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Transport Profiles Induced by Radially Localized Modes in Tokamak

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Abstract

We describe a new approach to the calculation of turbulent transport coefficients for radially localized modes. The theory takes into account the nonuniformity of the distribution of rational (resonant) magnetic surfaces in minor radius. This distribution function is proportional to the density of available states of excitation. The resulting density of states correction qualitatively changes the radial profile of the transport coefficients, as compared to the usual local diffusivity formulae. The correction factor calculated for the η_i-mode transport allows a much better agreement of χ_i with the experimental data than previously achieved.

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I. Introduction

Several important instabilities of the tokamak plasma can be described as the modes radially localized in the vicinity of the rational surfaces of the equilibrium magnetic field. These are the drift-wave, trapped-particle, interchange, and some of the ballooning-type perturbations. The turbulent transport induced by such modes is usually calculated for a single localized perturbation[1, 2, 3, 4], or for some finite, externally fixed number of neighboring modes.[5, 6] The transport induced by a single localized perturbation is, of course, significant only in the vicinity of the resonance and we will call it the *spike* transport, because it can be described by the spike on the profile of the transport coefficients. In the conventional approach[1-4,6,7] the global transport is calculated as the spike transport generated by the mode localized at a given radius. Since the rational surfaces are sufficiently dense, it is presumed that such a local mode always exists. (However, the latter is not always true and this question will be addressed in Sec. III.)

The local turbulent transport analysis results in the transport coefficient χ in the form

$$\chi = \chi_{sp}\left(r - r_\alpha, r_\alpha, \alpha\right) \ , \tag{1}$$

and then the conventional method of calculation the global profile of this transport coefficient is.

$$\chi(r) = \overline{\chi_{sp}}\left(r, q(r)\right) \tag{2}$$

with the average being over the first argument in Eq. (1). Here χ_{sp} is the rapidly varying function of the first argument, which is the distance from the mode rational surface $r = r_\alpha$ and $\overline{\chi_{sp}}$ is the average value of the local χ within the perturbation width; r is the minor radius, $\alpha = m/n$, m and n are the mode numbers. The rational points are defined by $q(r_\alpha) = \alpha = m/n$, where $q(r)$ is the safety factor. This averaging of χ_{sp} over the local region of support of the mode is described in some detail both analytically and numerically by Hamaguchi and Horton[6]. The local nonlinear interactions over the local mode width determine the amplitude of saturation and the phase shifts giving the final value of χ_{sp}.

In this letter we introduce a new approach to the calculation of χ. It is based on an assumption of the relative independence of fluctuations on different rational surfaces or an incoherent superposition of the local transport steps. However, this still allows the interaction of modes through local modification of the driving gradients.

In this case χ can be represented as the sum of the local transport coefficients:

$$\chi(r) = \sum_{\alpha} \chi_{sp} \left(r - r_\alpha, r_\alpha, \alpha\right) \ . \tag{3}$$

It is convenient to call each helical perturbation, producing a spike on the χ profile, *the state of excitation.* Then the distribution of significant resonances (where such an excitation is possible) in minor radius will be the *the density* of these states.

The value of χ calculated according to the model (3) will reflect the inhomogeneity in the radial distribution of the rational surfaces, and will have a qualitatively different radial profile as compared to the local scheme (2). There are two main causes for the inhomogeneity of χ:

1. The radial dependence of the "mean" density of states, due to physical and geometrical factors such as the nonuniformity of shear, toroidal or cylindrical geometry of the magnetic field, or the maximum wavenumber of unstable modes.

2. The inhomogeneity in the distribution of simple fractions $\alpha = m/n$ on the real axis, which is their inherent mathematical property.

These factors are described in detail in Secs. II and III of this work. In Sec. IV we define possible verification procedures and in Conlusion discuss some implications of the present approach.

II. Density of states and the mean transport

To justify equation (3) for evaluation of the transport coefficients we note that for any two functions ϕ and p that are single valued and periodic on the toroidal surface (θ, ζ), the average of their product in θ, ζ can be represented as

$$\chi \propto \langle \phi(\partial p/\partial\theta)\rangle_{\theta,\zeta} = \sum_{\alpha} \langle \alpha\phi_\alpha p'_\alpha\rangle_{\theta,\zeta} \ .$$

Here ϕ_α, p_α are the coefficients of the helical Fourier expansion $\phi(\theta,\zeta) = \sum_\alpha \phi_\alpha(\alpha\theta - \zeta)$, and are localized near the corresponding rational surfaces. The off-diagonal terms are zero because each $\phi_\alpha(\alpha\theta - \zeta)$ is periodic in both $m\theta$ and $n\zeta$, where m and n are the numerator and the denominator of the irreducible simple fraction α. If the saturated amplitude of the α-component does not depend significantly on the amplitudes of any other $\alpha_1 \neq \alpha$ components, then it can be evaluated by means of some local analysis. In the following we assume this analysis has been carried out, so that the values of χ_{sp} are known.

We proceed by describing a method for approximate evaluation of the sum over α in Eq. (3). We note that the α-sum is limited by the requirement that the rational surfaces r_α lie within the region defined by the rotational transform of the magnetic field, and that the range of available m numbers in $\alpha = m/n$ is limited to the excited modes only, which means, usually, $m < m_f$. Expression (3) can be rewritten as the more complex but nevertheless more convenient for evaluation double sum over m and n:

$$\sum_\alpha \chi_{sp}(\alpha) = \sum_{m=m_s}^{m_f} \sum_{n=n_s(m)}^{n_f(m)} \chi_{sp}(m/n) - \sum_{k>1}^{k\leq m_f/m_s} \sum_{m\geq m_s/k}^{m\leq m_f/k} \sum_{n\geq n_s(km)/k}^{n\leq n_f(km)/k} \chi_{sp}(m/n) \ . \qquad (4)$$

The first term above takes into account all significant rational surfaces in the calculation region (defined by m_s, n_s, m_f, n_f) but treats km/kn as different from m/n. The second term serves to eliminate these reducible fractions. The whole expression can be evaluated numerically, which will be discussed in the next section, but it is also possible to get some analytic estimates.

Let us assume that m_f, n_f are sufficiently large, so that a small interval $\delta r \ll r$ still contains many significant rational surfaces. The number of such resonances with fixed m can be estimated from

$$\begin{cases} nq = m \\ n\delta q + q\delta n = 0 \end{cases} \quad \Rightarrow \quad \delta n = -m\delta q/q^2 \; . \tag{5}$$

Here δ denotes the change of the quantity over the interval δr. Using $\delta q = q'(r)\delta r$, we get the density of rational surfaces with fixed m as

$$\frac{\delta n}{\delta r} = m\frac{|q'|}{q^2} \; . \tag{6}$$

This is not a derivative and the value has a meaning only in the average sense: by definition $\delta n \gg 1$ but $\delta r/r \ll 1$, which is consistent only for $m \gg 1$.

In the same limit we can take into account the second term in (4), i.e. eliminate from consideration all fractions where numerator and denominator are multiples of the same integer. Indeed, the ratio of the number of these irreducible fractions to the number of all simple fractions with denominators less than k tends to the constant $G = 6/\pi^2 \approx 0.608$ as $k \to \infty$.[1] We can now define

$$f_m(r) = G\frac{\delta n}{\delta r} \approx m\frac{6}{\pi^2}\frac{|q'|}{q^2} \tag{7}$$

to be the *mean density of states* with a fixed poloidal wavenumber m. With a good accuracy f_m is the radial density of significant resonances and we can use it to calculate the profile of the transport coefficient:

$$\chi(r) = \sum_{m=1}^{m_f} f_m(r) \int_{-\infty}^{+\infty} \chi_{sp}(r - r_\alpha, r_\alpha, \alpha) d(r - r_\alpha) \; . \tag{8}$$

Here the integral limits are taken to be much larger than the width of localization for χ_{sp}.

The knowledge of $\chi_{sp}(m)$ is necessary to evaluate expression (8). It can be obtained only from the explicit local turbulence analysis. However, for two groups of such theories we may

[1] The proof of this property is presented in Appendix.

use an approximation

$$\int_{-\infty}^{+\infty} \chi_{sp}(r - r_\alpha, r_\alpha, \alpha) d(r - r_\alpha) = \overline{\chi_{sp}} \Delta x_m \ , \tag{9}$$

where Δx_m is the characteristic width of the local perturbations. It is well known that for m below some m_f the widths of eigenfunctions behave as $\Delta x_m \approx \Delta x_b / m$ for ballooning and interchange modes, and $\Delta x_m \approx \Delta x = \text{const.}$ for drift wave turbulence. In this zero order approximation the value of $\overline{\chi_{sp}}$ is independent of m. Using these models to perform the summation in Eq. (8) we get formulas for the global χ:

$$\chi_{dw}(r) = f_{m_f} \frac{m_f + 1}{2} \Delta x \cdot \overline{\chi_{sp}} \equiv F_{dw} \cdot \overline{\chi_{sp}} \ , \tag{10}$$

and

$$\chi_b(r) = f_{m_f} \Delta x_b \cdot \overline{\chi_{sp}} \equiv F_b \cdot \overline{\chi_{sp}} \ . \tag{11}$$

Here F_{dw} and F_b are defined as the correction coefficients, which determine the relationship between the transport coefficients obtained from the local analysis $\overline{\chi_{sp}}$ and the global χ calculated from expression (8).

The short-wavelength limit of the instability can be expressed as $k_y \rho_i < k_{\max}$ for the η_i mode or $k_r \rho_i < k_{\max}$ for the applicability of the fluid model to the description of ballooning modes. Here ρ_i is the ion Larmor radius and is used here for normalization purposes, although ρ_s is another choice more appropriate for $T_e \gg T_i$ plasmas. Since $k_y = m/r$ and $k_r \approx 1/\Delta x_m$ in our geometry, we have $m_f|_{dw} = k_{\max} r / \rho_i$ and $m_f|_b = k_{\max} \Delta x_b / \rho_i$. Assuming that $m_f|_{dw} \gg 1$ we finally arrive at the definite values of the correction factors:

$$F_{dw} = k_{\max}^2 \frac{3|q'|}{\pi^2 q^2} \left(\frac{r}{\rho_i} \right)^2 \Delta x_{dw}, \tag{12}$$

and

$$F_b = k_{\max} \frac{6|q'|}{\pi^2 q^2} \frac{\Delta x_b^2}{\rho_i} \ . \tag{13}$$

Here $k_{\max}$ is a quantity of the order of or less than 1 and is defined by the precise instability boundary or by the limit of the applicability of χ_{sp} formulas (typically $k_{\max} \sim$ few tenths); $\Delta x_{dw} \sim \rho_i$ and Δx_b are the characteristic radial widths of the eigenfunctions.

In this section we have shown that the use of the superposition formula (3) instead of (2) leads to the appearance of the new corrections, proportional to the density of available states for the excitations. These factors qualitatively change the radial dependence of the calculated transport coefficients and may cause the modification of scaling laws with temperature and magnetic field.

III. Small-scale modifications of transport

In this section we consider some of the effects, which have been neglected above and try to assess their importance for the confinement.

The approach of Sec. II treats the density of states as a continuous variable, which it is not. While counting the rational surfaces with $m < m_f$ we have assumed that the resulting density will be a simple sum of densities for each fixed m. This is not entirely correct. To illustrate this statement we have plotted in Fig. 1(a) the number of rational values within the interval $x \leq n/m < x + \delta x$ as a function of x. The maximum m is taken to be 100 and $\delta x = 0.002$. The result is far from being constant in x. This apparent discrepancy is related to the initial requirement of Sec. II for the δx step to be large enough. Obviously, if we take $\delta x = 1$ then the result will be constant.

The ratio of the number of the irreducible fractions among all simple fractions of the form n/m has been taken to be its averaged limit G, while in reality it is dependent on the value of the denominator and its proximity to the low m rational numbers. The distribution function of the rational numbers with the multiples kn/km taken out is plotted in Fig. 1(b). One can see that while in the mean sense this curve lies $\sim 40\%$ below the curve (a), its small-scale behaviour is quite different.

Finally, in the transition from a direct superposition described by equation (4) to a kind of partial integration in Eq. (8) we have implicitly assumed that the spikes of the transport are mutually overlapping and that the local radial averaging of the transport coefficient is hence a legitimate procedure. Again, in most interesting cases this assumption is valid, but the reverse case is also possible and we should write criteria for the applicability of the formulae (8)-(13).

A. Non-overlapping distributions

We can estimate the limiting density of the significant resonances necessary to ensure the overlapping of the transport spikes by means of the relation

$$\sum_{m=1}^{m_f} f_m \Delta x_m > 1 \ , \tag{14}$$

which states that the integral radial width of all transport spikes is sufficient to cover the whole interval of the minor radius. However, if we take into account the nonuniformity of the real distribution of resonances, it becomes clear that the above condition slightly underestimates the necessary m_f at least at some points in radius (far from the low-mode resonances).

Relation (14) can be evaluated in the particular cases of the drift and ballooning modes to yield:

$$\rho_i < r \frac{|3rq'|^{1/2}}{\pi q} k_{\max} \tag{15}$$

for the drift, and

$$\rho_i < \Delta x_b^2 \frac{6|q'|}{\pi^2 q^2} k_{\max} \tag{16}$$

for the ballooning case.

It is now clear that the continuous approximation may not be valid for the drift instabilities close to the magnetic axis (where $rq' \to 0$) and for certain types of the resistive ballooning modes with sufficiently small area of localization.

B. Numerical evaluation

Instead of the approximate analytic approach discussed in the previous section we can always evaluate expression (4) by direct numerical methods. Direct evaluation will take into account all effects of discretization and give us a measure of their relative importance.

Figure 2 presents some results for such a calculation. We have used the model toroidal current distribution such that $q(r) = q_0/(1+2r^2) + 3.2r^2$ with the central and the edge values of the safety factor $q_0 = 0.91$ and $q_e = 3.51$ respectively. We also assume the constant temperature case $\rho_i =$ const. , so that it is easy to separate physically different effects from each other. The spike form is chosen to be $\mathcal{X}_{sp}(r - r_\alpha) = \mathcal{X}_0 \exp(-((r - r_\alpha)/\rho_i)^2)$, $k_{\max} = 1$. Here $\rho_i = 0.0025 r_e$, typical for the TFTR supershot.

Figure 3 shows solution for the same model but with the four times higher value of ρ_i. For comparison, the smooth curve described by the analytic estimate (12) is also plotted.

We can see that although the analytic curve accurately describes the mean behavior of the numerical result, there are significant fluctuations of the numerical curve around this mean value especially close to the axis ($r = 0$) and around the low mode-number resonances.

C. Averaging of transport

At this point it is logical to consider the importance of the sharp peaks and drops on the overall profile of the transport coefficient. Unfortunately, it is impossible to get an exact answer on this question without considering a particular model of the turbulence. However, most instabilities are driven by the gradients of the same parameter that is being transported as a result of the instability. As a consequence, each peak in $\mathcal{X}(r)$ will cause a local drop in the absolute value of the gradient and thus in the instability drive, which, in turn, is likely to decrease the level of fluctuations and the local value of $\overline{\mathcal{X}_{sp}}$. In the case of a local drop in the transport coefficient the same quasilinear mechanism should work in the opposite direction. In short, the gradient driven systems possess a negative feedback channel, which serves to eliminate the local inhomogenities of χ_{eff}.

The above argument is not applicable to the case of the non-overlapping resonances, because in this limit there are no modes to be excited in the gaps between resonant surfaces, and the quasilinear saturation will work to bring the local gradient to its threshold value rather than to its mean value as in the overlapping case.

Assuming that the local turbulence simply brings the local gradient down to its threshold value over the width of each spike, we can derive the effective transport coefficient for this model. If the mean value of the gradient is fixed by an external source and is a given parameter, then this mean gradient $|\nabla P_0|$ can be expressed as

$$|\nabla P_0| = \frac{\delta P_0}{\delta x} = \frac{\delta P + \delta x \sum_m f_m |\nabla P_{cr}| \cdot \Delta x_m}{\delta x} \; ; \tag{17}$$

here ∇P_{cr} is the threshold gradient and δP is the change in P over the region of δx unaffected by the turbulence where the transport coefficient is equal to the background coefficient χ_0. The resulting increase in the radial flux of P can be described by the introduction of the new effective coefficient

$$\chi_{\text{eff}} = \chi_0 \frac{1 - \sum_m (P'_{cr}/P'_0) f_m \Delta x_m}{1 - \sum_m f_m \Delta x_m} \; . \tag{18}$$

Since for the non-overlapping case condition (14) is not satisfied, both numerator and denominator of this ratio are positive. If the critical gradient is relatively independent of m the expression above can be rewritten as

$$\chi_{\text{eff}} = \chi_0 \frac{1 - F(P'_{cr}/P'_0)}{1 - F} \; , \tag{19}$$

and for our model cases the values of F can be calculated using equations (12) and (13).

IV. Verification by simulations

We have established several important consequences of the incoherent superposition approach to the calculation of the transport profiles. However the justification of the initial assumptions of this method is uncertain and at this point we consider possible verification procedures. To define such procedures we first consider the proper formulation of the local problem, which would be compatible with our derivation of the transport profiles. As the next step we discuss the possibility to use existing models with different formulations for our purpose.

We have agreed to define as a spike the transport effects produced by the single-helicity radially localized perturbation. We want to consider it in a fixed-gradient environment produced by other mutually overlapping perturbations. The latter requirement is often incompatible with traditional formulations of the local problem. Indeed, our approach requires the knowledge of $\overline{\chi_{sp}}$ dependence on the value of the driving gradient in the mode localization region, while in the traditional numerical simulations only the mean gradient in the calculation region is fixed and the driving gradient is allowed to relax during quasilinear saturation. In this respect the single-helicity results from Ref. [6], for example, are similar to the non-overlapping limit.

We can calculate the necessary combination of parameters from results with locally relaxed gradients using the following approximate formula:

$$\overline{\chi_{sp}}(p')\Delta_x = \chi_0 L_x \frac{\Lambda(P'/p') - 1}{\Lambda(\Lambda - 1)} \ , \tag{20}$$

where χ_0 is the background radial diffusion coefficient, L_x is the width of the calculation region, Δ_x is the mode width, and $\Lambda \equiv L_x/\Delta_x > 1$; $P' = \delta P/L_x$ is the mean gradient and $p' < P'$ is the local gradient at the mode position. If all these values are known and no external source term is used in the simulation, then this formula is sufficient as a starting point for our model.

Another existing approach to the numerical calculation of the transport coefficients is the 3-D simulation, which involves several helical modes. As a rule, the density of states in such models is externally fixed and different from that relevant to experimental situations. However, if enough data is presented to calculate the effectively used density of states, then it is possible to recover the single-spike transport and make the simulation data useful for our model. Unfortunately, such simulations often employ artificial source terms to maintain constant levels of the gradient. These terms make the interpretation uncertain.

An ideal numerical verification of the applicability of our approach would be a combined series of simulations for a single-helicity model, providing spike values, and a 3-D model with known density of states and overlapping mode distributions. No source terms are necessary and the output of the simulations should be the time- and θ-averaged gradient profiles. This would be sufficient for determining to what extent the weak-interaction assumption underlying our model is justified. No such simulations are available at this time.

V. Conclusion

Introduction of correction factors related to the density of states may qualitatively change existing theoretical scalings and radial profiles of diffusion coefficients in the tokamak plasma.

For example, the global drift wave formula (12) with Δx_{dw} scaling as ρ_i or ρ_s now gives a Bohm-like variation of the global $\chi_{dw}(r)$ when the local χ_{sp} is taken as the well known local drift wave diffusion $(\rho_s/L_n)(cT/eB)$. The magnitude of $\chi_{dw}(r)$ is considerably below the Bohm value due to other factors in Eq. (12) and is still compatible with experiments.

Decrease in the theoretical χ_{sp} with radius is currently the most contradictory feature of most η_i-based models of local turbulence, because it is not compatible with experimental data, which indicates a consistent increase in the transport coefficients toward the edge. In agreement with experimental behavior, the density of states rises dramatically with radius (see Fig. 2) and, consequently, so does the diffusion coefficient, calculated from the incoherent superposition (3) transport model.

The above example may indicate that the approach to the calculation of global transport coefficients may explain important features of tokamak operation. However, further work is needed to verify initial assumptions and confirm the results of this new theoretical model for transport.

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Appendix

The number of the irreducible simple fractions (those without common divisors of numerator and denominator) can be expressed in terms of the Euler phi function $\varphi(n)$, which is defined as *the number of integers not exceeding and relatively prime to n.* Thus, the number of proper irreducible fractions with denominators less than n is

$$\Phi(n) = \sum_{k=1}^{n} \varphi(k) \ .$$

At the same time the number of all simple fractions with such limitations is

$$N(n) = \sum_{k=1}^{n} k = n(n+1)/2 \ .$$

Using asymptotic formula for $\varphi(n)$ from the Handbook of Mathematical Functions [8]

$$\frac{1}{n^2} \sum_{k=1}^{n} \varphi(k) = \frac{3}{\pi^2} + \mathcal{O}\left(\frac{\ln n}{n}\right) \ , \tag{21}$$

we get the relative number of irreducible fractions as

$$G = \lim_{n \to \infty} \left(\Phi(n)/N(n)\right) = 6/\pi^2 \ . \tag{22}$$

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Figures

FIG. 1. The number of proper simple fructions (a) and irreducible simple fructions (b) with denominators less than $m_f = 100$ in intervals of $\delta x = 0.002$.

FIG. 2. The transport profile calculated as the superposition of local spikes for the model distribution of current; $\rho_i = 0.0025 r_e$.

FIG. 3. The transport profile calculated as in FIG. 2 with $\rho_i = 0.01 r_e$; the smooth curve represents the same profile calculated with continuous density of states (Eq. (12)).

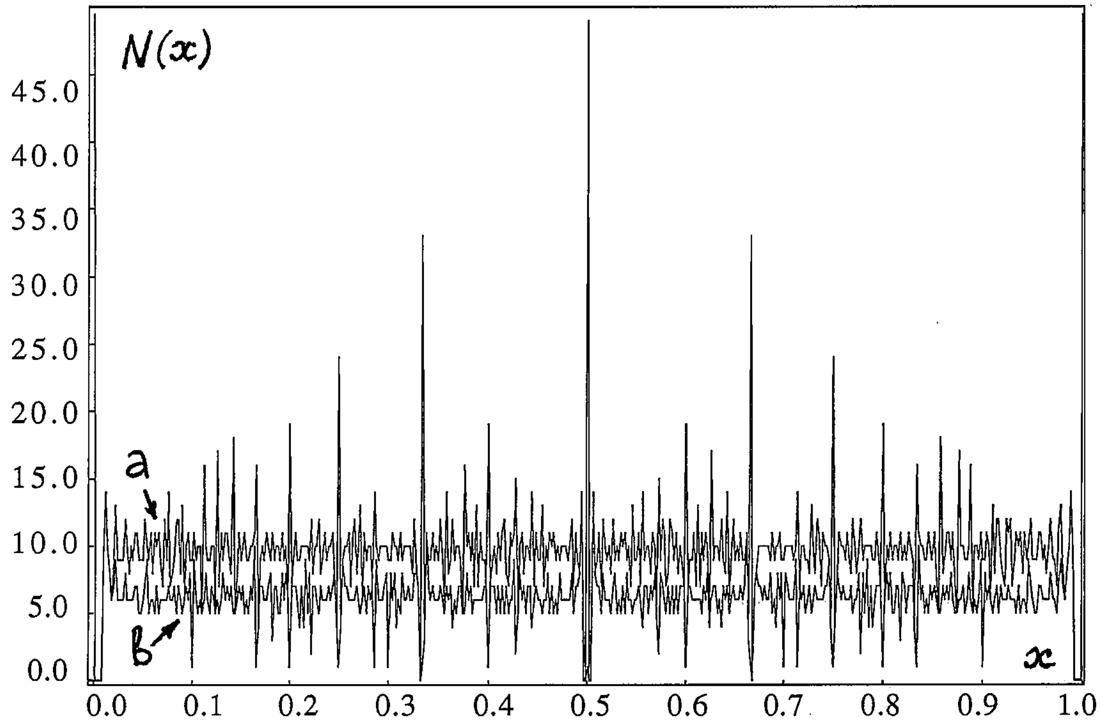
N(x)
45.0
40.0
35.0
30.0
25.0
20.0
15.0
10.0
5.0
0.0
a
b
x
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0

Fig. 1

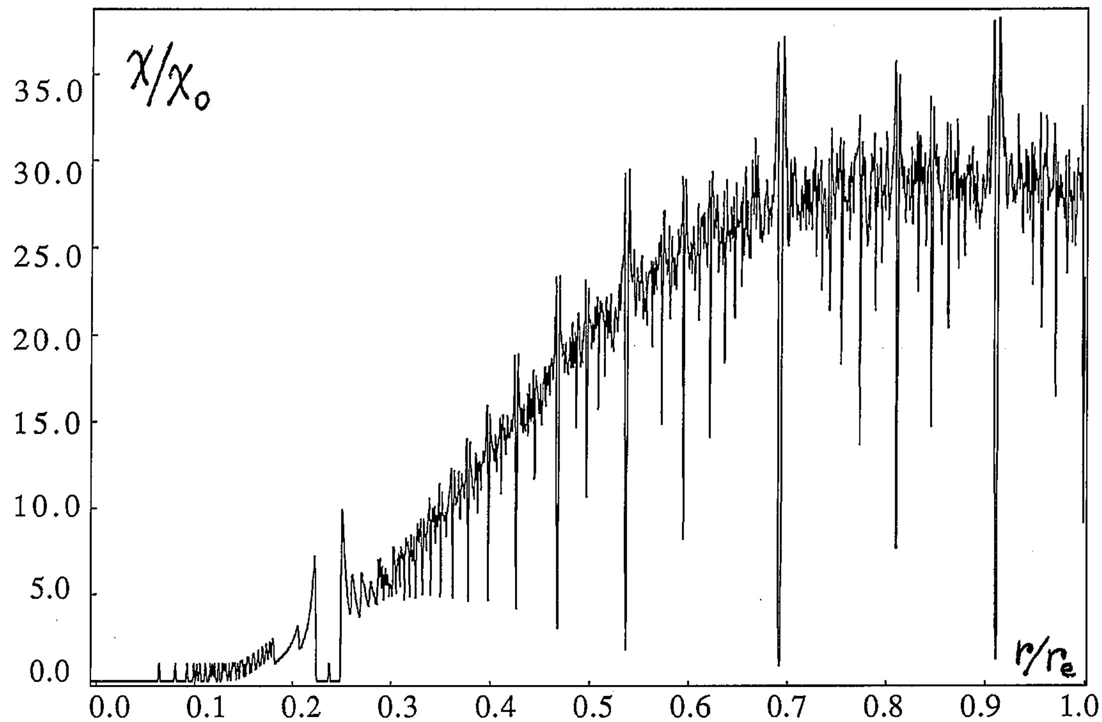
χ/χ₀
35.0
30.0
25.0
20.0
15.0
10.0
5.0
0.0
r/rₑ
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0

Fig. 2

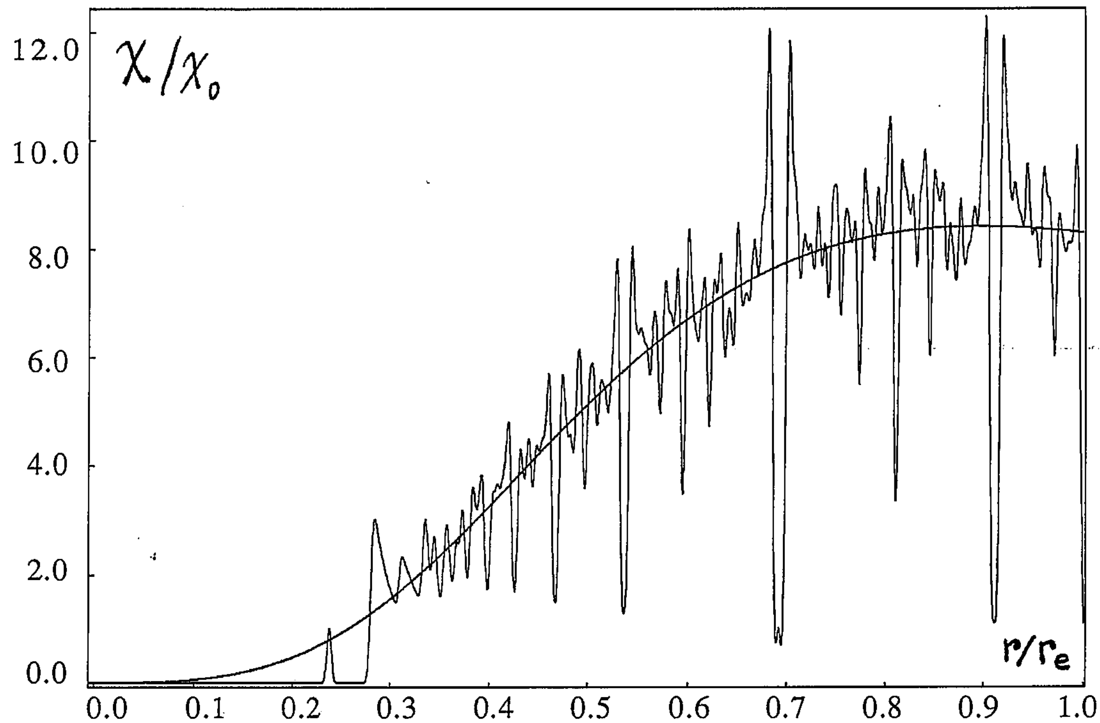

Fig. 3