UNIFIED KINETIC THEORY IN TOROIDAL SYSTEMS

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ABSTRACT

A kinetic theory for toroidal systems which includes the effects of collisions as well as instabilities is constructed. This yields a pair of evolution equations; one for the spectrum and one for the distribution function. In addition, this theory yields a toroidal generalization of the usual collision operator which is shown to have many similar properties, conservation laws, H theorem, to the usual collision operator.

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I. Introduction

In the mid 1960's the plasma kinetic theory for infinite homogeneous plasmas was characterized by two approaches; the Balescu-Guernsey-Lenard equation[1] valid for strongly stable modes and the quasilinear approach of Drummond and Pines[2] for unstable modes. The first approach has convergence problems as the Landau decrement approaches zero while the second approach does not relax to thermal equilibrium. In addition, the connection between the regimes was poorly understood. This connection was elucidated by Rogister and Oberman[3] who succeeded in unifying the transport due to both stable and unstable modes in a single framework. This removed the divergences in the earlier theories. In addition, it made possible a systematic calculation of electric field fluctuation amplitudes when the modes are weakly stable.

In the 1970's a similar situation exists with regard to kinetic theory in axisymmetric systems, especially tokamaks. On the one hand, there is neoclassical transport theory[4] involving only stable modes; on the other hand, there are various quasilinear theories in toroidal geometry of which the most useful appears to be that of Kaufmann.[5]

Our purpose is to produce a unified kinetic theory for axisymmetric systems. We shall employ Kaufmann's coordinates $\underset{\sim}{J}$, $\underset{\sim}{\theta}$ and use the techniques of Rogister and Oberman. In the next section we shall discuss some mathematical preliminaries. In Sec. III we shall derive the equation for the fluctuation amplitude. In Sec. IV we shall derive the particle kinetic equation. In Sec. V we discuss some properties of the particle kinetic equation. Finally, in Sec. VI we discuss our results and offer some speculations about future work.

II. Mathematical Preliminaries

We shall approach the problem of toroidal kinetic theory using Kaufmann's[5,6] coordinates $\underset{\sim}{J},\underset{\sim}{\theta}$ where

$$\underset{\sim}{J} = (M,P,J) \tag{1}$$

with

$$M \equiv \frac{m^2c}{e}\mu \quad , \quad P = m\dot{\zeta}R^2 - \frac{e}{c}\psi \quad , \quad J = \frac{e}{c}\oint \frac{\tilde{\alpha}d\beta}{2\pi} \quad ,$$

$$\tilde{\alpha} = \alpha(\beta;H_0,p,M) \tag{2}$$

Here $\dot{\zeta}$ is the toroidal angular velocity, α is the toroidal flux function, β is the corresponding poloidal angle, ψ is the poloidal flux function, μ is the usual magnetic moment, and R is the major radius. In addition, $\underset{\sim}{\theta}$ is defined by

$$\underset{\sim}{\theta} \equiv (\Theta_g,\varphi,\Theta) \quad ,$$

where Θ_g,φ and Θ are the angles corresponding to M, P, and J. For a further discussion of the coordinates see Ref. 6.

In terms of $\underset{\sim}{J},\underset{\sim}{\theta}$ we define the Klimontovich function N

$$N = \sum_{i=0}^{n} \delta[J - J_i(t)]\,\delta[\underset{\sim}{\theta} - \underset{\sim}{\theta}_i(t)] \quad , \tag{3}$$

where n is the number of particles and $\underset{\sim}{J}_i(t)$, $\underset{\sim}{\theta}_i(t)$ are the $\underset{\sim}{J},\underset{\sim}{\theta}$ of the i^{th} particle at time t. The equation for N may be written as

$$\frac{\partial N}{\partial t} + \frac{\partial}{\partial \underset{\sim}{\theta}} \cdot \left(\frac{dH}{d\underset{\sim}{J}} N \right) - \frac{\partial}{\partial \underset{\sim}{J}} \cdot \left(\frac{\partial H}{\partial \underset{\sim}{\theta}} N \right) = 0 ,$$

or

$$\frac{\partial N}{\partial t} + \frac{\partial H}{\partial \underset{\sim}{J}} \cdot \frac{\partial N}{\partial \underset{\sim}{\theta}} - \frac{\partial H}{\partial \underset{\sim}{\theta}} \cdot \frac{\partial N}{\partial \underset{\sim}{J}} = 0 . \tag{4}$$

Now, we define the Liouville function $D(\{\underset{\sim}{J}_i\},\{\underset{\sim}{\theta}_i\},t)$ where $\{\underset{\sim}{J}_i\} \equiv (\underset{\sim}{J}_1,\underset{\sim}{J}_2, \ldots,\underset{\sim}{J}_n)$ and $\{\underset{\sim}{\theta}_i\} = (\underset{\sim}{\theta}_1,\underset{\sim}{\theta}_2, \ldots,\underset{\sim}{\theta}_n)$ subject to the normalization

$$\int d\{\underset{\sim}{J}_i\}\, d\{\underset{\sim}{\theta}_i\}\, D\left(\{\underset{\sim}{J}_i\},\{\underset{\sim}{\theta}_i\},t\right) = 1 . \tag{5}$$

We further define the s particle correlation functions

$$F^s\left(\underset{\sim}{J}_i^0,\ldots,\underset{\sim}{J}_s^0,\ \underset{\sim}{\theta}_i^0,\ldots,\underset{\sim}{\theta}_s^0\right) =$$

$$n^{\hat{s}} \int d\{\underset{\sim}{J}_i\} d\{\underset{\sim}{\theta}_i\} D \left(\{\underset{\sim}{J}_i\},\{\underset{\sim}{\theta}_i\},t\right) \prod_{j=1}^{s} \delta\left(\underset{\sim}{J}_i - \underset{\sim}{J}_i^0\right) \delta\left(\underset{\sim}{\theta}_i - \underset{\sim}{\theta}_i^0\right) . \tag{6}$$

Finally, we define the ensemble average $\langle\rangle$ by

$$\langle A \rangle = \int d\{\underset{\sim}{J}_i\} d\{\underset{\sim}{\theta}_i\} D\left(\{\underset{\sim}{J}_i\},\{\underset{\sim}{\theta}_i\}\right) A , \tag{7}$$

and

$$\delta A = A - \langle A \rangle . \tag{8}$$

Combining Eqs. (7) and (8) with the definition of N it is easy to see that

$$\langle N \rangle = F^1(\underset{\sim}{J}) \tag{9}$$

and

$$\delta N(\underset{\sim}{J}_1,\underset{\sim}{\theta}_1)\, \delta N(\underset{\sim}{J}_2,\underset{\sim}{\theta}_2) =$$

$$P(\underset{\sim}{J}_1,\underset{\sim}{J}_2,\underset{\sim}{\theta}_1 - \underset{\sim}{\theta}_2) + F^1(\underset{\sim}{J}_1)\, \delta(\underset{\sim}{J}_1 - \underset{\sim}{J}_2)\, \delta(\underset{\sim}{\theta}_1 - \underset{\sim}{\theta}_2) \tag{10}$$

where P is $F^2(\underset{\sim}{J}_1,\underset{\sim}{J}_2,\underset{\sim}{\theta}_1,\underset{\sim}{\theta}_2) - F^1(\underset{\sim}{J}_1,\underset{\sim}{\theta}_1)F^1(\underset{\sim}{J}_2,\underset{\sim}{\theta}_2)$, the irreducible part of the two-particle distribution function. Because the $\underset{\sim}{\theta}$'s are ignorable coordinates, F^1 cannot depend on $\underset{\sim}{\theta}$ while F^2 and P can only depend on $\underset{\sim}{\theta}_1 - \underset{\sim}{\theta}_2$.

In addition, we introduce the Fourier transform in the ignorable coordinates

$$f(\underset{\sim}{\theta}) = \sum_{\underset{\sim}{\ell}} \exp[i\underset{\sim}{\ell}\cdot\underset{\sim}{\theta}]\, f_{\underset{\sim}{\ell}}$$

with

$$f_{\underset{\sim}{\ell}} = \frac{1}{(2\pi)^3}\int d\underset{\sim}{\theta}\, \exp[-i\underset{\sim}{\ell}\cdot\underset{\sim}{\theta}]\, f(\underset{\sim}{\theta})$$

where $\underset{\sim}{\ell}$ is a vector with integer components. Thus, we can rewrite Eq. (10) as

$$< \delta N^*_{\underset{\sim}{\ell}}(\underset{\sim}{J}_1)\ \delta N_{\underset{\sim}{\ell}_1}(\underset{\sim}{J}_2)> = \left[P_{\underset{\sim}{\ell}}(\underset{\sim}{J}_1,\underset{\sim}{J}_2) + \delta\frac{(\underset{\sim}{J}_1-\underset{\sim}{J}_2)F^1(\underset{\sim}{J}_1)}{(2\pi)^3}\right]\delta_{\underset{\sim}{\ell},\underset{\sim}{\ell}_1} \quad (11)$$

Now, we follow the procedure of Case[7] and define the local time average operator

$$\overline{A} = \lim_{T\to\infty} \frac{1}{2T}\int_{t-T}^{t+T} A(t')dt' \quad ,$$

and find

$$\frac{\partial}{\partial t}\overline{F}^1 = \frac{\partial}{\partial t} <\overline{N}> = -\frac{\partial}{\partial \underset{\sim}{J}}\cdot\sum_{\underset{\sim}{\ell}} <\overline{\delta\dot{\underset{\sim}{J}}_{\underset{\sim}{\ell}}{}^*\delta N_{\underset{\sim}{\ell}}}> \quad , \quad (12)$$

and

$$\frac{\partial}{\partial t}\,\delta N_{\underset{\sim}{\ell}} + i\underset{\sim}{\ell}\cdot\underset{\sim}{\Omega}\,\delta N_{\underset{\sim}{\ell}} = -\delta\dot{\underset{\sim}{J}}_{\underset{\sim}{\ell}}\cdot\frac{\partial F^{1}}{\partial \underset{\sim}{J}} , \qquad (13)$$

with $\underset{\sim}{\Omega} = \partial H_0/\partial \underset{\sim}{J}$. Note that since F^1 is constant on the zeroth-order time scale, we may replace $\overline{F^1}$ by F^1 to this order. We note that our Eq. (13) is exactly the Fourier transform of Kaufmann's[5] Eq. (22).

III. The Spectrum of Fluctuations

In this section, we shall derive an equation for the time evolution of the spectrum of fluctuations in a torus in the case where mode-mode coupling is not important.

We start by Laplace transforming Eq. (13) to obtain

$$-i(\omega-\underset{\sim}{\ell}\cdot\underset{\sim}{\Omega})\ \delta\hat{N}_{\underset{\sim}{\ell}} = \delta N_{\underset{\sim}{\ell}}(t=0) - \frac{\underset{\sim}{\ell}}{\omega}\cdot\int d\,\underset{\sim}{x}'\ \underset{\approx}{j}_{\underset{\sim}{\ell}}(\underset{\sim}{x}'|\underset{\sim}{J})\cdot\hat{\underset{\sim}{E}}(\underset{\sim}{x}',\omega)\ \frac{\partial F^1}{\partial \underset{\sim}{J}}\ , \tag{14}$$

where we have, following Kaufmann, defined

$$\underset{\approx}{j}_{\underset{\sim}{\ell}}(\underset{\sim}{x}|\underset{\sim}{J}) = \frac{1}{(2\pi)^3}\int d\,\underset{\sim}{\theta}\ \exp[-i\underset{\sim}{\ell}\cdot\underset{\sim}{\theta}]\ e\underset{\sim}{v}(\underset{\sim}{J},\underset{\sim}{\theta})\ \delta[\underset{\sim}{x} - \underset{\sim}{r}(\underset{\sim}{J},\underset{\sim}{\theta})]\ ,$$

and $\hat{\underset{\sim}{E}}$ is the Laplace transform of the electric field, $\underset{\sim}{E}$.

The reason for the appearance of the $\underset{\approx}{j}_{\underset{\sim}{\ell}}$'s is that the natural coordinates for describing N are the $\underset{\sim}{J}$'s and $\underset{\sim}{\theta}$'s while the natural coordinate for describing the electric field is the position $\underset{\sim}{x}$. Note that in the infinite homogeneous medium the natural coordinates for N are $\underset{\sim}{x},\underset{\sim}{v}$ so this complication does not occur. In addition, we note that we have assumed the radiation gauge here.

We now compute the fluctuating current

$$\delta\underset{\sim}{j}(x) = (2\pi)^3\sum_{\underset{\sim}{\ell}}\left[i\int d\underset{\sim}{J}\ \frac{\underset{\approx}{j}^*_{\underset{\sim}{\ell}}(\underset{\sim}{x}|\underset{\sim}{J})\,\delta N_{\underset{\sim}{\ell}}(t=0)}{\omega-\underset{\sim}{\ell}\cdot\underset{\sim}{\Omega}}\right.$$

$$\left. - \frac{i}{\omega}\int d\underset{\sim}{x}'\ d\underset{\sim}{J}\ \frac{\underset{\approx}{j}^*_{\underset{\sim}{\ell}}(\underset{\sim}{x}|\underset{\sim}{J})\ \underset{\approx}{j}_{\underset{\sim}{\ell}}(\underset{\sim}{x}'|\underset{\sim}{J})}{(\omega-\underset{\sim}{\ell}\cdot\underset{\sim}{\Omega})}\ \underset{\sim}{\ell}\cdot\frac{\partial F^1}{\partial\underset{\sim}{J}}\cdot\hat{\underset{\sim}{E}}(\underset{\sim}{x}')\right]\ , \tag{15}$$

where the usual Landau prescription must be used to analytically continue the integral into the stable half plan in ω. We combine Eq. (15) with Maxwell's equations to obtain

$$\underset{\sim}{D}^0\hat{\underset{\sim}{E}} \equiv \nabla\wedge(\nabla\wedge\hat{\underset{\sim}{E}}) - \frac{\omega^2}{c^2}\hat{\underset{\sim}{E}}$$

$$- c_0\sum_{\underset{\sim}{\ell}}\int d\underset{\sim}{J}d\underset{\sim}{x}'\left(\frac{\underset{\sim}{j}_{\underset{\sim}{\ell}}^*(\underset{\sim}{x}|\underset{\sim}{J})\underset{\sim}{j}_{\underset{\sim}{\ell}}(\underset{\sim}{x}'|\underset{\sim}{J})}{\omega - \underset{\sim}{\ell}\cdot\underset{\sim}{\Omega}}\underset{\sim}{\ell}\cdot\frac{\partial F^1}{\partial\underset{\sim}{J}}\right)\cdot\hat{\underset{\sim}{E}}(\underset{\sim}{x}')$$

$$= - c_0\omega\sum_{\underset{\sim}{\ell}}\int d\underset{\sim}{J}\frac{\underset{\sim}{j}_{\underset{\sim}{\ell}}^*(\underset{\sim}{x}|\underset{\sim}{J})}{\omega - \underset{\sim}{\ell}\cdot\underset{\sim}{\Omega}}\delta N_{\underset{\sim}{\ell}}(t=0) \quad , \qquad (16)$$

$$c_0 = \frac{4\pi}{c^2}(2\pi)^3 .$$

Note that if $\delta N_{\underset{\sim}{\ell}}(t=0)$ were an analytic function, then the inhomogeneous term in Eq. (16) would give no contribution to the Laplace inversion of $\hat{\underset{\sim}{E}}$; however, this is not the case here.

It is clear that $\underset{\sim}{D}^0$ is a very complicated operator; however, two things are clear. First, if ω is nearly real $\underset{\sim}{D}^0$ is approximately a Hermitian operator and thus has orthonormal eigenfunctions. Second, in the short wavelength limit the eigenfunctions of $\underset{\sim}{D}^0$ are local Fourier modes and thus are approximately orthogonal. Therefore, we shall assume that there exists a set of eigenfunctions $\underset{\sim}{E}_a(\underset{\sim}{x})$ such that

$$\underset{\sim}{D}^0\underset{\sim}{E}_a = \Lambda(a,\omega)\underset{\sim}{E}_a \quad ,$$

and

$$(\underset{\sim}{E}_a|\underset{\sim}{E}_{a'}) = \int d\underset{\sim}{x}\,\underset{\sim}{E}_a^*(\underset{\sim}{x})\cdot\underset{\sim}{E}_{a'}(\underset{\sim}{x}) = \delta_{a,a'} \quad , \qquad (17)$$

The assumption of a scalar eigenvalue equation in Eq. (17) permits considerable simplifications in the notation. This assumption does not preclude the existence of several branches to the dispersion relation, since this can be treated by the freedom in the function label "a". If a tensor eigenvalue equation were required, the entire analysis would be exactly the same as what follows except that $1/\Lambda$ would be replaced by $1/(\det|\Lambda)\cdot\underset{\approx}{\Lambda}^{CA}$ where $\underset{\approx}{\Lambda}^{CA}$ is the transpose of the cofactor matrix of the tensor $\underset{\approx}{\Lambda}$. Since $\underset{\approx}{\Lambda}$ is assumed to be regular, the only singularities in the Laplace inversion would come from zeros of the determinant. A property of the $\underset{\sim}{E}_a$'s which is easy to verify is that

$$(\underset{\sim}{D}_A\underset{\sim}{E}_a|\underset{\sim}{E}_{a'}) = 0 \qquad a \neq a' , \tag{18}$$

where $\underset{\sim}{D}_A$ is the anti-Hermitian part of $\underset{\sim}{D}^0$.

We now define

$$(\underset{\sim}{E}_a|\underset{\sim}{j}^*_{\underset{\sim}{\ell}}) = \lambda^*(a,\underset{\sim}{\ell},\underset{\sim}{J}) ,$$

and

$$\alpha(a,\underset{\sim}{\ell},\underset{\sim}{J}) = |\lambda(a,\underset{\sim}{\ell},\underset{\sim}{J})|^2 , \tag{19}$$

and expand

$$\hat{\underset{\sim}{E}}(\underset{\sim}{x},\omega) = \sum_a \underset{\sim}{E}_a\hat{\psi}_a(\omega) .$$

Therefore, Eq. (16) becomes

$$\Lambda(a,\omega)\hat{\psi}_a(\omega) = -c_0 \sum_{\underset{\sim}{\ell}} \omega \int d\underset{\sim}{J} \frac{\lambda^*(a,\underset{\sim}{\ell},\underset{\sim}{J})\,\delta N_{\underset{\sim}{\ell}}(\underset{\sim}{J},t=0)}{\omega - \underset{\sim}{\ell}\cdot\underset{\sim}{\Omega}} , \quad (20)$$

or, after the Laplace inversion

$$\psi_a(t) = \sum_{\sigma=\pm 1} \phi_a^{\sigma}(0)\,\exp[-i\omega_a^{\sigma} t] + ic_0 \times \sum_{\underset{\sim}{\ell}} \int d\underset{\sim}{J} \frac{\lambda^*(a,\underset{\sim}{\ell},\underset{\sim}{J})\,\delta N_{\underset{\sim}{\ell}}(\underset{\sim}{J},t=0)\,\underset{\sim}{\ell}\cdot\underset{\sim}{\Omega}\exp[-i\underset{\sim}{\ell}\cdot\underset{\sim}{\Omega}t]}{\Lambda(a,\underset{\sim}{\ell}\cdot\underset{\sim}{\Omega})} \quad (21)$$

with

$$\phi_a^{\sigma}(0) = ic_0 \sum_{\underset{\sim}{\ell}} \int d\underset{\sim}{J} \frac{\lambda^*(a,\underset{\sim}{\ell},\underset{\sim}{J})\,\delta N_{\underset{\sim}{\ell}}(\underset{\sim}{J},t=0)\,\omega_a^{\sigma}}{(\omega_a^{\sigma} - \underset{\sim}{\ell}\cdot\underset{\sim}{\Omega})\,[\partial\Lambda(a,\omega)/\partial\omega]_{\omega=\omega_a^{\sigma}}} ,$$

where ω_a^{σ} is the most unstable (least stable) root of $\Lambda(a,\omega) = 0$, with $\mathrm{Re}(\omega_a^{\sigma})/|\mathrm{Re}(\omega_a^{\sigma})| = \sigma$.

We note that to the order of interest, all of the terms effecting the slow time scale evolution of the plasma will be of the form $\langle\overline{\delta t^*\delta t}\rangle$ or $\langle\overline{\delta\underset{\sim}{\xi}^*\delta N}\rangle$. However, terms involving products of ϕ_a^{1} and ϕ_a^{-1} will always be oscillating and will be anihillated by the time averaging operator. Therefore, the effects of ϕ_a^{1} and ϕ_a^{-1} on the slow time scale evolution of the plasma are independent to second order. This corresponds to the fact, in infinite homogeneous medium , that waves with positive phase velocity and waves with negative phase velocity contribute independently to quasi-linear diffusion.[3] It is convenient to also separate the positive and negative "phase velocity" parts of the second term in Eq. (21). Therefore, we define ψ_a^{σ} :

$$\psi_a^\sigma(t) = \phi_a^\sigma \exp(-i\omega_a^\sigma t) + ic_0$$

$$\times \sum_{\underset{\sim}{\ell}} \int_{D^\sigma} \frac{d\underset{\sim}{J}\lambda^*(a,\underset{\sim}{\ell},\underset{\sim}{J})\,\delta N_{\underset{\sim}{\ell}}(\underset{\sim}{J},t=0)\,\underset{\sim}{\ell}\cdot\underset{\sim}{\Omega}\exp(-i\underset{\sim}{\ell}\cdot\underset{\sim}{\Omega}t)}{\Lambda(a,\underset{\sim}{\ell}\cdot\underset{\sim}{\Omega})} . \tag{22}$$

Where D^σ is the part of $\underset{\sim}{J}$ space with $\underset{\sim}{\ell}\cdot\underset{\sim}{\Omega}/|\underset{\sim}{\ell}\cdot\underset{\sim}{\Omega}| = \sigma$. This corresponds to Rogister and Oberman's[3] division of velocity space into regions with $\underset{\sim}{k}\cdot\underset{\sim}{v}$ greater than or less than zero. (The region with $\underset{\sim}{\ell}\cdot\underset{\sim}{\Omega} = 0$ can be harmlessly included in either S_1 or S_{-1} so long as we assume that $\omega_a^\sigma \neq 0$.) It is easy to check that this division of $\underset{\sim}{J}$ space is sensible. For example, for far untrapped particles $\underset{\sim}{\ell}\cdot\underset{\sim}{\Omega}$ is approximately $k_\| v_\| + \ell_g\Omega_g$ so the division is into particles with parallel velocities greater or less than $\ell_g\Omega_g/k_\|$. We now take the time derivation of Eq. (22) and obtain

$$\frac{d\psi_a^\sigma}{dt} =$$

$$-i\omega_a\psi_a^\sigma + ic_0\sum_{\underset{\sim}{\ell}}\int_{D^\sigma} d\underset{\sim}{J}\,\frac{\lambda^*(a,\underset{\sim}{\ell},\underset{\sim}{J})\,\delta N_{\underset{\sim}{\ell}}(\underset{\sim}{J},t=0)\,(\underset{\sim}{\ell}\cdot\underset{\sim}{\Omega})\,i(\omega_a^\sigma-\underset{\sim}{\ell}\cdot\underset{\sim}{\Omega})\exp[-i\underset{\sim}{\ell}\cdot\underset{\sim}{\Omega}t]}{\Lambda(a,\underset{\sim}{\ell}\cdot\underset{\sim}{\Omega})} . \tag{23}$$

Note that the integrand in Eq. (23) is regular as γ_a approaches zero. Now we multiply Eq. (23) by $\psi_a^{\sigma *}$, ensemble and time-average the resulting equation, and add it to its complex conjugate to obtain

$$\frac{d}{dt} I_a^\sigma = \frac{d}{dt}\overline{\langle \psi_a^{\sigma *} \psi_a^\sigma \rangle} = 2\gamma_a I_a^\sigma$$

$$+\sum_\sigma \Bigg\langle c_0^2 \sum_{\underset{\sim}{\ell}} \int_{D^\sigma} d\underset{\sim}{J} \sum_{\underset{\sim}{\ell}_1} \int_{D^\sigma} d\underset{\sim}{J}_1\, \delta N^*_{\underset{\sim}{\ell}_1}(\underset{\sim}{J}_1, t=0)\, \delta N_{\underset{\sim}{\ell}}(\underset{\sim}{J}, t=0)\, \lambda(a, \underset{\sim}{\ell}_1, \underset{\sim}{J}_1)\, \lambda^*(a, \underset{\sim}{\ell}, \underset{\sim}{J})$$

$$\times \quad \underset{\sim}{\ell}\cdot\underset{\sim}{\Omega} \frac{i(\omega_a^\sigma - \underset{\sim}{\ell}\cdot\underset{\sim}{\Omega})}{\Lambda(a^\sigma, \underset{\sim}{\ell}\cdot\underset{\sim}{\Omega})} \exp[-i\underset{\sim}{\ell}\cdot\underset{\sim}{\Omega}t]$$

$$\times \left[\left(\frac{\exp\left[i\omega_a^{\sigma *} t\right] \; \omega_a^{\sigma *}}{\left(\omega_a^{\sigma *} - \underset{\sim}{\ell}_1\cdot\underset{\sim}{\Omega}_1\right)\left(\frac{\partial \Lambda^*}{\partial \omega}(a,\omega)\right)_{\omega = \omega_a^{\sigma *}}} + \frac{\exp[-i\underset{\sim}{\ell}_1\cdot\underset{\sim}{\Omega}_1 t]\, \underset{\sim}{\ell}_1\cdot\underset{\sim}{\Omega}_1}{\Lambda^*(a, \underset{\sim}{\ell}_1\cdot\underset{\sim}{\Omega}_1)} \right)\right] \Bigg\rangle$$

$$+ \quad c.c. \; , \tag{24}$$

where we have abbreviated $\underset{\sim}{\Omega}_1 = \underset{\sim}{\Omega}(\underset{\sim}{J}_1)$ and so on. (Note that, as mentioned before, the product of $\sigma = \pm 1$ terms does not contribute to the slow evolution of the plasma because these terms are always oscillatory.) The crucial point is that the quantity in the brackets in Eq. (24) is regular as γ_a approaches zero. Therefore, we have the freedom to deform the contour in any consistent way. We choose the deformation S^σ (see Fig. 1) below $\omega_a^{\sigma *}$ and above ω_a^σ. This deformation allows us to show that the terms proportional to $\exp[-i(\underset{\sim}{\ell}\cdot\underset{\sim}{\Omega} - \omega_a^{\sigma *})t]$ or $\exp[i(\underset{\sim}{\ell}\cdot\underset{\sim}{\Omega} - \omega_a^\sigma)t]$ decay to zero as t becomes large. This argument is due to Rogister and Oberman[3]. Consider the term in the brackets proportional to $\exp[-i(\underset{\sim}{\ell}\cdot\underset{\sim}{\Omega} - \omega_a^{\sigma *})t]$. It is regular at $\underset{\sim}{\ell}\cdot\underset{\sim}{\Omega} = \omega_a^\sigma$ and the contour passes below $\underset{\sim}{\ell}\cdot\underset{\sim}{\Omega} = \omega_a^{\sigma *}$.

Therefore, we can push the contour into the lower half $\underset{\sim}{\ell}\cdot\underset{\sim}{\Omega}$ plane without getting any residue from $\omega_a^\sigma = \underset{\sim}{\ell}\cdot\underset{\sim}{\Omega}$. It is easy to see that the integral along this contour, which avoids the singularities of the integrand, is strongly damped.

In addition, the terms in Eq. (24) which are proportional to $P_{\underset{\sim}{\ell}}$ [see Eq. (10)] are also strongly damped. Thus, we finally obtain

$$\frac{\partial I_a^\sigma}{\partial t} = 2\gamma_a I_a^\sigma - \frac{c_0^2}{(2\pi)^3} \sum_{\underset{\sim}{\ell}} \int_{S^\sigma} d\underset{\sim}{J}\, \frac{2\gamma_a (\underset{\sim}{\ell}\cdot\underset{\sim}{\Omega})^2 \alpha(a,\underset{\sim}{\ell},\underset{\sim}{J}) F^1(\underset{\sim}{J})}{|\Lambda(a,\underset{\sim}{\ell}\cdot\underset{\sim}{\Omega})|^2} \quad . \qquad (25)$$

We note that to this order, the difference between $F^1(\underset{\sim}{J},t=0)$ and $F^1(\underset{\sim}{J},t)$ is unimportant and we resynchronize F^1 to $F^1(\underset{\sim}{J},t)$.

Equation (25) is a fundamental result of the theory. It predicts the spectrum of fluctuations including both the emission of waves by the particles and the growth of the waves. In the case where all $\gamma < 0$ it implies in steady state

$$I_a^\sigma = \frac{c_0^2}{(2\pi)^3} \sum_{\underset{\sim}{\ell}} \int_{S^\sigma} d\underset{\sim}{J}\, \frac{\alpha(a,\underset{\sim}{\ell},\underset{\sim}{J})}{|\Lambda(a,\underset{\sim}{\ell}\cdot\underset{\sim}{\Omega})|^2} (\underset{\sim}{\ell}\cdot\underset{\sim}{\Omega})^2 F^1(\underset{\sim}{J}) \quad , \qquad (26)$$

and S^σ gives the proper deformation of the contour as $\gamma_a \to 0$. We note that this sort of effect could be crucial for understanding cyclotron emission from plasmas due to the presence of the nearly undamped Bernstein modes.

IV. The Particle Kinetic Equation

We now turn our attention to the derivation of the particle kinetic equation. This equation will contain the unstable modes of Kaufmann's[5] paper as one special case and the toroidal generalization of the Balescu Guernsey Lenard equation as another. The arguments will be similar in many ways to those of the last section.

We start by writing the expression for $\delta N_{\underset{\sim}{\ell}}(t)$:

$$\delta N_{\underset{\sim}{\ell}}(t) = \delta N_{\underset{\sim}{\ell}}(t=0)\ \exp[-i\underset{\sim}{\ell}\cdot\underset{\sim}{\Omega}t] + \sum_{a,\sigma} \exp[-i\underset{\sim}{\ell}\cdot\underset{\sim}{\Omega}t]\int_0^t dt' \left[\underset{\sim}{\ell}\lambda(a,\underset{\sim}{\ell},\underset{\sim}{J})\left(\frac{\phi_a^\sigma(0)\ \exp[-i\omega_a^\sigma t']}{\omega_a^\sigma} + ic_0 \sum_{\underset{\sim}{\ell}_1}\int_{D^\sigma} d\underset{\sim}{J}_1\ \frac{\delta N_{\underset{\sim}{\ell}_1}(\underset{\sim}{J}_1, t=0)}{\Lambda(a,\underset{\sim}{\ell}_1\cdot\underset{\sim}{\Omega}_1)}\ \lambda^*(a,\underset{\sim}{\ell}_1,\underset{\sim}{J}_1) \times \exp[-i\underset{\sim}{\ell}_1\cdot\underset{\sim}{\Omega}_1 t']\right) \exp[i\underset{\sim}{\ell}\cdot\underset{\sim}{\Omega}t']\right] \cdot \frac{dF^1}{d\underset{\sim}{J}}, \tag{27}$$

Now, we compute

$$\frac{\partial}{\partial t}\bar{F}^1 = -\frac{\partial}{\partial \underset{\sim}{J}} \cdot \langle \overline{\delta\dot{\underset{\sim}{J}}^*\ \delta N} \rangle = -\frac{\partial}{\partial \underset{\sim}{J}} \cdot \underset{\sim}{\Gamma}, \tag{28}$$

where

where

$$\underset{\sim}{\Gamma} = \frac{1}{2T}\int_{t-T}^{t+T} dt \left\langle \left\{ \sum_{a_1,\sigma} \underset{\sim}{\ell}\lambda^*(a_1,\underset{\sim}{\ell},\underset{\sim}{J}) \left[\frac{\phi_{a_1}^{\sigma *}}{\omega_{a_1}^{\sigma *}} \exp[i\omega_{a_1}^{\sigma *} t] \right.\right.\right.$$

$$\left.\left. - ic_0 \sum_{\underset{\sim}{\ell}_1} \int_{D^\sigma} d\underset{\sim}{J}_1 \, \delta N_{\underset{\sim}{\ell}_1}^*(\underset{\sim}{J}_1, t=0) \frac{\lambda(a_1,\underset{\sim}{\ell}_1,\underset{\sim}{J}_1)}{\Lambda^*(a_1,\underset{\sim}{\ell}_1,\cdot\underset{\sim}{\Omega}_1)} \exp[i\underset{\sim}{\ell}_1\cdot\underset{\sim}{\Omega}_1 t] \right] \right\}$$

$$\left\{ \delta N_{\underset{\sim}{\ell}}(\underset{\sim}{J},t=0) \exp[-i\underset{\sim}{\ell}\cdot\underset{\sim}{\Omega}t] - \sum_{a_2,\sigma'} \underset{\sim}{\ell}\lambda(a_1,\underset{\sim}{\ell},\underset{\sim}{J}) \int_0^t dt' \exp[-i\underset{\sim}{\ell}\cdot\underset{\sim}{\Omega}(t-t')] \right.$$

$$\times \left[\frac{\phi_{a_2}^{\sigma'} \exp[-i\omega_{a_2}^{\sigma} t']}{\omega_{a_2}^{\sigma}} + c_0 i \sum_{\underset{\sim}{\ell}_2} \int_{D^{\sigma'}} d\underset{\sim}{J}_2 \frac{\delta N_{\underset{\sim}{\ell}_2}(\underset{\sim}{J}_2, t=0)\lambda^*(a_2,\underset{\sim}{\ell}_2,\underset{\sim}{J}_2)}{\Lambda(a_2,\underset{\sim}{\ell}_2\cdot\underset{\sim}{\Omega}_2)} \right.$$

$$\left.\left.\left. \times \exp[-i\underset{\sim}{\ell}_2\cdot\underset{\sim}{\Omega}_2 t'] \right] \cdot \frac{\partial F^1}{\partial \underset{\sim}{J}} \right\} \right\rangle . \qquad (29)$$

Note that the time-average operator will annihilate all terms in Γ which do not satisfy $\sigma = \sigma'$, therefore we will not consider these terms further and pull the single σ sum outside the ensemble average operator. We treat the contribution from the first term in the second { } in Eq. (29), the term due directly to the initial value of $\delta N_{\underset{\sim}{\ell}}$, $\partial/\partial\underset{\sim}{J}\cdot\underset{\sim}{\Gamma}_{iv}$,

$$\underset{\sim}{\Gamma}_{iv} =$$

$$\sum_{a_1,\sigma} \frac{1}{2T} \int_{t-T}^{t+T} dt' \underset{\sim}{\ell} \lambda^*(a_1, \underset{\sim}{\ell}, \underset{\sim}{J}) \left\{ \frac{\langle \phi_{a_1}^{\sigma *}(0)\, \delta N_{\underset{\sim}{\ell}}(\underset{\sim}{J}, t=0) \rangle}{\omega_{a_1}^{\sigma *}} \exp\left[i\left(\omega_{a_1}^{\sigma *} - \underset{\sim}{\ell}\cdot\underset{\sim}{\Omega}\right) t'\right] \right.$$

$$\left. + c_0 \sum_{\underset{\sim}{\ell}_1} \int d\underset{\sim}{J}_1 \frac{\langle \delta N_{\underset{\sim}{\ell}_1}(\underset{\sim}{J}, t=0)\, \delta N_{\underset{\sim}{\ell}}(\underset{\sim}{J}, t=0) \rangle \lambda(a_1, \underset{\sim}{\ell}_1, \underset{\sim}{J}_1)}{\Lambda^*(a_1, \underset{\sim}{\ell}_1 \cdot \underset{\sim}{\Omega}_1)} \exp[i \underset{\sim}{\ell}_1 \cdot \underset{\sim}{\Omega}_1 t'] \right\} , \tag{30}$$

$$\underset{\sim}{\Gamma}_{iv} = \sum_{a_1,\sigma} \frac{1}{2T} \int_{t-T}^{t+T} dt' \underset{\sim}{\ell} \lambda^*(a_1, \underset{\sim}{\ell}, \underset{\sim}{J}) c_0 \cdot \frac{F^1(\underset{\sim}{J})}{(2\pi)^3} \lambda(a_1, \underset{\sim}{\ell}, \underset{\sim}{J})$$

$$\times \left\{ \frac{\exp\left[i\left(\omega_{a_1}^{\sigma *} - \underset{\sim}{\ell}\cdot\underset{\sim}{\Omega}\right) t\right]}{\left(\omega_{a_1}^{\sigma *} - \underset{\sim}{\ell}\cdot\underset{\sim}{\Omega}\right) \frac{d\Lambda^*}{d\omega}} + \frac{1}{\Lambda^*(a_1, \underset{\sim}{\ell}\cdot\underset{\sim}{\Omega})} \right\} , \tag{31}$$

where we have used Eq. (11) and neglected the strongly decaying contributions from P . Note that the term in the large parentheses in Eq. (31) is regular as γ_{a_1} approaches zero and that the first term is negligible for stable modes. Therefore, by arguments analogous to those in the previous section we see that the first term provides the analytic continuation of the second as γ_{a_1} passes through zero from the stable side. In the case where a_1 is a continuous variable this corresponds to deforming the contour in a_1 space above the poles of $\Lambda^*(a_1, \omega^*)$. In the case of a discrete spectrum we replace f_{a_1} by $\underline{f}(a_1, \Delta)$ with

$$\underline{f}(a,\Delta) \equiv \sum_{a_j} f_{a_j} \frac{1}{(2\pi\Delta)^{\frac{1}{2}}} \exp\left[-\frac{1}{2\Delta}\left(a - a_j\right)^2\right] \qquad (32)$$

and perform the same manipulation as in the continuous case and then take the limit as $\Delta \to 0$.

Therefore, we can finally write the first term, $\underset{\sim}{\Gamma}_{iv}$, as

$$\underset{\sim}{\Gamma}_{iv} = \sum_{a_1}^{S} \sum_{\underset{\sim}{\ell}} \frac{\ell\alpha(a_1, \underset{\sim}{\ell}, \underset{\sim}{J}) F^1(\underset{\sim}{J})}{\Lambda^*(a_1 \underset{\sim}{\ell} \cdot \underset{\sim}{\Omega})} \left(\frac{4\pi}{c^2}\right) , \qquad (33)$$

where the $\lfloor S \rfloor$ superscript on the summation indicates that we have used the contour prescription already discussed. This term is the toroidal, electromagnetic, generalization of the polarization drag felt by a particle due both to particle discreteness and the stimulated emission of unstable waves.

We now turn to the remaining terms in $\underset{\sim}{\Gamma}$. With this definition we can write the second diffusion term

$$\Gamma_D(a_1,a_2) = \sum_{\sigma} \ell\lambda^*(a_1,\ell,J)\,\ell\lambda(a_2,\ell,J)\,\frac{1}{2T}\int_{t-T}^{t+T} dt'$$

$$\Bigg\langle \left[\frac{\phi_{a_1}^{\sigma*}}{\omega_{a_1}^{\sigma*}} \exp[i\omega_{a_1}^{\sigma*} t] + c_0 \sum_{\ell_1} \int_{D^\sigma} dJ_1\, \delta N_{\ell_1}^*(J_1, t=0) \right.$$

$$\left. \times \frac{\lambda(a_1,\ell_1,J_1)}{\Lambda^*(a_1,\ell_1\cdot\Omega_1)} \exp[i\ell\cdot\Omega t] \right] \times \left[\frac{\phi_{a_2}^{\sigma}}{\omega_{a_2}^{\sigma}} \left(\frac{\exp[-i\omega_{a_2}^{\sigma} t] - \exp[-i\ell\cdot\Omega t]}{i(\ell\cdot\Omega - \omega_{a_2})} \right) \right.$$

$$+ c_0 \sum_{\ell_2} \int_{D^\sigma} dJ_2\, \delta N_{\ell_2}(J_2, t=0)\, \frac{\lambda^*(a_2,\ell_2,J_2)}{\Lambda(a_2,\ell_2,\cdot\Omega_2)}$$

$$\left. \times \left(\frac{\exp[-i(\ell_2\cdot\Omega_2)t] - \exp[-i\ell\cdot\Omega t]}{i(\ell\cdot\Omega - \ell_2\cdot\Omega_2)} \right) \right] \Bigg\rangle \cdot \frac{\partial F^1}{\partial J} . \qquad (34)$$

We can effect a considerable reduction in the algebra by defining the operators

$$g^+_{\ell,\omega} = \frac{1}{i(\ell\cdot\Omega - \omega) + \delta} , \qquad (35)$$

since then the second terms in each of the parentheses in Eq. (34) can be shown to give a rapidly decaying contribution to the equation. We note that the introduction of g^+ does not change the equation because each of the parentheses in Eq. (34) is regular at the resonance. Therefore, the diffusion term becomes

$$
\begin{aligned}
\Gamma_D(a_1,a_2) = {} & \sum_{\boldsymbol{\ell},\sigma} \boldsymbol{\ell}\boldsymbol{\ell}\,\lambda(a_1,\boldsymbol{\ell},\boldsymbol{J})\,\lambda(a_2,\boldsymbol{\ell},\boldsymbol{J})\,\frac{1}{2T}\int_{t-T}^{t+T} dt \left\{ \frac{<\psi_{a_1}^{\;*}\psi_{a_2}>}{\omega_{a_1}^{\sigma *}\omega_{a_2}^{\sigma}} \right. \\
& \times \exp\left[i\left(\omega_{a_1}^{\sigma *} - \omega_{a_2}^{\sigma}\right)t\right] g^{+}_{\boldsymbol{\ell},\omega_{a_2}^{\sigma}} + \frac{c_0^2}{(2\pi)^3}\sum_{\boldsymbol{\ell}_1}\int_{D^{\sigma}} d\boldsymbol{J}_1 F^1(\boldsymbol{J}_1)\,\lambda(a_1,\boldsymbol{\ell}_1,\boldsymbol{J}_1) \\
& \times \lambda^{*}(a_2,\boldsymbol{\ell}_1,\boldsymbol{J}_1)\left[\frac{\exp\left[-i\omega_{a_1}^{\sigma}t\right]}{\Lambda(a_2,\boldsymbol{\ell}_1\cdot\boldsymbol{\Omega}_1)}\left(\frac{\boldsymbol{\ell}_1\cdot\boldsymbol{\Omega}_1}{\omega_{a_1}^{\sigma *}\Lambda^{*}(a_1,\boldsymbol{\ell}_1\cdot\boldsymbol{\Omega}_1)}\right.\right. \\
& \left. + \frac{1}{\left(\omega_{a_1}^{\sigma *} - \boldsymbol{\ell}_1\cdot\boldsymbol{\Omega}_1\right)\left[\frac{\partial\Lambda^{*}}{\partial\omega}(a_1,\omega)\right]_{\omega=\omega_{a_1}^{\sigma *}}}\right)\left(\exp\left[i\boldsymbol{\ell}_1\cdot\boldsymbol{\Omega}_1 t\right] g^{+}_{\boldsymbol{\ell},\boldsymbol{\ell}_1\cdot\boldsymbol{\Omega}_1}\right. \\
& \left. - \frac{\boldsymbol{\ell}_1\cdot\boldsymbol{\Omega}_1}{\omega_{a_2}^{\sigma}}\exp\left[-i\omega_{a_2}^{\sigma}t\right] g^{+}_{\boldsymbol{\ell},\omega_{a_2}^{\sigma}}\right) + \frac{\exp\left[-i\omega_{a_2}^{\sigma}t\right]}{\Lambda^{*}(a_1,\boldsymbol{\ell}_1\cdot\boldsymbol{\Omega}_1)} \\
& \times \left(\frac{\boldsymbol{\ell}_1\boldsymbol{\Omega}_1}{\omega_{a_2}^{\sigma}\Lambda(a_2,\boldsymbol{\ell}_1\cdot\boldsymbol{\Omega}_1)} + \frac{1}{\left(\omega_{a_2}^{\sigma} - \boldsymbol{\ell}_1\cdot\boldsymbol{\Omega}_1\right)\left[\frac{\partial\Lambda(a_2,\omega)}{\partial\omega}\right]_{\omega=\omega_{a_2}^{\sigma}}}\right) \\
& \times \left(\exp\left[i\boldsymbol{\ell}_1\cdot\boldsymbol{\Omega}_1 t\right] g^{+}_{\boldsymbol{\ell}g\boldsymbol{\ell}_1\cdot\boldsymbol{\Omega}_1} - \frac{\boldsymbol{\ell}_1\boldsymbol{\Omega}_1}{\omega_{a_1}^{\sigma *}}\exp\left[i\omega_{a_1}^{\sigma *}t\right] g^{+}_{\boldsymbol{\ell},\omega_{a_2}^{\sigma}}\right) \\
& + \frac{1}{\Lambda^{*}(a_1,\boldsymbol{\ell}_1\cdot\boldsymbol{\Omega}_1)\,\Lambda(a_2,\boldsymbol{\ell}_1\cdot\boldsymbol{\Omega}_1)}\left(\exp\left[i\boldsymbol{\ell}_1\cdot\boldsymbol{\Omega}_1 t\right] - \frac{\boldsymbol{\ell}_1\cdot\boldsymbol{\Omega}_1}{\omega_{a_1}^{\sigma *}}\exp\left[i\omega_{a_1}^{\sigma *}\right]\right) \\
& \left.\left. \times \left(\exp\left[-i\boldsymbol{\ell}_1\cdot\boldsymbol{\Omega}_1 t\right] g^{+}_{\boldsymbol{\ell},\boldsymbol{\ell}_1\cdot\boldsymbol{\Omega}_1} - \frac{\boldsymbol{\ell}_1\cdot\boldsymbol{\Omega}_1}{\omega_{a_2}^{\sigma}}\exp\left[-i\omega_{a_2}^{\sigma}t\right] g^{+}_{\boldsymbol{\ell},\omega_{a_2}^{\sigma}}\right)\right]\cdot\frac{\partial F^1}{\partial\boldsymbol{J}}\right\} . \qquad (36)
\end{aligned}
$$

As in the derivation of the spectrum we note that the quantity in the brackets is analytic in $\omega_{a_1}^{\sigma *}$, $\omega_{a_2}^{\sigma}$ as γ_{a_1} or γ_{a_2} approaches zero. As before, we deform the contour in $\underset{\sim}{J}_1$ to be always above $\omega_{a_2}^{\sigma}$ and below $\omega_{a_1}^{\sigma *}$. Then, we can write

$$\underset{\sim}{\Gamma}_D(a_1,a_2) = \sum_{\underset{\sim}{\ell},\sigma} \underset{\sim}{\ell}\underset{\sim}{\ell}\lambda^*(a_1,\underset{\sim}{\ell},\underset{\sim}{J})\lambda(a_2,\underset{\sim}{\ell},\underset{\sim}{J}) \left[\frac{I_{a_1}^{\sigma}}{|\omega_{a_1}^{\sigma}|^2} \delta_{a_1,a_2} g^+_{\underset{\sim}{\ell},\omega_{a_2}^{\sigma}} \right.$$

$$+ \frac{c_0^2}{(2\pi)^3} \sum_{\underset{\sim}{\ell}_1} \int_{S^\sigma} d\underset{\sim}{J}_1 F^2(\underset{\sim}{J}_1) \frac{\lambda(a_1,\underset{\sim}{\ell}_1,\underset{\sim}{J}_1)\lambda_2^*(a_2,\underset{\sim}{\ell}_1,\underset{\sim}{J}_1)}{\Lambda^*(a_1,\underset{\sim}{\ell}_1\cdot\underset{\sim}{\Omega}_1)\Lambda(a_2,\underset{\sim}{\ell}_1\cdot\underset{\sim}{\Omega}_1)}$$

$$\left. \left(g^+_{\underset{\sim}{\ell},\underset{\sim}{\ell}_1\cdot\underset{\sim}{\Omega}_1} - \frac{(\underset{\sim}{\ell}_1\cdot\underset{\sim}{\Omega}_1)^2}{|\omega_{a_1}^{\sigma *}|^2} g^+_{\underset{\sim}{\ell},\omega_{a_1}^{\sigma}} \delta_{a_1,a_2} \right) \right] \cdot \frac{\partial F^1}{\partial \underset{\sim}{J}} + \Psi , \qquad (37)$$

where Ψ is a strongly decaying transient due to terms like $\int F^1(\underset{\sim}{J}_1)\exp[i(\underset{\sim}{\ell}\cdot\underset{\sim}{\Omega} - \omega_{a_2}^{\sigma})t]$. We have also used the fact that the time average of $\exp[i(\omega_{a_1}^{\sigma} - \omega_{a_2}^{\sigma})t]$ is a delta function in a_1, a_2 thus for simplicity assuming nondegeneracy of the modes.

We now combine Eqs. (28), (29), and (37) to obtain

$$
\begin{aligned}
\frac{\partial \bar{F}^1}{\partial t}(\mathbf{J}) = -\frac{d}{d\mathbf{J}} \cdot \Bigg(& \sum_a^S \sum_{\boldsymbol{\ell}} \frac{\alpha(a,\boldsymbol{\ell},\mathbf{J})}{\Lambda^*(a,\boldsymbol{\ell}\cdot\boldsymbol{\Omega})} \frac{4\pi}{c^2} \bar{F}^1(\mathbf{J}) \\
& - \Bigg\{ \sum_{a,\boldsymbol{\ell},\sigma} \boldsymbol{\ell}\boldsymbol{\ell}\,\alpha(a,\boldsymbol{\ell},\mathbf{J}) \Bigg[\frac{I_a^\sigma}{|\omega_a|^2} g^+_{\boldsymbol{\ell},\omega_a^\sigma} + \left(\frac{4\pi}{c^2}\right)^2 \sum_{\boldsymbol{\ell}_1} \int_{S^\sigma} d\mathbf{J}_1 \bar{F}^1(\mathbf{J}_1) \\
& \times \frac{\alpha(a,\boldsymbol{\ell}_1,\mathbf{J}_1)}{|\Lambda(a,\boldsymbol{\ell}_1\cdot\boldsymbol{\Omega}_1)|^2} \left(g^+_{\boldsymbol{\ell},\boldsymbol{\ell}_1\cdot\boldsymbol{\Omega}_1} - \frac{(\boldsymbol{\ell}_1\cdot\boldsymbol{\Omega}_1)^2}{|\omega_a^\sigma|^2} g^+_{\boldsymbol{\ell},\omega_a^\sigma} \right) \Bigg] \Bigg\} \cdot \frac{\partial \bar{F}^1}{\partial \mathbf{J}}(\mathbf{J}) \\
& + \sum_a \sum_{\boldsymbol{\ell},\sigma} \sum_{a_1 \neq a} \boldsymbol{\ell}\boldsymbol{\ell}\,\lambda^*(a,\boldsymbol{\ell},\mathbf{J})\,\lambda(a_1,\boldsymbol{\ell},\mathbf{J}) \left(\frac{4\pi}{c^2}\right)^2 (2\pi)^3 \sum_{\boldsymbol{\ell}_1} \int_{S^\sigma} d\mathbf{J}_1 \\
& \times \bar{F}^1(\mathbf{J}_1) \frac{\lambda(a,\boldsymbol{\ell}_1,\mathbf{J}_1)\,\lambda^*(a_1,\boldsymbol{\ell},\mathbf{J})}{\Lambda^*(a,\boldsymbol{\ell}_1\cdot\boldsymbol{\Omega}_1)\,\Lambda(a_1,\boldsymbol{\ell}_1\cdot\boldsymbol{\Omega}_1)} g^+_{\boldsymbol{\ell},\boldsymbol{\ell}_1\cdot\boldsymbol{\Omega}_1} \cdot \frac{\partial \bar{F}^1}{\partial \mathbf{J}}(\mathbf{J}) \Bigg) . \qquad (38)
\end{aligned}
$$

Equation (38) is sufficiently complicated to warrant some discussion. The first drag term has been discussed before. The term proportional to I_a^σ is Kaufmann's diffusion due to the unstable waves. The second term in the bracket is the scattering of particles by other particles. However, note that this term is well behaved as the damping rate approaches zero. The off-diagonal a, a_1 sum is again particle-particle scattering and results because the natural coordinates for the electric field and the natural coordinates for F are not the same; that is, it corresponds to inner products of the form $(2\pi)^{-3} \int d\boldsymbol{\theta}\; \mathbf{E}_a^*(\mathbf{J},\boldsymbol{\theta})\; \mathbf{E}_{a'}(\mathbf{J},\boldsymbol{\theta})$ which are not zero even though the usual x-space inner product $(\mathbf{E}_a|\mathbf{E}_{a'}) = \delta_{a,a'}$.

Equation (38), used in conjunction with the time evolution of I_a^σ , [Eq. (25)], provides the complete description of plasma kinetic theory and transport in a regime where mode-mode coupling is unimportant. Typically, the terms proportional to I_a^σ will be dominant if unstable modes are present; however, as saturation is approached the other terms may again become important. We note in addition that in the absence of unstable modes Eq. (38) in conjunction with the adiabatic solution of Eq. (25) [see Eq. (26)] yields the toroidal generalization of the usual collision operator. We shall discuss this further in the next section.

V. Properties of the Particle Kinetic Equation

In this section we shall focus on the form of the particle kinetic equation when there are no unstable modes. Kaufmann[5] has discussed the terms proportional to I_a^σ.

In this case the drag term may be combined with the diffusive term to yield

$$\frac{\partial F^1}{\partial t} = -\frac{\partial}{\partial \underset{\sim}{J}} \cdot \sum_{\underset{\sim}{\ell}, \underset{\sim}{\ell}_1} \int d\underset{\sim}{J}_1 \;\; \underset{\sim}{\ell} C_{\underset{\sim}{\ell}, \underset{\sim}{\ell}_1} \;\; \delta(\underset{\sim}{\ell} \cdot \underset{\sim}{\Omega} - \underset{\sim}{\ell}_1 \cdot \underset{\sim}{\Omega}_1) \left(\underset{\sim}{\ell}_1 \cdot \frac{\partial}{\partial \underset{\sim}{J}_1} - \underset{\sim}{\ell} \cdot \frac{\partial}{\partial \underset{\sim}{J}} \right) F^1(\underset{\sim}{J}_1) F^1(\underset{\sim}{J})$$

$$= C(F^1, F^1) \quad , \tag{39}$$

with

$$C_{\underset{\sim}{\ell}, \underset{\sim}{\ell}_1} = R_e \sum_{a_1, a_2} (2\pi)^3 \left(\frac{4\pi}{c^2}\right)^2 \frac{\lambda^*(a_1, \underset{\sim}{\ell}, \underset{\sim}{J})\, \lambda(a_2, \underset{\sim}{\ell}, \underset{\sim}{J})\, \lambda(a_1, \underset{\sim}{\ell}_1, \underset{\sim}{J}_1)\, \lambda^*(a_2, \underset{\sim}{\ell}_1, \underset{\sim}{J}_1)}{\Lambda^*(a_1, \underset{\sim}{\ell}_1 \cdot \underset{\sim}{\Omega}_1) \;\; (a_2, \underset{\sim}{\ell}_1 \cdot \underset{\sim}{\Omega}_1)} \pi \tag{40}$$

We note that $C_{\underset{\sim}{\ell}, \underset{\sim}{\ell}_1} = C_{\underset{\sim}{\ell}_1, \underset{\sim}{\ell}}$. Equation (39) is quite reminiscent of the usual Balescu-Guernsey-Lenard operator or the Landau form of the collision operator. It is easy to see that particle conservation

$$\int d\underset{\sim}{J} d\underset{\sim}{\theta} \; C(F^1, F^1) = 0 \tag{41}$$

energy conservation

$$\int d\underset{\sim}{J} d\underset{\sim}{\theta}\; H_0 C(F^1, F^1) = 0 \;, \tag{42}$$

H theorem

$$\int d\underset{\sim}{J} d\underset{\sim}{\theta}\; \ell n F^1 C(F^1, F^1) = \frac{d}{dt} S > 0 \;, \tag{43}$$

where

$$S = -\int d\underset{\sim}{J} F^1(\underset{\sim}{J})\; \ell n F^1(\underset{\sim}{J}) \;.$$

Further, if $F = \exp[-H_0/T]$, then $C(F,F) = 0$ and $dS/dt = 0$ and this is the only F with $dS/dt = 0$. These arguments are exactly the same as the usual ones in the infinite homogeneous medium. We note that they depend on C being bilinear with certain $(\underset{\sim}{\ell}, \underset{\sim}{\ell}_1)$ symmetry. However, they do not depend on the details of Λ , or λ , although the actual transport, of course, does.

VI. Conclusion

We have developed a complete kinetic theory for a plasma in an axisymmetric toroidal system. This theory smoothly joins the stable and unstable regimes and is valid when non-linear effects are unimportant. Thus the unification discussed at the outset has been accomplished.

The consequences of this theory, however, remain to be explored. One important point is that the translation of the $\underset{\sim}{J}$ space fluxes to radial fluxes is quite subtle. We have discussed this point elsewhere.[6] In addition, the relationship of the stable version of this theory to neoclassical theory discussed in Sec. V needs to be explored. We note that because of the H theorem the kinetic equation admits a variational principle. Thus, it should be possible to recover all of neoclassical theory from our theory. This will be explored in future work.

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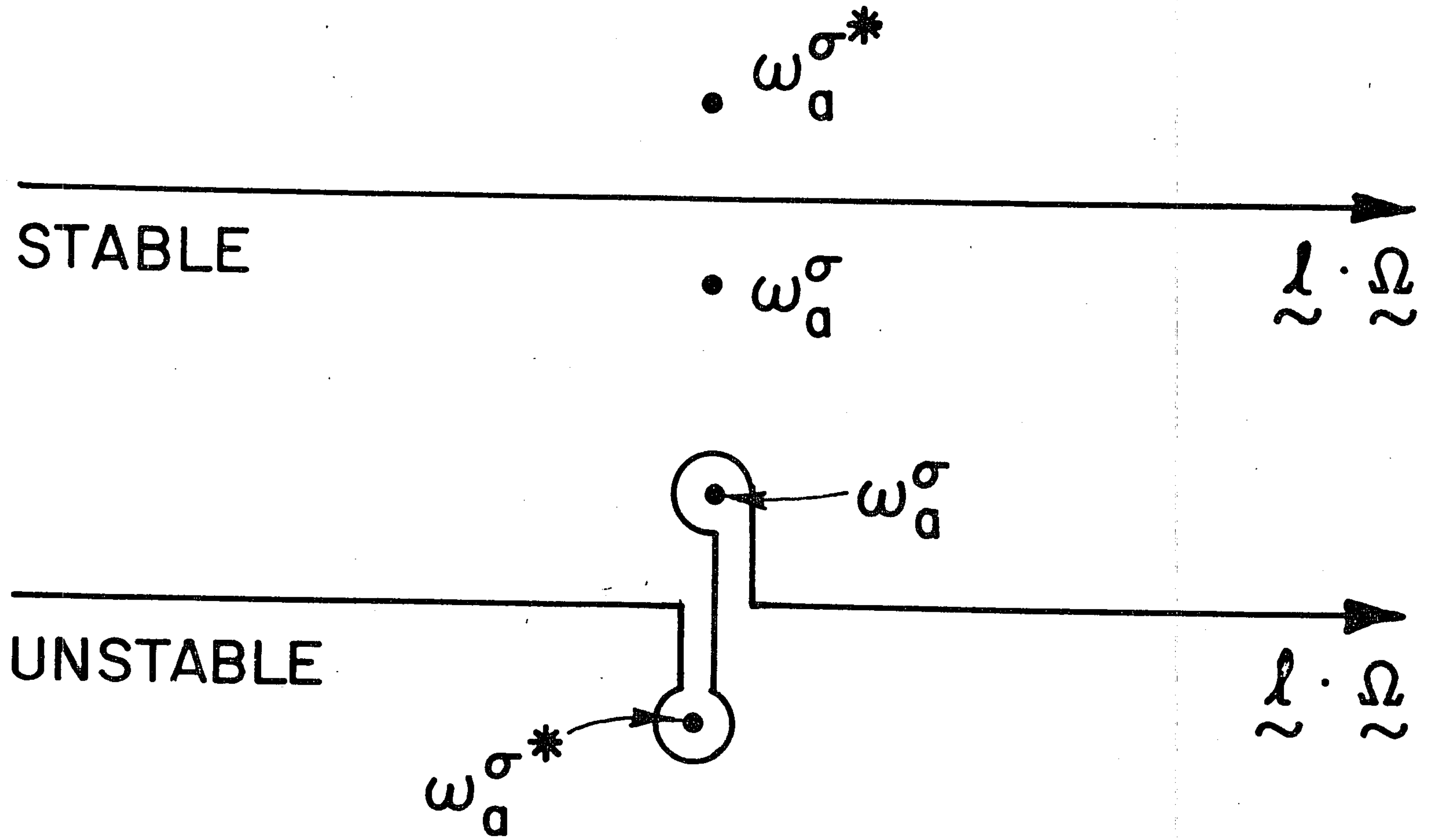
FIGURE CAPTIONS

Fig. 1. Contours of integration in the $\underset{\sim}{\ell}\cdot\underset{\sim}{\Omega}$ plane for stable and unstable modes.


$\omega_a^{\sigma *}$
STABLE
ω_a^{σ}
$\underset{\sim}{\ell} \cdot \underset{\sim}{\Omega}$
ω_a^{σ}
UNSTABLE
$\omega_a^{\sigma *}$
$\underset{\sim}{\ell} \cdot \underset{\sim}{\Omega}$