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Abstract

A method is presented for constructing exact solutions to a system of nonlinear plasma fluid equations that combines the physics of reduced magnetohydrodynamics and the electrostatic drift-wave description of the Charney-Hasegawa-Mima equation. The system has nonlinearities that take the form of Poisson brackets involving the fluid field variables. The method relies on modifying a class of simple equilibrium solutions, but no approximations are made. A distinguishing feature is that the original nonlinear problem is reduced to the solution of two linear partial differential equations, one fourth-order and the other first-order. The first-order equation has Hamiltonian characteristics and is easily integrated, supplying information about the general structure of solutions.

I. Introduction and Overview

In this paper the construction of exact analytic solutions for a system of nonlinear plasma fluid equations is discussed. The equations occur in a fluid model[1] which combines the physics of reduced magnetohydrodynamics[2,3] (RMHD) and the Charney-Hasegawa-Mima (CHM) equation.[4] The combined model is of interest because RMHD is an important tool for the interpretation of experimental results and for the prediction and theoretical analysis of nonlinear plasma fluid behavior in tokamaks. (To date, most of this work has been done numerically.) In the context of plasma physics, the CHM equation has been used in the study of electrostatic fluctuations in hot, turbulent plasmas; it incorporates the physics of electrostatic drift waves, which is not described by RMHD.

The solutions admitted by this nonlinear system are physically interesting because they are fully electromagnetic, like many disturbances seen in tokamak experiments; and they can take the form of solitary waves, which can be long-lived and very stable to perturbations. Hence they could describe plasma behavior that might be detected experimentally. The method by which a class of solutions is obtained here is also of intrinsic mathematical interest: the nonlinear system of governing partial differential equations (pde's) is reduced to a linear system that is in principle exactly soluble by standard techniques. (If one wishes to be single-mindedly practical, the analytic solutions could also serve as a means of verifying the computer codes used for RMHD calculations.)

Here is an overview of what follows. In Section II the fluid equations are presented and their physical content is briefly discussed. Their nonlinear character is manifested by Poisson brackets involving the fluid field variables.

Section III is concerned with finding solutions to the fluid equations for the case of a perfectly conducting plasma. First the construction of exact solutions for the equilibrium form of the equations is considered. A simple change of dependent variables is used to eliminate the Poisson brackets and reduce the problem to solving a single linear pde. This provides a foundation and motivation for the more general problem of constructing exact solutions to the nonequilibrium equations. Next, by a slight modification of the change of dependent variables for the equilibrium case, the nonlinear, nonequilibrium equations are reduced to a pair of linear pde's, one first-order and the other fourth-order. An algorithm for constructing solutions based on this reduction is presented. Finally, the first-order pde is integrated by the method of characteristics; this will be seen to provide information about the general structure of the solutions.

In Section IV a summary is presented. The limitations of our method, possible modifications to it, and areas for further work are discussed.

II. Fluid Equations

A. Geometry and Coordinates

What follows is a description of the geometry and the coordinates used. First of all, the presumed geometry is toroidal, that of a tokamak with a circular cross section. However, the parameter beta for the plasma is assumed to be small — this excludes pressure-driven dynamics and magnetic curvature from the physics described by the fluid equations, thus making them applicable to cylindrical and slab geometries, also. Let us introduce a set of normalized coordinates:

$$x = \frac{R - R_0}{a} , \quad y = \frac{Z}{a} , \quad \text{and} \quad z = -\zeta . \tag{1}$$

Here (R, ζ, Z) represent cylindrical coordinates centered on the symmetry axis of the tokamak: R measures radial displacements away from the symmetry axis, ζ is the toroidal angle, and Z measures vertical displacements above or below the horizontal symmetry plane of the tokamak. R_0 is the major radius of the tokamak. In the context of RMHD, a is the tokamak's minor radius and is thus a scale characterizing fluid motions transverse to the magnetic field.

If a is taken to be of the order of the ion Larmor radius, it serves as a useful length scale for the description of electrostatic drift-wave physics in the context of the CHM equation. Hence (x, y, z) is a right-handed set of local poloidal coordinates useful for describing plasma behavior on different length scales within the torus.

B. Important Physical Quantities and Their Orderings

We next introduce the three normalized field variables that appear in the equations: ϕ, ψ, and χ. The quantity ϕ represents the electrostatic potential; ψ represents the parallel component of the magnetic vector potential, or the poloidal magnetic flux; and χ represents a small perturbation of the plasma density. The unperturbed plasma density, denoted by n_c, is assumed to be constant in both space and time. The vacuum magnetic field is assumed to be purely toroidal and to dominate any magnetic fields due to the plasma. Thus ψ represents the addition, due to the plasma, to the vacuum field.

The dimensionless ordering parameter is ε, the inverse aspect ratio of the tokamak:

$$\varepsilon \equiv \frac{a}{R_0} \ll 1 \; . \tag{2}$$

The electric and magnetic fields are ordered using ε to express the presumed dominance of the vacuum magnetic field: the scalar and vector potentials for the electromagnetic fields generated by the plasma are assumed $\mathcal{O}(\varepsilon)$ compared to that for the vacuum magnetic field. The plasma beta is $\mathcal{O}(\varepsilon^2)$, a "low beta" ordering. The plasma density is assumed to deviate from n_c by a quantity $\mathcal{O}(\varepsilon)$. A normalized time coordinate τ is defined by

$$\tau \equiv \varepsilon \, \frac{t v_A}{a} \; , \tag{3}$$

which is appropriate for the slow, shear-Alfvén fluid motions of interest. Here t is the usual time coordinate; v_A, a constant, is a measure of the Alfvén speed for the plasma. Thus all the important physical quantities are ordered in terms of ε.

In terms of the ε orderings described above, the component of the fluid velocity perpendicular to the magnetic field is[1]

$$\mathbf{v}_\perp = \varepsilon\, v_A\, \hat{\mathbf{z}} \times \nabla_\perp \phi + \mathcal{O}(\varepsilon^2) \; . \tag{4}$$

Here $\nabla_\perp$ is the poloidal component of the normalized gradient operator $a\nabla$:

$$\nabla_\perp \equiv \hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} \; , \tag{5}$$

where $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are just the appropriate unit vectors. The first term on the right-hand side of (4) represents the usual $\mathbf{E} \times \mathbf{B}$ fluid drift, and the factor εv_A emphasizes that the fluid motions considered are very slow compared to the Alfvén speed.

C. The Reduced Fluid Equations

To obtain the reduced fluid equations for the combined system, the ε ordering scheme summarized above is incorporated into the appropriate exact, resistive MHD equations. To arrive at the approximate equations given below, the terms of lowest order in ε are kept. A complete derivation of the equations is available elsewhere.[1] The following short description is provided to make the physical content and the mathematical symbolism more transparent.

Before proceeding with the presentation of the fluid equations, we introduce two quantities that will appear quite often below. The first is

$$U \equiv \nabla_\perp^2 \phi \; , \tag{6}$$

the parallel component of the fluid vorticity. The second is

$$J \equiv \nabla_\perp^2 \psi \; , \tag{7}$$

the parallel component of the plasma current. To make the fluid equations more compact, it is also useful to introduce the Poisson bracket defined by

$$[f,g] \equiv \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} = \hat{\mathbf{z}} \cdot \nabla_\perp f \times \nabla_\perp g \; . \tag{8}$$

The first of the equations is the "shear-Alfvén law,"

$$\frac{\partial U}{\partial \tau} + [\phi, U] = -\left(\frac{\partial J}{\partial z} - [\psi, J]\right) . \tag{9}$$

The left-hand side represents the convective time derivative of U: the second term, $[\phi, U]$, represents convection of U due to the $\mathbf{E} \times \mathbf{B}$ fluid drift. Acting on J in the right-hand side of (9) is the operator $\frac{\partial}{\partial z} - [\psi, \quad]$, which is essentially $\mathbf{B} \cdot \nabla$. The physical content of the right-hand side of (9) is thus current-driven dynamics, such as kink modes.

The second of the equations comes from the parallel component of a modified Ohm's law:

$$\frac{\partial \psi}{\partial \tau} + \frac{\partial \phi}{\partial z} - [\psi, \phi] = \hat{\eta}\, J + \alpha \left(\frac{\partial \chi}{\partial z} - [\psi, \chi]\right) . \tag{10}$$

Here the left-hand side represents the parallel component of the electric field. The quantity $\hat{\eta}$ is a normalized collisional resistivity. The last quantity in parentheses on the right-hand side represents pressure effects on parallel electron flow. In the combined model it is assumed that electrons almost exclusively carry the parallel current. The constant α is defined by

$$\alpha^2 = \frac{\rho_s^2}{a^2} , \tag{11}$$

where

$$\rho_s^2 = \frac{T_e}{m_i\, \Omega_i^2} . \tag{12}$$

Here m_i is the ion mass, T_e is the constant *electron* temperature in energy units, and Ω_i is the ion Larmor frequency:

$$\Omega_i = \frac{e\, B_T}{m_i\, c} , \tag{13}$$

where B_T is a constant that measures the strength of the vacuum magnetic field. In the combined model, α represents the marriage of RMHD and electrostatic drift-wave physics.

The last equation we consider,

$$\frac{\partial \chi}{\partial \tau} + [\phi, \chi] + \frac{\partial J}{\partial z} - [\psi, J] = 0 , \tag{14}$$

is derived from the equation for electron conservation and quasineutrality. The second term on the left side is just $\mathbf{E} \times \mathbf{B}$ convection of the plasma density. Electron parallel mobility is explicit in the last two terms: these come from the divergence of the parallel electron fluid velocity, which is essentially proportional to the parallel plasma current J.

III. Construction of Exact Solutions

A. Framework

Having introduced the fluid equations, we next discuss a method for arriving at exact solutions of them.

We denote the partial derivative of a quantity by a subscript, e.g., $\frac{\partial U}{\partial \tau} \equiv U_\tau$. Then, after rearranging the terms of (9) and (10) and subtracting (14) from (9), we can write

$$U_\tau + [\phi, U] + J_z + [J, \psi] = 0 \ , \tag{15}$$

$$\psi_\tau + (\phi - \alpha X)_z + [\phi - \alpha X, \psi] = 0 \ , \tag{16}$$

and

$$(U - X)_\tau + [\phi, U - X] = 0 \ . \tag{17}$$

This is the nonlinear system we will study. Note that we are taking $\hat{\eta} = 0$ in (16); the resistivity of the plasma is neglected for all that follows.

To satisfy (17) we take

$$X = g(z) + U \ , \tag{18}$$

where g is an arbitrary function of z. This is by no means the general solution to (17); it is simply a special case which satisfies (17) with little effort. Defining

$$\xi \equiv \phi - \alpha\, g(z) \tag{19}$$

and recasting (15) and (16) in terms of ξ gives

$$U_\tau + [\xi, U] + J_z + [J, \psi] = 0 \tag{20}$$

and

$$\psi_\tau + (\xi - \alpha U)_z + [\xi - \alpha U, \psi] = 0 \ , \tag{21}$$

where (18) has been used. We note in passing that from (19) and (6), the definition of U, we have

$$U = \nabla_\perp^2 \xi \ , \tag{22}$$

a relation that will be used often in what follows.

Now we have to find solutions to (20) and (21). Let us first consider the simpler case of axisymmetric equilibrium.

B. Axisymmetric Equilibrium

Under the assumption of axisymmetric equilibrium, $\frac{\partial}{\partial \tau} \equiv 0$ and $\frac{\partial}{\partial z} \equiv 0$, (20) and (21) reduce to

$$[\xi, \nabla_\perp^2 \xi] - [\psi, \nabla_\perp^2 \psi] = 0 \tag{23}$$

and

$$[\xi, \psi] - \alpha [U, \psi] = 0 \ . \tag{24}$$

(Here $J \equiv \nabla_\perp^2 \psi$ and $U = \nabla_\perp^2 \xi$ were used.) We take

$$\psi = \gamma \xi \ , \tag{25}$$

where γ is an arbitrary constant. Then (23) and (24) reduce to

$$(1 - \gamma^2)[\xi, \nabla_\perp^2 \xi] = 0 \tag{26}$$

and

$$\alpha\gamma[\xi, U] = 0 \ . \tag{27}$$

In the same spirit as (25), we take

$$U = \delta\xi \ , \tag{28}$$

where δ is an arbitrary constant. This choice has the virtue of satisfying both (26) and (27) with little effort. In addition it imposes the constraint that

$$\nabla_\perp^2 \xi = \delta\xi \ . \tag{29}$$

Thus finding some solutions of the *nonlinear* pde's (23) and (24) has been reduced to solving the *linear* pde (29): the troublesome nonlinear Poisson brackets have been eliminated with the *ansätze* $\psi = \gamma\xi$ and $U = \delta\xi$. Knowing ξ as determined by (29), one can then easily find the field variables ϕ, ψ, and X.

This class of solutions for axisymmetric equilibrium has an interesting physical interpretation. For the low beta case being considered, the magnetic field in the tokamak takes the form[1]

$$\mathbf{B} = \frac{B_T}{1 + \varepsilon\, x}\, \hat{\mathbf{z}} - \varepsilon\, B_T\, \hat{\mathbf{z}} \times \nabla_\perp \psi + \mathcal{O}(\varepsilon^2) \ . \tag{30}$$

The second term on the right-hand side represents the poloidal magnetic field, $\mathbf{B}_p$. Operating on the relation $\psi = \gamma\xi$ with $\varepsilon v_A\, \hat{\mathbf{z}} \times \nabla_\perp$ gives[5]

$$\varepsilon v_A\, \hat{\mathbf{z}} \times \nabla_\perp \psi = \gamma[\varepsilon v_A\, \hat{\mathbf{z}} \times \nabla_\perp \phi] \ . \tag{31}$$

Comparing this with (30) and the relation (4) for $\mathbf{v}_\perp$, one can see that the left side is proportional to $\mathbf{B}_p$ and the right side is essentially proportional to $\mathbf{v}_\perp$. Thus (31) can be rewritten more suggestively as

$$\mathbf{v}_\perp = \frac{-v_A}{\gamma B_T}\, \mathbf{B}_p + \mathcal{O}(\varepsilon^2) \ . \tag{32}$$

This result is similar to the fluid velocity for a nonlinear Alfvén wave found by Walén.[6]

C. Allowing for τ and z Dependence

Next we complicate the previous discussion somewhat with the addition of τ and z dependence to ξ. As for the case of axisymmetric equilibrium, we continue to take $\psi = \gamma\xi$ and $U = \delta\xi$ and use these relations in (20) and (21) to arrive at the linear equations

$$\xi_\tau + \gamma\xi_z = 0 \tag{33}$$

and

$$\gamma\xi_\tau + (1 - \alpha\delta)\xi_z = 0 \ . \tag{34}$$

These two first-order pde's in ξ will be consistent with each other if we take

$$1 - \alpha\delta = \gamma^2 \ . \tag{35}$$

From (33) one can see that the solution for ξ must be of the form $\xi = \xi(x, y, z - \gamma\tau)$, which corresponds to a structure propagating toroidally.

In addition to the first-order equation (33), ξ must once again satisfy $\nabla_\perp^2 \xi = \delta\xi$ because of the *ansätz* $U = \delta\xi$. Even though there are now two equations to solve for ξ, they are linear and therefore much more tractable than the nonlinear equations (20) and (21).

D. Addition of Perturbation Terms

A more general class of solutions can be obtained with the *ansätze*

$$\psi = \gamma\xi(x, y, z, \tau) + f(x, y, z, \tau) \tag{36}$$

and

$$U = \delta\xi(x, y, z, \tau) + h(x, y, z, \tau) \ . \tag{37}$$

As before, γ and δ are arbitrary constants; f and h are arbitrary functions. One can view f and h as perturbations on the forms $\psi = \gamma\xi$ and $U = \delta\xi$ used previously. However, no assumption is made about their sizes: f and h need not be small.

Let us proceed with the construction of solutions. First, note that

$$J = \gamma\delta\xi + \gamma h + \nabla_\perp^2 f \tag{38}$$

from the definition $J \equiv \nabla_\perp^2 \psi$, (36), and (37). Using (36)–(38) in the nonlinear equations (20) and (21), one obtains the following two equations linear in ξ:

$$\xi_\tau + \gamma\xi_z + \left[\xi,\ \frac{1-\gamma^2}{\delta}\, h + \gamma f - \frac{\gamma}{\delta}\, \nabla_\perp^2 f\right]$$

$$= \frac{1}{\delta}\left\{[f, \gamma h + \nabla_\perp^2 f] - (\gamma h + \nabla_\perp^2 f)_z - h_\tau\right\} \tag{39}$$

and

$$\xi_\tau + \frac{1-\alpha\delta}{\gamma}\,\xi_z + \left[\xi,\ \frac{1-\alpha\delta}{\gamma}\, f + \alpha h\right] = \frac{1}{\gamma}\,\{\alpha h_z - [f, \alpha h] - f_\tau\}\ . \tag{40}$$

Next we require that these two equations be redundant. This is by no means a necessary constraint, and in fact we have found a class of solutions where (39) and (40) are not redundant. However, requiring redundancy does lead to interesting solutions.

After some manipulation the conditions for redundancy are found to be

$$\gamma = \frac{1-\alpha\delta}{\gamma}\ , \tag{41}$$

$$\nabla_\perp^2 f = -\frac{\delta}{\gamma}\, p(z, \tau) \tag{42}$$

and

$$\left[f,\ \frac{1}{\gamma} h\right] - \frac{1}{\gamma}\,\{h - \delta p(z,\tau)\}_z + \left(\frac{\delta}{\gamma} f - h\right)_\tau = 0\ . \tag{43}$$

Here p is an arbitrary function of z and τ. The application of these conditions reduces (39) and (40) to the single equation

$$\xi_\tau + \gamma\xi_z + [\xi, \gamma f + \alpha h] = \frac{\alpha\delta}{\gamma}\, p_z - (\gamma f + \alpha h)_\tau\ , \tag{44}$$

a linear, first-order pde in ξ. From $U = \nabla_\perp^2 \xi$ and $U = \delta\xi + h$, we obtain the additional relation

$$\nabla_\perp^2 \xi = \delta\xi + h \ . \tag{45}$$

Thus the two nonlinear equations in ξ, (20) and (21), have been transformed into the two linear equations (44) and (45); these linear equations in ξ are supplemented by the three redundancy conditions, (41)–(43).

Next we must ensure that (44) and (45) are compatible with one another. For convenience we introduce the operator $\mathcal{L}$ defined by

$$\mathcal{L} \equiv -\gamma \frac{\partial}{\partial z} + [\gamma f + \alpha h, \quad] \ . \tag{46}$$

Operating on (44) with $\nabla_\perp^2$ and making use of (45) and (46), one obtains the compatibility condition

$$\delta\xi_\tau + h_\tau - \nabla_\perp^2 \mathcal{L}\xi = -\nabla_\perp^2(\gamma f + \alpha h)_\tau \ . \tag{47}$$

Introducing the commutator $(\nabla_\perp^2, \mathcal{L})$ defined by

$$(\nabla_\perp^2, \mathcal{L}) \equiv \nabla_\perp^2 \mathcal{L} - \mathcal{L}\nabla_\perp^2 \ , \tag{48}$$

and using (45), one can rewrite (47) in the more interesting form

$$\delta\xi_\tau + h_\tau - \mathcal{L}(\delta\xi + h) - (\nabla_\perp^2, \mathcal{L})\xi = -\nabla_\perp^2(\gamma f + \alpha h)_\tau \ . \tag{49}$$

In the interest of simplicity, we impose the constraint that

$$(\nabla_\perp^2, \mathcal{L})\xi \equiv 0 \tag{50}$$

for every "well-behaved" function ξ. This condition can be shown to require that

$$\gamma f + \alpha h = H \ , \tag{51}$$

where H is defined by

$$H \equiv \frac{1}{2} a(z,\tau)(x^2 + y^2) + b(z,\tau)y + c(z,\tau)x + d(z,\tau) \ . \tag{52}$$

Here a, b, c, and d are arbitrary functions of z and τ. Note that f can now be eliminated in favor of h and H through the relation

$$f = \frac{1}{\gamma}(H - \alpha h) , \tag{53}$$

which follows directly from (51).

With (50) and (51), the first-order equation (44), and the explicit definition of $\mathcal{L}$ as given by (46), we can reexpress (49) as

$$\left(\frac{\alpha\delta^2}{\gamma} p + \gamma h\right)_z + (h - \delta H + \nabla_\perp^2 H)_\tau = \gamma[f, h] . \tag{54}$$

This relation can be further simplified with the application of (53) and the redundancy conditions (41) and (43). The result is

$$\nabla_\perp^2 H_\tau = -\frac{\delta}{\gamma} p_z , \tag{55}$$

a much more compact form for the compatibility condition.

There are two more pde's to consider in addition to those for ξ. Using (53) and (55) in the redundancy relations (42) and (43), we obtain the following pair of equations for h:

$$\alpha \nabla_\perp^2 h = \nabla_\perp^2 H + \delta p \tag{56}$$

and

$$h_\tau - \mathcal{L}h = -(\gamma^2 \nabla_\perp^2 H - \delta H)_\tau . \tag{57}$$

The compatibility of these equations is treated in much the same way as for the ξ equations: taking $\nabla_\perp^2$ of both sides of (57) and making use of (50), (55), and (56) in the result, one obtains the condition

$$\nabla_\perp^2 H_z = -\frac{\delta}{\gamma} p_\tau . \tag{58}$$

It is more enlightening to rewrite (55) and (58) together using

$$\nabla_\perp^2 H = 2a(z, \tau) , \tag{59}$$

which follows from the definition of H. Doing this, one obtains the complementary relations

$$2a_\tau = -\frac{\delta}{\gamma} p_z \tag{60}$$

and

$$2a_z = -\frac{\delta}{\gamma} p_\tau \ . \tag{61}$$

From these equations it is easily found that

$$a = a_1(z+\tau) + a_2(z-\tau) \tag{62}$$

and

$$p = -\frac{2\gamma}{\delta} \left[a_1(z+\tau) - a_2(z-\tau)\right] + \kappa \ , \tag{63}$$

where κ is an arbitrary constant and a_1 and a_2 are arbitrary functions. Thus consideration of the compatibility of the pde's for h and ξ has yielded information about the structure of p and a and the relationship between them.

We next distill our four pde's for ξ and h to two essential equations for ξ alone. We first collect the four equations for ξ and h. Recall from (45) that the second-order equation for ξ can be expressed as

$$h = \nabla_\perp^2 \xi - \delta\xi \ . \tag{64}$$

Using (46), (51), and (60), we can rewrite the first-order equation for ξ, (44), as

$$\xi_\tau - \mathcal{L}\xi = -(2\alpha a + H)_\tau \ . \tag{65}$$

With (59) and (63) the second-order relation for h, (56), becomes

$$\alpha \nabla_\perp^2 h = 2(1-\gamma)a_1 + 2(1+\gamma)a_2 + \delta\kappa \ . \tag{66}$$

The first-order relation (57) becomes

$$h_\tau - \mathcal{L}h = -(2\gamma^2\, a - \delta H)_\tau \tag{67}$$

with the use of (59).

Now we can eliminate h from (66) using (64) to obtain

$$\alpha\nabla_\perp^2(\nabla_\perp^2\,\xi - \delta\xi) = 2(1-\gamma)a_1 + 2(1+\gamma)a_2 + \delta\kappa\ , \tag{68}$$

a fourth-order relation for ξ. With the help of (64) and (65), it is easy to show that (67) in fact reduces to an identity. We also note that it is not difficult to directly ascertain that (65) and (68) are compatible with each other as they stand; no further constraints are needed to ensure their compatibility.

At this point we have only to integrate the linear equations (65) and (68) for ξ to obtain a complete, explicit solution for our original system of nonlinear fluid equations. The following algorithm summarizes the results of this section.

E. An Algorithm for Constructing Solutions

1. Choose values for the constants δ and α and thus determine γ from the relation

$$\gamma = \sigma\sqrt{1-\alpha\delta}\ , \tag{69}$$

which follows from (41). Here $\sigma = \pm 1$. The parameter α is determined by physical considerations; recall (11).

2. Specify

$$H \equiv \frac{1}{2}\left[a_1(z+\tau) + a_2(z-\tau)\right](x^2+y^2) + b(z,\tau)y + c(z,\tau)x + d(z,\tau) \tag{70}$$

by choosing the functions $a_1, a_2, b, c,$ and d. This form for H is obtained after incorporating into (52) the information we obtained about the function a from (60) and (61).

3. Choose a value for the constant κ and then find ξ by integrating

$$\xi_\tau + \sigma\sqrt{1-\alpha\delta}\,\xi_z + [\xi, H] = -\left(2\alpha[a_1+a_2] + H\right)_\tau \tag{71}$$

and

$$\alpha\nabla_\perp^2(\nabla_\perp^2\,\xi - \delta\xi) = 2\left(1 - \sigma\,\sqrt{1-\alpha\delta}\right)a_1 + 2(1+\sigma\sqrt{1-\alpha\delta}\,)a_2 + \delta\kappa\ . \tag{72}$$

Equations (71) and (72) follow from (65), (68), and (69).

4. Choose the function $g(z)$. The solutions for the field variables ϕ, ψ, and χ readily follow: from (19),

$$\phi = \xi + \alpha g\ ; \tag{73}$$

from (36), (53), and (64),

$$\psi = \frac{1}{\sigma\sqrt{1-\alpha\delta}}\left(\xi - \alpha\nabla_\perp^2\,\xi + H\right)\ ; \tag{74}$$

and from (18) and (22),

$$\chi = g + \nabla_\perp^2\,\xi\ . \tag{75}$$

Other physical quantities of interest are the vorticity,

$$U = \nabla_\perp^2\,\xi\ , \tag{22}$$

and the parallel current,

$$J = \sigma\,\sqrt{1-\alpha\delta}\ \nabla_\perp^2\,\xi + 2(a_1 - a_2) - \frac{\delta}{\sigma\,\sqrt{1-\alpha\delta}}\,\kappa\ , \tag{76}$$

which follows from the definition $J \equiv \nabla_\perp^2\,\psi$, (74), and (72).

Next the integration of the first-order equation (71) is considered in more detail to study the general structure of the solutions.

F. Integration of the First-Order Equation for ξ

We integrate the first-order pde for ξ, (71), by the method of characteristics. The characteristics are determined by integrating the following system of ordinary differential equations associated with (71):

$$\frac{dx}{d\tau} = H_y = a(z,\tau)y + b(z,\tau)\ , \tag{77}$$

$$\frac{dy}{d\tau} = -H_x = -a(z,\tau)x - c(z,\tau) \; , \tag{78}$$

$$\frac{dz}{d\tau} = \gamma \; , \tag{79}$$

and

$$\frac{d\xi}{d\tau} = -(2\alpha a + H)_\tau \; . \tag{80}$$

Relation (69) has been used to make (79) more compact for the sake of the work to follow; a is given by (62).

We can readily integrate (79) to find

$$z = z_0 + \gamma\tau \; , \tag{81}$$

where z_0 is a constant of integration.

We can use (81) to replace z wherever it occurs in (77), (78), and (80) to facilitate the integration of these equations. For the moment let us focus upon (77) and (78). From (77) we obtain

$$\frac{dx}{d\tau} = H_y = a(z_0 + \gamma\tau, \tau)y + b(z_0 + \gamma\tau, \; \tau) \; ; \tag{82}$$

from (78),

$$\frac{dy}{d\tau} = -H_x = -a(z_0 + \gamma\tau, \tau)x - c(z_0 + \gamma\tau, \; \tau) \; . \tag{83}$$

Note that this pair of equations is Hamiltonian in structure, with H playing the role of the Hamiltonian function that governs the dynamics of x and y.

Writing the *homogeneous* form of (82) and (83) in terms of matrices, we have

$$\frac{d}{d\tau}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & a(z_0 + \gamma\tau, \tau) \\ -a(z_0 + \gamma\tau, \tau) & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \; . \tag{84}$$

It is interesting to note that if we define a position vector $\mathbf{r} \equiv x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$ then (84) can be written in the form

$$\frac{d\mathbf{r}}{d\tau} = \mathbf{r} \times \hat{\mathbf{z}}\, a(z_0 + \gamma\tau, \tau) \; , \tag{85}$$

which describes a gyration in the xy-plane with a *time-dependent* frequency $a(z_0 + \gamma\tau, \tau)$. Consequently it is not surprising that a fundamental matrix for (84) is

$$\boldsymbol{\mu}(\tau; z_0) = \begin{bmatrix} \cos\left(\int_0^\tau a(z_0 + \gamma\tau', \tau')d\tau'\right) & \sin\left(\int_0^\tau a(z_0 + \gamma\tau', \tau')d\tau'\right) \\ -\sin\left(\int_0^\tau a(z_0 + \gamma\tau', \tau')d\tau'\right) & \cos\left(\int_0^\tau a(z_0 + \gamma\tau', \tau')d\tau'\right) \end{bmatrix} , \qquad (86)$$

which reduces to the identity matrix at $\tau = 0$.

With the fundamental matrix $\boldsymbol{\mu}$ at our disposal, we can write the solution to (82) and (83) as

$$\mathbf{r} = \boldsymbol{\mu}(\tau; z_0)\mathbf{r}(0) + \boldsymbol{\mu}(\tau; z_0) \int_0^\tau \boldsymbol{\mu}^{-1}(\tau'; z_0)\ \mathbf{f}(\tau'; z_0 + \gamma\tau')d\tau' \ . \qquad (87)$$

Here

$$\mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix} ; \qquad (88)$$

$$\mathbf{r}(0) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} , \qquad (89)$$

where x_0 and y_0 are arbitrary constants of integration; and

$$\mathbf{f}(\tau; z_0 + \gamma\tau) = \begin{bmatrix} b(z_0 + \gamma\tau,\ \tau) \\ -c(z_0 + \gamma\tau,\ \tau) \end{bmatrix} . \qquad (90)$$

To construct the most general solution of the homogeneous form of (71), we need three functionally independent first integrals. One of these first integrals is

$$z_0(x, y, z, t) = z - \gamma\tau \ , \qquad (91)$$

which follows from (81). Two additional first integrals are

$$\begin{bmatrix} x_0(x, y, z, \tau) \\ y_0(x, y, z, \tau) \end{bmatrix} = \boldsymbol{\mu}^{-1}(\tau; z - \gamma\tau) \begin{bmatrix} x \\ y \end{bmatrix} - \int_0^\tau \boldsymbol{\mu}^{-1}(\tau'; z - \gamma\tau)\,\mathbf{f}(\tau'; z - \gamma[\tau - \tau'])d\tau' \ , \qquad (92)$$

which follow from (87) upon replacing z_0 with $z - \gamma\tau$. The general solution to the homogeneous form of (71) is thus $\xi_h(x_0, y_0, z_0)$, an arbitrary function of the first integrals.

A "particular integral" which satisfies the inhomogeneous equation (71) may be obtained by integrating (80). For brevity we simply denote this particular integral as $\xi_p(x, y, z, \tau)$. Thus a solution for (71) is

$$\xi = \xi_h(x_0, y_0, z_0) + \xi_p(x, y, z, t) \ . \tag{93}$$

G. Examples

Here we consider some special cases of (93) obtained by specializing a, b, c, and d in H of (70). Our choices for these four functions will determine the structure of the first integrals x_0 and y_0 through (92). For the cases we consider, their structure will be easy to discern and will give some insight into the behavior of ξ.

Case (i)

The first case we consider is a rather drastic simplification of the general result (92): *we take a, b, c, and d all to be zero, getting rid of H entirely.* Then we are simply left with

$$x_0 = x \qquad \text{and} \qquad y_0 = y \ . \tag{94}$$

Thus in this case the general solution for ξ is of the form

$$\xi = \xi(x, y, z - \gamma\tau) \ , \tag{95}$$

which corresponds to a structure propagating toroidally with speed γ.

Case (ii)

Next let us take a to be a nonzero constant, still keeping b, c, and d zero. Then

$$x_0 = x \cos a\tau - y \sin a\tau \tag{96}$$

and

$$y_0 = x \sin a\tau + y \cos a\tau \ . \tag{97}$$

If we introduce the poloidal polar coordinates r and θ such that

$$x = r \cos\theta \qquad \text{and} \qquad y = r \sin\theta \ , \tag{98}$$

then (96) and (97) may be written as

$$x_0 = r\cos(\theta + a\tau) \qquad \text{and} \qquad y_0 = r\sin(\theta + a\tau) \ . \tag{99}$$

Consequently the general solution for ξ can take the form

$$\xi = \xi(r, \theta + a\tau,\ z - \gamma\tau) \ . \tag{100}$$

Thus ξ represents a structure that rotates poloidally with speed a, in addition to propagating toroidally.

Case (iii)

We take a, b, c, and d all to be nonzero constants. Then we find that

$$x_0 = \left(x + \frac{c}{a}\right) \cos a\tau - \left(y + \frac{b}{a}\right) \sin a\tau - \frac{c}{a} \tag{101}$$

and

$$y_0 = \left(x + \frac{c}{a}\right) \sin a\tau + \left(y + \frac{b}{a}\right) \cos a\tau - \frac{b}{a} \ . \tag{102}$$

We can drop the constants at the end of each of these relations to obtain another perfectly good pair of first integrals,

$$\widehat{x}_0 = \left(x + \frac{c}{a}\right) \cos a\tau - \left(y + \frac{b}{a}\right) \sin a\tau \tag{103}$$

and

$$\widehat{y}_0 = \left(x + \frac{c}{a}\right) \sin a\tau + \left(y + \frac{b}{a}\right) \cos a\tau \ . \tag{104}$$

Thus we see that as for Case (ii) above, the general solution for ξ represents a structure exhibiting poloidal rotation with speed a, except that now the rotation occurs about the origin with (x, y) coordinates $\left(-\frac{c}{a}, -\frac{b}{a}\right)$. To make this more explicit, we introduce the polar coordinates r_1 and θ_1 such that

$$x + \frac{c}{a} = r_1 \cos\theta_1 \qquad \text{and} \qquad y + \frac{b}{a} = r_1 \sin\theta_1 \; . \tag{105}$$

Then the general solution for ξ will take the form

$$\xi = \xi(r_1, \theta_1 + a\tau, \; z - \gamma\tau) \; . \tag{106}$$

Case (iv)

If we take $a = 0$ *and* b, c, *and* d *nonzero constants*, then we obtain the first integrals

$$x_0 = x - b\tau \qquad \text{and} \qquad y_0 = y + c\tau \; . \tag{107}$$

Therefore the general solution for ξ takes the form

$$\xi = \xi(x - b\tau, \; y + c\tau, \; z - \gamma\tau) \; , \tag{108}$$

representing a structure that propagates rectilinearly through space with the velocity $b\hat{\mathbf{x}} - c\hat{\mathbf{y}} + \gamma\hat{\mathbf{z}}$.

The simple cases we have considered above are sufficient to show one how the parameters a, b, c, and d determine the structure of the general solution for ξ, even when one generalizes to the case where a, b, c, and d depend on z and τ. The parameter a determines the speed of rotation about some origin in the poloidal, xy-plane; b and c determine the center or origin for that poloidal rotation. In Case (iv) with only $a = 0$, the finite shift becomes a rectilinear propagation with speed and direction determined by b, c, and γ.

Note that for the cases we have considered, with a, b, c, and d all constants, the first-order equation for ξ given by (71) is homogeneous: the source term $-(2\alpha a + H)_\tau = 0$. Thus the remarks of the preceding paragraph — appropriately generalized for a, b, c, and d depending on z and τ — only apply to the homogeneous part $\xi_h(x_0, y_0, z_0)$ of the general solution for ξ given in (93). Note also that d plays no role in determining the structure of the homogeneous solution — d only appears in the source term $-(2\alpha a + H)_\tau$ of (71). Thus d only plays a role in determining the particular integral for (71).

The next step is to integrate the fourth-order equation (72) to completely determine ξ. We prefer to defer this to a future publication.

IV. Summary and Discussion

We have constructed exact analytic solutions to a system of nonlinear plasma fluid equations which combine RMHD and CHM drift dynamics. (The resistivity of the plasma was neglected: $\hat{\eta} = 0$.) Our method reduces the original system to a pair of linear pde's, one first-order, the other fourth-order. In Section III, Part G, from the integration of the first-order equation, it is seen that solutions for some special cases take the form of fully electromagnetic structures propagating through the plasma, i.e. solitary waves. These special cases were obtained by making choices for four arbitrary functions (of z and τ) that determine the motion of the wave. For the simple choices considered, this motion can take the form of purely toroidal propagation with a constant speed; it can also include a poloidal rotation at constant speed coupled with the toroidal propagation; or it can consist of a rectilinear propagation at constant speed in an arbitrary direction. For more general choices the motion can be much more complicated. (Even though our efforts have largely been directed at constructing propagating solutions, the method can be used to construct nonlinear equilibrium solutions that may be of physical interest as well.)

We conclude with a discussion of the limitations of our method for constructing solutions and where it might be modified to obtain classes of solutions distinct from the ones thus far presented.

First of all, one should note that the nolinearities in the fluid equations — (15)–(17) in Section III, Part A — exhibit a special structure: they exclusively take the form of the *Poisson brackets* defined by (8) involving the field variables ϕ, ψ, χ, and U and J. The key feature of our method is the elimination of these nonlinear Poisson brackets, leaving only linear equations to integrate. Thus this approach is by no means a general method of constructing solutions for any given nonlinear system of equations.

Now recall that the starting point for our method is the *ansätze* (18), (36), and (37), which reduce the nonlinear system (15)–(17) to two equations linear in ξ, (39) and (40). However, the way we have chosen to proceed after this starting point is not unique; there are at least two principal points in the development that can be modified. First, recall that having (39) and (40) be redundant is a matter of choice: it is possible to construct a class of solutions distinct from the one hitherto discussed without this condition. Next note that after imposing the redundancy of (39) and (40), having the commutator vanish in (50) is also a matter of choice. Relaxing this constraint will result in another distinct class of solutions.

Recall also that our *ansätze* (36) and (37) can be viewed as perturbations on the forms $\psi = \gamma\xi$ and $U = \delta\xi$. These latter forms were used to solve the system (15)–(17) under the assumption of axisymmetric equilibrium in Section III, Part B. This particular class of equilibrium solutions, which serves as the origin of our construction, is an especially simple case of the equilibria possible for the nonlinear system.[5] Perhaps even more interesting equilibrium solutions could be "perturbed" as the starting point for constructing other solutions having r and z dependence.

Finally, we note that it might also be interesting to investigate how our means of constructing solutions fits into the framework of symmetry (Lie group) methods for the integration of systems of pde's. Perhaps the explicit application of these methods to the nonlinear system we have considered would also yield physically interesting solutions.

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