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Self-Consistent Theory for Ion Gyroresonance

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Abstract

For describing ion gyroresonance processes, a complete set of self-consistent Vlasov-Maxwell equations is derived by systematically transforming a self-consistent action principle from particle coordinates to guiding-center/oscillation-center coordinates. They include the oscillation-center Vlasov equation; the equations for wave-particle resonant interactions; and Maxwell equations for the background electromagnetic fields. These equations satisfy local conservation laws for energy, momentum, and angular momentum, constructed by using the Noether algorithm. A heuristic interpretation based

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on the theory of linear mode conversion in ray phase space is also presented for ion gyroresonance processes, which suggests a method for obtaining analytic solutions in general geometry.

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I. Introduction

The main purpose of this paper is to develop a general formalism for the self-consistent description of ion gyroresonance processes. Ion gyroresonance is the physical mechanism behind the so-called *ion cyclotron resonant heating* scheme, which is an important method for raising the plasma temperature in a tokamak in order to achieve thermonuclear fusion. Besides its practical significance, ion gyroresonance is also very interesting as a basic physical problem. The literature on this subject is huge and rapidly growing.[1–3] In the present work we derive a complete set of self-consistent equations that describe not only the coupling between the resonant particles and the wave fields, but also the slower evolution of the background plasma and electromagnetic fields. We demonstrate the self-consistency of these equations by deriving the conservation laws for energy, momentum, and angular momentum. We also propose a heuristic interpretation for the ion gyroresonance processes based on the theory of linear mode conversion in ray phase space. This interpretation allows us to obtain analytic solutions of the gyroresonance coupling equations.[4–7]

The results presented in this paper are the fruit of a long line of development made in the last decade by our group and many other workers. A historical account of the evolution of ideas and an outline of techniques have been given in Ref. 8. Below we shall briefly discuss some of the important mathematical methods that are used in this paper so as to provide some background materials. They are: (1) the self-consistent action principle for the Vlasov-Maxwell equations; (2) the Lagrangian guiding-center and oscillation-center theory; (3) resonance crossings; (4) the ray phase-space eikonal (also called WKB) theory; and (5) the theory of linear mode conversion in ray phase space. For practicality we shall cite only references that are of immediate relevance to our present work and, if possible, refer readers to review articles where they may find more complete bibliography.

The *self-consistent action principle* for the Vlasov-Maxwell equations has been known for a long time.[9,10] For the guiding center-Maxwell system (without perturbation) two different action principles were introduced by Similon[11] and Pfirsch.[12,13] Similon's approach is based on the particle orbit dynamics (the Lagrangian picture) and uses a Lorentz-covariant formulation, while Pfirsch's approach is based on the Hamilton-Jacobi equation (thus closer to the Eulerian picture). Both authors use Littlejohn's phase-space guiding-center Lagrangian.[14] Boghosian[15] extended Similon's method to include nonresonant interactions between particles and an eikonal perturbation; he also uses the covariant notation. In this paper we extend Boghosian's work to include the resonant particles. We shall follow Similon's methodology, but since we are aiming at providing formulas that are useful to plasma physicists (in particular fusion researchers), we shall use more conventional notations and consider only nonrelativistic particles. A comprehensive discussion of these two and several other versions of self-consistent action principles for Vlasov-Poisson and Vlasov-Maxwell equations (and their relationships to each other) can be found in Ref. 16. Action principles provide a natural vehicle for carrying out a *systematic approximation scheme.*[17] When approximations are performed on the action, the effects will spread out into the evolution equations after the variation, and self-consistency is attained automatically. Furthermore, Noether's theorem provides an algorithm that makes the construction of conservation laws almost routine.

The *Lagrangian theory of guiding centers* was developed by Littlejohn.[14] The theory of *oscillation centers* was first introduced by Dewar[18] and further developed by Cary[19] for the unmagnetized plasmas, and later generalized to the magnetized plasmas by Grebogi et al.[20,21] In this process a powerful new method, called the *Lagrangian Lie transform*, has also been developed[22,23] (also see Appendix A). It allows us to carry out systematic perturbation expansions using noncanonical variables such as the guiding-center coordinates. With the exception of Ref. 18, all of the above references (including 15) have only dealt with

the nonresonant wave-particle interactions. McDonald et al[24] extended Dewar's work to magnetized plasmas with fixed background fields, derived the coupled plasma-wave kinetic equations in which the resonance effects appear as dissipation, and proved the H-theorem for these equations. In this paper we continue this line of development by including the resonant wave-particle interactions and the self-consistent background electromagnetic fields.

The idea of *resonance crossing* is based on the following observation:[25] In the particle phase space there are resonance regions where particles will suffer jumps in their adiabatic invariants when they cross. Outside the resonance regions particles interact with the wave fields adiabatically. For small perturbations, the jump in adiabatic invariants can be calculated by integrating the perturbed Hamiltonian along the unperturbed particle orbits across the resonance region. This local averaging procedure leads to *window functions*[18] that devide the particle phase space into resonant and nonresonant regions in a natural way. The validity criteria for the resonance crossing treatment are given in Ref. 25. For ion gyroresonance in an inhomogeneous magnetic field, the gyroresonance regions are usually well-separated due to the spatial variation of the ion gyrofrequency.

The above three tools are essentially sufficient for us to derive the set of self-consistent equations that govern the wave-particle resonant interactions. The following two methods pertain mainly to the solving of the linearized gyroresonance coupling equations.

The *ray phase space (x-k space) eikonal theory* was developed by McDonald[26] to overcome the caustic singularity problem in the traditional x-space eikonal theory. The use of *Wigner functions* allows one to deal with wave equations in a representation independent fashion, both for coherent and incoherent perturbations. The method of *Weyl symbol calculus* systematizes the asymptotic analysis.[27,28] In our calculations we will be using the Fourier representation for the wave fields; but the end results are expressed in terms of Wigner functions so that any other representations can be substituted into the final formulas.

Similarly to the ray phase space eikonal theory on which it is based, the *ray phase space mode conversion theory* was developed as a generalization of its traditional x-space counterpart.[29,30] Often the operators in the linear wave coupling equations can be simplified by canonical transformations of the ray phase space coordinates, thus enabling us to construct analytic solutions in the new coordinates. The theory of *metaplectic transformations* was developed by Littlejohn[31] to describe the transformation of the *wave functions* themselves as the ray phase space coordinates undergo a linear canonical transformation. The two taken together allows one to obtain analytic solutions of single mode conversion problems. Friedland[32,33] also developed the *congruent reduction algorithm* as a systematic way of reducing the number of components of the wave fields. We interpret ion gyroresonance processes as *pairwise* linear mode conversion, so as to apply these powerful methods to obtain analytic solutions. The notion of pairwise mode conversion was first introduced by Cairns and Lashmore-Davies.[34,35]

This paper is organized as follows. In Section II we review the self-consistent action principle for the Vlasov-Maxwell equations in particle coordinates. In Section III we transform the particle Lagrangian to the guiding-center coordinates, and obtain the unperturbed guiding-center equations of motion. In Section IV we further transform the guiding-center Lagrangian to the oscillation-center coordinates to eliminate the fast-scale wave-particle interactions in the nonresonant regions of particle phase space. The ponderomotive Hamiltonian and the Vlasov equation in the oscillation-center coordinates are obtained. In Section V we discuss the ray phase space eikonal theory and the general theory of linear mode conversion. In Section VI we derive the self-consistent Maxwell equations for the background electromagnetic fields. In Section VII we first linearize the coupling equations between the resonant particles and the wave fields, then present a heuristic interpretation for gyroresonance processes. In Section VIII we discuss other potential applications of our formalism.

Finally in Section IX we conclude. Appendix A is a review of the Lagrangian Lie transform technique.

II. Action Principle for the Vlasov-Maxwell Equations

In this section we review the phase-space action principle for the self-consistent Vlasov-Maxwell equations.[11,16] We shall start from the beginning with the particle coordinates. In this paper we consider only nonrelativistic particles. The extension to treat relativistic particles is straightforward, but is most conveniently done in Lorentz-covariant notation.[15]

Let us denote the particle coordinates in six-dimensional particle phase space by $\mathbf{z}_i = (\mathbf{r}_i, \mathbf{v}_i)$, where i is the particle label. Also denote the electromagnetic field by $(\mathbf{E}, \mathbf{B})(\mathbf{x}, t)$, and the associated four-potential by $(\mathbf{A}, \Phi)(\mathbf{x}, t)$, so that

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\Phi, \qquad \mathbf{B} = \nabla \times \mathbf{A}. \tag{1}$$

Then the *action functional* S takes the form

$$S = \sum_i \int dt\, L(\mathbf{z}_i(t), \dot{\mathbf{z}}_i(t), t) + \int d^4x\, \mathcal{L}_F(\mathbf{x}, t), \tag{2}$$

where $d^4x \equiv dt\, d^3\mathbf{x}$, L is the single-particle Lagrangian (we shall omit the particle label whenever there is no danger of confusion)

$$L = \left[m\mathbf{v} + \frac{q}{c}\mathbf{A}(\mathbf{r}, t)\right] \cdot \dot{\mathbf{r}} - \left[\frac{1}{2}m\mathbf{v}^2 + q\Phi(\mathbf{r}, t)\right], \tag{3}$$

and $\mathcal{L}_F$ is the Lagrangian density for the fields

$$\mathcal{L}_F = \frac{1}{8\pi}(\mathbf{E}^2 - \mathbf{B}^2) + \frac{1}{c}\mathbf{J}_{ext} \cdot \mathbf{A} - \rho_{ext}\Phi. \tag{4}$$

We allow for a given external source of electrical charge and current densities $(\mathbf{J}_{ext}, \rho_{ext})$. The field Lagrangian density $\mathcal{L}_F$ is standard, but the particle Lagrangian L has many variants, among them the traditional quadratic form in $\dot{\mathbf{r}}$. Our choice is called the *phase-space*

Lagrangian, which allows the most general transformations of the particle coordinates.[14] This property arises from the geometrical character of $L\,dt$ as a *one-form* in the (extended) particle phase space (see Appendix A).

It should be emphasized here that both $\mathbf{r}$ and $\mathbf{v}$ are to be varied independently. We also note that the end points of a particle's path $\mathbf{z}(t)$ are held fixed in the variation. As a consequence the particle Lagrangian L is determined only up to a total time derivative. In other words, the following transformation

$$L \mapsto L + \frac{dF}{dt} \tag{5}$$

does not affect the resulting equations of motion. This *Lagrangian gauge* freedom plays an essential role in the theory of canonical transformations.[36] It plays a similar role in the theory of Lie transforms which is a powerful generalization of canonical transformations (see Appendix A). Lie transforms will be used extensively in the next two sections.

Carrying out the variation of S with respect to the particle path $\mathbf{z}(t)$ we obtain the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{z}}}\right) - \frac{\partial L}{\partial \mathbf{z}} = 0. \tag{6}$$

Upon substituting into it the particle Lagrangian (3) we obtain the familiar Lorentz force law

$$\dot{\mathbf{r}} = \mathbf{v}, \qquad \dot{\mathbf{v}} = \frac{q}{m}\left[\mathbf{E}(\mathbf{r},t) + \frac{1}{c}\mathbf{v}\times\mathbf{B}(\mathbf{r},t)\right]. \tag{7}$$

Particles with the same mass m and charge q are identical and can be gathered into a group called *species*, and for each species (labeled by s) we define a *Klimontovich distribution* function

$$f_s(\mathbf{z},t) = \sum_{i\in s} \delta^6(\mathbf{z} - \mathbf{z}_i(t)). \tag{8}$$

Then the Lorentz equations (7) can be written equivalently as a single partial differential equation

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{r}} + \frac{q_s}{m_s}\left[\mathbf{E}(\mathbf{r},t) + \frac{1}{c}\mathbf{v} \times \mathbf{B}(\mathbf{r},t)\right] \cdot \frac{\partial f_s}{\partial \mathbf{v}} = 0. \tag{9}$$

This equation becomes the *Vlasov equation* if we ignore the discreteness of the plasma and think of f_s as a smooth solution of Eq. (9). Formally this smoothing can be done from the outset by labeling the particles with a continuous variable $\mathbf{z}_0$ instead of i, and replacing

$$\sum_i \quad \text{by} \quad \int d^6\mathbf{z}_0 \, f_{s0}(\mathbf{z}_0), \tag{10}$$

where $f_{s0}(\mathbf{z}_0)$ is a reference distribution function in the label space.

Variation of S with respect to the four-potential $(\mathbf{A}, \Phi)$ yields the Euler-Lagrange equations for the fields

$$\nabla \cdot \mathbf{D} = 4\pi\rho_f, \qquad \nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} = \frac{4\pi}{c}\mathbf{J}_f, \tag{11}$$

where $\mathbf{D}$, $\mathbf{H}$, $\mathbf{J}_f$, and ρ_f (the subscript f stands for "free") are defined as *functional derivatives* of S with respect to the explicit fields and the potentials (i.e., treating $(\mathbf{E}, \mathbf{B})$ and $(\mathbf{A}, \Phi)$ as if they were independent)

$$\begin{aligned} \mathbf{D} &\equiv 4\pi\left(\frac{\delta S}{\delta \mathbf{E}}\right), \qquad & \rho_f &\equiv -\left(\frac{\delta S}{\delta \Phi}\right), \\ \mathbf{H} &\equiv -4\pi\left(\frac{\delta S}{\delta \mathbf{B}}\right), \qquad & \mathbf{J}_f &\equiv c\left(\frac{\delta S}{\delta \mathbf{A}}\right). \end{aligned} \tag{12}$$

These functional derivatives are functions of $(\mathbf{x}, t)$, and can be calculated by the following formula

$$\delta S = \int d^4x \left[\left(\frac{\delta S}{\delta \mathbf{E}}\right) \cdot \delta\mathbf{E} + \left(\frac{\delta S}{\delta \mathbf{B}}\right) \cdot \delta\mathbf{B} + \left(\frac{\delta S}{\delta \mathbf{A}}\right) \cdot \delta\mathbf{A} + \left(\frac{\delta S}{\delta \Phi}\right)\delta\Phi\right], \tag{13}$$

where δ stands for arbitrary variations. Working it out using Eqs. (2), we find

$$\begin{aligned} \rho_f &= \rho_{ext} + \sum_s q_s \int d^6\mathbf{z}\, \delta^3(\mathbf{x} - \mathbf{r}) f_s(\mathbf{z}, t), \\ \mathbf{J}_f &= \mathbf{J}_{ext} + \sum_s q_s \int d^6\mathbf{z}\, \delta^3(\mathbf{x} - \mathbf{r})\mathbf{v} f_s(\mathbf{z}, t). \end{aligned} \tag{14}$$

Here we have replaced the sum over particles in a species by a phase-space integral weighted with the distribution function (8). Since the particle Lagrangian (3) does not depend on $\mathbf{E}$ and $\mathbf{B}$ explicitly, we have $\mathbf{D} = \mathbf{E}$, $\mathbf{H} = \mathbf{B}$. Thus we see that Eq. (11) are indeed the correct nonhomogeneous Maxwell equations. The other pair of Maxwell's equations are already implicit in Eq. (1), but we list them here for completeness:

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0. \tag{15}$$

Taking these together with Eqs. (9) and (11), we have a complete set of equations describing the self-consistent evolution of the Vlasov-Maxwell system.

Thus far we have shown that the Vlasov-Maxwell equations are derivable from a single action principle. A fundamental property of the Euler-Lagrange equations (6) and (11) is their covariance with respect to change of variables. That is to say that under coordinate transformations the *form* of these equations will remain unchanged. We will rely on this fact in the following sections to construct the self-consistent Vlasov-Maxwell equations in the guiding-center/oscillation-center representation.

The hallmark of self-consistency of the evolution equations is the conservation laws that they satisfy. The conservation laws can be derived by the Noether algorithm (see e.g. Ref. 37). The time-space translational invariance of the action (2) leads to the conservation of energy and momentum. For a system described by the Lagrangians (3) and (4), the law of energy conservation reads

$$\frac{\partial U}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{J}_{ext} \cdot \mathbf{E}, \tag{16}$$

with the energy density and energy flux density defined by

$$\begin{aligned} U &= \frac{1}{8\pi}(\mathbf{E}^2 + \mathbf{B}^2) + \sum_s \frac{m_s}{2} \int d^6\mathbf{z}\, \delta^3(\mathbf{x} - \mathbf{r}) v^2 f_s(\mathbf{z}, t), \\ \mathbf{S} &= \frac{c}{4\pi}(\mathbf{E} \times \mathbf{B}) + \sum_s \frac{m_s}{2} \int d^6\mathbf{z}\, \delta^3(\mathbf{x} - \mathbf{r}) v^2 \mathbf{v} f_s(\mathbf{z}, t). \end{aligned} \tag{17}$$

(Actually because of the presence of external sources, it is perhaps more appropriate to call this the "law of energy balance," but we will ignore this distinction.) Similarly the law of momentum conservation reads

$$\frac{\partial \mathbf{g}}{\partial t} + \nabla \cdot \mathcal{T} = -\left[\rho_{ext}\mathbf{E} + \frac{1}{c}\mathbf{J}_{ext} \times \mathbf{B}\right], \tag{18}$$

with the momentum density and momentum flux density defined by

$$\begin{aligned} \mathbf{g} &= \frac{1}{4\pi c}(\mathbf{E} \times \mathbf{B}) + \sum_s m_s \int d^6\mathbf{z}\, \delta^3(\mathbf{x} - \mathbf{r})\mathbf{v} f_s(\mathbf{z}, t), \\ \mathcal{T} &= \frac{1}{4\pi}\left[\frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)\mathbf{1} - \mathbf{E}\mathbf{E} - \mathbf{B}\mathbf{B}\right] \\ &\quad + \sum_s m_s \int d^6\mathbf{z}\, \delta^3(\mathbf{x} - \mathbf{r})\mathbf{v}\mathbf{v} f_s(\mathbf{z}, t). \end{aligned} \tag{19}$$

We have used $\mathbf{1}$ to denote the unit tensor, i.e., $\mathbf{1} \cdot \mathbf{V} = \mathbf{V} \cdot \mathbf{1} = \mathbf{V}$ for any vector $\mathbf{V}$. These conservation laws will be derived again in Section VI for the guiding-center/oscillation-center Vlasov-Maxwell equations.

III. Lagrangian Guiding Center Theory

In this section we specialize to the situation where the electromagnetic fields are composed of two parts

$$\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1, \qquad \mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1, \tag{20}$$

where $(\mathbf{E}_0, \mathbf{B}_0)$ are the background fields, slowly varying on the particle gyration scale, and $(\mathbf{E}_1, \mathbf{B}_1)$ are the perturbed fields. Since our aim is to study the gyroresonance processes, we shall assume that the perturbed fields are of high frequencies and short wavelengths, on the same scale as the particle gyration, so that the *eikonal approximation* is appropriate.

However we will not introduce the usual (coherent) eikonal ansatz, but instead use the general Fourier representation for the perturbed four-potential

$$\begin{aligned} \mathbf{A}_1(x) &= \int \frac{d^4k}{(2\pi)^4} \mathbf{A}_1(k) e^{ik\cdot x}, \\ \Phi_1(x) &= \int \frac{d^4k}{(2\pi)^4} \Phi_1(k) e^{ik\cdot x}. \end{aligned} \tag{21}$$

Here we have used the short-hand notations $x = (t, \mathbf{x})$ and $k = (-\omega, \mathbf{k})$, thus $k\cdot x = \mathbf{k}\cdot\mathbf{x} - \omega t$. Eq. (1) gives the Fourier components of the perturbed electromagnetic fields

$$\begin{aligned} \mathbf{E}_1(k) &= \frac{i\omega}{c}\mathbf{A}_1(k) - i\mathbf{k}\Phi_1(k), \\ \mathbf{B}_1(k) &= i\mathbf{k}\times\mathbf{A}_1(k). \end{aligned} \tag{22}$$

Note that the reality of the fields in x-space implies a symmetry condition in k-space, e.g. $\mathbf{E}_1^*(k) = \mathbf{E}_1(-k)$, where the star denotes complex conjugate. By "high frequency" we mean that the perturbations vanish as $\omega \to 0$. As we will see momentarily, this assumption allows us to deal with the electrical perturbation $\mathbf{E}_1$ only, resulting in significant simplifications.

Substituting $\mathbf{A} = \mathbf{A}_0 + \mathbf{A}_1$, $\Phi = \Phi_0 + \Phi_1$ into the particle Lagrangian (3) we find that $L = L_0 + L_1$, where

$$L_0 = \left(m\mathbf{v} + \frac{q}{c}\mathbf{A}_0\right)\cdot\dot{\mathbf{r}} - \left(\frac{1}{2}m\mathbf{v}^2 + q\Phi_0\right), \tag{23}$$

and

$$L_1 = \frac{q}{c}\mathbf{A}_1\cdot\dot{\mathbf{r}} - q\Phi_1. \tag{24}$$

Consider the perturbed Lagrangian L_1 for a moment. Inserting Eq. (21) into L_1 and using the relationship (22) to eliminate $\mathbf{A}_1(k)$ in favor of $\mathbf{E}_1(k)$, we can rewrite it as following:

$$L_1 = \left[q\int \frac{d^4k}{(2\pi)^4}\frac{\mathbf{E}_1}{i\omega}(k)e^{ik\cdot r}\right]\cdot\dot{\mathbf{r}} + \frac{d}{dt}\left[q\int\frac{d^4k}{(2\pi)^4}\frac{\Phi_1}{i\omega}(k)e^{ik\cdot r}\right], \tag{25}$$

where r denotes $(t, \mathbf{r})$ so $k\cdot r = \mathbf{k}\cdot\mathbf{r} - \omega t$. The second term is a total time derivative and can be omitted, owing to the Lagrangian gauge freedom discussed in the previous section. We note

that L_1 has then also become manifestly gauge invariant. The field part of the perturbed action can also be expressed in terms of $\mathbf{E}_1(k)$ only, but we will defer that discussion to Section V.

Before proceeding to the guiding-center transformation, let us make a change of variables that removes the perturbations from the $\dot{\mathbf{r}}$ term. Define

$$\mathbf{u} = \mathbf{v} + \frac{q}{m} \int \frac{d^4k}{(2\pi)^4} \frac{\mathbf{E}_1}{i\omega}(k) e^{ik\cdot r}, \tag{26}$$

and eliminate $\mathbf{v}$ in favor of $\mathbf{u}$, then the whole Lagrangian L becomes

$$L(\mathbf{r}, \mathbf{u}, \dot{\mathbf{r}}, t) = L_0(\mathbf{r}, \mathbf{u}, \dot{\mathbf{r}}, t) - H_1(\mathbf{u}, \mathbf{r}, t) - H_2(\mathbf{r}, t), \tag{27}$$

where L_0 has the same form as in Eq. (23) except that $\mathbf{v}$ is now replaced by $\mathbf{u}$. H_1 and H_2 are the perturbed Hamiltonians, given by

$$\begin{aligned} H_1 &= -q\mathbf{u} \cdot \int \frac{d^4k}{(2\pi)^4} \frac{\mathbf{E}_1}{i\omega}(k) e^{ik\cdot r}, \\ H_2 &= \frac{q^2}{2m} \left| \int \frac{d^4k}{(2\pi)^4} \frac{\mathbf{E}_1}{i\omega}(k) e^{ik\cdot r} \right|^2 . \end{aligned} \tag{28}$$

This simple transformation will greatly simplify the averaging calculations to be discussed in Section IV. This is because averaging the Hamiltonian requires only the *Hamiltonian Lie transform*, a much simpler special case of the *Lagrangian Lie transform* (see Appendix A). This fact was exploited in Ref. 21 but not in a manifestly gauge invariant form. The Lagrangian Lie transform method was used in Refs. 15,38. This transformation has another deeper significance with respect to the Poisson bracket which will be discussed later in this section. We note in passing that the same simplification can also be achieved for short-wavelength but low-frequency perturbations, i.e., for perturbations that vanish as $|\mathbf{k}| \to 0$ but not necessarily as $\omega \to 0$. In this case we can express the perturbed Lagrangian L_1 as

$$\begin{aligned} L_1 = &- \left[\frac{q}{c} \int \frac{d^4k}{(2\pi)^4} \frac{\mathbf{k} \times \mathbf{B}_1}{i\mathbf{k}^2}(k) e^{ik\cdot r} \right] \cdot \dot{\mathbf{r}} + q \int \frac{d^4k}{(2\pi)^4} \frac{\mathbf{k} \cdot \mathbf{E}_1}{i\mathbf{k}^2}(k) e^{ik\cdot r} \\ &+ \frac{d}{dt} \left[\frac{q}{c} \int \frac{d^4k}{(2\pi)^4} \frac{\mathbf{k} \cdot \mathbf{A}_1}{i\mathbf{k}^2}(k) e^{ik\cdot r} \right] \end{aligned} \tag{29}$$

Thus a transformation similar to Eq. (26) can again be made and perturbations be moved into the Hamiltonian. But note that in this case we have to deal with both $\mathbf{E}_1$ and $\mathbf{B}_1$.

Now we are ready to carry out the guiding-center transformation. Our presentation below is slightly different from Refs. 14,39. Let $B \equiv |\mathbf{B}_0|$ and $\mathbf{b} \equiv \mathbf{B}_0/B$. Also assume that $|\mathbf{E}_0 \cdot \mathbf{b}|$ is small so that the guiding center transformation is applicable,[40] and that the drift velocity associated with the perpendicular electric field

$$\mathbf{v}_E \equiv c\mathbf{E}_0 \times \mathbf{b}/B \tag{30}$$

is of the same order as the particle thermal speed. The guiding-center transformation can be achieved in three steps. The first step is to separate the parallel and perpendicular components of $\mathbf{u}$. Define $u_{||} \equiv \mathbf{u} \cdot \mathbf{b}$, $\mathbf{u}_\perp \equiv \mathbf{u} - u_{||}\mathbf{b} - \mathbf{v}_E$, $u_\perp \equiv |\mathbf{u}_\perp|$, and $\mathbf{c} \equiv \mathbf{u}_\perp/u_\perp$, where $\mathbf{b}$ and $\mathbf{v}_E$ are evaluated at $\mathbf{r}$, so we can write

$$\mathbf{u} = u_{||}\mathbf{b}(\mathbf{r},t) + u_\perp \mathbf{c} + \mathbf{v}_E(\mathbf{r},t). \tag{31}$$

By definition $\mathbf{c}$ is a unit vector perpendicular to $\mathbf{b}$, so it must satisfy the following two constraints

$$\mathbf{c} \cdot \mathbf{c} = 1, \qquad \mathbf{c} \cdot \mathbf{b}(\mathbf{r},t) = 0. \tag{32}$$

$\mathbf{c}$ can be used as an independent variable as long as these constraints are taken into account. Notice that, besides the rapid gyration, $\mathbf{c}$ also has a slower variation imposed on it by the constraints.

In the second step we make a preparatory transformation from the particle position $\mathbf{r}$ to the guiding-center position $\mathbf{R}$, defined by

$$\mathbf{R} \equiv e^{-\rho\mathbf{a}\cdot\nabla}\mathbf{r} = \mathbf{r} - \rho\mathbf{a} + \frac{1}{2}\rho\mathbf{a}\cdot\nabla(\rho\mathbf{a}) + \cdots, \tag{33}$$

where $\rho \equiv u_\perp/\Omega$ is the (signed) gyroradius, $\Omega \equiv qB/mc$ is the (signed) gyrofrequency, and $\mathbf{a} \equiv \mathbf{b} \times \mathbf{c}$, all evaluated at $\mathbf{r}$. In Eq. (33) we hold $(u_{||}, u_\perp, \mathbf{c})$ constant when calculating ∇.

The use of the exponential operator not only makes the inverse transformation apparent, but more importantly it also preserves the simple form of the constraints (32), which become

$$\mathbf{c}\cdot\mathbf{c}=1, \qquad \mathbf{c}\cdot\mathbf{b}(\mathbf{R},t)=0. \tag{34}$$

The inverse transformation of Eq. (33) is

$$\mathbf{r}=e^{\rho\mathbf{a}\cdot\nabla}\,\mathbf{R}=\mathbf{R}+\rho\mathbf{a}+\frac{1}{2}\rho\mathbf{a}\cdot\nabla(\rho\mathbf{a})+\cdots, \tag{35}$$

where the right-hand side is evaluated at $\mathbf{R}$. Now we substitute Eqs. (31) and (35) into L_0 and make a Taylor expansion in powers of ρ. For most practical purposes, we need to keep only the zeroth-order terms in $\mathbf{b}$ and $\mathbf{v}_E$, and the first-order terms in the four-potential $(\mathbf{A}_0,\Phi_0)$. The calculation is lengthy but straightforward, and the result is

$$\begin{aligned} L_0=&\left[m(u_\parallel\mathbf{b}+\mathbf{v}_E)+\frac{q}{c}\mathbf{A}_0\right]\cdot\dot{\mathbf{R}}+\frac{mu_\perp^2}{2\Omega}\mathbf{c}\cdot\dot{\mathbf{a}}-H_0\\ &+m(u_\parallel\mathbf{b}+\mathbf{v}_E)\cdot\frac{d}{dt}(\rho\mathbf{a})+\frac{d}{dt}\left[\frac{q\rho}{c}\mathbf{a}\cdot\mathbf{A}_0+\frac{q\rho^2}{2c}\mathbf{aa}:\nabla\mathbf{A}_0\right]+\cdots, \end{aligned} \tag{36}$$

where H_0 is called the *guiding-center Hamiltonian*, defined by

$$H_0=\frac{1}{2}m(u_\parallel^2+u_\perp^2+\mathbf{v}_E^2)+q\Phi_0; \tag{37}$$

the colon denotes trace operation $\mathbf{A}:\mathbf{B}=A_{ij}B_{ji}$; and the overdot stands for total time derivative. Note that the right-hand sides of Eqs. (36) and (37) are evaluated at $\mathbf{R}$. Again the last term in Eq. (36) is a total time derivative and can be omitted. Similarly we substitute Eqs. (31) and (35) into the perturbed Hamiltonians, expand only the exponent to first order in ρ, and obtain

$$\begin{aligned} H_1&=-q(u_\parallel\mathbf{b}+\mathbf{v}_E+u_\perp\mathbf{c})\cdot\int\frac{d^4k}{(2\pi)^4}\,\frac{\mathbf{E}_1}{i\omega}(k)e^{i\mathbf{k}\cdot R}e^{i\mathbf{k}\cdot\rho\mathbf{a}},\\ H_2&=\frac{q^2}{2m}\left|\int\frac{d^4k}{(2\pi)^4}\,\frac{\mathbf{E}_1}{i\omega}(k)e^{i\mathbf{k}\cdot R}e^{i\mathbf{k}\cdot\rho\mathbf{a}}\right|^2, \end{aligned} \tag{38}$$

where R denotes $(t,\mathbf{R})$, so $k\cdot R=\mathbf{k}\cdot\mathbf{R}-\omega t$.

The third and final step of the guiding-center transformation is to perform a *gyroaveraging* to eliminate the rapid oscillations from the Lagrangian. To do this we need to introduce the gyrophase θ to separate the fast and the slow variations of $\mathbf{c}$. Choosing a local perpendicular unit vector $\mathbf{e}_1(\mathbf{x}, t)$ and defining $\mathbf{e}_2 \equiv \mathbf{b} \times \mathbf{e}_1$, so that $(\mathbf{b}, \mathbf{e}_1, \mathbf{e}_2)$ form a slowly varying local triad, we can then define θ by

$$\mathbf{c} = \mathbf{e}_1(\mathbf{R}, t) \cos\theta - \mathbf{e}_2(\mathbf{R}, t) \sin\theta. \tag{39}$$

The relation $\mathbf{a} = \mathbf{b} \times \mathbf{c}$ is preserved by the exponential operator, yielding

$$\mathbf{a} = \mathbf{e}_1(\mathbf{R}, t) \sin\theta + \mathbf{e}_2(\mathbf{R}, t) \cos\theta. \tag{40}$$

Thus $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ also form a (gyrating) triad. Notice that Eq. (39) manifestly satisfies the constraints (34), so θ can be treated as a free variable. However, it is clear that the value of θ depends on the choice of $\mathbf{e}_1$. This freedom of choosing the basis vector $\mathbf{e}_1$ is yet another gauge freedom, called the *gyrogauge*, whose subtle effects on the guiding-center motion have been studied by Littlejohn.[41]

For the rest of this section we shall concentrate on the unperturbed guiding-center Lagrangian L_0. The averaging of the perturbed Hamiltonians H_1 and H_2 is more involved and will be dealt with in the next section. The gyroaveraging of L_0 can be carried out to arbitrary orders by using Lie transform techniques, as shown in detail in Ref. 15, but for our purpose we only need the lowest order contributions. We retain the notations $(\mathbf{R}, u_\parallel, u_\perp, \theta)$ for the averaged guiding-center variables, but keep in mind that their defining equations (26), (31), and (33) are no longer exact—there will be higher order gyroradius corrections. The actual relationship between the particle and the guiding-center coordinates is not needed in this paper; interested readers can find them in Refs. 14,15. The lowest order gyroaveraged Lagrangian then reads

$$\bar{L}_0 = \left[m(u_\parallel \mathbf{b} + \mathbf{v}_E) + \frac{q}{c}\mathbf{A}_0\right] \cdot \dot{\mathbf{R}} + p_g \dot{\theta} - p_g \dot{\mathbf{e}}_1 \cdot \mathbf{e}_2 - H_0. \tag{41}$$

Here we have eliminated $u_\perp$ in favor of the *gyromomentum* p_g which is defined by

$$p_g \equiv \frac{mu_\perp^2}{2\Omega(\mathbf{R},t)}. \tag{42}$$

Since p_g is the canonical momentum conjugate to θ, and θ is an ignorable coordinate in (41), p_g is a constant of motion in the unperturbed system. The *magnetic moment* μ of a guiding center can be defined by

$$\mu \equiv \left(\frac{q}{mc}\right) p_g. \tag{43}$$

The set $\mathbf{Z} = (\mathbf{R}, u_\parallel, p_g, \theta)$ is called the *standard guiding-center variables* by Littlejohn.[14] The guiding-center Hamiltonian H_0 now reads

$$H_0 = \frac{1}{2}m(u_\parallel^2 + \mathbf{v}_E^2) + p_g\Omega + q\Phi_0. \tag{44}$$

The $\dot{\mathbf{e}}_1 \cdot \mathbf{e}_2$ term in Eq. (41) is the gyrogauge term. Although it is usually small, it must be kept because it compensates the $\dot{\theta}$ term to make L_0 as a whole gyrogauge invariant. To see its effects more clearly, let us rewrite it as

$$-p_g\dot{\mathbf{e}}_1 \cdot \mathbf{e}_2 = \frac{q}{c}\mathbf{\Lambda} \cdot \dot{\mathbf{R}} - q\Lambda_0, \tag{45}$$

where $\mathbf{\Lambda} \equiv -(cp_g/q)\nabla\mathbf{e}_1 \cdot \mathbf{e}_2$, $\Lambda_0 \equiv (p_g/q)\partial_t\mathbf{e}_1 \cdot \mathbf{e}_2$. Thus $(\mathbf{\Lambda}, \Lambda_0)$ acts like a four-potential, with the associated fields given by

$$\begin{aligned} \mathbf{E}_\Lambda &= -\frac{1}{c}\frac{\partial \mathbf{\Lambda}}{\partial t} - \nabla\Lambda_0 = \frac{p_g}{q}\nabla\mathbf{b} \cdot (\partial_t\mathbf{b} \times \mathbf{b}), \\ \mathbf{B}_\Lambda &= \nabla \times \mathbf{\Lambda} = -\frac{cp_g}{2q}\epsilon_{ijk}b_i\nabla b_j \times \nabla b_k, \end{aligned} \tag{46}$$

where ϵ_{ijk} is the Levi-Civita symbol: $\epsilon_{123} = 1 = -\epsilon_{321}$, etc. We shall see in Section VI that the gyrogauge term also contributes to the magnetization vector (and in turn to the current density and the energy-momentum tensor). It is also essential for angular momentum conservation.

Applying the Euler-Lagrange equation (6) to Eq. (41), we obtain the lowest order guiding-center equations of motion

$$\begin{aligned}
\dot{\mathbf{R}}^{(0)} &= u_{\|}\mathbf{b} + \frac{c}{qB_{\|}^{*}}\left[q\mathbf{E}^{*} + \frac{q}{c}u_{\|}\mathbf{b}\times\mathbf{B}^{*} - p_g\nabla\Omega - \frac{1}{2}m\nabla\mathbf{v}_E^2\right]\times\mathbf{b}, \\
\dot{u}_{\|}^{(0)} &= \frac{\mathbf{B}^{*}}{mB_{\|}^{*}}\cdot\left[q\mathbf{E}^{*} + \frac{q}{c}u_{\|}\mathbf{b}\times\mathbf{B}^{*} - p_g\nabla\Omega - \frac{1}{2}m\nabla\mathbf{v}_E^2\right], \\
\dot{\theta}^{(0)} &= \Omega + \dot{\mathbf{e}}_1^{(0)}\cdot\mathbf{e}_2, \\
\dot{p}_g^{(0)} &= 0,
\end{aligned} \tag{47}$$

where $(\mathbf{E}^{*}, \mathbf{B}^{*})$ are the "effective" electromagnetic fields

$$\begin{aligned}
\mathbf{E}^{*} &= \mathbf{E}_0 - \frac{m}{q}(u_{\|}\partial_t\mathbf{b} + \partial_t\mathbf{v}_E) + \mathbf{E}_{\Lambda}, \\
\mathbf{B}^{*} &= \mathbf{B}_0 + \frac{mc}{q}(u_{\|}\nabla\times\mathbf{b} + \nabla\times\mathbf{v}_E) + \mathbf{B}_{\Lambda},
\end{aligned} \tag{48}$$

and $B_{\|}^{*} \equiv \mathbf{b}\cdot\mathbf{B}^{*}$. The quantity in the square bracket in Eq. (47) can clearly be interpreted as the effective force that acts on a guiding center.

In the future we will need to calculate the Poisson bracket in the guiding-center coordinates. The *fundamental Poisson brackets* can be obtained straightforwardly from the guiding-center Lagrangian $\bar{L}_0$, by first computing the fundamental Lagrange brackets, then inverting. The details can be found in Ref. 14 (also Appendix A). Here we only display the results:

$$\begin{aligned}
&\{\mathbf{R},\mathbf{R}\} = \frac{c}{qB_{\|}^{*}}\mathbf{b}\times\mathbf{1} = \frac{c}{qB_{\|}^{*}}\mathbf{1}\times\mathbf{b}, \qquad \{\mathbf{R}, u_{\|}\} = \frac{\mathbf{B}^{*}}{mB_{\|}^{*}}, \\
&\{\mathbf{R},\theta\} = \{\mathbf{R},\mathbf{R}\}\cdot(\nabla\mathbf{e}_1\cdot\mathbf{e}_2), \qquad \{u_{\|},\theta\} = -\{\mathbf{R},u_{\|}\}\cdot(\nabla\mathbf{e}_1\cdot\mathbf{e}_2), \\
&\{\theta, p_g\} = 1, \qquad \text{all others} = 0.
\end{aligned} \tag{49}$$

Another set of variables commonly used in guiding-center theory is $(\mathbf{R}, E, p_g, \theta)$, where $u_{\|}$ is replaced by the energy $E \equiv H_0$. This is especially useful when the background fields are stationary, because then E is an invariant of motion

$$\dot{E}^{(0)} = \dot{\mathbf{R}}^{(0)}\cdot\left[-\frac{\partial}{\partial t}\left(m(u_{\|}\mathbf{b} + \mathbf{v}_E) + \frac{q}{c}\mathbf{A}_0\right) + e\mathbf{E}_{\Lambda}\right] = 0, \tag{50}$$

where ∂_t is taken with E held constant. It should be emphasized that, although partial derivatives such as ∇ may have different meanings in different coordinate systems, the Poisson bracket between scalar functions is always the same.

Also in Ref. 14 it has been pointed out that the higher order gyroradius corrections that were left out in Eq. (41) can all be put into the Hamiltonian H_0. Since the Poisson bracket does not depend on the Hamiltonian, Eq. (49) is in fact correct to all orders in ρ. Moreover, since the perturbations have been moved into the Hamiltonian by Eq. (26) before the guiding-center transformation, the Poisson bracket is also correct to all orders of the perturbation amplitudes.

(The above points are so basic that they are worthy of further comment. In the study of Hamiltonian systems, we must distinguish the *Hamiltonian nature* from the *Hamiltonian itself.* The equations of motion of course depend on the Hamiltonian, but their Hamiltonian nature are solely determined by the Poisson bracket structure. When the Hamiltonian is modified, the equations of motions will also change but their Hamiltonian nature remains the same. Therefore although the guiding-center equations per se are approximate, they have captured the exact Hamiltonian nature of the original particle dynamics. In this sense we can think of the guiding-center coordinates as a *representation* rather than an approximation.)

The Hamiltonian nature of the guiding-center equations (47) are manifested in the Liouville theorem. By noting that the fundamental Poisson brackets transform like a contravariant tensor

$$\{Z^i, Z^j\} = \frac{\partial Z^i}{\partial z^k} \{z^k, z^l\} \frac{\partial Z^j}{\partial z^l}, \tag{51}$$

we can easily compute the Jacobian of the guiding-center transformation

$$\mathcal{J} \equiv \det\left(\frac{\partial \mathbf{z}}{\partial \mathbf{Z}}\right) = (\det\{\mathbf{Z}, \mathbf{Z}\})^{-1/2} = \frac{q}{c} B_{\parallel}^*. \tag{52}$$

Here we have used $\det\{\mathbf{z},\mathbf{z}\} = 1$. We emphasize again that this Jacobian is *exact*. The Liouville theorem on the phase-space volume conservation can be expressed as

$$\frac{\partial \mathcal{J}}{\partial t} + \frac{\partial}{\partial \mathbf{Z}} \cdot (\dot{\mathbf{Z}}^{(0)} \mathcal{J}) = 0, \tag{53}$$

which can be verified by direct calculations. This conservation law is a property of the Poisson bracket. It holds true if the higher order gyroradius corrections, or more importantly for our purpose, the perturbations, are included in the Hamiltonian.

The Klimontovich distribution (8), when expressed in terms of the guiding-center variables, becomes

$$f_s(\mathbf{Z},t) = \mathcal{J}_s^{-1} \sum_{i \in s} \delta^6(\mathbf{Z} - \mathbf{Z}_i(t)). \tag{54}$$

This is also an exact relationship. Just as in the particle case, we can rewrite the lowest order guiding-center equations of motion (47) equivalently as a single partial differential equation

$$\left(\frac{d}{dt}\right)_0 f_s \equiv \left(\frac{\partial}{\partial t} + \dot{\mathbf{Z}}^{(0)} \cdot \frac{\partial}{\partial \mathbf{Z}}\right) f_s = 0, \tag{55}$$

which is known as the *drift kinetic equation.* In obtaining Eq. (55) we have used the Liouville theorem (53). By ignoring the discreteness effects, we can interpret it as the Vlasov equation in the guiding-center representation.

To summarize: In this section we carried out the guiding-center transformation, defined with respect to the background electromagnetic fields. We obtained the unperturbed guiding-center equations of motion, and the (exact) Poisson bracket in guiding-center coordinates. The perturbed Hamiltonians have also been transformed into the guiding-center coordinates; their averaging is the topic of next section.

IV. Oscillation Center Theory

Now let us turn to the perturbed guiding-center Hamiltonian (38). Expanding the wave vector $\mathbf{k}$ in terms of the local triad,

$$\mathbf{k} = k_{\|}\mathbf{b} + k_{\perp}(\mathbf{e}_1 \cos\lambda + \mathbf{e}_2 \sin\lambda), \tag{56}$$

where $\lambda(\mathbf{k};\mathbf{x},t)$ is the angle between $\mathbf{k}_\perp$ and $\mathbf{e}_1(\mathbf{x},t)$, and using Eq. (40), we have $\mathbf{k}\cdot\rho\mathbf{a} = k_\perp\rho\sin(\theta+\lambda)$. Then by using the familiar formula

$$e^{i\xi\sin(\theta+\lambda)} = \sum_\ell J_\ell(\xi)e^{i\ell(\theta+\lambda)}, \tag{57}$$

we can express the perturbed Hamiltonians as Fourier series in θ:

$$H_1 = \sum_\ell \int \frac{d^4k}{(2\pi)^4}\,\mathbf{J}_\ell^*(u_{\|},\mu,R,k)e^{i\psi_\ell(\theta,R,k)}\cdot\mathbf{E}_1(k), \tag{58}$$

$$H_2 = \frac{q^2}{2m}\left|\sum_\ell \int \frac{d^4k}{(2\pi)^4}\,\frac{J_\ell(k_\perp\rho)}{i\omega}e^{i\psi_\ell(\theta;R,k)}\,\mathrm{E}_1(k)\right|^2, \tag{59}$$

where the star denotes complex conjugate. The phase function ψ_ℓ and the vector $\mathbf{J}_\ell$ are defined by

$$\psi_\ell(\theta;R,k) \equiv \mathbf{k}\cdot\mathbf{R} - \omega t + \ell[\theta + \lambda(\mathbf{k},\mathbf{R},t)], \tag{60}$$

$$\mathbf{J}_\ell(u_{\|},\mu,R,k) \equiv \frac{iq}{\omega}\left[(u_{\|}\mathbf{b}+\mathbf{v}_E)J_\ell + \frac{\rho\Omega}{\sqrt{2}}(J_{\ell-1}e^{i\alpha}\mathbf{e}_- + J_{\ell+1}e^{-i\alpha}\mathbf{e}_+)\right], \tag{61}$$

where $\mathbf{e}_\pm \equiv \frac{1}{\sqrt{2}}(\mathbf{e}_1 \pm i\mathbf{e}_2)$, and the argument of the Bessel functions is $k_\perp\rho$ with $\rho \equiv (2p_g/m\Omega)^{1/2}$. Using Eq. (56) we can also express $\mathbf{J}_\ell$ in a manifestly gyrogauge invariant form:

$$\mathbf{J}_\ell(u_{\|},\mu,R,k) = \frac{iq}{\omega}\left[\left(u_{\|}\mathbf{b}+\mathbf{v}_E+\frac{\ell\Omega}{k_\perp^2}\mathbf{k}_\perp\right)J_\ell - i\frac{\rho\Omega}{2k_\perp}(\mathbf{b}\times\mathbf{k})J_\ell'\right]. \tag{62}$$

In order to carry out averaging transformations on H_1 and H_2, we first need to identify their oscillatory parts that vary on the same scale as the particle gyration. In the eikonal

approximation we assume that their rate of change is primarily determined by the phase factor $e^{i\psi_\ell}$. Define

$$D_\ell \equiv \left(\frac{d}{dt}\right)_0 \psi_\ell = \mathbf{k}\cdot\dot{\mathbf{R}}^{(0)} - \omega + \ell\left[\Omega + \frac{k_\parallel}{k_\perp^2}(\mathbf{k}\times\mathbf{b})\cdot\dot{\mathbf{b}}^{(0)}\right]. \tag{63}$$

Clearly if D_ℓ is far from zero then the ℓth term in H_1 is oscillatory. The condition $D_\ell \approx 0$ determines the *resonance regions* in the particle phase space, whose width can be estimated as follows. The phase change as a particle goes through the resonance region is roughly given by $\Delta\psi_\ell \approx \dot{D}_\ell(\Delta t)^2$. The time it takes a particle to cross a resonance region can thus be estimated by setting $\Delta\psi_\ell \approx 1$, which yields $\Delta t \approx \dot{D}_\ell^{-1/2}$. So the width ΔD_ℓ of a resonance region is approximately $\dot{D}_\ell \Delta t \approx \dot{D}_\ell^{1/2}$. Let us introduce a window function, $w(D_\ell)$, which is close to unity inside the resonance region ($|D_\ell| < |\dot{D}_\ell|^{1/2}$) and quickly falls to zero outside of it. The exact form of the window function may be tailored to the specific problem at hand, but it should be strictly unity at the center of a resonance, i.e., $w(0) = 1$.[18] The non-oscillatory (resonant) part of H_1 can then be defined as

$$K_1 = \sum_\ell \int \frac{d^4k}{(2\pi)^4}\, w(D_\ell)\mathbf{J}_\ell^*(u_\parallel, \mu, R, k)e^{i\psi_\ell(\theta;R,k)}\cdot\mathbf{E}_1(k), \tag{64}$$

Now we are ready to remove the oscillatory (nonresonant) part of H_1 by a near-identity Lie transform. We shall work only to the second order of the perturbation. The relevant Lie transform formulas are (see Appendix A)

$$\left(\frac{d}{dt}\right)_0 F_1 = H_1 - K_1, \tag{65}$$

$$K_2 = \left\langle H_2 + \frac{1}{2}\{F_1, H_1 + K_1\}\right\rangle, \tag{66}$$

where F_1 is a Lie generating function, $\{\cdot,\cdot\}$ is the guiding-center Poisson bracket as given by Eq. (49), and K_2 is the so-called *ponderomotive Hamiltonian.* The angular bracket here denotes "slowly varying part." The generating function F_1 can be solved by integrating

Eq. (65) along the characteristics of $(d/dt)_0$; since the right-hand side is purely oscillatory, we are safe from secularities. Fourier expand F_1 in both R and θ:

$$F_1(R, u_\parallel, p_g, \theta) = \sum_\ell \int \frac{d^4k}{(2\pi)^4} F_\ell(u_\parallel, p_g, k) e^{i\psi_\ell(\theta;R,k)}. \tag{67}$$

and substitute it into Eq. (65). The dominant contribution to $(dF_1/dt)_0$ is from the phase factor, since p_g is an invariant, and in practice there is usually another invariant of the unperturbed motion. Two common situations are: (1) the background fields are quasi-stationary, in which case we can use the energy E instead of $u_\parallel$ (see Eq. (50)); (2) the background fields are axisymmetric (a good approximation for tokamaks), in which case the toroidal angular momentum p_ϕ is an invariant and can be used in place of $u_\parallel$. In either case F_ℓ will be constant. However, it is not necessary to make an explicit change of variables here, because in the end F_ℓ only appears inside a Poisson bracket that can be calculated in any coordinates (see the remarks below Eq. (50)).

Thus, after equating the integrands on both sides of Eq. (65) we obtain

$$iD_\ell(u_\parallel, p_g, R, k) F_\ell(u_\parallel, p_g, k) = (1 - w_\ell) \mathbf{J}_\ell^*(u_\parallel, p_g, R, k) \cdot \mathbf{E}_1(k), \tag{68}$$

where $w_\ell \equiv w(D_\ell)$. To be precise we should interpret the variable R in D_ℓ and $\mathbf{J}_\ell^*$ as an operator: $R \to i\partial/\partial k$, which is required for equating the integrands. But for nonresonant particles, and within the eikonal approximation, the R variation is unimportant. (In the case when the perturbations are global so that the eikonal approximation does not apply, action-angle variables may be used[42,43] to solve Eq. (68). A more general method is perhaps to employ a symbolic integral operator that may be implemented later by other means, such as an ergodicity argument.[44]) Therefore we have

$$F_1 = \sum_\ell \int \frac{d^4k}{(2\pi)^4} \frac{(1 - w_\ell)}{iD_\ell} \mathbf{J}_\ell^* \cdot \mathbf{E}_1(k) e^{i\psi_\ell(\theta;R,k)}. \tag{69}$$

Appendix A discusses how to calculate the actual coordinate transformation from F_1. But to find the ponderomotive Hamiltonian K_2, the knowledge of F_1 itself is all we need. Using Eqs. (66) and (69) we obtain

$$K_2 = \left\langle \frac{1}{2} \sum_{\ell\ell'} \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4k'}{(2\pi)^4} \, \mathcal{K}_{\ell\ell'}(k,k') e^{i[\psi_{\ell'}(k') - \psi_\ell(k)]} : \mathbf{E}_1(k') \mathbf{E}_1^*(k) \right\rangle, \tag{70}$$

where

$$\begin{aligned} \mathcal{K}_{\ell\ell'}(k,k') = \Bigg[& \frac{q^2}{m} \frac{J_\ell}{\omega}(k) \frac{J_{\ell'}}{\omega'}(k') \mathbf{1} + \left\{ \frac{(1-w_\ell)}{D_\ell} \mathbf{J}_\ell(k), \psi_{\ell'}(k') \right\} (1+w_{\ell'}) \mathbf{J}_{\ell'}^*(k') \\ & + \frac{(1-w_\ell)}{D_\ell} \mathbf{J}_\ell(k) \left\{ (1+w_{\ell'}) \mathbf{J}_{\ell'}^*(k'), \psi_\ell(k) \right\} \Bigg]. \end{aligned} \tag{71}$$

Here for clarity only the dependence on k and ℓ is displayed. In this paper we shall not consider the nonlinear effect of particle-beat wave gyroresonant interactions (called *induced scattering*[45]), so we drop all $\ell \neq \ell'$ terms, which depend on the gyrophase θ, reducing the double sum to a single one. The result reads

$$K_2 = \left\langle \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4k'}{(2\pi)^4} \, \mathcal{K}(k,k') e^{i(k'-k)\cdot R} : \mathbf{E}_1(k') \mathbf{E}_1^*(k) \right\rangle, \tag{72}$$

with the two-point kernel given by

$$\mathcal{K}(k,k') = \sum_\ell \mathcal{K}_{\ell\ell}(k,k') e^{i\ell[\lambda(k') - \lambda(k)]}. \tag{73}$$

Note that $\lambda(k') - \lambda(k) = \arccos(\mathbf{k}_\perp \cdot \mathbf{k}'_\perp)$ is independent of the unit vector $\mathbf{e}_1$. Therefore $\mathcal{K}$ is gyrogauge invariant as it should be.

It is now straightforward to reexpress the ponderomotive Hamiltonian in terms of $\mathbf{E}_1(x)$, by simply substituting

$$\mathbf{E}_1(k) = \int d^4x \, \mathbf{E}_1(x) e^{-ik\cdot x} \tag{74}$$

into Eq. (72). The form of K_2 will remain the same

$$K_2 = \left\langle \frac{1}{2} \int d^4x \int d^4x' \, \mathcal{K}(x,x') : \mathbf{E}_1(x') \mathbf{E}_1^*(x) \right\rangle, \tag{75}$$

only the kernel is Fourier transformed, as given by

$$\mathcal{K}(x,x') = \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4k'}{(2\pi)^4}\, \mathcal{K}(k,k') e^{i[k\cdot(x-R)-k'\cdot(x'-R)]}. \tag{76}$$

Thus we say that K_2 is *form invariant* with respect to change of representations. Using the formalism of *Weyl symbols* and *Wigner functions*, we can cast it into a form that manifestly exhibits this symmetry between the x- and k-space. Let $k = \bar{k} + \frac{1}{2}\sigma$, $k' = \bar{k} - \frac{1}{2}\sigma$, then Eq. (72) becomes

$$K_2 = \left\langle \frac{1}{2} \int \frac{d^4\bar{k}}{(2\pi)^4} \int \frac{d^4\sigma}{(2\pi)^4}\, \mathcal{K}(\bar{k}+\tfrac{1}{2}\sigma, \bar{k}-\tfrac{1}{2}\sigma) e^{-i\sigma\cdot R} : \mathbf{E}_1(\bar{k}-\tfrac{1}{2}\sigma)\mathbf{E}_1^*(\bar{k}+\tfrac{1}{2}\sigma) \right\rangle. \tag{77}$$

Henceforth we drop the bar over the dummy variable $\bar{k}$. Defining the Weyl symbol of the two-point kernel $\mathcal{K}(k,k')$ by

$$\mathcal{K}(x,k) = \int \frac{d^4\sigma}{(2\pi)^4}\, \mathcal{K}(k+\tfrac{1}{2}\sigma, k-\tfrac{1}{2}\sigma) e^{i\sigma\cdot x}, \tag{78}$$

and the Wigner tensor of $\mathbf{E}_1(k)$ by

$$\mathcal{W}_E(x,k) = \left\langle \frac{1}{2} \int \frac{d^4\sigma}{(2\pi)^4}\, \mathbf{E}_1(k+\tfrac{1}{2}\sigma)\mathbf{E}_1^*(k-\tfrac{1}{2}\sigma) e^{i\sigma\cdot x} \right\rangle, \tag{79}$$

we obtain the ponderomotive Hamiltonian in the form

$$K_2 = \int d^4x \int \frac{d^4k}{(2\pi)^4}\, \mathcal{K}(x-R,k) : \mathcal{W}_E(x,k), \tag{80}$$

where we have used the eikonal assumption that $\mathcal{K}(x,k)$ is slowly varying to move it out of the angular bracket. It is appropriate to think of the Weyl symbol and the Wigner tensor as creatures living in the *ray phase space* (i.e., the x-k space) since they do not depend on either representation. For instance we can express $\mathcal{W}_E(x,k)$ in terms of $\mathbf{E}_1(x)$ in a form almost identical to Eq. (79):

$$\mathcal{W}_E(x,k) = \left\langle \frac{1}{2} \int d^4s\, \mathbf{E}_1(x+\tfrac{1}{2}s)\mathbf{E}_1^*(x-\tfrac{1}{2}s) e^{-is\cdot k} \right\rangle. \tag{81}$$

In fact $\mathcal{K}(x,k)$ and $\mathcal{W}_E(x,k)$ can be defined without using an explicit representation.[31]

To further simplify the expression for K_2, let us expand the Wigner tensor in Eq. (80) about R. To the lowest order in the eikonal approximation we can replace x by R in $\mathcal{W}_E(x,k)$. Then the x integral acts only on the kernel $\mathcal{K}(x-R,k)$ and can be easily carried out using Eq. (78), yielding

$$K_2 = \int \frac{d^4k}{(2\pi)^4} \mathcal{K}(k,k'=k) : \mathcal{W}_E(R,k). \tag{82}$$

The diagonal kernel $\mathcal{K}(k,k)$, which can be evaluated from Eqs. (73) and (71), has the following concise form:

$$\mathcal{K}(k,k) = \frac{q^2}{m\omega^2}\mathbf{1} + \sum_\ell \left\{ \frac{(1-w_\ell^2)}{D_\ell} \mathbf{J}_\ell \mathbf{J}_\ell^*, \psi_\ell \right\}(k). \tag{83}$$

In obtaining (83) we have used the Bessel function identity $\sum_\ell J_\ell^2 = 1$.

As an example, let us calculate the ponderomotive Hamiltonian for an eikonal wave $\mathbf{E}_1(x) = \mathbf{a}(x)e^{i\Theta(x)} + c.c.$, where $\mathbf{a}(x)$ is the slowly varying amplitude. Inserting this expression into Eq. (81) we obtain the Wigner tensor, to lowest order in the eikonal approximation,

$$\mathcal{W}_E(x,k) = \frac{(2\pi)^4}{2}[\mathbf{a}\mathbf{a}^*\delta^4(k-\partial\Theta) + \mathbf{a}^*\mathbf{a}\delta^4(k+\partial\Theta)]. \tag{84}$$

On substituting it into Eq. (82) we find

$$K_2 = \mathbf{a}^*(R) \cdot \mathcal{K}(u_{||}, p_g, R, k = \partial\Theta(R)) \cdot \mathbf{a}(R), \tag{85}$$

where $\mathcal{K}$ is given by Eq. (83). This result has been obtained previously[15,20,21] where the eikonal wave form is assumed from the outset.

Adding K_1 and K_2 to the guiding-center equation (47), we obtain the oscillation-center equations of motion:

$$\dot{\mathbf{Z}} = \dot{\mathbf{Z}}^{(0)} + \{\mathbf{Z}, K_1 + K_2\}. \tag{86}$$

As mentioned in the previous section, the Liouville theorem (53) remains valid if we replace $\dot{\mathbf{Z}}^{(0)}$ by $\dot{\mathbf{Z}}$. Furthermore, the oscillation-center Vlasov distribution is the same as in Eq. (54)—we just need to reinterpret $\mathbf{Z}_i(t)$ as the oscillation-center variables. Finally, the oscillation-center Vlasov equation reads

$$\left(\frac{d}{dt}\right)_0 f_s + \{f_s, K_1 + K_2\} = 0, \tag{87}$$

where K_1 represents resonance and K_2 represents nonresonant ponderomotive effects.

To recapitulate this section: In the nonresonance regions we have transformed the coordinates to the oscillation-center representation, and obtained the ponderomotive Hamiltonian. In the resonance region no transformation was performed so the wave-particle interaction there is still fully nonlinear.

V. Wave Equations and Ray Phase Space Eikonal Theory

Now that we have the oscillation-center Lagrangian $L = \bar{L}_0 - K_1 - K_2$, from Eqs. (41), (64), and (82), we can use it in the action principle (2) to obtain self-consistent equations for the electromagnetic fields. The background fields and the perturbations have different scales and can be varied independently. In this section we will concentrate on the equation for the perturbed fields (the *wave equation*).

The whole action functional is given by

$$S = \sum_i \int dt(\bar{L}_0 - K_1 - K_2) + \int d^4x(\mathcal{L}_0 + \mathcal{L}_2), \tag{88}$$

where $\mathcal{L}_0$ is similar to Eq. (4) but with the four-potential replaced by $(\mathbf{A}_0, \Phi_0)$. (We have assumed that the external sources $(\rho_{ext}, \mathbf{J}_{ext})$ are slowing varying so they affect only the background fields.) The perturbed field part of the action is quadratic in wave fields

$$\int d^4x \mathcal{L}_2 = \int d^4x \frac{\mathbf{E}_1^2 - \mathbf{B}_1^2}{8\pi}(x) = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\epsilon_{vac}(k) : \mathbf{E}_1(k)\mathbf{E}_1^*(k), \tag{89}$$

where we have used Faraday's law $\omega \mathbf{B}_1(k) = c\mathbf{k} \times \mathbf{E}_1(k)$ to eliminate the magnetic perturbation. $\epsilon_{vac}(k)$ is called the *vacuum dispersion tensor*, defined by

$$\epsilon_{vac}(k) = \frac{c^2}{4\pi\omega^2}\left[\left(\frac{\omega^2}{c^2} - \mathbf{k}^2\right)\mathbf{1} + \mathbf{kk}\right]. \tag{90}$$

Another quadratic contribution comes from the ponderomotive Hamiltonian K_2. The whole second order action can be written in the general form

$$S_2 = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\int \frac{d^4k'}{(2\pi)^4}\, \mathcal{D}(k,k') : \mathbf{E}_1(k')\mathbf{E}_1^*(k), \tag{91}$$

where $\mathcal{D}(k,k')$ is called the *two-point plasma dispersion tensor.* Using the Wigner-Weyl formalism introduced in the last section, we can rewrite S_2 as

$$S_2 = \int d^4x \int \frac{d^4k}{(2\pi)^4}\, \mathcal{D}(x,k) : \mathcal{W}_E(x,k), \tag{92}$$

where $\mathcal{D}(x,k)$ is the Weyl symbol of $\mathcal{D}(k,k')$ and is called the *local plasma dispersion tensor.* From Eqs. (88), (82), and (83), we obtain

$$\mathcal{D}(x,k) = \epsilon_{vac}(k) - \sum_s \int d^6\mathbf{Z}\, \mathcal{J}_s \delta(\mathbf{x}-\mathbf{R})\left[\frac{q^2}{m\omega^2}\mathbf{1} + \sum_\ell \left\{\frac{(1-w_\ell^2)}{D_\ell}\mathbf{J}_\ell\mathbf{J}_\ell^*, \psi_\ell\right\}\right] f_s. \tag{93}$$

Here we have replaced the sum over particles in a species by an integral over the particle phase space. The second term in Eq. (93) represents the linear *nonresonant* plasma response to the electromagnetic perturbation. It is called the *linear susceptibility* and is usually denoted by $\boldsymbol{\chi}(x,k)$. The intimate relationship it has with the ponderomotive Hamiltonian K_2 is known as the K-χ *theorem.*[19,46] The dispersion tensor $\mathcal{D}(x,k)$ embodies all the information about the (nonresonant) plasma linear response to the electromagnetic perturbations. It should be emphasized that $\mathcal{D}(x,k)$ does not contain the resonant particle contribution and is thus Hermitian.

Variation of the action (88) with respect to $\mathbf{E}_1^*(k)$, which is equivalent to varying $\mathbf{A}_1^*(k)$ since $\omega \neq 0$, yields the linear wave equation

$$\int \frac{d^4k'}{(2\pi)^4}\, \mathcal{D}(k,k') \cdot \mathbf{E}_1(k') = \sum_s \mathbf{j}_s(k), \tag{94}$$

where the right-hand side comes from K_1 and represents the current density due to the resonant particles

$$\mathbf{j}_s(k) = \sum_\ell \int dt d^6\mathbf{Z}\, \mathcal{J}_s f_s(\mathbf{Z},t) w_\ell \mathbf{J}_\ell(\mathbf{Z},t,k) e^{-i\psi_\ell(\theta;R,k)}. \tag{95}$$

Variation with respect to $\Phi_1^*(k)$ yields only the $\mathbf{k}$ component of the same equation and is thus redundant. Again Eq. (94) is form-invariant and looks just the same in the x-representation:

$$\int d^4x'\, \mathcal{D}(x,x') \cdot \mathbf{E}_1(x') = \sum_s \mathbf{j}_s(x), \tag{96}$$

where $\mathcal{D}(x,x')$ is the Fourier transform of $\mathcal{D}(k,k')$:

$$\mathcal{D}(x,x') = \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4k'}{(2\pi)^4}\, \mathcal{D}(k,k') e^{i(k\cdot x - k'\cdot x')}; \tag{97}$$

similarly for $\mathbf{j}_s(x)$. Eqs. (94) and (96) are two different representations of the same linear wave equation that describes the linear plasma response to a given source, which in this case is the resonant particles. In Ref. 27 a similar equation is considered when the source is due to discrete particle effects.

Let us concentrate on the left-hand side of (94) for the rest of this section. For asymptotic analyses it is useful to turn the integral equation (94) into a pseudodifferential equation. For this purpose we can employ the *Weyl symbol calculus*, as decribed in detail in Ref. 28. We first replace k' by $k - \sigma$, then use the displacement operator to obtain

$$\int \frac{d^4\sigma}{(2\pi)^4}\, \mathcal{D}(k,k-\sigma) \cdot \mathbf{E}_1(k-\sigma) = \int \frac{d^4\sigma}{(2\pi)^4}\, \mathcal{D}(k,k-\sigma) e^{-\sigma\cdot\partial_k} \cdot \mathbf{E}_1(k). \tag{98}$$

Define the *common symbol* $\mathbf{d}(x,k)$ by[28]

$$\mathbf{d}(x,k) = \int \frac{d^4\sigma}{(2\pi)^4}\, \mathcal{D}(k,k-\sigma) e^{i\sigma\cdot x}, \tag{99}$$

then Eq. (98) is seen to be equivalent to

$$\mathbf{d}(x \mapsto i\partial_k, k) \cdot \mathbf{E}_1(k) = 0 \tag{100}$$

with the rule that the derivative acts only on the wave field. In practice, we can first expand $\mathbf{d}(x,k)$ in a power series of x, arrange it so that k is before (to the left of) x in every term, then replace x by the operator $i\partial_k$. A more descriptive name for $\mathbf{d}(x,k)$ is the "k-before-x symbol." In one dimension the rule can be expressed concisely as

$$\mathbf{d}(x,k) \mapsto \sum_n \frac{1}{n!}\partial_x^n \mathbf{d}(x=0,k)(i\partial_k)^n. \tag{101}$$

Clearly the same technique applies to the x-space equation (96) as well. It is straightforward to show that the common symbol for $\mathcal{D}(x,x')$, also called the "x-before-k symbol" and denoted by $\mathbf{d}^\dagger(x,k)$, can be defined in terms of $\mathcal{D}(k,k')$ as

$$\mathbf{d}^\dagger(x,k) = \int \frac{d^4\sigma}{(2\pi)^4}\,\mathcal{D}(k-\sigma,k)e^{-i\sigma\cdot x}. \tag{102}$$

Therefore, since $\mathcal{D}(k,k')$ is Hermitian, the x-space common symbol $\mathbf{d}^\dagger(x,k)$ is the adjoint of its k-space counterpart, $\mathbf{d}(x,k)$, hence the notations.

Note that unlike the Weyl symbols, a common symbol is typically non-Hermitian. Using the same technique as above, we can easily obtain the following relationships between these symbols

$$\mathbf{d} = e^{\frac{i}{2}\partial_x\cdot\partial_k}\mathcal{D}, \qquad \mathbf{d}^\dagger = e^{-\frac{i}{2}\partial_x\cdot\partial_k}\mathcal{D}. \tag{103}$$

The inverses of Eq. (103) are

$$\mathcal{D} = e^{-\frac{i}{2}\partial_x\cdot\partial_k}\mathbf{d} = e^{\frac{i}{2}\partial_x\cdot\partial_k}\mathbf{d}^\dagger. \tag{104}$$

These rules are useful for finding the operators that correspond to given dispersion functions. Let us illustrate the rules with an example. Suppose that we want to find the x-space operator associated with the Weyl symbol $D(x,k) = xk$ in a one-dimensional problem. Applying Eq. (103), we find that $d^\dagger(x,k) = xk - \frac{i}{2}$, so the desired operator is

$$D(x,k) \mapsto x(-i\partial_x) - \tfrac{i}{2} = -\tfrac{i}{2}(x\partial_x + \partial_x x). \tag{105}$$

Similarly we can find the k-space operator by computing $d(x,k) = kx + \frac{i}{2}$, therefore

$$D(x,k) \mapsto k(i\partial_k) + \tfrac{i}{2} = \tfrac{i}{2}(k\partial_k + \partial_k k). \tag{106}$$

These operators are Hermitian, which is a general property of real Weyl symbols.

As we have seen in Eq. (100), a pseudodifferential equation is in general an infinite-order partial differential equation and is often intractable analytically. In the eikonal approximation one assumes that the plasma, represented here by the dispersion tensor $\mathcal{D}(x,k)$ (or by a symbol such as $\mathbf{d}(x,k)$), is slowly varying on the characteristic scales of the wave fields. Then it is possible to truncate the series expansion of the symbols and reduce Eq. (100) to a more manageable lower order partial differential equation.[27,28] In the following we review those elements of the ray phase space eikonal theory that are related to the theory of linear mode conversion. These ideas will be applied in Section VII to the ion gyroresonance processes.

The lowest order eikonal equation is the dispersion relation, usually written as

$$\det \mathcal{D}(x, k = \partial\Theta) = 0. \tag{107}$$

The close resemblance of Eq. (107) to the Hamilton-Jacobi equation of classical mechanics leads to the concept of *rays*, defined as the trajectories of the corresponding mechanical system. And just as in the case of classical mechanics, the most natural setting of the eikonal theory is the ray phase space, i.e., the x-k space. Let $D(x,k)$ be an eigenvalue of $\mathcal{D}(x,k)$, then the dispersion relation $D(x,k) = 0$ defines a *dispersion surface* in the eight-dimensional ray phase space that is analogous to the energy surface in mechanics. The rays are confined to this surface [Figure 1] and obey the Hamilton equations

$$\frac{dx}{d\tau} = -\frac{\partial D}{\partial k}, \qquad \frac{dk}{d\tau} = \frac{\partial D}{\partial x}, \tag{108}$$

where τ is an arbitrary parametrization along the rays. If $\partial_\omega D \neq 0$, then we can solve the dispersion relation $D(x,k) = 0$ to obtain $\omega = \omega(\mathbf{x}, \mathbf{k}; t)$. There is usually more than one

solution; each corresponds to *a wave* (or *a mode*). The ray equations for a given wave in the reduced six-dimensional ray phase space (the **x**-**k** space) become

$$\frac{d\mathbf{x}}{dt} = \frac{\partial\omega}{\partial\mathbf{k}}, \qquad \frac{d\mathbf{k}}{dt} = -\frac{\partial\omega}{\partial\mathbf{x}}, \qquad \frac{d\omega}{dt} = \frac{\partial\omega}{\partial t}. \tag{109}$$

The third equation is a trivial consequence of the first two. The dispersion surface in this reduced ray phase space can be defined only if $\partial\omega/\partial t = 0$, by assigning $\omega(\mathbf{x}, \mathbf{k})$ a given value such as the frequency of the antenna.

The next order eikonal equation is a differential equation that describes how the wave action and polarization change as they propagate along the rays. A general, Lorentz-covariant eikonal theory has been developed in Ref. 47, which contains a concise formula for polarization transport. It is in this order that the traditional x-space eikonal theory encounters the so-called *caustic singularities*, where the solution of the eikonal equation becomes infinite at certain places. The caustics occur when the rays focus in x-space. This focusing can not happen in the ray phase space because of the Liouville theorem associated with the ray equations (109). Indeed it is possible to construct a uniform eikonal approximation in the wave space which is free of caustic singularities.[28,31] There are other advantages in taking the ray phase space point of view, and we will argue below that this approach becomes essential when dealing with linear mode conversion.

A medium such as a magnetized plasma can support many waves simultaneously. Normally, each wave propagates on its own dispersion surface and the coupling between waves is negligible, at least in linear theory. However, when two dispersion surfaces come close to or cross each other, then a degeneracy occurs in the original wave equation, leading to a phenomenon known as *linear mode conversion*. Figure 2 schematically depicts such an event. One wave traveling on its own dispersion surface and passing through the crossing can linearly convert part of its energy to the other wave, which would then propagate away on *its* dispersion surface. As we have seen in the discussion of the ray equations (109),

to determine the dispersion relation of a wave it is necessary to diagonalize the dispersion tensor $\mathcal{D}(x,k)$ by reducing the number of the components of the wave fields down to one. But in a mode conversion region, because of the degeneracy, this reduction usually requires division by a small quantity, results in a rapid variation of the wave polarization and causes the eikonal approximation to break down. This scenario has been analyzed in Refs. 32,33. Therefore in reducing the number of the components of the wave fields in the mode conversion region, one would come to a point when all the elements of the remaining dispersion tensor have become small. The dispersion tensor is then only *block diagonalized*, but this is as far as one can go without violating the eikonal assumption. A general theory for the block diagonalization of the dispersion tensor, called the *congruent reduction algorithm*, has been developed by Friedland,[32,33] which makes systematic use of congruence transformations and Weyl symbol calculus.

The essential technique of ray phase space linear mode conversion is the following. In the ray phase space, mode conversions typically occur pairwise at isolated locations. In this case of two waves crossing each other, the equations can be reduced to two coupled first order partial differential equations (i.e., the dispersion tensor can be reduced to a 2×2 block whose entries are linear functions of x and k). These equations can be reduced to a set of one first order ordinary differential equation plus one algebraic equation, by a linear canonical transformation of the ray phase space.[29,30] Thus to construct an analytic solution in the new coordinates becomes a trivial matter. The theory of *metaplectic transformations*[31] can then be applied to transform the found solution back to the x- or k-representation. The different mode conversion regions, which are separated in ray phase space, often coincide when projected onto x-space. Therefore to describe the same problem in the traditional x-space mode conversion theory, one must use higher order differential equations.

These arguments apply to any linear wave equation. In Section VII we derive the gyroresonance coupling equations by linearizing the oscillation-center Vlasov equation (87)

and the wave equation (94). Then we give a heuristic interpretation of the gyroresonance processes based on the ray phase space mode conversion theory, so that the above techniques may be employed to construct analytic solutions.

VI. Maxwell's Equations and Conservation Laws

Varying the oscillation-center action (88) with respect to $(\mathbf{A}_0, \Phi_0)$ yields the Maxwell equations for the background fields that have the same general form as Eq. (11). The functional derivatives $\mathbf{D}$, $\mathbf{H}$, ρ_f, and $\mathbf{J}_f$ can be calculated from Eq. (12) in a straightforward manner.

However, the guiding-center Lagrangian $\bar{L}_0$ as given by Eq. (41) is not yet suitable for computing $\mathbf{H}$. Recall that the gyrophase θ, by its definition, has an implicit dependence on $\mathbf{b}$ that must be removed before one can vary $\mathbf{A}_0$. For this purpose let us take a step back and use the unit vector $\mathbf{c}$ with constraints (34). Introducing two Lagrange multipliers λ_1 and λ_2 for the two constraints, we then obtain a modified Lagrangian

$$\bar{L}_0 = \left[m(u_\parallel \mathbf{b} + \mathbf{v}_E) + \frac{q}{c}\mathbf{A}_0\right] \cdot \dot{\mathbf{R}} + p_g(\mathbf{c} \times \mathbf{b}) \cdot \dot{\mathbf{c}} - \lambda_1(\mathbf{c}^2 - 1) - \lambda_2(\mathbf{c} \cdot \mathbf{b}) - H, \qquad (110)$$

where $H = H_0 + K_1 + K_2$. Not surprisingly, $\bar{L}_0$ has become manifestly gyrogauge invariant.

The two Lagrange multipliers can be calculated as follows. The Euler-Lagrange equation for p_g yields

$$(\mathbf{c} \times \mathbf{b}) \cdot \dot{\mathbf{c}} = \frac{\partial H}{\partial p_g}. \qquad (111)$$

By enforcing the constraints (34) we obtain

$$\dot{\mathbf{c}} = (\mathbf{c} \times \mathbf{b})\frac{\partial H}{\partial p_g} - (\mathbf{c} \cdot \dot{\mathbf{b}})\mathbf{b}, \qquad (112)$$

which is equivalent to the equation for $\dot{\theta}$. The Euler-Lagrange equation for $\mathbf{c}$ yields

$$\frac{d}{dt}(p_g \mathbf{c} \times \mathbf{b}) - p_g \mathbf{b} \times \dot{\mathbf{c}} + 2\lambda_1 \mathbf{c} + \lambda_2 \mathbf{b} + \frac{\partial K_1}{\partial \mathbf{c}} = 0. \qquad (113)$$

Taking dot products of this equation with $\mathbf{c}$ and $\mathbf{b}$ respectively, and using Eqs. (34) and (112), we then find

$$\lambda_1 = p_g \frac{\partial H}{\partial p_g} - \frac{\mathbf{c}}{2} \cdot \frac{\partial K_1}{\partial \mathbf{c}}, \qquad \lambda_2 = -p_g \mathbf{a} \cdot \dot{\mathbf{b}} - \mathbf{b} \cdot \frac{\partial K_1}{\partial \mathbf{c}}. \tag{114}$$

For ready reference we also list here the Poisson bracket of $\mathbf{c}$ with other guiding-center variables. They can be obtained directly from Eqs. (39) and (49) by the chain rule:

$$\begin{aligned} \{\mathbf{R}, \mathbf{c}\} &= -\{\mathbf{R}, \mathbf{b}\} \cdot \mathbf{cb}, \qquad \{p_g, \mathbf{c}\} = \mathbf{b} \times \mathbf{c}, \\ \{u_\parallel, \mathbf{c}\} &= -\{u_\parallel, \mathbf{b}\} \cdot \mathbf{cb}, \qquad \{\mathbf{c}, \mathbf{c}\} = 0. \end{aligned} \tag{115}$$

Now substituting Eq. (110) into the action (88), and working out all the functional derivatives, we obtain $\mathbf{D} = \mathbf{E}_0 + 4\pi\mathbf{P}$ and $\mathbf{H} = \mathbf{B}_0 - 4\pi\mathbf{M}$, where $\mathbf{P}$ and $\mathbf{M}$ are the *polarization* and *magnetization* vectors arising from the explicit dependence of the oscillation-center Lagrangian on $\mathbf{E}_0$ and $\mathbf{B}_0$; they are given by

$$\mathbf{P} = \frac{\mathbf{cb}}{B} \times \sum_s \int d^6\mathbf{Z}\, \mathcal{J}_s \delta^3(\mathbf{x} - \mathbf{R}) \left[m\dot{\mathbf{R}} - \frac{\partial H}{\partial \mathbf{v}_E} \right] f_s, \tag{116}$$

$$\begin{aligned} \mathbf{M} = \sum_s \int d^6\mathbf{Z}\, \mathcal{J}_s \delta^3(\mathbf{x} - \mathbf{R}) \Bigg[&-\mathbf{b} \left(\frac{m\mathbf{v}_E}{B} \cdot \dot{\mathbf{R}} + \frac{\partial H}{\partial B} \right) \\ &+ \frac{\mathbf{1} - \mathbf{bb}}{B} \cdot \left(m u_\parallel \dot{\mathbf{R}} + \frac{mc}{B} \dot{\mathbf{R}} \times \mathbf{E}_0 - p_g \mathbf{b} \times \dot{\mathbf{b}} - \frac{\partial H}{\partial \mathbf{b}} + \mathbf{cb} \cdot \frac{\partial K_1}{\partial \mathbf{c}} \right) \Bigg] f_s. \end{aligned} \tag{117}$$

Also the "free" charge and current densities are found to be

$$\begin{aligned} \rho_f &= \rho_{ext} + \sum_s q_s \int d^6\mathbf{Z}\, \mathcal{J}_s \delta^3(\mathbf{x} - \mathbf{R}) f_s, \\ \mathbf{J}_f &= \mathbf{J}_{ext} + \sum_s q_s \int d^6\mathbf{Z}\, \mathcal{J}_s \delta^3(\mathbf{x} - \mathbf{R}) \dot{\mathbf{R}} f_s. \end{aligned} \tag{118}$$

Here we have again replaced the summation over particles within a species by an integral over the particle phase space. Comparing the unperturbed part of the above results with those obtained by Similon[11] and by Pfirsch and Morrison[13] (for neither included the wave fields), we find that there is a new term in our magnetization vector. This is the $p_g \mathbf{b} \times \dot{\mathbf{b}}$

in Eq. (117), which comes from the $\dot{\mathbf{c}}$ term in the Lagrangian (110) and is associated with the gyrogauge. Also, by dropping the resonant contributions, our results would agree with Ref. 15. Many of the terms in $\mathbf{P}$ and $\mathbf{M}$ can be explained physically by examining the shape of the particle gyroorbits.[48]

The total charge and current densities are given by $\rho = \rho_f - \nabla \cdot \mathbf{P}$ and $\mathbf{J} = \mathbf{J}_f + c\nabla \times \mathbf{M} + \partial_t \mathbf{P}$. They must agree with Eq. (14). However, to derive Eqs. (116), (117), and (118) by direct transformation from the particle representation would be extremely laborious.

The conservation laws for energy and momentum can be derived by the Noether theorem, just as in the particle coordinates. The general form of Eqs. (16) and (18) are still usable if we change $(\mathbf{E}, \mathbf{B})$ to $(\mathbf{E}_0, \mathbf{B}_0)$ on the right-hand side, but the definitions of U, $\mathbf{S}$, $\mathbf{g}$, and $\mathcal{T}$ must be replaced by the following:

$$
\begin{aligned}
U_{oc} &= \frac{1}{8\pi}(\mathbf{E}_0^2 + \mathbf{B}_0^2) + \mathbf{P} \cdot \mathbf{E}_0 + \sum_s \int d^6\mathbf{Z}\, \mathcal{J}_s \delta^3(\mathbf{x} - \mathbf{R}) K f_s + U_w, \\
\mathbf{S}_{oc} &= \frac{c}{4\pi}(\mathbf{E}_0 \times \mathbf{H}) + \sum_s \int d^6\mathbf{Z}\, \mathcal{J}_s \delta^3(\mathbf{x} - \mathbf{R}) \dot{\mathbf{R}} K f_s + \mathbf{S}_w, \\
\mathbf{g}_{oc} &= \frac{1}{4\pi c}(\mathbf{D} \times \mathbf{B}_0) + \sum_s \int d^6\mathbf{Z}\, \mathcal{J}_s \delta^3(\mathbf{x} - \mathbf{R}) m_s (u_\parallel \mathbf{b} + \mathbf{v}_E) f_s + \mathbf{g}_w, \qquad (119) \\
\mathcal{T}_{oc} &= \left[\left(\frac{1}{8\pi}(\mathbf{E}_0^2 + \mathbf{B}_0^2) - \mathbf{B}_0 \cdot \mathbf{M} \right) \mathbf{1} - \frac{1}{4\pi}(\mathbf{B}_0 \mathbf{H} + \mathbf{D}\mathbf{E}_0) \right] \\
&\quad + \sum_s \int d^6\mathbf{Z}\, \mathcal{J}_s \delta^3(\mathbf{x} - \mathbf{R}) \dot{\mathbf{R}} m_s (u_\parallel \mathbf{b} + \mathbf{v}_E) f_s + \mathcal{T}_w,
\end{aligned}
$$

where $K \equiv H - q\Phi_0$. The subscript oc denotes "oscillation center," and the subscript w denotes contributions from the wave fields which will be discussed in a moment. (It must be said that in the oscillation-center coordinates, the distinction between the particle and the wave contributions is fuzzy at best. We use "wave contribution" to denote those terms that come from the variations of the wave fields in the action principle.) These formulas can also be compared with Refs. 11,13,15.

The gyrogauge term in $\mathbf{M}$ also contributes to $\mathbf{S}_{oc}$ and $\mathcal{T}_{oc}$; in particular it makes $\mathcal{T}_{oc}$ asymmetric. This asymmetry can be attributed to the "spin" angular momentum that the

guiding centers possess. By applying the Noether theorem to the rotational symmetry of the oscillation-center action (88), we obtain the conservation law for angular momentum:

$$\epsilon_{ijk}(\mathcal{T}_{oc}^{ij} - \mathcal{T}_{oc}^{ji}) = \frac{\partial s_k}{\partial t} + \frac{\partial \mathcal{S}_{lk}}{\partial x^l}, \tag{120}$$

where $\mathbf{s}$ and $\mathcal{S}$ denote the spin density and spin flux density, given by

$$\begin{aligned} \mathbf{s} &= \sum_s \int d^6\mathbf{Z}\, \mathcal{J}_s \delta^3(\mathbf{x} - \mathbf{R}) \mathbf{b} p_g f_s + \mathbf{s}_w, \\ \mathcal{S} &= \sum_s \int d^6\mathbf{Z}\, \mathcal{J}_s \delta^3(\mathbf{x} - \mathbf{R}) \dot{\mathbf{R}} \mathbf{b} p_g f_s + \mathcal{S}_w. \end{aligned} \tag{121}$$

Eq. (120) has a nice interpretation: the left-hand side is the external torque on a plasma element which causes the intrinsic angular momentum of that element to change; the net rate of change is the right-hand side.

As is well known, the energy-momentum tensor is not uniquely defined. In particular since $\mathcal{T}_{oc}$ is asymmetric, it can not be equal to the particle momentum flux density (19). We can however symmetrize it by the standard procedure. The symmetrized version of momentum density and momentum flux density are given by

$$(\mathbf{g}_{oc})_s = \mathbf{g}_{oc} + \frac{1}{2}\nabla \times \mathbf{s}, \tag{122}$$

$$(\mathcal{T}_{oc}^{ij})_s = \mathcal{T}_{oc}^{ij} - \frac{1}{2}\epsilon^{ijk}\partial_t s_k + \frac{1}{2}\partial_k \left(\mathcal{S}^{il}\epsilon^{lkj} + \mathcal{S}^{jl}\epsilon^{lki} - \mathcal{S}^{kl}\epsilon^{lij}\right). \tag{123}$$

It is easy to verify that $(\mathcal{T}_{oc})_s$ is symmetric, and that

$$\partial_t(\mathbf{g}_{oc})_s + \nabla \cdot (\mathcal{T}_{oc})_s = -\left[\rho_{ext}\mathbf{E}_0 + \frac{1}{c}\mathbf{J}_{ext} \times \mathbf{B}_0\right]. \tag{124}$$

The derivation of the wave contributions to the conservation laws is quite involved, primarily because we have avoided using the usual eikonal form for the perturbations that would make the wave fields in the oscillation-center Lagrangian *local.* Kull et al[49] have developed a method for Lagrangians that contain arbitrary order derivatives of the wave fields. Their method can be easily generalized to infinite order and thus be applied to the

nonlocal case that we have. (An integral can be written as infinite derivatives, as is done in Eq. (98).) We shall only display the results here. Denote the wave energy-momentum tensor by $\mathcal{T}_w^{\mu\nu}$ (we use the convention that a Greek index runs from 0 to 3 and a Latin index runs from 1 through 3), so that $\mathcal{T}_w^{00} \equiv U_w$, $\mathcal{T}_w^{i0} \equiv S_w^i$, and $\mathcal{T}_w^{0i} \equiv g_w^i$. Then we have

$$\mathcal{T}_w^{\mu\nu}(x) = g^{\mu\nu}\mathcal{L}_2(x) + \int d^4s\, s^\mu \mathbf{I}(x-s,x)\alpha(s\cdot\overleftarrow{\partial} + s\cdot\overrightarrow{\partial})\cdot\partial^\nu \mathbf{E}_1(x)$$
$$- \frac{i}{2}\int \frac{d^4s d^4k}{(2\pi)^4}\frac{\partial\mathcal{D}}{\partial k_\mu}(x+\tfrac{1}{2}s,k)e^{-ik\cdot s} : \mathbf{E}_1(x+s)\alpha(-\tfrac{1}{2}s\cdot\overleftarrow{\partial} - \tfrac{1}{2}s\cdot\overrightarrow{\partial})\partial^\nu \mathbf{E}_1(x), \tag{125}$$

where $g^{\mu\nu}$ is the Minkowski metric, $\mathbf{I}(x,x')$ is defined by

$$\mathbf{I}(x,x') = \sum_s \int d^6\mathbf{Z}\, \mathcal{J}_s \delta^3(\mathbf{x}-\mathbf{R}) \sum_\ell \left[\int \frac{d^4k}{(2\pi)^4} w_\ell \mathbf{J}_\ell^* e^{i\psi_\ell(\theta;R,k)-ik\cdot x'}\right] f_s, \tag{126}$$

and α denotes a power series given by

$$\alpha(z) \equiv \int_0^1 d\tau e^{\tau z} = \frac{e^z - 1}{z}. \tag{127}$$

In Eq. (125), $\overleftarrow{\partial}$ acts on every x to its left and $\overrightarrow{\partial}$ acts on every x to its right. Similarly, denote the wave spin density and spin flux density by $\mathcal{S}_w^{\mu i}$ so that $\mathcal{S}_w^{0i} \equiv s_w^i$; then

$$(\mathcal{S}_w^{\mu i})(x) = -\int d^4s\, s^\mu \mathbf{I}(x-s,x)\alpha(s\cdot\overleftarrow{\partial} + s\cdot\overrightarrow{\partial}) \times \mathbf{E}_1(x)$$
$$+ \frac{i}{2}\int \frac{d^4s d^4k}{(2\pi)^4}\frac{\partial\mathcal{D}}{\partial k_\mu}(x+\tfrac{1}{2}s,k)e^{-ik\cdot s}\cdot \mathbf{E}_1(x+s)\alpha(-\tfrac{1}{2}s\cdot\overleftarrow{\partial} - \tfrac{1}{2}s\cdot\overrightarrow{\partial}) \times \mathbf{E}_1(x). \tag{128}$$

For an eikonal wave field $\mathbf{E}_1(x) = \mathbf{a}(x)e^{i\Theta(x)} + c.c.$, the nonresonant contributions to Eqs. (125) and (128) yield, to lowest order, the following eikonal expressions:

$$\mathcal{T}_w^{\mu\nu}(x) = \mathbf{a}^*(x)\cdot\left[g^{\mu\nu}\epsilon_{vac}(k(x)) - k^\nu(x)\frac{\partial\mathcal{D}}{\partial k_\mu}(x,k(x))\right]\cdot\mathbf{a}(x), \tag{129}$$

$$(\mathcal{S}_w^{\mu i})(x) = -i\mathbf{a}^*(x)\times\frac{\partial\mathcal{D}}{\partial k_\mu}(x,k(x))\cdot\mathbf{a}(x) + c.c., \tag{130}$$

where $k(x) \equiv \partial\Theta(x)$. Eq. (129) is previously known,[8] but Eq. (130) is a new result.

In summary: We have now obtained the general results of this paper. We have derived the oscillation-center Vlasov equation (87), the wave equation coupled to the resonant particles (94) or (96), and the Maxwell equations for the background electromagnetic fields (11) with definitions (116) through (118). We have also constructed the conservation laws for energy, momentum, and angular momentum, as given by Eqs. (16), (18), and (120) with definitions (119), (121), (125), and (128). These equations form a complete set that describes the self-consistent evolution of the Vlasov-Maxwell system.

VII. Linear Gyroresonance Equations

In this section we consider the *linearized* interaction of the resonant particles with the wave fields, and derive the coupling equations that can be used to describe linear ion gyroresonance problems.

We break the oscillation-center distribution function into two parts $f = \bar{f} + f_1$, where $\bar{f}$ represents the nonresonant particles and is assumed to be independent of the gyrophase θ, while f_1 represents the resonant particle distribution. Fourier expanding f_1 in both R and θ:

$$f_1 = \sum_\ell \int \frac{d^4k}{(2\pi)^4} f_\ell(u_\parallel, p_g, k) e^{i\psi_\ell(\theta;R,k)}, \tag{131}$$

and substituting it into the linearized Vlasov equation

$$\left(\frac{d}{dt}\right)_0 f_1 + \{\bar{f}, K_1\} = 0, \tag{132}$$

we obtain (see the justification above Eq. (68)), with K_1 from Eq. (64),

$$D_\ell(u_\parallel, p_g, R, k) f_\ell(u_\parallel, p_g, k) = (d_\ell \bar{f})\, w_\ell \mathbf{J}_\ell^*(u_\parallel, p_g, R, k) \cdot \mathbf{E}_1(k), \tag{133}$$

where D_ℓ is given by Eq. (63), and $d_\ell \bar{f}$ is defined by

$$d_\ell \bar{f} \equiv \{\psi_\ell, \bar{f}\} = \mathbf{k} \cdot \{\mathbf{R}, \bar{f}\} + \ell \left[\frac{\partial \bar{f}}{\partial p_g} + \frac{k_\parallel}{k_\perp^2}(\mathbf{k} \times \mathbf{b}) \cdot \{\mathbf{b}, \bar{f}\}\right]. \tag{134}$$

In Eq. (133) D_ℓ should be interpreted as an operator with $R \mapsto i\partial/\partial k$ according to Eq. (101). (The R-dependence of $d_\ell \bar{f}$ and $\mathbf{J}_\ell$ is normally unimportant, but if necessary it can also be interpreted as an operator in the same way.) Similarly, substituting $f = \bar{f} + f_1$ and Eq. (131) for f_1 into Eqs. (94) and (95), we obtain

$$\int \frac{d^4k'}{(2\pi)^4}\, \mathcal{D}(k,k') \cdot \mathbf{E}_1(k') = 2\pi \sum_{s,\ell} \int du_{\|} dp_g\, \mathcal{J}_s w_\ell \mathbf{J}_\ell(u_{\|}, p_g, R, k) f_\ell(u_{\|}, p_g, k). \qquad (135)$$

The factor 2π is the result of integrating over gyrophase θ. Eqs. (133) and (135) are the basic equations that describe the linear resonant coupling between the plasma and the electromagnetic waves.

In the following we consider only *gyroresonance*, for which $\ell \neq 0$. The $\ell = 0$ case is for *Landau resonance* and is physically quite different. We now present a heuristic interpretation for gyroresonance processes that can serve a guide in seeking analytic solutions for Eqs. (133) and (135).

In the usual approach to plasma wave problems, one eliminates the linear plasma response entirely and considers only the equation for the wave fields. Consequently one must deal with a non-Hermitian wave equation containing the plasma dispersion function. Such an equation can be very difficult to solve analytically. According to the philosophy of the congruent reduction algorithm, f_ℓ is not reducible since in the gyroresonance region $D_\ell \approx 0$, so we must solve Eqs. (133) and (135) together. On the other hand, by keeping the resonant particles out of the dispersion tensor $\mathcal{D}$ we may be able make better approximations for it, as is often the case.

In our mode conversion approach, we shall call the waves associated with f_ℓ the *ℓth harmonic gyroresonant ballistic waves*, since they are the eigenfunctions of the Vlasov equation *in the absence* of electromagnetic perturbations. They are also known as the *streaming functions* and are closely related to the Case-van Kampen waves.[50] The dispersion relation for

the gyroresonant ballistic waves is just the local gyroresonance condition $D_\ell = 0$. Applying the Hamilton ray equations (109) to D_ℓ as given by Eq. (63), we obtain

$$\begin{aligned} \dot{\mathbf{x}} &= \dot{\mathbf{R}}^{(0)} + \ell \frac{\partial}{\partial \mathbf{k}} \left[\frac{k_\parallel}{k_\perp^2} (\mathbf{k} \times \mathbf{b}) \cdot \dot{\mathbf{b}}^{(0)} \right], \\ \dot{\mathbf{k}} &= -\nabla \dot{\mathbf{R}}^{(0)} \cdot \mathbf{k} - \ell \nabla \left[\Omega + \frac{k_\parallel}{k_\perp^2} (\mathbf{k} \times \mathbf{b}) \cdot \dot{\mathbf{b}}^{(0)} \right]. \end{aligned} \tag{136}$$

Figure 3 depicts the ray of a gyroresonant ballistic wave and its projections onto the $\mathbf{x}$- and $\mathbf{k}$-space. The ray equations show that a gyroresonant ballistic wave travels in both $\mathbf{x}$ and $\mathbf{k}$ directions in the ray phase space. Consider the simplest one-dimensional (slab) model with the resonance condition $D_\ell(x) \equiv \omega - \ell\Omega(x) = 0$, then the ray equations become simply

$$\dot{x} = 0, \qquad \dot{k}_x = -\ell \Omega'(x). \tag{137}$$

Therefore in this case the phase change of the gyroresonant ballistic wave is entirely due to its propagation in the k-space. Staying in the x-space, one can see a stationary wave packet, with varying phase velocity. Unlike Landau resonance, the inhomogeneity of the magnetic field is of crucial importance here. Besides giving rise to $\dot{\mathbf{k}}$, it also effectively separates the gyroresonance regions of different harmonics.

Another important property of the gyroresonant ballistic waves is their kinetic character: they form a continuum, parametrized by $u_\parallel$ and p_g, and can superpose to produce collective waves such as the *Bernstein waves*. In this respect they are very much like the Case-van Kampen waves. The Hermiticity of equations (133) and (135) (after a trivial rescaling) makes them easier to solve; and it is from the *solutions* of these equations one can obtain the collective Bernstein waves by an appropriate superposition.

A clear picture of the nature of a gyroresonance process has emerged from the these considerations, as depicted by Figure 4. The upper surface represents the dispersion surface of an electromagnetic wave that comes in and traverses the gyroresonance region. There it excites a continuum of gyroresonant ballistic waves, which then travel in the ray phase space

according to Eq. (136). They may come across another dispersion surface of an electromagnetic wave. If they do, each of them can excite a wave on the second dispersion surface, as represented by the many rays there. Therefore, in the case when the two dispersion surfaces correspond to incident and reflected branches of the same wave, a single incident ray can generate a continuum of reflected rays, which then interfere to produce the *phasing effect.*[5] It is perhaps more interesting if the incident wave is a wave packet. Then because of the finite velocity of the gyroresonant ballistic waves propagating in the ray phase space, the reflected waves will emerge after a certain time lapse, giving rise to the phenomenon of *linear cyclotron echo.*[51,52]

For the purpose of obtaining analytic solutions to the gyroresonance equations, this picture suggests that we treat the transmission and reflection of an electromagnetic wave as two separate mode conversions, connected by the propagation of gyroresonant ballistic waves in between. For each mode conversion the dispersion function of the electromagnetic wave can be linearized around the respective mode conversion region, yielding a *first order* differential equation that is analytically soluble.[4-7]

To conclude this section let us construct the action principle for Eqs. (133) and (135). The action is a quadratic form, defined by

$$\begin{aligned} S_q = \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4k'}{(2\pi)^4} \Bigg[& \mathcal{D}(k,k') : \mathbf{E}_1(k')\mathbf{E}_1^*(k) \\ & + \sum_{s,\ell} \int du_\parallel dp_g \, \mathcal{J}_s \frac{D_\ell(u_\parallel, p_g, k, k')}{(d_\ell \bar{f}_s)/2\pi} f_\ell(u_\parallel, p_g, k') f_\ell^*(u_\parallel, p_g, k) \Bigg] \\ & - \sum_{s,\ell} \int \frac{d^4k}{(2\pi)^4} \left[2\pi \int du_\parallel dp_g \, \mathcal{J}_s w_\ell \mathbf{J}_\ell(u_\parallel, p_g, R, k) \cdot \mathbf{E}_1^*(k) f_\ell(u_\parallel, p_g, k) \right], \end{aligned} \tag{138}$$

where

$$D_\ell(u_\parallel, p_g, k, k') = D_\ell(u_\parallel, p_g, R \to i\partial/\partial k, k)\, \delta^4(k - k') \tag{139}$$

is the two-point dispersion function whose Weyl symbol is $D_\ell(u_\parallel, p_g, R, k)$. It can be readily checked that variations of S_q with respect to $\mathbf{E}_1^*$ and f_ℓ^* yield the correct equations. The

action principle makes the Hermiticity of the gyroresonance equations more apparent. It also suggests a definition of the wave action density associated with the gyroresonant ballistic waves. Eq. (138) shows that the actual dispersion function for f_ℓ is $2\pi D_\ell/d_\ell\bar{f}$, leading to the following definition of wave action density associated with f_ℓ by

$$2\pi \int du_{\|} dp_g \, \mathcal{J} \left[\frac{\partial}{\partial \omega} \left(\frac{D_\ell}{d_\ell \bar{f}} \right) W_\ell \right] (u_{\|}, p_g, x, k) \tag{140}$$

in analogy to $(\partial \mathcal{D}/\partial\omega) : \mathcal{W}_E$, which is the wave action density for an eletromagnetic wave.[28] W_ℓ is the Wigner function of f_ℓ, defined by

$$W_\ell(u_{\|}, p_g, x, k) = \left\langle \frac{1}{2} \int \frac{d^4\sigma}{(2\pi)^4} f_\ell(u_{\|}, p_g, k + \tfrac{1}{2}\sigma) f_\ell^*(u_{\|}, p_g, k - \tfrac{1}{2}\sigma) e^{i\sigma\cdot x} \right\rangle . \tag{141}$$

These formulas may be derivable directly from an Eulerian action principle[16,53] that is linearized and uses the guiding-center/oscillation-center coordinates.

In summary: In this section we have derived the linear gyroresonance coupling equations. We then interpreted gyroresonance processes as mode conversions between the electromagnetic waves and the gyroresonant ballistic waves. We have also shown how the mode conversion approach may lead to analytic solutions of ion gyroresonance problem in general geometry.

VIII. Discussion

Although this paper has been prepared with the ion gyroresonance problem in mind, the method that we use here is general enough so that the results obtained can be applied to many other problems as well. Below we discuss a few of the possible applications.

1. As long as relativistic effects are unimportant, our formalism can be applied directly to the problems of electron gyroresonance.
2. The heuristic interpretation of gyroresonance processes in Section VII applies only to *gyroresonance*, i.e., $\ell \neq 0$. Other than that we have not made any distinction between

gyroresonance and *Landau resonance* ($\ell = 0$). Our main results can be used for both cases. However, since Landau resonances often occur for frequencies much lower than the particle gyrofrequency, we can significantly simplify our formalism by dropping all the $\ell \neq 0$ terms in K_1.

3. By keeping only the $\ell = 0$ terms our formalism becomes essentially the *gyrokinetic theory*,[38] applicable to frequencies much lower than the particle gyrofrequency. The only essential restriction we have put on the perturbed fields is that they are of short wavelength so that the eikonal approximation is appropriate. The assumption that $\omega \neq 0$ is solely for the convenience of calculations. To remove it we need to go back to Section III and use Eq. (29) instead of Eq. (25) for the perturbed Lagrangian L_1. The whole calculation that follows can be carried out in just the same way as we have done. It is noteworthy that we although our calculations require the eikonal approximation, we have not used the usual eikonal representation $\mathbf{a}(x)e^{i\Theta(x)} + c.c.$ for (coherent) wave fields. In fact our formulas are fully nonlocal, so in principle one can use any representation.
4. Moreover, we do not even require the wave fields to be coherent. The Wigner function formalism is well suited for handling statistically random perturbations. (Indeed Wigner introduced it originally as a classical analogue of the quantal probability function.[54]) In this case the angular bracket in the definition (79) or (81) has the meaning of ensemble average. Ref. 27 contains a detailed discussion of the use of Wigner function for random perturbations.

IX. Conclusions

In conclusion, we have derived a complete set of self-consistent Vlasov-Maxwell equations in the guiding-center/oscillation-center coordinates. They include the Vlasov equation (87), the wave equation (94) or (96), and the Maxwell equations (11) with definitions (116), (117), and (118). Within this formalism, the macroscopic quantities such as the charge and current

densities are unambiguously defined. If the knowledge of exact particle orbits is not needed (such a knowledge may be required for studying particle loss to the wall, for example), we can treat these equations as the basic equations. There is no need to transform back to the particle coordinates in order to compute e.g. current.

We have also derived the conservation laws for energy, momentum, and angular momentum for the whole system (Eqs. (16), (18), and (120) with definitions (119), (121), (125), and (128)). An essential ingredient of our formalism is to treat the resonant and nonresonant regions of particle phase space separately. The coordinates in the nonresonant regions are transformed so that particles there respond linearly to the rapidly varying perturbations and nonlinearly to the ponderomotive effects, while the coordinates in the resonance region are not transformed and particles there can behave very nonlinearly.

Then we have derived the linearized gyroresonance coupling equations (133) and (135). These equations are interpreted as mode conversion equations between the electromagnetic perturbations and the gyroresonant ballistic waves. A heuristic picture is introduced that can be used as a guide in constructing analytic solutions.

Besides these results, we have also discussed some of the new mathematical methods that have been developed in plasma physics in the last decade; they include the self-consistent action principles for the Vlasov-Maxwell equations, the Lie transform technique, the Lagrangian guiding-center theory, the oscillation-center theory, the ray phase space eikonal theory; and the linear mode conversion in the ray phase space (plus the congruent reduction algorithm). This paper demonstrates how these powerful tools can be brought together to tackle a practical problem of plasma physics.

We have applied the formalism developed in this article to the analytic solution of the one-dimensional model of ion cyclotron heating of plasmas. We have obtained explicit formulas for the transmission, reflection, conversion, and absorption, for both the majority

ion second harmonic and minority ion fundamental gyroresonances.[6] This work is being prepared for publication.

Appendix A. Lagrangian Lie Transform

The traditional theory of canonical transformations is based on the mixed-variable generating functions. It is often rather awkward to use because it provides no systematic ways that can help in developing expansion series. The Hamiltonian Lie transform theory, developed in the nineteen-seventies, recognizes the fact that canonical transformations form a group, and by exploiting this group structure it proves itself far superior to the mixed-variable method (see Ref. 55 for a review of this method). But it is still limited to canonical transformations. The Lagrangian Lie transform theory removes this restriction by utlizing the intrinsic geometrical properties that underlies the system and the coordinate independent tools of differential geometry.

It thus allows us to use noncanonical but physical variables (e.g., the particle velocity $\mathbf{v}$ versus the gauge dependent canonical momentum $\mathbf{p}$), and the results will be manifestly gauge invariant. This is not insignificant because to prove the gauge invariance of a physical result, such as the ponderomotive Hamiltonian, that is obtained after a myriad of transformations, can be very difficult. Below we will summarize the main results; for their derivation and applications see Refs. 22,23. Ref. 15 also provides an easy-to-follow introduction to the subject, including the basics of differential geometry, with a good number of examples. For explicitness we shall write everything out in components.

The starting point is the phase space Lagrangian for single particle motion in an electromagnetic field, as introduced in Eqs. (3) in Section II. More generally the Lagrangian takes the following form:

$$L(\mathbf{z}, \dot{\mathbf{z}}, t) = \gamma_i(\mathbf{z}, t)\dot{z}^i - H(\mathbf{z}, t), \tag{A.1}$$

where i runs from 1 to $2N$, N being the number of degrees of freedom. It is convenient to include t as a coordinate, so as to make the problem autonomous. Thus we write $L\,dt = \gamma_\mu(z)dz^\mu$, with $z^\mu \equiv (t, \mathbf{z})$, and $\gamma_\mu \equiv (-H, \boldsymbol{\gamma})$. γ is called the *Lagrangian one-form* (an older name for one-form is *covector*), and its exterior derivative is called the *symplectic two-form* (also called the *Lagrange bracket*), usually denoted by ω:

$$\omega \equiv d\gamma = \frac{1}{2}\omega_{\mu\nu}\, dz^\mu \wedge dz^\nu, \tag{A.2}$$

where $\omega_{\mu\nu} \equiv \partial_\mu\gamma_\nu - \partial_\nu\gamma_\mu$. The equations of motion in this $(2N+1)$-dimensional extended phase space are Hamiltonian: $\omega_{\mu\nu}dz^\nu/d\lambda = 0$, where λ is an arbitrary parameter. Setting $\lambda = t$, this equation becomes

$$\dot{z}^i = J^{ij}\,\omega_{tj} = J^{ij}\,(\partial_t\gamma_j + \partial_j H), \tag{A.3}$$

where J^{ij} is the inverse of ω_{ij} and is called the *Poisson tensor*, because it defines the Poisson bracket

$$\{A, B\} = (\partial_i A)\, J^{ij}\, (\partial_j B). \tag{A.4}$$

Direct calculation can verify that the Poisson bracket thus defined satisfies the Jacobi identity. It can also be verified that Eq. (A.3) is indeed the correct Euler-Lagrange equation.

The Lagrangian Lie transform is a near identity coordinate transformation. It uses a generating vector field $g^\mu(z)$, and can be represented symbolically by

$$z \mapsto e^{\epsilon\mathcal{L}_g} z, \tag{A.5}$$

where ϵ is the small parameter, and the exponential is short-hand for the corresponding power series. $\mathcal{L}_g$ is called the *Lie derivative*, defined by the following rules:

when acting on a scalar:	$\mathcal{L}_g S = g^\mu\partial_\mu S;$
when acting on a vector:	$(\mathcal{L}_g V)^\mu = g^\nu\partial_\nu V^\mu - V^\nu\partial_\nu g^\mu;$
when acting on a one-form:	$(\mathcal{L}_g\gamma)_\mu = g^\nu\omega_{\nu\mu} + \partial_\mu(g^\nu\gamma_\nu).$

The use of the exponential map indicates that the inverse transform can be obtained by simply reversing the sign of the exponent. The Lie transformed scalar, vector, and one-form are, respectively,

$$S = e^{-\epsilon \mathcal{L}_g} s, \qquad V = e^{-\epsilon \mathcal{L}_g} v, \qquad \Gamma = e^{-\epsilon \mathcal{L}_g} \gamma. \tag{A.6}$$

Note that these are functional relationships: both sides are evaluated in the same coordinates.

Now suppose that the Lagrangian one-form that we start with is itself a power series in ϵ:

$$\gamma = \gamma^{(0)} + \epsilon \gamma^{(1)} + \epsilon^2 \gamma^{(2)} + \cdots. \tag{A.7}$$

Correspondingly we can make a succession of Lie transforms, represented by

$$z \mapsto \left[e^{\epsilon \mathcal{L}_{g_1}} e^{\epsilon^2 \mathcal{L}_{g_2}} \cdots \right] z. \tag{A.8}$$

The Lie transformed Lagrangian one-form then becomes

$$\Gamma = \cdots e^{-\epsilon^2 \mathcal{L}_{g_2}} e^{-\epsilon \mathcal{L}_{g_1}} \gamma. \tag{A.9}$$

Expanding it in powers of ϵ we obtain, order by order,

$$\begin{aligned} \Gamma_\mu^{(0)} &= \gamma_\mu^{(0)}, \\ \Gamma_\mu^{(1)} &= \gamma_\mu^{(1)} - g_1^\nu \omega_{\nu\mu}^{(0)} + \partial_\mu F_1, \\ \Gamma_\mu^{(2)} &= \gamma_\mu^{(2)} - g_2^\nu \omega_{\nu\mu}^{(0)} - \frac{1}{2} g_1^\nu \left(\omega^{(1)} + \Omega^{(1)} \right)_{\nu\mu} + \partial_\mu F_2, \end{aligned} \tag{A.10}$$

etc. We have added a perfect differential to each order except the lowest, which amounts to making a near identity Lagrangian gauge change. As has been pointed out in Section II, this has no effect on the equations of motion. Eq. (A.10) are useful for the purpose of averaging. By appropriately choosing the vector generators, we can remove, order by order, the oscillatory parts from γ, so that Γ is slowly varying. In principle this can be carried out to any order; for our purpose, the second order is sufficient.

Consider the $\Gamma^{(1)}$ equation in Eq. (A.10). Separating out the time component we have

$$\Gamma_i^{(1)} = \gamma_i^{(1)} - g_1^j \omega_{ij}^{(0)} - g_1^t \omega_{ti}^{(0)} + \partial_i F_1, \tag{A.11}$$

$$K_1 = H_1 + g_1^i \omega_{it}^{(0)} - \partial_t F_1, \tag{A.12}$$

where we have used $\gamma_t^{(1)} = -H_1$ and $\Gamma_t^{(1)} = -K_1$. There is considerable freedom in solving these equations. It is natural to set $g_1^t = 0$, so that time remains unchanged. Also we can set $\Gamma_i^{(1)} = 0$ and obtain

$$\left(\frac{d}{dt}\right)_0 F_1 = H_1 - \dot{z}^{(0)i}\gamma_i^{(1)} - K_1, \tag{A.13}$$

$$g_1^i = (\partial_j F_1 + \gamma_j^{(1)}) J^{ji}, \tag{A.14}$$

where J^{ij} now denotes the inverse of $\omega_{ij}^{(0)}$, and $(d/dt)_0 \equiv \partial_t + \dot{z}^{(0)i}\partial_i$ denotes the total time derivative along the unperturbed orbit. In obtaining Eqs. (A.13) and (A.14) we have used the unperturbed equations of motion (A.3). Now we choose K_1 to absorb the non-oscillatory part of the first order perturbation, if there is any,

$$K_1 = \left\langle H_1 - \dot{z}^{(0)i}\gamma_i^{(1)} \right\rangle, \tag{A.15}$$

where the angular bracket stands for "slowly varying part." F_1 is then solved by integrating Eq. (A.13) along the unperturbed orbit; since the non-oscillatory part has been removed from the right hand side by K_1, we can be sure that F_1 is free from secularity. Eq. (A.14) then yields the generating vector field g_1, which can be used in Eq. (A.5) to calculate the actual coordinate transformation.

Similar treatment of the $\Gamma^{(2)}$ equation leads to the following:

$$\left(\frac{d}{dt}\right)_0 F_2 = H_2 - \dot{z}^{(0)i}\gamma_i^{(2)} - K_2 + \frac{1}{2} g_1^i \left(\omega^{(1)} + \Omega^{(1)}\right)_{it} + \frac{1}{2} g_1^i \omega_{ji}^{(1)} \dot{z}^{(0)i}, \tag{A.16}$$

$$g_2^i = \left(\partial_j F_2 + \gamma_j^{(2)} - \frac{1}{2} g_1^k \omega_{kj}^{(1)}\right) J^{ji}. \tag{A.17}$$

Now K_2 has to absorb the non-oscillatory parts from both the second order perturbation, and the quadratic terms of the first order perturbation:

$$K_2 = \left\langle H_2 - \dot{z}^{(0)i}\gamma_i^{(2)} + \frac{1}{2}g_1^i\left(\omega^{(1)} + \Omega^{(1)}\right)_{it} + \frac{1}{2}g_1^j\,\omega_{ji}^{(1)}\,\dot{z}^{(0)i}\right\rangle. \tag{A.18}$$

Here we see how a lower order oscillatory perturbation shows up as a higher order correction after averaging. For the purpose of this paper, we only need to find the ponderomotive K_2, so it is not necessary to calculate g_2.

If the original perturbation is in the Hamiltonian only, then these formulas simplify considerably. Now that $\gamma_i^{(1)} = \gamma_i^{(2)} = 0$, Eqs. (A.13), (A.14), and (A.18) become

$$\left(\frac{d}{dt}\right)_0 F_1 = H_1 - K_1, \tag{A.19}$$

$$g_1^i = \{F_1, z^i\}, \tag{A.20}$$

$$K_2 = \left\langle H_2 - \frac{1}{2}\{F_1, H_1 + K_1\}\right\rangle. \tag{A.21}$$

In this case the Lie transform is called the *Hamiltonian Lie transform.*[55] Eqs. (A.19) and (A.21) were used in Section IV.

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References

[1] "Radio-frequency Power in Plasmas," AIP Conf. Proc. **190**, Irvine, CA (1989).

[2] "Radio-frequency Power in Plasmas," Ninth Topical Conference, Charleston, August (1991), to be published as AIP Conf. Proc.

[3] R. A. Cairns, "Radiofrequency Heating of Plasmas" (A. Hilger, Bristol, England, 1991.)

[4] H. Ye and A. N. Kaufman, Phys. Rev. Lett. **61**, 2762 (1988).

[5] H. Ye and A. N. Kaufman, in "A Variety of Plasmas," eds. A. Sen and P. Kaw (Indian Academy of Sciences, Bangalore, 1991), p.273.

[6] H. Ye, Ph.D. Dissertation, University of California at Berkeley (1990).

[7] A. N. Kaufman, A. J. Brizard, D. R. Cook, E. R. Tracy, and H. Ye, in "Radio-Frequency Powers in Plasmas," (Ref. 2).

[8] A. N. Kaufman, in "Nonlinear and Chaotic Phenomena in Plasmas, Solids, and Fluids," eds. W. Rozmus and J. Tuszynski (World Scientific, Singapore, 1990), p.160.

[9] F. E. Low, Proc. Roy. Soc. **A248**, 282 (1958).

[10] R. L. Dewar, Austr. J. Phys. **30**, 533 (1977).

[11] P. L. Similon, Phys. Lett. A **112**, 33 (1985).

[12] D. Pfirsch, Z. Naturforsch. **39a**, 1 (1984).

[13] D. Pfirsch and P. J. Morrison, Phys. Rev. A **32**, 1714 (1985).

[14] R. G. Littlejohn, J. Plasma Phys. **29**, 111 (1983).

[15] B. M. Boghosian, Ph.D. Dissertation, University of California at Davis (1987).

[16] H. Ye and P. J. Morrison, to appear in Phys. Fluids B, (1992).

[17] R. R. Dominguez and H. L. Berk, Phys. Fluids **27**, 1142 (1984).

[18] R. L. Dewar, Phys. Fluids **16**, 1102 (1973).

[19] J. R. Cary and A. N. Kaufman, Phys. Fluids **24**, 1238 (1981).

[20] C. Grebogi, A. N. Kaufman, and R. G. Littlejohn, Phys. Rev. Lett. **43**, 1668 (1979).

[21] C. Grebogi and R: G. Littlejohn, Phys. Fluids **27**, 1996 (1984).

[22] R. G. Littlejohn, J. Math. Phys. **23**, 742 (1982).

[23] J. R. Cary and R. G. Littlejohn, Ann. Phys. **151**, 1 (1983).

[24] S. W. McDonald, C. Grebogi, and A. N. Kaufman, Phys. Lett. A **111**, 19 (1985).

[25] C. Grebogi and A. N. Kaufman, in "Long-Time Prediction in Dynamics," (C. W. Horton, L. Reichl, and V. Szebehely eds., Wiley, New York, 1983), p.247.

[26] S. W. McDonald, Phys. Rev. Lett. **54**, 1211 (1985).

[27] S. W. McDonald and A. N. Kaufman, Phys. Rev. A **32**, 1708 (1985).

[28] S. W. McDonald, Phys. Reports **158**, 337 (1989).

[29] L. Friedland, G. Goldner, and A. N. Kaufman, Phys. Rev. Lett. **58**, 1392 (1987).

[30] A. N. Kaufman and L. Friedland, Phys. Lett. A **123**, 387 (1987).

[31] R. G. Littlejohn, Phys. Reports **138**, 193 (1986).

[32] L. Friedland and A. N. Kaufman, Phys. Fluids **30**, 3035 (1987).

[33] L. Friedland, Phys. Fluids **31**, 2615 (1988).

[34] R. A. Cairns and C. Lashmore-Davies, Phys. Fluids **26**, 1268 (1983).

[35] R. A. Cairns and C. Lashmore-Davies, Phys. Fluids **29**, 3639 (1986).

[36] H. Goldstein, "Classical Mechanics," second ed. (Addison-Wesley, New York, 1980).

[37] D. E. Soper, "Classical Field Theory" (Wiley, New York, 1978).

[38] A. J. Brizard, J. Plasma Phys. **41**, 541 (1989).

[39] R. G. Littlejohn, Contemporary Mathematics **28**, 151 (1984).

[40] T. G. Northrop, "The Adiabatic Motion of Charged Particles" (Interscience, New York, 1963).

[41] R. G. Littlejohn, Phys. Rev. A **38**, 6034 (1988).

[42] A. N. Kaufman, Phys. Fluids **15**, 1063 (1972).

[43] R. D. Hazeltine, S. M. Mahajan, and D. A. Hitchcock, Phys. Fluids, **24**, 1164 (1978).

[44] H. L. Berk and D. Pfirsch, Phys. Fluids **31**, 1532 (1988).

[45] J. D. Crawford, S. Johnston, A. N. Kaufman, and C. Oberman, Phys. Fluids **29**, 3219 (1986).

[46] A. N. Kaufman, Phys. Rev. A **36**, 982 (1987).

[47] A. N. Kaufman, H. Ye, and Y. Hui, Phys. Lett. A **120**, 327 (1985).

[48] A. N. Kaufman, Phys. Fluids **29**, 1736 (1986).

[49] H. J. Kull, H. L. Berk, and P. J. Morrison, Phys. Fluids B **1**, 55 (1989).

[50] Z. Sedláček, Czech. J. Phys. **B22**, 439 (1972).

[51] D. B. Muldrew and E. L. Hagg, Canadian J. Phys. **44**, 925 (1966).

[52] H. Ye and A. N. Kaufman, Phys. Rev. Lett. **60**, 1642 (1988).

[53] J. Larsson, submitted to J. Plasma Phys. (1992).

[54] E. P. Wigner, Phys. Rev. **40**, 749 (1932).

[55] J. R. Cary, Phys. Reports **79**, 131 (1981).

Figure Captions

Figure 1. A dispersion surface in eight-dimensional ray phase space and a ray that propagates on it.

Figure 2. A diagram depicting a single linear mode conversion in the eight-dimensional ray phase space. The intersection of the two dispersion surfaces, represented here by a line, is itself a surface. The mode conversion occurs around a point, shown here by the dot, where a ray of wave *a* crosses the dispersion surface of wave *b*.

Figure 3. A diagram showing a ray of gyroresonant ballistic waves in six-dimensional ray phase space and its projections in **x**- and **k**-space. The projection in **x**-space is essentially the guiding-center orbit; the projection in **k**-space shows the propagation due to magnetic field inhomogeneity.

Figure 4. A schematic diagram depicting a typical gyroresonance process as viewed in ray phase space. A ray of the incident electromagnetic wave comes in on the upper dispersion surface. As it crosses the gyroresonance layer, its excites a continuum of gyroresonant ballistic waves. The gyroresonant ballistic waves then propagate in six-dimensional ray phase-space and intersect the dispersion surface of the reflection branch (the lower surface). Each gyroresonant ballistic wave mode-converts to a reflected wave. In general there is a continuum of reflected waves that interfere and cause the phasing effect.

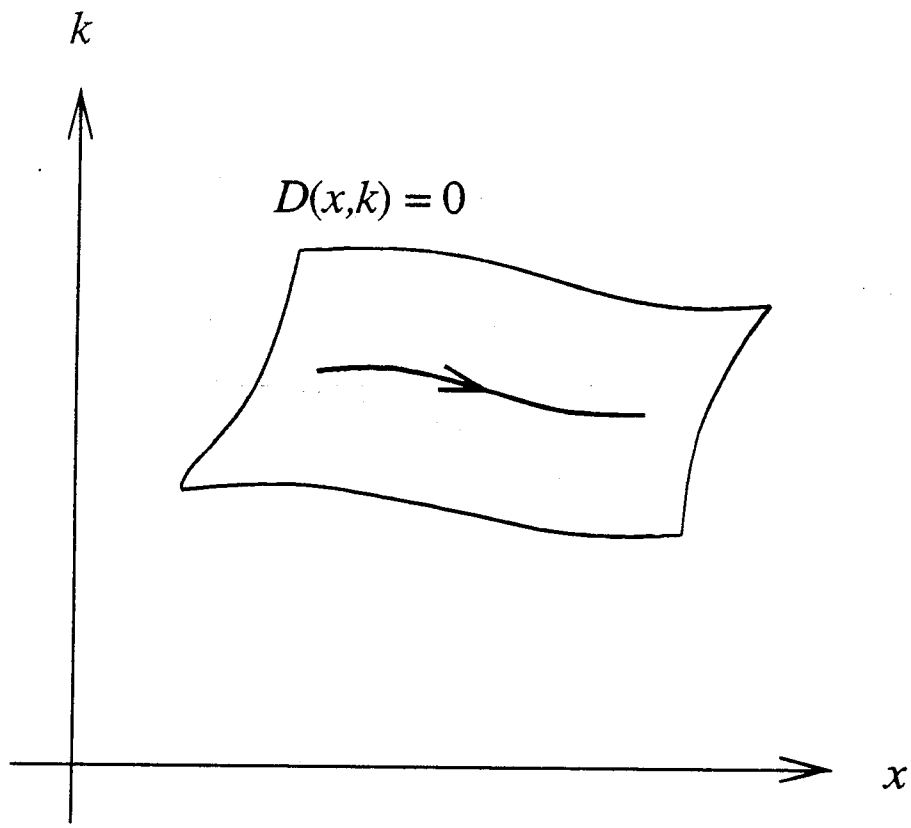

Figure 1

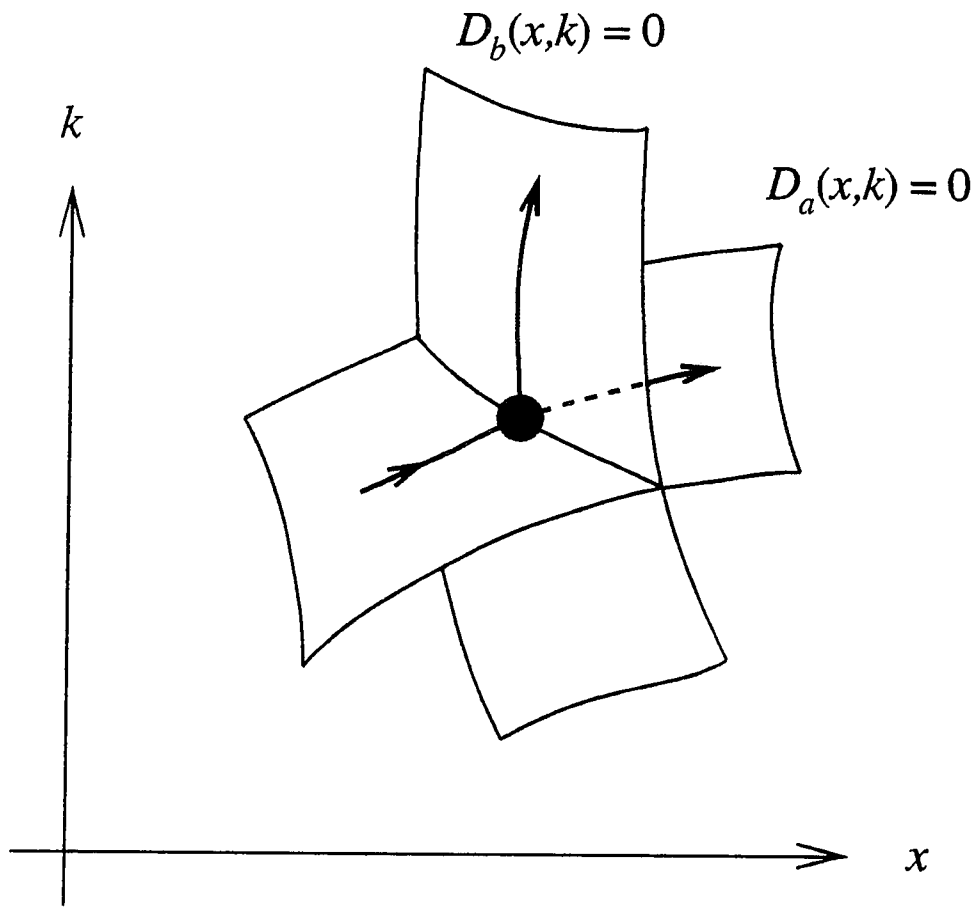

Figure 2

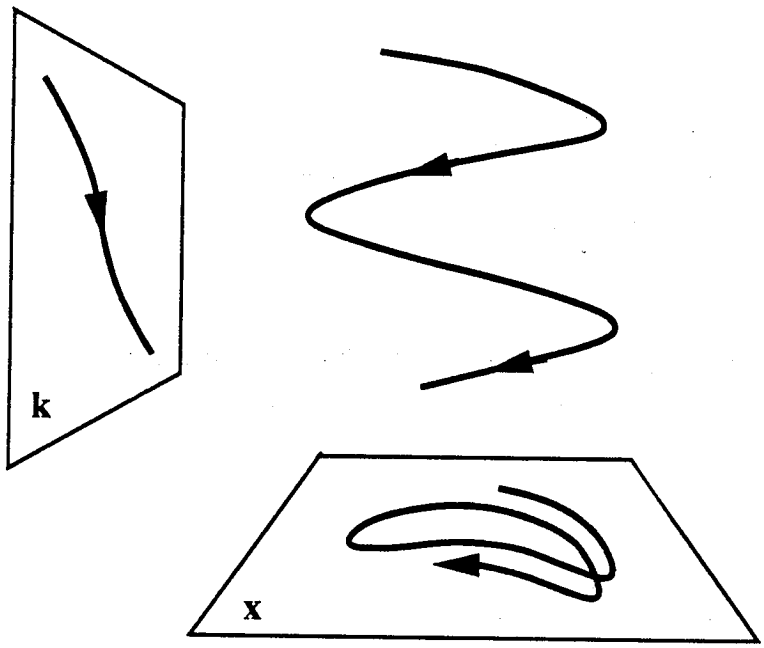

Figure 3

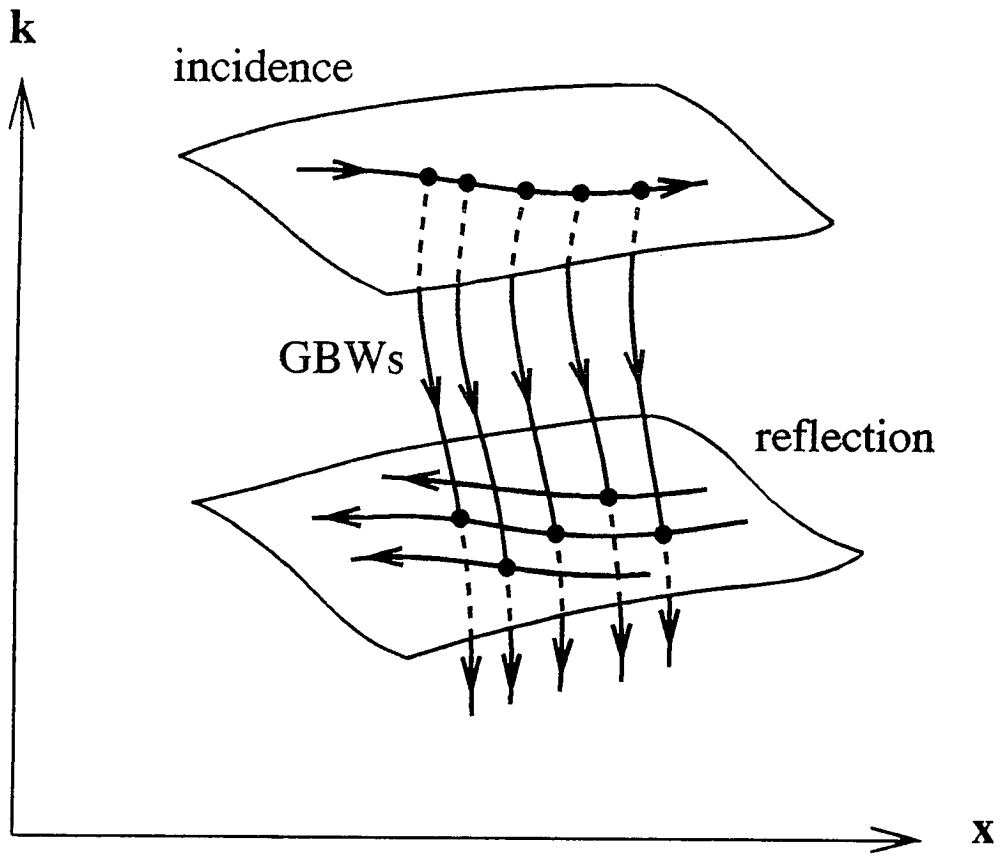

Figure 4