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Abstract

Equilibria with minimum energy are constructed from a variational principle in which the energy of a plasma is minimized subject to a recently proposed set of global invariants. The equilibrium equation is solved in axisymmetric, toroidal geometry. In order to compute toroidal equilibria, we exploit the variational principle to obtain a reduced set of ordinary differential equations which we solve numerically. We find tokamak-like and pinch-like solutions of minimum energy in toroidal geometry. Based on the ideal and resistive stability studies of the cylindrical limit of these solutions, it is argued that some of these equilibria should have robust stability to modes of low m- and n-number.

## I. INTRODUCTION

Recently[1,2,3], we have developed a variational principle in which the total energy of a plasma is minimized subject to a set of global invariants. The choice of the appropriate set of global invariants is based on a model of turbulent relaxation dominated by a tearing mode of single helicity. Since these invariants are preserved rigorously by all ideal motions and approximately by the class of nonideal motions of the plasma permitted by the model, those relaxed states which have minimum energy may be expected to be at once ideally and resistively stable to the modes for which the model holds good. In previous papers, we have solved the equilibrium equation in cylindrical geometry for a plasma with zero pressure and vanishing current density on the boundary assuming that equilibrium quantities depend only on radius. A study[3] of the ideal and resistive stability of these equilibria has shown that some of them are stable to a wide band of modes of low m- and n-number.

In this paper, we present solutions to the equilibrium equation, which we have derived from the variational principle, in axisymmetric, toroidal geometry. The variational principle itself offers a natural starting point for the numerical computation of toroidal equilibria. The procedure is essentially the same as that used in a recent paper[4] on the computation of three-dimensional equilibria, except that it employs a different variational principle. Specializing to a pressureless plasma, we find that, as in the case of equilibrium solutions in cylindrical geometry, there are both tokamak-like and pinch-like solutions of minimum energy with q-profiles respectively increasing and decreasing from the magnetic axis to the wall. On the basis of our exploratory results on the ideal and resistive stability of the cylindrical solutions, we argue that the toroidal equilibria computed in the present paper should be stable to a wide band of current-driven instabilities of low mode number. The extent of the band of stability may be made precise by subjecting these equilibria to a detailed stability test by numerical codes such as PEST[5] or PEST2.[6]

We now explain briefly the plan of this paper. In Section II we state the variational principle and obtain the Euler-Lagrange equations describing static equilibria in axisymmetric toroids. In Section III we derive from the variational principle an approximate set of reduced equations which describe these equilibria accurately and obtain numerical solutions. Section IV contains a description of the toroidal solutions to the equilibrium equation derived from Taylor's model[7] of relaxation of toroidal discharges, and Section V is a detailed discussion of toroidal equilibria in the present model.

## II. ENERGY PRINCIPLE WITH GLOBAL INVARIANTS[1,2]

We seek minima of the potential energy functional

$$W = \int_{V_0} d\tau \left[ \frac{B^2}{2} + \frac{p}{\gamma - 1} \right] , \tag{1}$$

for a toroidal plasma of total volume $V_0$ bounded by a perfectly conducting wall; $\vec{B}$ is the magnetic field, $p$ the pressure, and $\gamma$ the specific heat of the plasma. The magnetic field $\vec{B}$ in a toroidal plasma with nested surfaces may be represented as[8]

$$\vec{B} = \vec{\nabla}\zeta \times \vec{\nabla}\Psi + \vec{\nabla}\Phi \times \vec{\nabla}\theta , \tag{2}$$

in the magnetic coordinate system $(\Psi, \theta, \zeta)$. Here $\Psi$ is the poloidal flux function which labels magnetic surfaces, $\Phi$ the toroidal flux function, and $\theta$ and $\zeta$ are respectively the poloidal and toroidal angles parameterizing any given surface. It has been shown[2] that under the class $I$ of ideal Eulerian variations given by

$$\delta\Psi = -\vec{\xi}\cdot\vec{\nabla}\Psi , \tag{3a}$$

$$\delta\Phi = q(\Psi)\,\delta\Psi , \tag{3b}$$

$$\delta\rho = -\vec{\nabla}(\rho\vec{\xi}) , \tag{3c}$$

$$\delta p = -\gamma p \vec{\nabla}\cdot\vec{\xi} - \vec{\xi}\cdot\vec{\nabla}p , \tag{3d}$$

where $\rho$ is the density, $q$ the safety factor and $\vec{\xi}(\vec{r},t)$ is a virtual displacement of the plasma, the following global quantities are strictly conserved:

$$M[u_\alpha] = \int_{V_0} d\tau u_\alpha(\Psi,\Phi)\rho \quad , \tag{4}$$

$$S[v_\alpha] = \int_{V_0} d\tau v_\alpha(\Psi,\Phi) \frac{\rho}{\gamma - 1} \ln\left(\frac{p}{\rho^\gamma}\right) \quad , \tag{5}$$

and

$$K[\omega_\alpha] = \int_{V_0} d\tau \omega_\alpha(\Psi,\Phi) \frac{\vec{A}\cdot\vec{B}}{2} \quad . \tag{6}$$

Here, $u_\alpha(\Psi,\Phi)$, $v_\alpha(\Psi,\Phi)$ and $\omega_\alpha(\Psi,\Phi)$ are arbitrary sequences of basic functions. The vector potential $\vec{A}(\vec{B} = \vec{\nabla}\times\vec{A})$ is represented by

$$\vec{A} = \Phi\vec{\nabla}\theta - \Psi\vec{\nabla}\zeta \quad , \tag{7}$$

assuming that $\Phi = 0$ on the magnetic axis and $\Psi = 0$ on the wall. We now seek minima of the free energy functional $F \equiv W - \sum_\alpha[\lambda_\alpha K(\omega_\alpha) + \mu_\alpha M(u_\alpha) + T_\alpha S(v_\alpha)]$ , under independent variations $\delta\Psi$, $\delta\Phi$, $\delta\theta$, $\delta p$ and $\delta\rho$ . The Euler-Lagrange equations for variations with respect to $\Psi$, $\Phi$, $\theta$, $p$ and $\rho$, under the appropriate fixed-boundary constraints[2], are respectively given by

$$\vec{J}\cdot\nabla\zeta - \frac{q}{\mathcal{J}}\sum_\alpha \lambda_\alpha\left[\frac{\Psi}{2}\frac{\partial\omega_\alpha}{\partial\Psi} + \frac{\Phi}{2}\frac{\partial\omega_\alpha}{\partial\Phi} + \omega_\alpha\right] + \frac{\partial p}{\partial\Psi} = 0 \quad , \tag{8a}$$

$$\vec{J}\cdot\vec{\nabla}\theta - \frac{1}{\mathcal{J}}\sum_\alpha \lambda_\alpha\left[\frac{\Psi}{2}\frac{\partial\omega_\alpha}{\partial\Psi} + \frac{\Phi}{2}\frac{\partial\omega_\alpha}{\partial\Phi} + \omega_\alpha\right] - \frac{\partial p}{\partial\Phi} = 0 \quad , \tag{8b}$$

$$\vec{J}\cdot\vec{\nabla}\Phi = 0 \quad , \tag{8c}$$

$$p = \rho\sum_\alpha T_\alpha v_\alpha \quad , \tag{8d}$$

$$\sum_\alpha \mu_\alpha u_\alpha = \frac{1}{\gamma - 1}\left[\gamma - \ln\left(\frac{p}{\rho^\gamma}\right)\right]\sum_\alpha T_\alpha v_\alpha \tag{8e}$$

where $\mathcal{J} \equiv (\vec{\nabla}\Psi\cdot\vec{\nabla}\theta\times\vec{\nabla}\zeta)^{-1}$ is the Jacobian. Note that we have held the toroidal angle $\zeta$ fixed and identified it with the azimuthal angle of the cylindrical coordinate system $(R,\zeta,Z)$. If we do not fix $\zeta$, its variation leads to the Euler-Lagrange equation $\vec{J}\cdot\nabla\Psi = 0$, which is no more than a restatement of the equilibrium constraint already implied by equation (8c), that is, the current density $\vec{J}$ should be on flux surfaces. From equations (8a - 8c), we get[2]

$$\vec{J} = \sum_{\alpha}\lambda_{\alpha}\left[\frac{\Psi}{2}\frac{\partial\omega_{\alpha}}{\partial\Psi} + \frac{\Phi}{2}\frac{\partial\omega_{\alpha}}{\partial\Phi} + \omega_{\alpha}\right]B - \mathcal{J}\left[\frac{\partial p}{\partial\Psi}\vec{\nabla}\Psi\times\vec{\nabla}\theta - \frac{\partial p}{\partial\Phi}\vec{\nabla}\zeta\times\vec{\nabla}\Psi\right] . \quad (9)$$

It is readily seen that

$$\vec{J}\times\vec{B} = \vec{\nabla}p , \quad (10)$$

which implies that equations (8a - 8e) represent magnetostatic equilibria.

We remark that by treating the variations $\delta\Psi$, $\delta\Phi$, $\delta p$, and $\delta\rho$ as independent, we have enlarged the class of motions allowed to the plasma to one that includes the ideal class $I$ as a subclass. For example, by relaxing the requirement that $\delta\Phi$ be tied to $\delta\Psi$ everywhere through the equation (3b), we allow $\Phi$-surfaces to "slip" with respect to $\Psi$-surfaces and effectively remove the topological constraint that the "degree of knottedness"[9] of a field line remains fixed.

From here onwards, we will consider a pressureless plasma. The quasi-ideal model of Kadomtsev and Monticello, and some additional requirements specified in our earlier paper[2], leads us to the natural choice

$$\omega_\alpha = \chi^\alpha \quad , \tag{11}$$

where $\chi = q_s\Psi - \Phi$ is the helical flux preserved (approximately) by the dominant mode of pitch $q_s$ , and $\alpha$ is zero or a positive integer. Under this choice, equations (8a) and (8b) become respectively

$$\vec{J}\cdot\vec{\nabla}\zeta - \frac{q}{\mathcal{J}}\sum_\alpha \lambda_\alpha \frac{(\alpha + 2)}{2} \chi^\alpha = 0 \quad , \tag{12a}$$

$$\vec{J}\cdot\vec{\nabla}\theta - \frac{1}{\mathcal{J}}\sum_\alpha \lambda_\alpha \frac{(\alpha + 2)}{2} \chi^\alpha = 0 \quad . \tag{12b}$$

## III. AXISYMMETRIC TOROIDAL EQUILIBRIA

In order to compute the equilibria determined by equations (12a,b) in axisymmetric toroidal geometry, we use a variant of the method proposed in Ref. 4. We now discuss this method and derive the appropriate set of reduced equations.

In Fig. 1, we show the cylindrical coordinate system $(R,\zeta,Z)$ and the magnetic coordinate system $(v,\theta,\zeta)$, where $v$ is a radial label, not necessarily identical to $\Psi$. The magnetic field $\vec{B}$ is represented as[1]

$$\vec{B} = \vec{\nabla}\zeta \times \vec{\nabla}\Psi(v) + \vec{\nabla}\Phi(v) \times \vec{\nabla}\theta \quad . \tag{13}$$

Since the free energy $F$ is a scalar, it is independent of the coordinate system in which it is expressed. For axisymmetric systems, we will characterize equilibria by the inverse mapping,

$$R = R(v,\theta) \quad , \tag{14a}$$

$$Z = Z(v,\theta) \quad , \tag{14b}$$

whence, the roles of dependent and independent variables are interchanged, and $R$ and $Z$ become the new dependent variables. For a pressureless plasma, it may be shown by straightforward algebra that

$$F \equiv 2\pi\int_0^a dv \int_0^{2\pi} d\theta\, \mathcal{F}(R,R_v,R_\theta,Z_v,Z_\theta,\Psi,\Psi_v,\Phi,\Phi_v)$$

$$= 2\pi\int_0^a dv \int_0^{2\pi} d\theta \left[\frac{\Psi_v^2}{2}\frac{g_{\theta\theta}}{\sqrt{\|g\|}} + \frac{\Phi_v^2}{2}\frac{g_{\zeta\zeta}}{\sqrt{\|g\|}} - \sum_\alpha \frac{\lambda_\alpha}{2}\,\omega_\alpha(\Phi\Psi_v - \Psi\Phi_v)\right] , \tag{15}$$

where $v = a$ is the label for the outermost flux surface, in contact with the conducting wall. The nonvanishing elements of the metric tensor $g_{ij} \equiv (\partial\vec{r}/\partial x_i)\cdot(\partial\vec{r}/\partial x_j)$ for the transformation from $(x,y,z)$ coordinates to $(v,\theta,\zeta)$ coordinates are

$$g_{vv} = R_v^2 + Z_v^2 \ , \tag{16a}$$

$$g_{v\theta} = R_v R_\theta + Z_v Z_\theta = g_{\theta v} \ , \tag{16b}$$

$$g_{\theta\theta} = R_\theta^2 + Z_\theta^2 \ , \tag{16c}$$

$$g_{\zeta\zeta} = R^2 \ , \tag{16d}$$

and

$$\sqrt{\|g\|} \equiv \sqrt{\det\, g_{ij}} = R(R_\theta Z_v - R_v Z_\theta) \tag{16e}$$

($f_x$ denotes the partial derivative of $f$ with respect to $x$). The Euler-Lagrange equations for variations with respect to $R$ and $Z$ are respectively given by[4]

$$G_R \equiv RZ_\theta \left[ \frac{\Psi_v}{\sqrt{\|g\|}} \left( \frac{\partial}{\partial v} \frac{g_{\theta\theta}\Psi_v}{\sqrt{\|g\|}} - \frac{\partial}{\partial \theta} \frac{g_{v\theta}\Psi_v}{\sqrt{\|g\|}} \right) + \frac{\Phi_v}{\sqrt{\|g\|}} \frac{\partial}{\partial v} \frac{g_{\zeta\zeta}\Phi_v}{\sqrt{\|g\|}} - \frac{\Phi_v^2}{2\|g\|} \frac{\partial}{\partial v} g_{\zeta\zeta} \right] - \frac{\Phi_v^2}{2} RZ_v \frac{\partial}{\partial \theta} \frac{g_{\zeta\zeta}}{\|g\|} - \frac{\Phi_v^2 R}{\sqrt{\|g\|}} = 0 \ , \tag{17}$$

and

$$G_Z \equiv RR_\theta \left[ \frac{\Psi_v}{\sqrt{\|g\|}} \left( \frac{\partial}{\partial v} \frac{g_{\theta\theta}\Psi_v}{\sqrt{\|g\|}} - \frac{\partial}{\partial \theta} \frac{g_{v\theta}\Psi_v}{\sqrt{\|g\|}} \right) + \frac{\Phi_v}{\sqrt{\|g\|}} \frac{\partial}{\partial v} \frac{g_{\zeta\zeta}\Phi_v}{\sqrt{\|g\|}} - \frac{\Phi_v^2}{2\|g\|} \frac{\partial}{\partial v} g_{\zeta\zeta} \right] - \frac{\Phi_v^2}{2} RR_v \frac{\partial}{\partial \theta} \frac{g_{\zeta\zeta}}{\|g\|} = 0 \ . \tag{18}$$

It may be shown by linear combination of equations (17) and (18) that

$$\frac{\partial}{\partial \theta} \frac{g_{\zeta\zeta}}{\sqrt{\|g\|}} = 0 \ , \tag{19}$$

which may be incorporated in the variational form $F$ at the outset by defining the toroidal field function

$$I(v) \equiv \frac{g_{\zeta\zeta}}{\sqrt{\|g\|}} \Phi_v \ , \tag{20}$$

or equivalently, by representing $\vec{B}$ as

$$\vec{B} = \vec{\nabla}\zeta \times \vec{\nabla}\Psi + I\vec{\nabla}\zeta \ . \tag{21}$$

If we do so, the two equations (17) and (18) become each identical to

$$G \equiv \frac{1}{\sqrt{\|g\|}} \left( \frac{\partial}{\partial v} \frac{g_{\theta\theta}\Psi_v}{\sqrt{\|g\|}} - \frac{\partial}{\partial \theta} \frac{g_{v\theta}\Psi_v}{\sqrt{\|g\|}} \right) + \frac{II_\Psi}{R^2} = 0 \ , \tag{22}$$

which is the Grad-Shafranov equation in inverse variables for a pressureless plasma.

There are two remaining Euler-Lagrange equations which are found by varying $\Psi$ and $\Phi$ independently. These are

$$\left\langle \frac{\partial}{\partial v} \frac{\partial F}{\partial \Psi_v} - \frac{\partial F}{\partial \Psi} \right\rangle = 0 \ , \tag{23}$$

and

$$\left\langle \frac{\partial}{\partial v} \frac{\partial F}{\partial \Phi_v} - \frac{\partial F}{\partial \Phi} \right\rangle = 0 \quad , \tag{24}$$

where $<\;>$ is defined by

$$<A> \equiv \frac{1}{2\pi} \int_0^{2\pi} d\theta A(v,\theta) \quad . \tag{25}$$

Equations (23) and (24) give, respectively,

$$\left\langle \frac{\partial}{\partial v} \left( \Psi_v \frac{g_{\theta\theta}}{\sqrt{\|g\|}} \right) - \sum_\alpha \lambda_\alpha \left( \frac{\Psi}{2} \frac{\partial \omega_\alpha}{\partial \Psi} + \frac{\Phi}{2} \frac{\partial \omega_\alpha}{\partial \Phi} + \omega_\alpha \right) \Phi_v \right\rangle = 0 \quad , \tag{26}$$

and

$$\left\langle \frac{\partial}{\partial v} \left( \Phi_v \frac{g_{\zeta\zeta}}{\sqrt{\|g\|}} \right) + \sum_\alpha \lambda_\alpha \left( \frac{\Psi}{2} \frac{\partial \omega_\alpha}{\partial \Psi} + \frac{\Phi}{2} \frac{\partial \omega_\alpha}{\partial \Phi} + \omega_\alpha \right) \Psi_v \right\rangle = 0 \quad . \tag{27}$$

Multiplying equation (26) by $\Psi_v$ , equation (27) by $\Phi_v$ and adding the resultant equations, we get

$$\left\langle \Psi_v \frac{\partial}{\partial v} \left( \Psi_v \frac{g_{\theta\theta}}{\sqrt{\|g\|}} \right) + \Phi_v \frac{\partial}{\partial v} \left( \Phi_v \frac{g_{\zeta\zeta}}{\sqrt{\|g\|}} \right) \right\rangle = 0 \quad , \tag{28}$$

which is identical to the surface-averaged Grad-Shafranov equation

$$< \sqrt{\|g\|}\, G > = 0 \quad . \tag{29}$$

Therefore, once equations (17) and (18) [or the equivalent equations (19) and (22)] are given, only one of equations (26) and (27) is linearly independent.

Exploiting the periodicity of the inverse mapping [equations (14)] in $\theta$ and assuming the equilibria to be up-down symmetric, we write

$$R(v,\theta) = \sum_{m=0}^{\infty} R_m(v) \cos m\theta \quad , \qquad (30a)$$

$$Z(v,\theta) = \sum_{n=1}^{\infty} Z_m(v) \sin m\theta \quad . \qquad (30b)$$

Substituting the Fourier series [equations (30)] directly in equation (15) and varying each of the Fourier amplitudes $R_m(v)$ and $Z_n(v)$ independently, we obtain formally the following infinite set of coupled ordinary differential equations

$$\langle G_R \cos m\theta \rangle = 0 \qquad m = 0,1,2,\ldots \qquad (31)$$

$$\langle G_Z \sin n\theta \rangle = 0 \qquad n = 1,2,3\ldots \quad .(32)$$

We have not yet chosen any specific flux surface label $v$. Following Ref. 4, we make the choice $v \equiv -R_1$. We may not then vary the free energy $F$ with respect to $R_1$, but the variation of $F$ with respect to both $\Psi$ and $\Phi$ give independent conditions. Thus, equations (26) and (27), which imply equation (29), are now to be treated as independent equations.

For numerical computations, we truncate the infinite Fourier series for the inverse mapping to obtain a reduced set of equations. Indeed, it is found that the Fourier series converges rapidly (to be precise, the Fourier amplitudes $R_m$, $Z_n$ decay in magnitude exponentially as $m$, $n$ increase) and that for equilibria with circular boundaries, only one coefficient each in the Fourier series for R and Z proves to be adequate to obtain accurate numerical equilibria. Thus, we solve numerically (by collocation, in the manner indicated in Ref. 4), the two-point boundary-value system comprising the equations

$$\langle G_R \cos m\theta \rangle = 0 , \qquad m = 0 \tag{33}$$

$$\langle G_Z \sin n\theta \rangle = 0 , \qquad n = 1 \tag{34}$$

and equations (26) and (27). The boundary conditions are (see Ref. 4, and references therein):

$$R_{0v}(0) = 0 \tag{35a}$$

$$R_0(a) \equiv R_{0a} \tag{35b}$$

$$Z_1(0) = 0 \tag{36a}$$

$$Z_1(a) \equiv R_{0a} \tag{36b}$$

$$\Psi_v(0) = 0 \tag{37a}$$

$$\Psi(a) = 0 \tag{37b}$$

$$\Phi(0) = 0 \tag{38a}$$

$$\Phi(a) = \Phi_p \quad . \tag{38b}$$

Here $R_{0a}$ is the distance of the magnetic axis from origin of the cylindrical coordinate system. The total toroidal flux is $2\pi\Phi_p$ , and is a global invariant by virtue of the boundary conditions.

This is an appropriate point to emphasize the novelty of the present approach for constructing equilibria for toroidal plasmas. Typically, for pressureless, axisymmetric toroids one usually specifies the toroidal field function $F(v)$ [or $q(v)$] and solves the Grad-Shafranov equation numerically to determine equilibria. In the present formulation, however, $F(v)$ [or $q(v)$] is naturally determined from the variational principle for those equilibria which are extrema of the potential energy of the plasma.

## IV. TAYLOR'S MODEL

It has been conjected by Taylor[7] that in the presence of a small amount of resistivity the only surviving global invariant is

$$K_0 \equiv K[\omega_0] = \int_{V_0} d\tau \, \frac{\vec{A}\cdot\vec{B}}{2} \, , \tag{39}$$

which corresponds to keeping only the first member of the sequence [equation (11)]. Equations (26) and (27) become, in this special case,

$$\left\langle \frac{\partial}{\partial v}\left(\Psi_v \frac{g_{\theta\theta}}{\sqrt{\|g\|}}\right) - \lambda_0 \Phi_v \right\rangle = 0 \tag{40}$$

$$\left\langle \frac{\partial}{\partial v}\left(\Phi_v \frac{g_{\zeta\zeta}}{\sqrt{\|g\|}}\right) + \lambda_0 \Psi_v \right\rangle = 0 \, . \tag{41}$$

In the limit $a/R \to 0$, we recover the solutions in a straight cylinder with identified ends, with equilibrium quantities dependent only on $r$ in the usual $(r,\theta,z)$ coordinates. These solutions are[7]

$$B_\theta = B_0 J_0(\lambda_0 r) \tag{42a}$$

$$B_z = B_0 J_1(\lambda_0 r) \tag{42b}$$

where $B_0$ is a normalization constant. In this limit, the cylindrical solutions given by equations (42a-b) have only q-profiles which are monotonically decreasing with radius. But for finite values of the aspect ratio and small values of $\lambda_0 a$, Taylor's model permits q-profiles which are monotonically increasing and tokamak-like.[10] A closer examination of these tokamak-like solutions shows, however, that these tokamak-like solutions, apart from having a finite current density at the edge, have very weak shear. In Fig. 2 we show a typical q-profile ($\lambda_0 a = .35$, $a/R_{0a} = 1/5$). We note that $q$ increases from approximately 1.13 at the magnetic axis to 1.16 at the edge. Even if these equilibria are ideally and resistively stable, they are not very interesting from a practical point of view.

Continuing with our description of toroidal solutions in Taylor's model, we find that there is a transition from tokamak-like to pinch-like solutions at $\lambda_0 a \simeq 2.62a/R_{0a}$. At precisely the transition point, $q \simeq .75$ everywhere inside the plasma. Beyond this transition point, the q-profiles are all monotonically decreasing with radius, pinch-like, and show little difference from the cylindrical solutions, which have been well studied by Taylor[7,10] and Reiman.[11]

## V. PRESENT MODEL

In our earlier papers[1,2,3], we argued that the replacement of the entire infinite set of ideal invariants by a single invariant $K_0$ for a slightly non-ideal plasma is probably too drastic, and that a well-confined toroidal discharge preserves a few more invariants approximately on the timescale of evolution of tearing instabilities. Based on the Kadomtsev-Monticello model of magnetic reconnection[1,12], we have suggested that, apart from $K_0$, the global invariant

$$K_1 = \int_{V_0} d\tau \chi \frac{\vec{A}\cdot\vec{B}}{2} \tag{43}$$

is also approximately preserved. Equations (26) and (27) give, in our case,

$$\left\langle \frac{\partial}{\partial v} \left( \Psi_v \frac{g_{\theta\theta}}{\sqrt{\|g\|}} \right) - \left( \lambda_0 + \frac{3\lambda_1}{2} \chi \right) \Phi_v \right\rangle = 0 \tag{44}$$

and

$$\left\langle \frac{\partial}{\partial v} \left( \Phi_v \frac{g_{\zeta\zeta}}{\sqrt{\|g\|}} \right) + \left( \lambda_0 + \frac{3\lambda_1}{2} \chi \right) \Psi_v \right\rangle = 0 \ . \tag{45}$$

We examine those equilibria for which the current density vanishes on the wall.[1,2,3] These are obtained by simply requiring that

$$\frac{3\lambda_1}{2} = \frac{\lambda_0}{\Phi_p} \ . \tag{46}$$

In view of the importance[1,3] of the $m=1$, $n=1$ mode for tokamaks, we report the numerical results assuming the dominant mode to be m=1, n=1 ($q_s = 1$) for a device of aspect ratio $R_{0a}/a = 5$. As we have emphasized earlier[1,3], the qualitative features of this theory are not very sensitive to the choice of a particle dominant mode. Fig. 3 is a plot of $(a/R_{0a})(2\pi\Phi_p)^{-2}W(\lambda_0)$ vs. $(a/R_{0a})(2\pi\Phi_p)^{-2}K_0(\lambda_0)$, which is proportional to the amount of volt-seconds/toroidal flux in the system. As in the cylindrical case[3], there are two classes of solutions, "tokamak-like" (T) and "pinch-like" (P). For a given value of volt-seconds/toroidal flux, the plasma will relax into the status of lower energy indicated by the solid lines. The dashed lines indicate equilibria for which W is stationary but not minimum. In the same plot we exhibit the corresponding curve (marked by $\Delta$) for the axisymmetric solutions in Taylor's model. We note that the imposition of an additional constraint raises the energy of the system. In Fig. 4, we show a typical q-profile ($\lambda_0 a = 1$) from the minimum-energy T-branch. We note that q varies from approximately 1.16 at the axis to 2.16 at the edge, which has considerably larger shear than Taylor's tokamak-like equilibria, and is much closer to experimentally realized tokamak profiles. On the minimum-energy P-branch, the toroidal solutions show little difference from the cylindrical solutions, which have been studied in detail in Ref. 3. Experimental predictions of this model for self-reversal in pinches, embodied by the so-called F-θ diagram, are not sensitive to toroidal effects. Since they have been studied in the cylindrical limit in our earlier papers[1,3], we will not repeat them here.

## VI. CONCLUSION

We have constructed equilibria with minimum energy in an axisymmetric toroidal plasma with the current density vanishing on the wall. Though the numerical results reported here specialize to a plasma without pressure, the variational principle proposed here may be used to construct equilibria with nonzero pressure gradients. The construction of finite beta toroidal equilibria is left to future work.

Earlier work[3] on the ideal and resistive stability of the cylindrical solutions of our model has shown that there is a window of tokamak-like, minimum-energy equilibrium which has particularly robust stability. We argue that the toroidal solutions reported in the present paper will exhibit, on closer scrutiny by toroidal stability codes, a similar resilience to instabilities.

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References

1. A. Bhattacharjee, R. L. Dewar, and D. A. Monticello, Phys. Rev. Lett. 45, 347 (1980); Phys. Rev. Lett. 45, 1217(E) (1980).

2. A. Bhattacharjee and R. L. Dewar, Institute for Fusion Studies Report #19, The University of Texas at Austin, Austin, Texas 78712 USA, April 1981. Submitted to Phys. Fluids.

3. A. Bhattacharjee, R. L. Dewar, A. H. Glasser, M. S. Chance and J. C. Wiley, Institute for Fusion Studies Report #51, The University of Texas at Austin, Austin, Texas 78712 USA, January 1982. Submitted to Phys. Fluids.

4. A. Bhattacharjee, J. C. Wiley and R. L. Dewar, Institute for Fusion Studies Report #48, The University of Texas at Austin, Austin, Texas 78712 USA, December 1981. Submitted to J. Comput. Phys.

5. R. C. Grimm, J. M. Greene, and J. L. Johnson, in Methods in Computational Physics, Vol. 16, p. 253 (Academic Press, New York, 1976).

6. R. C. Grimm, R. L. Dewar, and J. Manickam, to be published.

7. J. B. Taylor, Phys. Rev. Lett. 35, 1139 (1974).

8. ~~J. M. Greene and J. L. Johnson, Phys. Fluids 5, 510 (1962).~~

9. H. K. Moffatt, J. Fluid Mech. 35, 117 (1969).

10. J. B. Taylor, in Proc. 5th Int. Conf. on Plasma Physics and Controlled Nuclear Fusion Research, Tokyo, Jpn., 1974 (IAEA, Vienna, 1975), Vol. I, p. 161.

11. A. Reiman, Phys. Fluids 23, 230 (1980).

12. B. Kadomtsev, Sov. J. Plasma Phys. 1, 389 (1975).

## Figure Captions

Fig. 1 The cylindrical coordinate system $(R,\zeta,Z)$ and the magnetic coordinate system $(v,\theta,\zeta)$; both are right-handed.

Fig. 2 A typical tokamak-like solution in Taylor's model $(\lambda_0 a = 0.35)$ with $q(0) \simeq 1.13$ and $q(a) \simeq 1.16$.

Fig. 3 Energy of axisymmetric equilibria in present model compared with those in Taylor's model (marked by $\Delta$). Arrows indicate direction of increasing $\lambda_0$. Labels P and T distinguish pinch-like and tokamak-like equilibria. Dashed lines indicate equilibrium for which energy is stationary but not minimum. The dominant mode is assumed to be $m=1$, $n=1$.

Fig. 4 A typical minimum-energy tokamak-like solutions in the present model $(\lambda_0 a = 1)$, with $q(0) \simeq 1.16$ and $q(a) \simeq 2.16$.

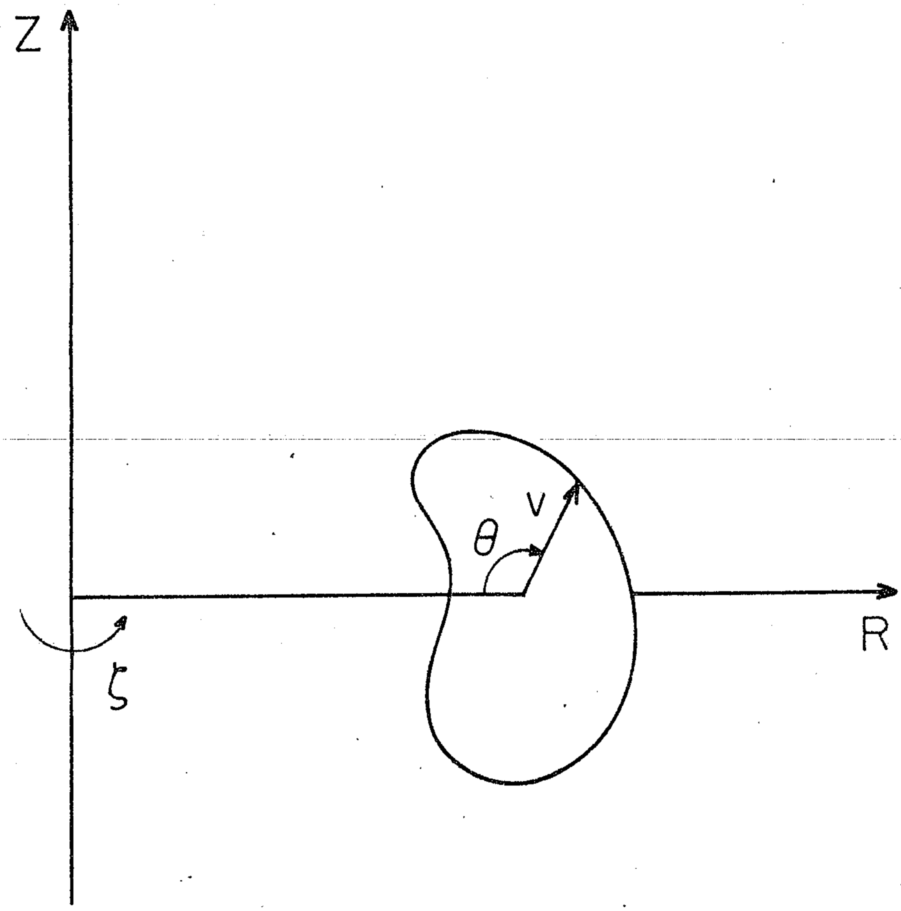


Fig. 1

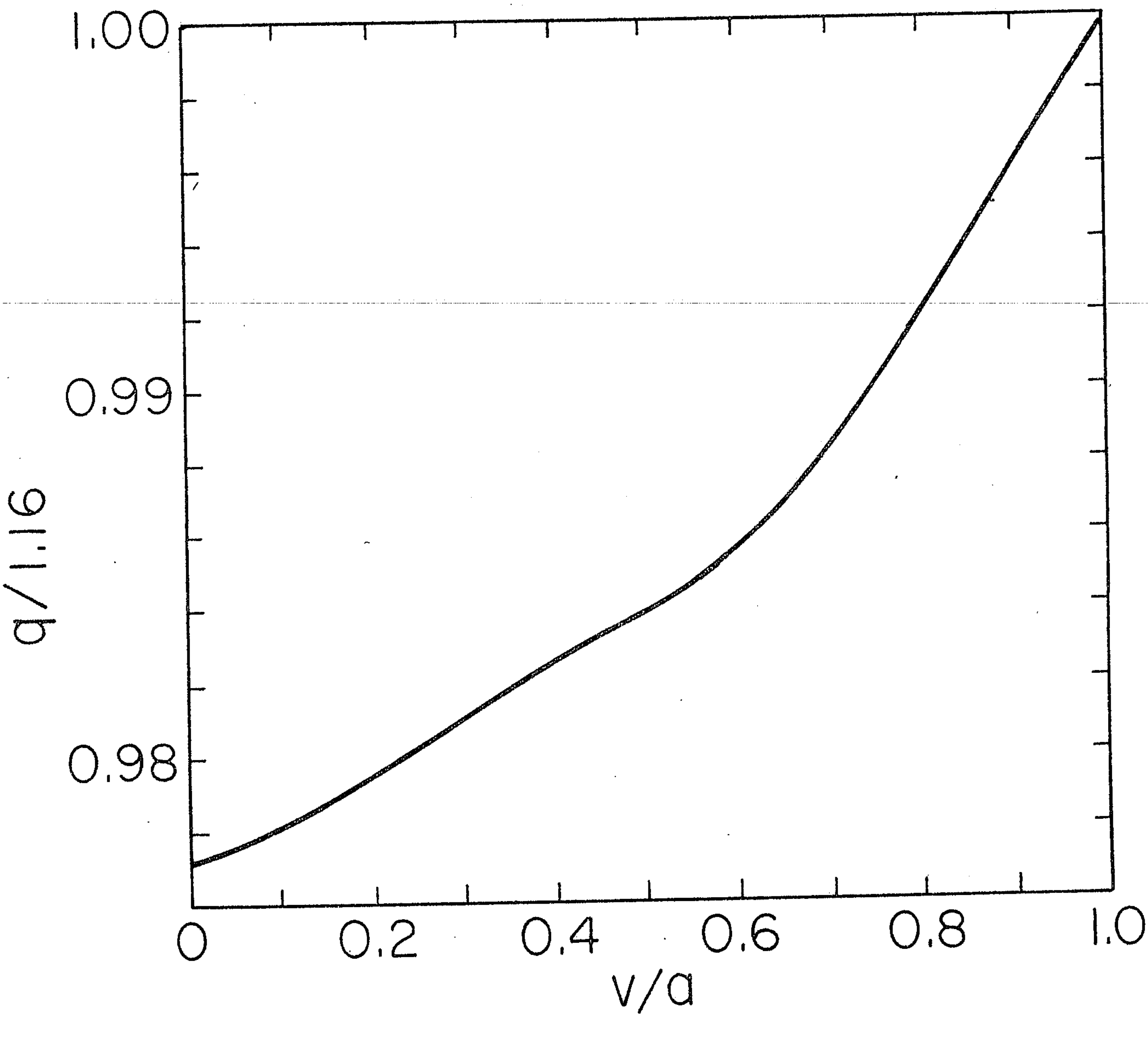


Fig. 2

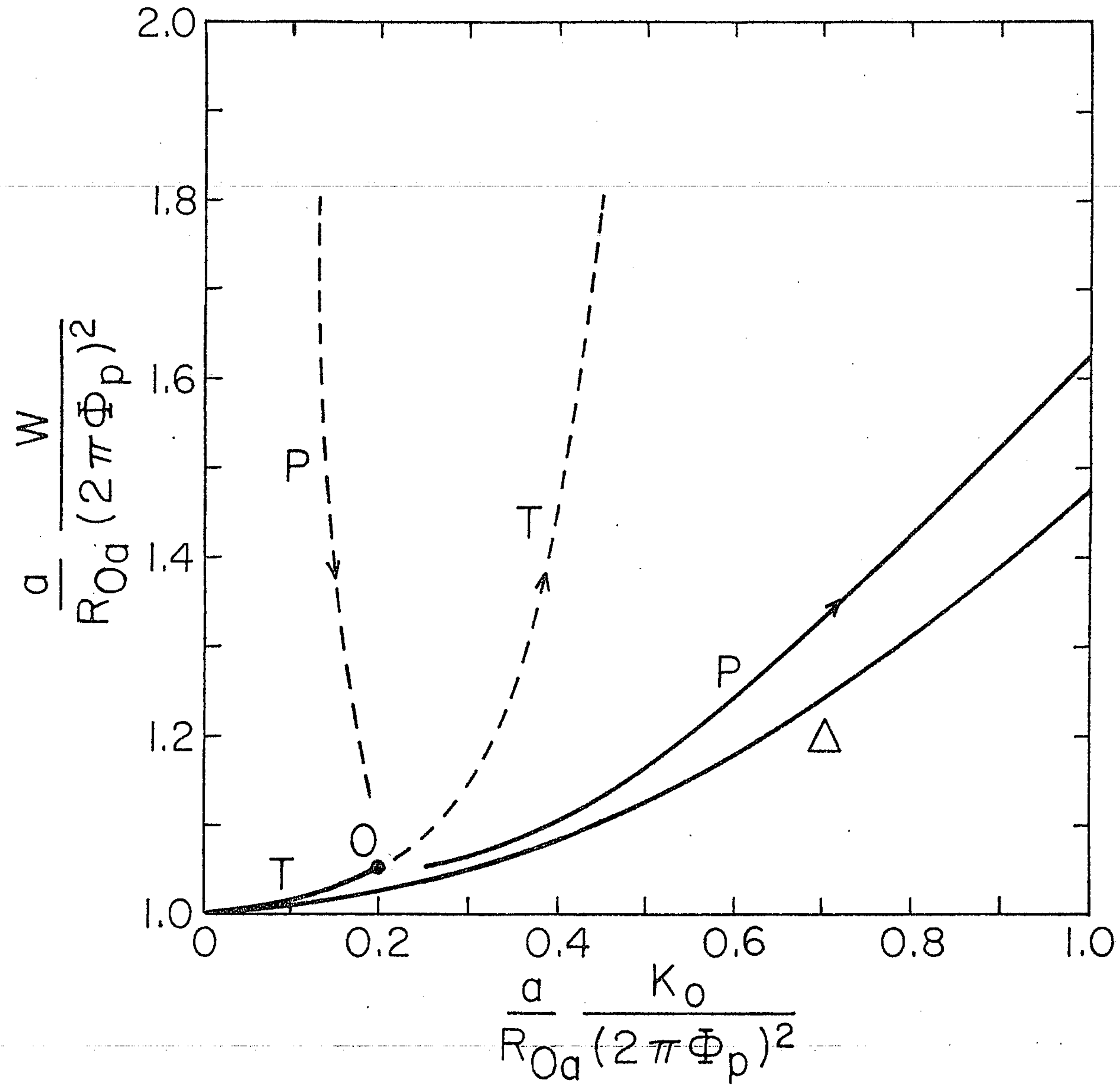
2.0
1.8
1.6
1.4
1.2
1.0
0
0.2
0.4
0.6
0.8
1.0
$\frac{a}{R_{0a}} \frac{W}{(2\pi\Phi_p)^2}$
$\frac{a}{R_{0a}} \frac{K_0}{(2\pi\Phi_p)^2}$
P
T
O
T
P
Δ

Fig. 3

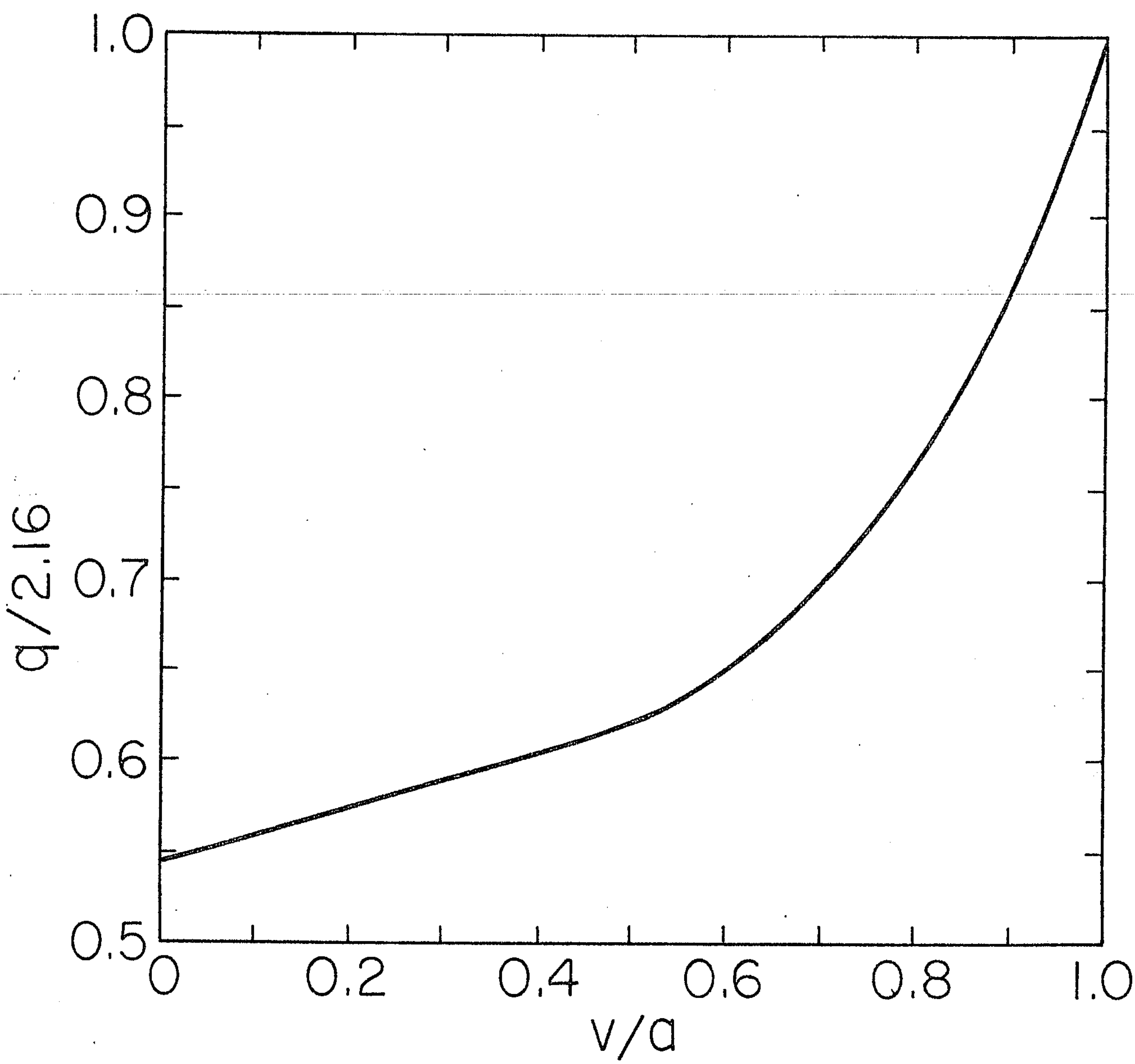


Fig. 4