 + (\alpha - 1) \partial_t \left(\frac{\dot{a}}{a}\right) \right) y\, \delta\rho + \partial_y\, \delta p + g \delta\rho = 0 \end{aligned} \tag{3}$$

$$\partial_x\, \delta v_x + \partial_y\, \delta v_y = 0 \tag{4}$$

where Eq. (1) is the mass conservation equation, Eq. (2) is the x momentum equation, Eq. (3) is the y momentum equation, and Eq. (4) is the equation of state.

To solve this system, we first eliminate δv_x from equation Eq. (2). This is accomplished by substituting from Eq. (4), then operating on the resulting equation with ∂_x, then substituting Eq. (4) again. The result is

$$\rho_0 \left(-\partial_t \, \partial_y \, \delta v_y + \alpha \frac{\dot{a}}{a} \partial_x \left(x \, \partial_y \, \delta v_0 \right) \right) - (\alpha - 1) \frac{\dot{a}}{a} \, y \, p_y^2 \, \delta v_y + \alpha \frac{\dot{a}}{a} \partial_y \, \delta v_y$$

$$+ \, \partial_x \left(\delta \rho \, h(\alpha, x, a) \right) + \partial_x^2 \, \delta p = 0 \tag{5}$$

where $h(\alpha, x, a) = \left(\alpha^2 \left(\frac{\dot{a}}{a} \right)^2 - \alpha \partial_t \left(\frac{\dot{a}}{a} \right) \right) x.$

Now we operate on Eq. (5) with ∂_y and on Eq. (3) with ∂_x^2. We then get two results for $\partial_x^2 \, \partial_y \, \delta p$. Equating them gives

$$\partial_y \left\{ \rho_0 \left(-\partial_t \, \partial_y \, \delta v_y + \alpha \frac{\dot{a}}{a} \partial_x (x \, \partial_y \, \delta v_y) - (\alpha - 1) y \frac{\dot{a}}{a} \partial_y^2 \, \delta v_y + \alpha \frac{\dot{a}}{a} \partial_y \, \delta v_y \right) \right\}$$

$$- \rho_0 \, \partial_x^2 \left\{ \partial_t \, \delta v_y - \alpha \frac{\dot{a}}{a} x \, \partial_x \, \delta v_y + (\alpha - 1) \frac{\dot{a}}{a} y \, \partial_y \, \delta v_y + (\alpha - 1) \frac{\dot{a}}{a} \delta v_y \right\}$$

$$+ \, \partial_y \, \partial_x \left(\delta \rho \, h(\alpha, x, a) \right) - \partial_x^2 \left[\left(n(\alpha, y, a) + g \right) \delta p \right] = 0 \tag{6}$$

where $n(\alpha, y, a) = \left((\alpha - 1)^2 \left(\frac{\dot{a}}{a} \right)^2 + (\alpha - 1) \partial_t \left(\frac{\dot{a}}{a} \right) \right) y.$

Thus far we have assumed: incompressible flow, $v_{z0} = \frac{\dot{a}}{a} z$, $v_{x0} = -\alpha \frac{\dot{a}}{a} x$, $v_{y0} = (\alpha - 1) \frac{\dot{a}}{a} y$; pressure to stabilize against gravity and to produce the above zero-order flows; $\rho_0 = \rho_0(y)$; and $\delta v_z = 0$. We now specify the system further. We assume two fluids with a sharp interface at $y = 0$. Each fluid has a constant density:

$$\rho(y) = \begin{cases} \rho_{0+} & y > 0 \\ \rho_{0-} & y < 0 \, . \end{cases}$$

Equation (6) is now greatly simplified. Away from the interface, $\partial_y \rho_0 = 0$. Also, $\delta\rho = 0$ except near the interface. This can be seen from the mass conservation equation

$$\partial_t \rho + \mathbf{v} \cdot \nabla \rho = 0 \longrightarrow \partial_t \, \delta\rho + \delta\mathbf{v} \cdot \nabla \rho_0 + \mathbf{v}_0 \cdot \nabla \delta\rho = 0 \ .$$

In an incompressible fluid, density perturbations can arise only from advection or from some insertion of density fluctuations as an initial condition. As long as $\delta\rho = 0$ everywhere initially, it will, in this system, remain zero everywhere except near the interface.

So, away from the interface, Eq. (6) becomes

$$\begin{aligned} -\partial_t \, \partial_y^2 \, \delta v_y + \alpha \frac{\dot a}{a} \, \partial_x(x \, \partial_y^2 \, \delta v_y) - (\alpha - 1)\frac{\dot a}{a} \, \partial_y^2 \, \delta v_y - (\alpha - 1) \frac{\dot a}{a} \, y \, \partial_y^3 \, \delta v_y + \alpha \frac{\dot a}{a} \, \partial_y^2 \, \delta v_y \\ -\partial_t \, \partial_x^2 \, \delta v_y + \alpha \frac{\dot a}{a} \, \partial_x^2(x \, \partial_x \, \delta v_y) - (\alpha - 1) \frac{\dot a}{a} \, y \, \partial_x^2 \, \partial_y \, \delta v_y + (\alpha - 1)\frac{\dot a}{a} \partial_x^2 \delta v_y = 0 \ . \end{aligned} \qquad (7)$$

Inspection will show that it is solved by any δv_y such that

$$\partial_x^2 \, \delta v_y = -\partial_y^2 \, \delta v_y \ .$$

Now we go back to Eq. (7) and integrate it across the interface

$$\left[\rho_0 \left(-\partial_t \, \partial_y \, \delta v_y + \alpha \frac{\dot a}{a} \, \partial_x(x \, \partial_y \, \delta v_y)\right) + \alpha \frac{\dot a}{a} \, \partial_y \, \delta v_y\right]_-^+ = \partial_x^2 \int_{-\epsilon}^{+\epsilon} \delta\rho \, g \, dy \ . \qquad (8)$$

What we would eventually like is a solution of the form $\delta v_y = \delta v_{y0}(t) f(x, y, t)$. If this can be obtained, the final equation for $\delta v_{y0}(t)$ will have no x dependent terms. So, we eliminate them here, if possible.

This is possible if $\delta v_y = e^{\pm ika^\alpha x} \delta v_{y0}(y, t)$, for then

$$-\partial_t \, \partial_y \, \delta v_y = \left(\pm ika^\alpha \frac{\dot a}{a} \, \alpha x e^{\pm ika^\alpha x} \partial_y \, \delta v_{y0} + e^{\pm ika^\alpha x} \partial_t \, \partial_y \, \delta v_{y0}\right)$$

and

$$\alpha \frac{\dot a}{a} \, \partial_x(x \, \partial_y \, \delta v_y) = \alpha \frac{\dot a}{a} \, \partial_y \, \delta v_{y0} \, e^{\pm ika^\alpha x} \pm ika^\alpha \frac{\dot a}{a} \, \alpha x \, e^{\pm ika^\alpha x} \, \partial_y \, \delta v_{y0} \ .$$

So Eq. (8) becomes

$$e^{\pm ika^\alpha x}\left[\rho_0\left(-\partial_t\,\partial_y\,\delta v_{y0}+2\alpha\,\frac{\dot a}{a}\,\partial_y\,\delta v_{y0}\right)\right]_-^+=\partial_x^2\int_{-\epsilon}^{\epsilon}\delta\rho\,g\,dy\ .$$

We assume the same x dependence for $\delta\rho$, so

$$\left[\rho_0\left(-\partial_t\,\partial_y\,\delta v_{y0}+2\alpha\,\frac{\dot a}{a}\,\partial_y\,\delta v_{y0}\right)\right]_-^+=-k^2\,a^{2\alpha}\int_{-\epsilon}^{\epsilon}\delta\rho_0\,g\,dy\ . \tag{9}$$

Now, remembering $\partial_x^2\,\delta v_y=-\partial_y^2\,\delta v_y$, we see

$$\delta v_{y0}=\delta v_{y0}(t)e^{\pm ka^\alpha y}\ .$$

If $\delta v_y=0$ at $y=\pm\infty$,

$$\delta v_{y0}=\delta v_{y0}(t)e^{-|ka^\alpha y|}\ .$$

So Eq. (8) is now

$$-(\rho_{0+}+\rho_{0-})|k|\left(-\partial_t+2\alpha\,\frac{\dot a}{a}\right)(|a^\alpha|\delta v_{y0}(t))=-k^2\,a^{2\alpha}\int_{-\epsilon}^{\epsilon}\delta\rho_0\,g\,dy\ .$$

Lastly, we eliminate $\delta\rho$ by making use of the mass conservation equation Eq. (1). The x dependence of $\delta\rho$ causes the x dependent terms in the equation to cancel. We then integrate over the interface:

$$\partial_t\int_{-\epsilon}^{\epsilon}\delta\rho_0\,dy+(\alpha-1)\frac{\dot a}{a}\int_{-\epsilon}^{\epsilon}y\,\partial_y\,\delta\rho_0\,dy+\delta v_y(t)(\rho_{0+}-\rho_{0-})=0\ .$$

Since the interface is at $y=0$:

$$\int_{-\epsilon}^{\epsilon}y\,\partial_y\,\delta\rho_0\,dy=\int_{-\epsilon}^{\epsilon}\partial_y(y\,\delta\rho_0)dy-\int_{-\epsilon}^{\epsilon}\delta\rho_0\,dy=-\int_{-\epsilon}^{\epsilon}\delta\rho_0\,dy\ .$$

We see

$$\left[\partial_t-(\alpha-1)\,\frac{\dot a}{a}\right]\int_{-\epsilon}^{\epsilon}\delta\rho_0\,dy=-\delta v_{y0}(t)(\rho_{0+}-\rho_{0-})\ . \tag{10}$$

So

$$\left[\partial_t - (\alpha - 1)\,\frac{\dot{a}}{a}\right]\left[\frac{1}{a^{2\alpha}}\,\left(-\partial_t + 2\alpha\,\frac{\dot{a}}{a}\right)(|a^\alpha|\delta v_y)\right] = -g|k|\,\frac{\rho_{0+} - \rho_{0-}}{\rho_{0+} + \rho_{0-}}\,\delta v_y\ , \tag{11}$$

or

$$\left[\partial_t - (\alpha - 1)\,\frac{\dot{a}}{a}\right]\left[\frac{1}{a^{2\alpha}}\,\left(\partial_t - 2\alpha\,\frac{\dot{a}}{a}\right)(|a^\alpha|\delta v_{y0}(t))\right] = g|k|\,\frac{\rho_{0+} - \rho_{0-}}{\rho_{0+} + \rho_{0-}}\,\delta v_{y0}(t)\ . \tag{12}$$

This equation is exactly solvable for certain cases of α and a. For instance, if we assume no z direction expansion ($\alpha = 0$), and we assume y direction expansion proportional to time along with corresponding z direction contraction ($a = t_0/t$) then

$$(\partial_t - 1/t)\,(\partial_t\,\delta v_{y0}(t)) = g|k|\,\frac{\rho_{0+} - \rho_{0-}}{\rho_{0+} + \rho_{0-}}\,\delta v_{y0}(t)\ .$$

This equation is solved by

$$d\delta v_{y0}(t) = c_1\tau\, I_1(\tau) + c_2\tau\, K_1(\tau)$$

where $\tau = \sqrt{g|k|\,\frac{\rho_{0+} - \rho_{0-}}{\rho_{0+} + \rho_{0-}}}\ t$, and I_1 and K_1 are hyperbolic Bessel and Basset functions respectively.

The function $\tau I_1(\tau)$ represents the growing mode of the instability. Asymptotically, $I_1(\tau)$ approaches $e^\tau/\sqrt{2\pi\tau}$ so $\delta v_{y0}(t)$ approaches $c_1\sqrt{\tau/2\pi}\,e^\tau$. The instability growth relative to the overall expansion is $\propto \delta v_{y0}/\tau \propto e^\tau/\sqrt{\tau}$. In absolute terms, we have super-exponential growth. Relative to the expansion, however, we have sub-exponential growth. The relative growths of the RT instability for a static fluid and for the expanding fluid we have studied here, are plotted in Fig. 1. It might be said that the expansion causes a "relative stabilization" of the fluid. A similar relative stabilization of the RT instability has also been found for incompressible, spherically expanding fluids [9].

B. Adiabatic Flow

Similar phenomena occur in compressible fluids with overall expansion. Bernstein and Book [5] derive eigenmodes and growth rates of Rayleigh-Taylor instabilities in a spherically expanding compressible system. Their system is RT stable. That is, all density gradients point in the same direction as the pressure gradients. We will study a system which is unstable to RT perturbations by generalizing the adiabatic equation of state to allow for a spatial dependence in temperature. First, we review some of the results of Book and Bernstein. They require self-similar expansion and an equation of state $p = \widehat{p}(\rho/\widehat{\rho})^\gamma$ where $\widehat{p}, \widehat{\rho}$, and γ are constants. These conditions fix the density profile:

$$\rho_0 = \widehat{\rho}\left[1 - \frac{\widehat{\rho}(\gamma-1)}{2\widehat{p}\gamma\tau^2} r_0^2\right]^{1/(\gamma-1)}$$

and lead to an equation for the expansion rate

$$\ddot{f}\, f^{1+\nu(\gamma-1)} = \tau^{-2}$$

where ν is the dimensionality of the expansion and τ is an arbitrary constant, the choice of which will fix the zero-order expansion rate and fluid quantity profiles. The perturbation equation of motion is found to be:

$$f^{3\gamma-1}\ddot{\boldsymbol{\xi}} = \boldsymbol{\nabla}\left\{\left[\frac{\gamma\widehat{p}\tau^2}{\widehat{\rho}} - \frac{(\gamma-1)r^2}{2}\right]\boldsymbol{\nabla}\cdot\boldsymbol{\xi}\right\} - \mathbf{r}\cdot\boldsymbol{\nabla}\boldsymbol{\xi} - \mathbf{r}\times(\boldsymbol{\nabla}\times\boldsymbol{\xi}) \qquad (13)$$

where $\boldsymbol{\xi}$ is the Lagrangian perturbation and all $\mathbf{r}$'s and $\boldsymbol{\nabla}$'s refer to the initial positions of each fluid element.

If we perturb the fluid in an incompressible, irrotational manner, then Eq. (13) is simplified to

$$f^{3\gamma-1}\ddot{\boldsymbol{\xi}} = -\mathbf{r}\cdot\boldsymbol{\nabla}\boldsymbol{\xi}\ . \qquad (14)$$

Solution of this equation leads to $\boldsymbol{\xi} = \boldsymbol{\nabla}\mathcal{X}(\mathbf{r},t)$, where $\mathcal{X}(\mathbf{r},t)$ solves the equations

$$\mathcal{X} = \sum_{\ell m}\left[X_+(\ell)r^\ell + X_-(\ell)r^{-\ell-1}\right]Y_{\ell m}$$

and

$$f^{3\gamma-1}\ddot{X}_{\pm} = [3/2 \mp (\ell + 1/2)]\, X_{\pm}\ .$$

Bernstein and Book then show that all possible modes (compressible and rotational included) grow no faster than the rate of overall expansion when $\gamma \neq 1$. When $\gamma = 1$, only the incompressible, irrotational modes grow faster than overall expansion. In this case, X_{+}/f and X_{-}/f diverge as $t \to \infty$, but they diverge extremely slowly.

Of interest here is the behavior of R-T modes in a decidedly unstable system; i.e. $\partial\rho/\partial r < 0$ and acceleration points inward, slowing the expansion. Such a system can be described by a more general equation of state:

$$p = \widehat{p}(\mathbf{r}) \left(\frac{\rho}{\widehat{\rho}}\right)^{\gamma}$$

where $\widehat{\rho}$ is a constant and $\mathbf{r}$ is the initial position of a fluid element. In other words, $\widehat{\rho}$ is the same for all fluid elements, whereas $\widehat{p}$ can vary from element to element but $\widehat{p}$ for a given element never changes. We now have the freedom to choose initial conditions such that we have a dense fluid surrounded by a thinner fluid, with higher pressure in the thinner fluid, thus slowing expansion of the system. The equation of expansion then becomes

$$\ddot{f}\, f^{1+\nu(\gamma-1)} = -\tau^{-2}\ . \tag{15}$$

The equation of motion of the perturbations is

$$f^{2+\nu(\gamma-1)}\ddot{\boldsymbol{\xi}} = \boldsymbol{\nabla}\left(\tau^2 \frac{\partial p_0}{\partial \rho_0} \boldsymbol{\nabla}\cdot\boldsymbol{\xi}\right) + \mathbf{r}\times(\boldsymbol{\nabla}\times\boldsymbol{\xi}) + \mathbf{r}\cdot\boldsymbol{\nabla}\boldsymbol{\xi} - \tau^2 \frac{p_0}{\rho_0}\boldsymbol{\nabla}\cdot\boldsymbol{\xi}\boldsymbol{\nabla}\,\ell n\widehat{p}$$

where again $\mathbf{r}$ and $\boldsymbol{\nabla}$ refer to the initial positions of each fluid element and p_0 are the original unperturbed pressure and density of each element.

We find $f(t)$ from Eq. (15). Reduction of order of this equation gives

$$\left(\frac{df}{dt}\right)^2 - \left(\frac{df}{dt}\right)^2_{t=0} = -\frac{2}{3-3\gamma}\left(f^{3-3\gamma}(t) - f^{3-3\gamma}(0)\right)\frac{1}{\tau^2} \quad , \text{ for } \quad \gamma \neq 1\ .$$

If we choose $f(0) = 1$ and $df/dt \to 0$ as $t \to \infty$ ("minimal escape velocity," speaking figuratively) then

$$f = \left(\frac{3\gamma - 1}{[2(3\gamma-3)]^{1/2}} \frac{t}{\tau} + 1 \right)^{\frac{2}{3\gamma-1}} \quad , \text{ for } \quad \gamma \neq 1 \ .$$

If we look at incompressible irrotational modes again, we find

$$f^{3\gamma-1}\ddot{X}_+ = (\ell - 1)X_+/\tau^2$$

$$f^{3\gamma-1}\ddot{X}_- = (\ell - 2)X_-/\tau^2 \ .$$

A change of variable $t' = \frac{3\gamma - 1}{[2(3\gamma-3)]^{1/2}} \frac{t}{\tau} + 1$ gives

$$t'^2 \ddot{X}_+ = \frac{(\ell - 1)}{\alpha^2} X_+ \tag{16}$$

$$t'^2 \ddot{X}_- = \frac{(-\ell - 2)}{\alpha^2} X_- \tag{17}$$

where $\alpha = \frac{3\gamma - 1}{[2(3\gamma-3)]^{1/2}}$. Eqs. (16) and (17) are solved by $X = t'^n$ where n solves the equations

$$n(n-1) = \frac{\ell - 1}{\alpha^2} \qquad \text{or} \qquad n(n-1) = \frac{-\ell - 2}{\alpha^2} \ .$$

Note that, for $\ell - 1/\alpha^2 \geq 0$, X_+ has one growing mode $(t'^n, n > 0)$ and one decaying mode $(t'^n, n < 0)$. So, we have unstable modes, but they grow as power laws of t'. The instability growth rate is sub-exponential, both relative to the overall expansion and in absolute terms.

Should $(\ell - 1)/\alpha^2 < 0$ or, as is always the case, $(-\ell - 2)/\alpha^2 < 0$, the solutions to the equations take the form t^{a+ib} and t^{a-ib}. It can be shown that $a = 1$ so the real solutions that can be constructed from these solutions are:

$$t \cos(b\ell n(t)) \quad , \quad t \sin(b\ell n(t)) \ .$$

We have analytically studied two different fluid systems undergoing overall expansion (with necessary contraction in the incompressible case). In each case, the growth of the R-T instability is sub-exponential relative to the overall expansion. This qualitative conclusion is an initial test which we have applied to our algorithm for simulating expanding fluid and MHD systems. We review the algorithm and the simulation results in the next section.

III. Computational Results

A. Algorithm

Our simulation model is a non-resistive, adiabatic, two-dimensional MHD-plasma. We simulate the plasma with a Lax-Wendroff-type algorithm used by Nakagawa, Steinolfson, and Wu [10], and first developed by Rubin and Burstein [11].

To facilitate studying the R-T instability in an expanding plasma, the algorithm is altered so that each grid point becomes a co-moving point, tracking a predetermined zero-order expansion of the plasma. The advantage of this approach is that, if we deal with an expanding system, it will prevent an expanding system from growing beyond the computational boundaries. Conversely, it will prevent a contracting system from shrinking to a size smaller than the grid spacing can handle.

To see how this alteration is made, we first look at how the MHD equations can be rewritten in terms of variables co-moving with a homogeneous expansion.

Our static coordinates are r_x, r_y, r_z, and t. They can be expressed in terms of the co-moving coordinates x, y, z, t':

$$r_x = a_x(t)x$$

$$r_y = a_y(t)y$$

$$r_z = a_z(t)z$$

$$t = t' \ .$$

The $a_i(t)$ are functions of time chosen beforehand, and are not unlike the elements of a time-dependent metric in general relativity. Since what we are doing here amounts to a simple change of variable, the $a_i(t)$ can be, in principle, anything we like. However, the most useful $a_i(t)$ will be functions that lead to spatial coordinates which track, or nearly track, the overall expansion of the system. So we determine the $a_i(t)$ beforehand by first solving the zero-order fluid or MHD equations of motion. If these equations lead to a homogeneous expansion of the fluid, *i.e.* a fluid element at (r_x, r_y, r_z) at $t = 0$ moves to $(f_x(t)r_x, f_y(t)r_y, f_z(t)r_z)$ at t, then $a_i(t) = f_i(t)$. If the fluid equations return an expansion which is not quite homogeneous, then we can still make some use of the algorithm by choosing the $a_i(t)$ to be large enough for the computational boundaries to always contain the entire system.

The velocities u, v, and w can be expressed in terms of velocities relative to the local motion of expansion:

$$u(r_x, r_y, r_z, t) = v_x(x, y, z, t') + \dot{a}_x(t')x$$

$$v(r_x, r_y, r_z, t) = v_y(x, y, z, t') + \dot{a}_y(t')y$$

$$w(r_x, r_y, r_z, t) = v_z(x, y, z, t') + \dot{a}_z(t')z \ .$$

So, for example, if (v_x, v_y, v_z) is equal to $(0, 0, 0)$ at some point (x, y, z), then, at that point, the fluid is moving exactly with the expanding coordinates.

The derivatives are rewritten

$$\partial_{r_x} = \frac{1}{a_x(t)}\frac{\partial}{\partial x}$$

$$\partial_{r_y} = \frac{1}{a_y(t)}\frac{\partial}{\partial y}$$

$$\partial_{r_z} = \frac{1}{a_z(t)}\frac{\partial}{\partial z}$$

$$\partial_t = \partial_t' - \frac{\dot{a}_x}{a_x}x\frac{\partial}{\partial x} - \frac{\dot{a}_y}{a_y}y\frac{\partial}{\partial y} - \frac{\dot{a}_z}{a_z}z\frac{\partial}{\partial z} \ .$$

The MHD equations then become

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}_r \cdot (\rho \mathbf{v}) + \sum_j \frac{\dot{a}_j}{a_j} \rho = 0 \tag{18}$$

$$\frac{\partial(\rho v_i)}{\partial t} + \boldsymbol{\nabla}_r \cdot (\rho \mathbf{v} v_i) + \frac{1}{a_i} \partial_i p + \frac{1}{4\pi} [(\boldsymbol{\nabla}_r \times \mathbf{B}) \times \mathbf{B}]_i +$$

$$\sum_j \frac{\dot{a}_j}{a_j} \rho v_i + \frac{\dot{a}_i}{a_i} \rho v_i + \rho \ddot{a}_i x_i = 0 \tag{19}$$

$$\frac{\partial B_i}{\partial t} + [\nabla_{\mathbf{r}} \times (\mathbf{v} \times \mathbf{B})]_i + B_i \sum_{j \neq i} \frac{\dot{a}_j}{a_j} = 0 \tag{20}$$

$$\frac{\partial}{\partial t} \left(\frac{p}{\gamma - 1} + \frac{\rho v^2}{2} + \frac{B^2}{8\pi} \right) + \boldsymbol{\nabla}_r \cdot \left[\mathbf{v} \left(\frac{\gamma p}{\gamma - 1} + \frac{\rho v^2}{2} \right) + \frac{1}{4\pi} \mathbf{B} \times (\mathbf{v} \times \mathbf{B}) \right]$$

$$+ \sum_j v_j \ddot{a}_j x_j \rho + \sum_j \frac{\dot{a}_j}{a_j} v_j^2 \rho + \left(\frac{\gamma p}{\gamma - 1} + \frac{\rho v^2}{2} \right) \sum_j \frac{\dot{a}_j}{a_j}$$

$$+ \frac{B_y^2 + B_z^2}{4\pi} \frac{\dot{a}_x}{a_x} + \frac{B_z^2 + B_x^2}{4\pi} \frac{\dot{a}_y}{a_y} + \frac{B_x^2 + B_y^2}{4\pi} \frac{\dot{a}_z}{a_z} = 0 \tag{21}$$

where $\boldsymbol{\nabla}_r = \left(\frac{1}{a_x} \frac{\partial}{\partial x}, \frac{1}{a_y} \frac{\partial}{\partial y}, \frac{1}{a_z} \frac{\partial}{\partial z} \right)$, and all summations are explicit, *i.e.* repeated indices do *not* imply summation. Our simulation is two-dimensional, but z direction terms have been retained here for completeness.

Now we must alter the computational algorithm to account for the "source terms" and the time-dependent coefficients appearing in the new equations. We now represent the MHD equations by

$$\partial_t U + \frac{1}{a_x} \partial_x F + \frac{1}{a_y} \partial_y G + S = 0 \, .$$

We first find mid-point values for U^{n+1}: Here, we closely follow the standard Lax-Wendroff algorithm, but we introduce two changes. First, the spatial derivatives are modified by the expansion factors a_x and a_y. For instance,

$$\frac{d}{dx} F^n \longrightarrow \frac{1}{a_x^n \Delta x} \left(F_{i+1,j}^n - F_{i,j}^n \right) \, .$$

Second, we account for the expansion source terms:

$$U^{n+1}_{i+1/2,j} = U^{n+1}_{i,j} + \Delta t \,\frac{dF^n}{dx} + \Delta t \,\frac{dG^n}{dy} - \frac{1}{2}\left(S^n_{i+1,j} + S^n_{i,j}\right) \ .$$

Midpoint values are then calculated for F^{n+1}, G^{n+1}, and S^{n+1}:

$$F^{n+1}_{i+1/2,j} = F\left(U^{n+1}_{i+1/2,j}\right) \qquad \text{etc.}$$

Lastly, grid point values are calculated for U^{n+1}:

$$U^{n+1}_{i,j} = U^n_{i,j} - \frac{\Delta t}{2}\left\{\left(\frac{\partial F}{\partial x}\right)^{n+1}_{i,j} + \left(\frac{\partial F}{\partial x}\right)^{n}_{i,j}\right\} - \frac{\Delta t}{2}\left\{\left(\frac{\partial G}{\partial y}\right)^{n+1}_{i,j} + \left(\frac{\partial G}{\partial y}\right)^{n}_{i,j}\right\}$$
$$- \frac{1}{2}\left(S^n_{i,j} + \overline{S}^{n+1}_{i,j}\right)$$

where $\overline{S}^{n+1}_{i,j} = \frac{1}{4}\left(S^{n+1}_{i+1/2,j} + S^{n+1}_{j+1/2}, S^{n+1}_{i-1/2,j} + S^{n+1}_{i,j-1/2}\right)$.

B. Simulation Results

We model a non-resistive MHD plasma. The adiabatic constant is $\gamma = 5/3$. The plasma is contained in a box with reflective boundary conditions. Physically, it is twice as long in the y direction as in the x direction. One hundred twenty two grids are evenly spaced along the y direction; thirty are spaced along the x direction. A dense plasma occupies the lower third of the box, a thin plasma the upper two thirds. The interface between them is a transition region with a width of about 10 grid spaces. A magnetic field $B(y)\hat{z}$ is imposed; it is relatively weak in the thick plasma and strong in the thin plasma.

In detail, the pressure, density, and magnetic field strength profiles are:

$$p = p_0 - p_1 \tanh\left(\frac{y - y_0}{\ell}\right)$$

$$\rho = \rho_0 - \rho_1 \tanh\left(\frac{y - y_0}{\ell}\right)$$

$$\mathbf{B} = \left(B_0^2 + B_0^2 \tanh\left(\frac{y - y_0}{\ell}\right)\right)^{1/2} \hat{z} \ .$$

y_0 is the center of the transition region.. ℓ is on the order of half the width of the transition region. p_0 and p_1 are chosen so that the thick region pressure (p_{tk}) is seven times that of the thin region pressure (p_{tn}). The variation in ρ is exactly proportional to the variation in p. The total pressure ($p + B^2/8\pi$), however, increases as we cross the transition region from the thick plasma into the thin plasma:

$$p_{tn} + \frac{2B_0^2}{8\pi} = 1.2 p_{tk} .$$

The plasma β in the thin region is 0.11. The overall $\beta (= 8\pi\, p_{tk}/2B_0^2)$ is 0.77.

The imbalance in the total pressure creates a force across the transition region directed toward the thick plasma. This system is R-T unstable.

To test the code, we first studied the growth of initial R-T perturbations in a plasma without expansion. Velocity perturbations took the form

$$\left.\begin{aligned} u &= v_0 \sin\left(\frac{2\pi x}{\lambda}\right)\left(\frac{y}{\ell_{tk}}\right) \\ v &= -v_0 \cos\left(\frac{2\pi x}{\lambda}\right)\left(\frac{y}{\ell_{tk}}\right) \end{aligned}\right\} \text{in the thick plasma} \tag{22}$$

$$\left.\begin{aligned} u &= v_0 \left(1 - 2\frac{y - \ell_{tk}}{\ell_{tr}}\right) \sin\left(\frac{2\pi x}{\lambda}\right) \\ v &= -v_0 \cos\left(\frac{2\pi x}{\lambda}\right) \end{aligned}\right\} \text{in the transition region} \tag{23}$$

$$\left.\begin{aligned} u &= v_0 \left(1 - \frac{y - \ell_{tk} + \ell_{tr})}{\ell_{tn}}\right) \sin\left(\frac{2\pi x}{\lambda}\right) \\ v &= -v_0 \left(1 - \frac{\ell_{tk} + \ell_{tr})}{\ell_{tn}}\right) \cos\left(\frac{2\pi x}{\lambda}\right) \end{aligned}\right\} \text{in the thin plasma} \tag{24}$$

(25)

where ℓ_{tk} is the size of the thick plasma, ℓ_{tr} is the size of the transition region, and ℓ_{tn} is the size of the plasma. λ, the perturbation wavelength, was set to twice the length of the x direction wall of the simulation box — v_0 is about 0.1 of the sound speed in the thick plasma . The transition region was also given a sinusoidal bend of the same wavelength having an amplitude of about one grid space.

The R-T instability grows qualitatively as standard theory and experimental results have indicated. Namely, the sinusoidal perturbation of the interface grows and then becomes cycloid-like, sending "spikes" of dense plasma into the thin plasma [12].

We have also compared the early growth rate of the instability to a "ball park" obtained from the standard expression for the growth rate of an incompressible, two-fluid R-T instability in gravity, namely:

$$n = \sqrt{gk \frac{\rho_2 - \rho_1}{\rho + 2 + \rho_1}} ,$$

where k is the perturbation wavenumber, g is the gravitational acceleration, and ρ_2 and ρ_1 are the fluid densities above and below the interface, respectively. For g, we substitute a number approximately equal to the average acceleration in the transition region:

$$g = \frac{(2B_0^2/8\pi + \Delta p)/\ell_{tr}}{\rho(y_{tr})}$$

where $\Delta p < 0$. Given that

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{2\ell_x} = \frac{2\pi}{\ell_y} ,$$

we find

$$n_{\text{class}} = 5.17 \frac{c_s}{\ell_y} ,$$

where c_s is the dense plasma sound speed and ℓ_y is the y direction box length. This is, indeed, within the "ball park" of the simulation results:

$$n = 3.05 \frac{c_s}{\ell_y} ,$$

or

$$n = 0.58 n_{\text{class}} .$$

The main result of interest from these simulations is the somewhat stabilizing effect of the expansion on the instability. Let the amplitude of the "bend" in the interface be denoted by $I(t)$. Then the amplitude relative to expansion is

$$s(t) = I(t)/a_y(t) .$$

If $a_y(t)$ is a constant, then our present problem reduces to the static case with some exponential growth rate

$$I(t) \propto e^{nt} \propto s(t) .$$

However, if $a_y(t)$ is a function of time, such as $\alpha t + 1$, the simulations shows that $s(t)$ is some sub-exponential function of time. The time evolutions of the density contours of static and expanding plasmas is shown in Fig. 2. Note that, after equal times, the interface of the expanding plasma remains much more planar than that of the static plasma. The $s(t)$ for two different expansions, $a(t) = t + 1$ and $a(t) = 4t + 1$ (t being normalized to the time needed for a wave traveling at the thick plasma sound speed to traverse the y extent of the computational grid) along with the results of the static plasma are shown in Fig. 3. The plasma instability growth is slowed relative to the overall expansion of the plasma. In particular, the early-time growth of the instability is slowed by 7% when $a(t) = t + 1$ and by 14% when $a(t) = 4t + 1$. Also, the instability appears to saturate and enter a non-linear regime earlier with faster expansion. When $a(t) = 1$, fall-off from linear growth begins at about 0.17 of the classical growth time. This occurs at about 0.15 of the classical growth time when $a(t) = t + 1$ and at about 0.1 of the classical growth time when $a(t) = 4t + 1$.

We take a moment here to review the relationships between the two systems we studied in Sec. 2 and the system we have simulated. In each of the Sec. 2 systems, the instability growth rate, relative to the overall expansion, is sub-exponential. However there are significant differences in the makeup of the two systems: The incompressible instability under consideration is driven by a constant gravitational force while the expansion (and accompanying contraction) of the fluid is chosen to be constant. On the other hand, the compressible instability is driven by the same pressure gradient that cause the expansion to slow down;

it is precisely this deceleration that destabilizes the system. The incompressible system has a slab geometry; the compressible system has a spherical geometry.

The system we have simulated stands "half-way between" the other two in certain respects. It is a compressible fluid in a slab geometry. Its overall expansion rate in the y direction is chosen to be constant but, since the fluid is compressible, there is no need for any contraction. The instability driving force is created by a pressure imbalance at the interface of the heavy and light plasmas. This imbalance is not responsible for any change in the overall expansion rate but, as in the compressible case, it is depleted by the expansion. Despite the differences in these systems, fluid expansion slows RT growth in them all. This can be taken as a successfully passed, qualitative test of the simulation algorithm.

IV. Conclusions

A large number of specific expanding fluid systems give rise to slowed, or even sub-exponential relative growth of Rayleigh-Taylor instabilities. We have presented analytical results for two new systems which exhibit this phenomenon: an incompressible two-fluid system and an adiabatic fluid system. The adiabatic system is a generalization of Book and Bernstein [5]. They studied an expanding adiabatic fluid with a pressure completely dependent on fluid density. This led to a spherical system with both density and pressure falling off with radius. The density and pressure gradients were necessarily in the same direction and the system was RT stable. We have introduced a spatial variation of temperature into the equation of state, making possible a rise in pressure together with a fall-off in density. This makes the system RT unstable.

Results from a new MHD fluid code written specifically for simulating fluids undergoing overall expansion (or contraction) confirm this result for a simple slab geometry system undergoing a simple linear expansion.

It should be noted that fluid expansion will not always lead to stabilization of instabilities.

Modes developing along magnetic field lines, called Parker modes or ballooning instabilities, behave in a marked way in an expanding gas in their nonlinear stages [13]. Expansion could contribute to nonlinear destabilization through mass motion along the field lines.

The computational algorithm presented here may find application in the study of astrophysical phenomena, such as certain stages of supernova behavior. For instance, it might be used to study the effects of the interstellar medium or magnetic field swept up in the leading edge of the explosion. It might also be usefully applied to problems in inertial confinement fusion. For example, fluid expansion and contraction might significantly affect the results of such research as Emery *et al.* [1] and Kull [2].

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References

[1] M. Emery, J.P. Dahlburg, J.H. Gardner, Phys. Fluids **31**, 1007 (1988).

[2] H.J. Kull, Phys. Fluids B **1**, 170(1989).

[3] D. Book and I. Bernstein, Phys. Fluids **22**, 79 (1979).

[4] R.A. Chevalier, Ap. J. **207**, 872 (1976).

[5] I. Bernstein and D. Book, Ap. J. **225**, 633 (1978).

[6] K. Holcomb and T. Tajima, Phys. Rev. D **40**, 3809 (1989).

[7] F.S. Felber, Phys. Fluids **25**, 643 (1982).

[8] T. Tajima, S. Eliezer, R. Kulsrud, *AIP Conf. Proc. 181*, "Muon Catalyzed Fusion," eds. Jones S.E. *et al.*, p. 423.

[9] K.O. Mikaelin, Phys. Rev. Lett. **65**, 992 (1990).

[10] Y. Nakagawa, R.S. Steinolfson, S.T. Wu, Solar Physics **47**, 193 (1976).

[11] E.L. Rubin, S.Z. Burstein, J. Comp. Phys. **2**, 178 (1967).

[12] G. Tryggvason, J. Comp. Phys. **75**, 253 (1988); B.S. Daly, Phys. Fluids **12**, 1340 (1969).

[13] K. Shibata, T. Tajima, and R. Matsumoto, Phys. Fluids B **2**, 1989 (1990).

Figure Captions

1. Growth of the two-fluid RT instability for a static fluid (dashed curve) and the expanding fluid studied in this section. $s(t)$ is the growth of the instability relative to the instantaneous size of the fluid system. τ is time in terms of the instability growth time in the static fluid: $(gk(\rho_+ - \rho_-)/(\rho_+ + \rho_-)^{1/2}$.

2. a) Density contours of the expanding MHD plasma undergoing RT unstable motion. Time is in terms of the classically calculated static RT growth time $|\nabla p_{\rm tot}/\rho(y_{tr})|k(\rho_{tk} - \rho_{tn})/(\rho_{tk} + \rho_{tn})$ evaluated at $t = 0$. Expansion is in the y direction only. The expansion factor is $a_y(t) = 4.t/t_{cs} + 1$, where t_{cs} is the t length of the box in the y direction (at $t = 0$) divided by the thick region soundspeed (at $t = 0$). Initial thin region density (at the top of the box) was 1/7 that of the thick region. Proportions remain the same through the simulation, but absolute density is cut in each frame by a factor of $1/a_y(t)$. b) Density contours of the static MHD plasma undergoing RT unstable motion. Note much larger relative perturbation growth for static plasma.

3. Relative growth of RT instabilities from simulations of static plasma (solid curve) expanding plasma with $a_y(t) = t/t_{cs} + 1$ (long dashes) and expanding plasma with $a_y(t) = 4t/t_{cs} + 1$ (short dashes). Time is in terms of classically calculated RT growth time for static plasma.

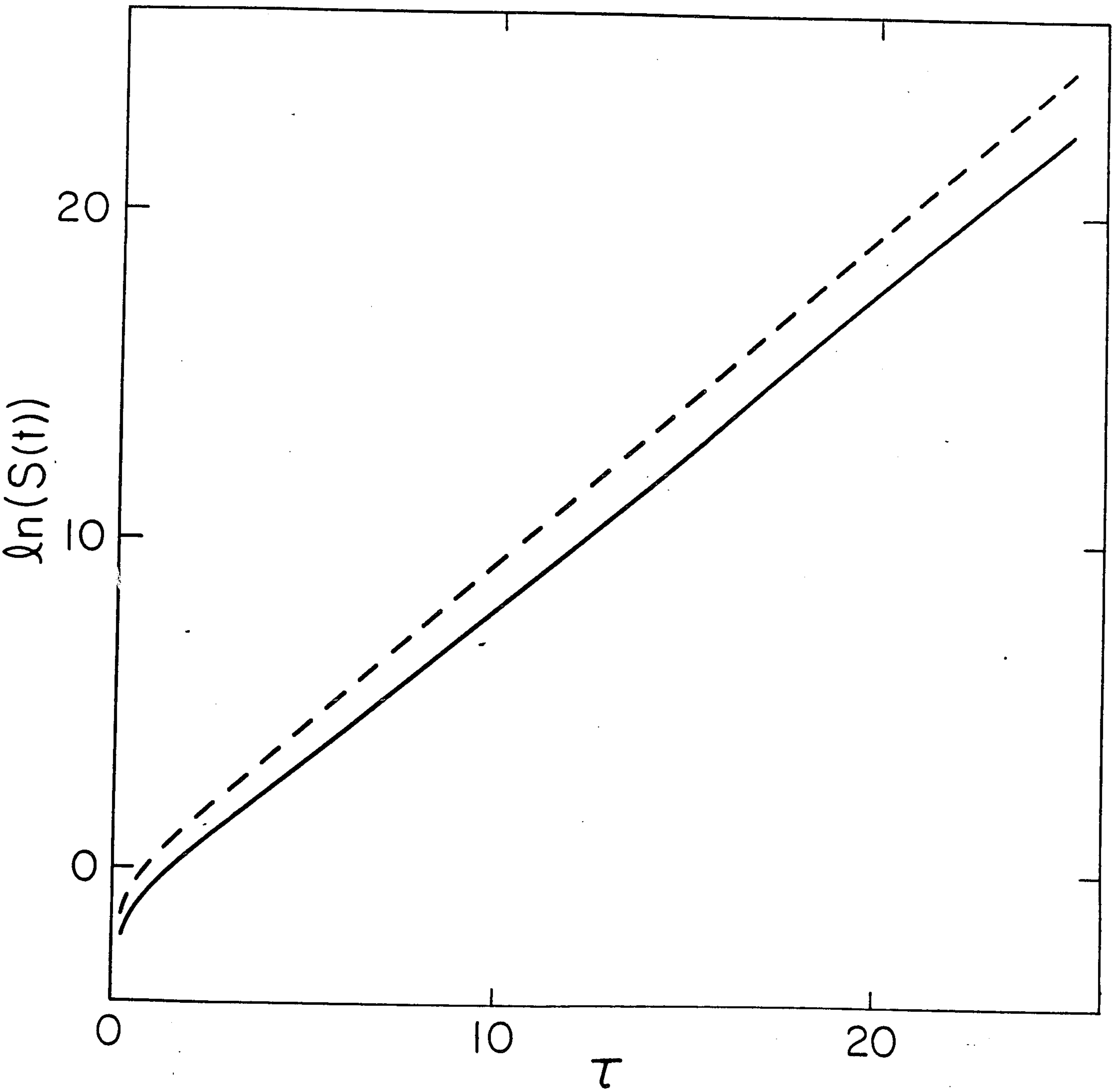

Fig. 1

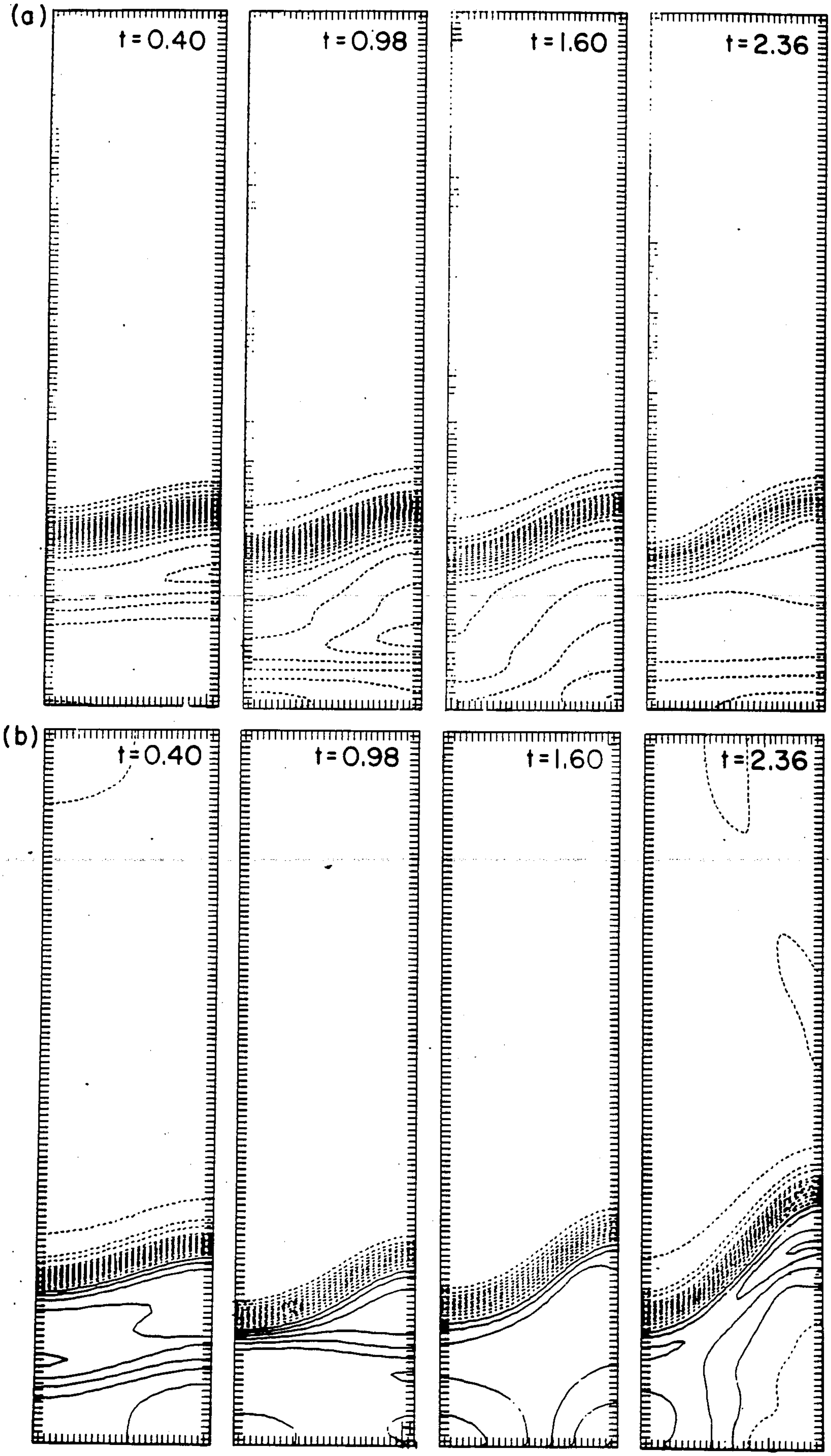

Fig. 2

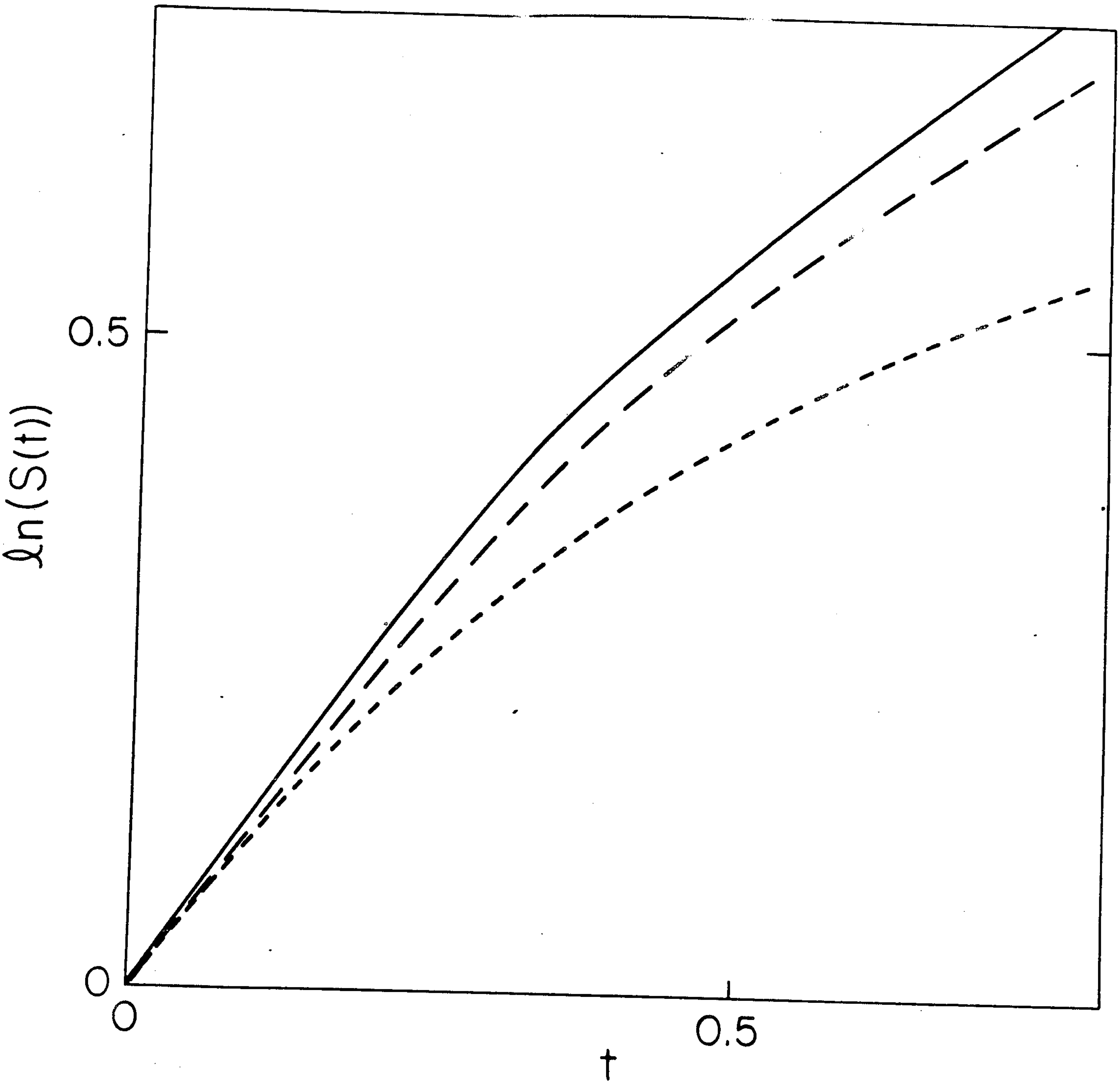

Fig. 3