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Steady-State Dynamo and Current Drive in a Nonuniform Bounded Plasma

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ABSTRACT

Current drive due to helicity injection and the dynamo effect are examined in an inhomogeneous bounded plasma. Averaged over a magnetic surface, there is in general no dynamo effect independent of resistivity - contrary to the results found previously for an unbounded plasma. The dynamo field is calculated explicitly for an incompressible visco-resistive fluid in the plane-slab model. In accord with our general conclusion, outside the Alfvén resonant layer it is proportional to the resistivity. Within the resonant layer there is a contribution which is enhanced, relative to its value outside the layer, by a factor $(\omega a^2/(\eta + \nu))$, where ω is the wave frequency, a the plasma radius, η the magnetic diffusivity, and ν the kinematic viscosity. However, this contribution vanishes when integrated across the layer. The average field in the layer is enhanced by a factor $(\omega a^2/(\eta + \nu))^{2/3}$ and is proportional to the shear in the magnetic field and the cube root of the gradient of the Alfvén speed. These results are interpreted in terms of helicity balance, and reconciled with the infinite medium calculations.

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I. INTRODUCTION

Several previous studies[1-7] have investigated helicity injection and current drive by plasma waves in an unbounded plasma. These studies showed that non-uniformity of the Alfvén speed and non-resistive damping, in addition to resistive damping, could contribute to the dynamo effect produced by oscillating fields. This raises the possibility of large current drive from small amplitude waves in a plasma with vanishing resistivity. However, these studies do not realistically represent a steady-state situation with fluctuations generated at a plasma boundary.

Recently, Bellan and Shalit[8] examined steady-state current drive in a bounded plasma, but this study was restricted to a uniform plasma column. The present study focuses on current drive in a bounded inhomogeneous plasma.

The plan of the paper is as follows. The basic equations are described in Sec. II and some general results concerning the dynamo effect are derived in Sec. III. Current drive in a one-dimensional slab is discussed in Secs. IV and V and the connection with helicity injection is discussed in Sec. VI. Our conclusions, and the relation of the present results to those obtained previously are given in Sec. VII.

II. BASIC EQUATIONS

Our study is based on the incompressible magnetohydrodynamic (MHD) fluid model with a scalar resistivity and viscosity, and an Ohm's law,

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = \mu_0 \eta \, \mathbf{J}. \tag{1}$$

Then

$$\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{B} - (\mathbf{B}\cdot\nabla)\mathbf{V} - \eta \nabla^2 \mathbf{B} = 0, \tag{2}$$

$$\rho \frac{\partial \mathbf{V}}{\partial t} + \rho\,(\mathbf{V}\cdot\nabla)\mathbf{V} - \frac{1}{\mu_0}(\mathbf{B}\cdot\nabla)\mathbf{B} + \nabla \mathcal{P} - \mu \nabla^2 \mathbf{V} = 0, \tag{3}$$

$$\nabla \cdot \mathbf{V} = 0. \tag{4}$$

Here, **V** and ρ are the fluid velocity and density, **B** is the magnetic flux, and $\mathcal{P}$ is the total pressure. The parameters μ and η represent viscosity and magnetic diffusivity. We express **B**, **V**, ρ, and $\mathcal{P}$ as the sum of an equilibrium contribution (subscript zero) and a first order fluctuating component (tilde) with zero mean. Then, with $\mathbf{V}_0 = 0$, the fluctuating components satisfy

$$\frac{\partial \tilde{\mathbf{b}}}{\partial t} + (\tilde{\mathbf{v}} \cdot \nabla)\mathbf{B}_0 - (\mathbf{B}_0 \cdot \nabla)\tilde{\mathbf{v}} - \eta \nabla^2 \tilde{\mathbf{b}} = 0, \tag{5}$$

$$\rho_0 \frac{\partial \tilde{\mathbf{v}}}{\partial t} - \frac{1}{\mu_0}(\mathbf{B}_0 \cdot \nabla)\tilde{\mathbf{b}} - \frac{1}{\mu_0}(\tilde{\mathbf{b}} \cdot \nabla)\mathbf{B}_0 + \nabla \tilde{\mathcal{P}} - \mu \nabla^2 \tilde{\mathbf{v}} = 0, \tag{6}$$

$$\nabla \cdot \tilde{\mathbf{v}} = 0. \tag{7}$$

III. THE DYNAMO EFFECT

In an MHD fluid, current may be driven by fluctuations through the so-called "dynamo effect." If Ohm's law, Eq. (1), is averaged over the fluctuating fields, then

$$\langle \mathbf{E} \rangle + \langle \tilde{\mathbf{v}} \times \tilde{\mathbf{b}} \rangle = \mu_0 \eta \langle \mathbf{J} \rangle \tag{8}$$

and

$$\mathbf{E} = \langle \tilde{\mathbf{v}} \times \tilde{\mathbf{b}} \rangle \tag{9}$$

is the dynamo electric field. For useful current drive, we need the component of **E** parallel to the equilibrium magnetic field,

$$\mathbf{E} \cdot \mathbf{B}_0 = \mathbf{B}_0 \cdot \langle \tilde{\mathbf{v}} \times \tilde{\mathbf{b}} \rangle. \tag{10}$$

We consider first a perfectly conducting plasma ($\eta = 0$) with fluctuations superimposed on a slowly varying background. In this case the fluctuations satisfy

$$\frac{\partial \tilde{\mathbf{b}}}{\partial t} = \nabla \times (\tilde{\mathbf{v}} \times \mathbf{B}_0), \tag{11}$$

and if we define $\tilde{\mathbf{s}} \equiv \tilde{\boldsymbol{\xi}} \times \mathbf{B}_0$ with $\partial \tilde{\boldsymbol{\xi}}/\partial t \equiv \tilde{\mathbf{v}}$,

$$\mathbf{B}_0 \cdot (\tilde{\mathbf{v}} \times \tilde{\mathbf{b}}) = -\tfrac{1}{2}\left\{\nabla \cdot (\tilde{\mathbf{s}} \times \frac{\partial \tilde{\mathbf{s}}}{\partial t}) + \frac{\partial}{\partial t}[\tilde{\mathbf{s}} \cdot (\nabla \times \tilde{\mathbf{s}})]\right\}. \tag{12}$$

Then, for steady fluctuations, the dynamo field is

$$\mathbf{E} \cdot \mathbf{B}_0 = -\tfrac{1}{2} \nabla \cdot < \tilde{\mathbf{s}} \times \frac{\partial \tilde{\mathbf{s}}}{\partial t} >, \tag{13}$$

which may also be written

$$\mathbf{E} \cdot \mathbf{B}_0 = \tfrac{1}{2} \nabla \cdot [\mathbf{B}_0 (< \tilde{\boldsymbol{\xi}} \times \frac{\partial \tilde{\boldsymbol{\xi}}}{\partial t} > \cdot \mathbf{B}_0)]. \tag{14}$$

Consequently, in a perfectly conducting plasma, the average dynamo field over an equilibrium magnetic flux surface Ψ vanishes:

$$\oint \frac{dS}{|\nabla \Psi|} \mathbf{E} \cdot \mathbf{B}_0 = 0. \tag{15}$$

If the fluctuating field is non-stationary, Eq. (15) is replaced by

$$\int_{-\infty}^{\infty} dt \oint \frac{dS}{|\nabla \Psi|} \mathbf{E}\cdot\mathbf{B}_0 = 0. \tag{16}$$

Equations (15) and (16) may appear to be in conflict with earlier results,[3-5] which found that a dynamo effect could be produced in non-resistive plasmas by wave helicity injection. The apparent conflict may be resolved by noting that if one integrates only from $t = 0$, then Eq. (16) is replaced by

$$\int_{0}^{\infty} dt \oint \frac{dS}{|\nabla \Psi|} \mathbf{E}\cdot\mathbf{B}_0 = \frac{1}{2} \oint \frac{dS}{|\nabla \Psi|} < \tilde{\mathbf{s}}\cdot(\nabla \times \tilde{\mathbf{s}}) > \Big|_{t=0}, \tag{17}$$

representing a dynamo effect due to an initial input of helicity. Similarly, if one integrates over an arbitrary volume, instead of a flux surface, Eq. (15) is replaced by

$$\int_V dV\, \mathbf{E}\cdot\mathbf{B}_0 = \frac{1}{2} \oint_S d\mathbf{S}\cdot\mathbf{B}_0 < \tilde{\boldsymbol{\xi}} \times \frac{\partial \tilde{\boldsymbol{\xi}}}{\partial t} > \cdot\mathbf{B}_0\,. \tag{18}$$

(Note that contributions to Eq. (18) arise only on the part of the surface that cuts magnetic field lines.) This represents a dynamo effect due to input of helicity *along* the magnetic field, from a source within the magnetic surface.

We next consider the dynamo field in a resistive plasma. In particular, we consider the effect of monochromatic fluctuations, such as might be generated by an antenna. In this case we may put

$$\tilde{\mathbf{b}}(\mathbf{x}, t) = \mathbf{b}_\omega(\mathbf{x}) e^{-i\omega t} + \text{c.c.}\,, \tag{19}$$

$$\tilde{\mathbf{v}}(\mathbf{x}, t) = \mathbf{v}_\omega(\mathbf{x}) e^{-i\omega t} + \text{c.c.}\,. \tag{20}$$

Then, including the effect of resistivity,

$$-i\omega \mathbf{b}_\omega = \nabla\times(\mathbf{v}_\omega\times\mathbf{B}_0) + \eta\nabla^2\mathbf{b}_\omega, \tag{21}$$

and the dynamo electric field can be written

$$\mathbf{E}\cdot\mathbf{B}_0 = \frac{1}{i\omega}\left\{\tfrac{1}{2}\mathbf{B}_0\cdot\nabla[\mathbf{B}_0\cdot(\mathbf{v}_\omega\times\mathbf{v}_\omega^*)] + \eta\,\mathbf{B}_0\cdot(\mathbf{v}_\omega\times\nabla^2\mathbf{b}_\omega^*)\right\} + \text{c.c.}\,. \tag{22}$$

The first term, which persists when $\eta = 0$, is due to wave-helicity flow parallel to the magnetic field. Only the second term contributes to the average over an equilibrium flux surface:

$$\oint\frac{dS}{|\nabla\Psi|}\mathbf{E}\cdot\mathbf{B}_0 = \frac{\eta}{i\omega}\oint\frac{dS}{|\nabla\Psi|}\mathbf{B}_0\cdot(\mathbf{v}_\omega^*\times\nabla^2\mathbf{b}_\omega) + \text{c.c.}\,. \tag{23}$$

In agreement with the preceding discussion, this vanishes when $\eta = 0$. Note that this result depends only on the resistive Ohm's law, Eq. (1), and therefore other plasma properties such as compressibility or viscosity enter only implicitly through $\mathbf{v}_\omega$ and $\mathbf{b}_\omega$. It shows that only the *resistive* damping is essential for the surface average dynamo field.

It should be noted that for $\eta = 0$, Eq. (22) is in agreement with the effective field due to the ponderomotive force on a particle in the low-frequency limit (or alternatively in the limit of small particle mass). See, for example, Refs. 9 or 10.

IV. CURRENT DRIVE IN A PLANE SLAB

We now turn to the main subject of the present work, the investigation of current drive in a bounded, inhomogeneous plasma. As a simple geometrical

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layer arises from helicity $\eta\mu_0\tilde{\mathbf{j}}\cdot\tilde{\mathbf{b}}$ "dissipated" by the fluctuations *within* the layer, not from helicity flux into the layer.

In conclusion, it appears that in a plasma with a simple resistive Ohm's law, there is no dynamo field independent of resistivity, such as was found in an unbounded plasma, which can be excited from a boundary. If one is to obtain such a field, one must depart from the simple Ohm's law used in this paper - possibly by the introduction of the Hall current.

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VII. CONCLUSIONS

We have examined the dynamo effect produced by wave-like fluctuations in a resistive-fluid plasma, with particular emphasis on the situation in a bounded, inhomogeneous plasma.

Previous calculations[1-5] for an infinite medium suggested that a dynamo field independent of resistivity (and hence giving a large current drive when $\eta \to 0$) could be obtained from wave helicity damping. However, our present results, while in accord with helicity conservation, show that this is only possible when the helicity is provided through an initial condition, or by injection parallel to the equilibrium magnetic field (both seemingly unrealistic for fusion devices). The flux of helicity *across* the equilibrium magnetic field is always itself of order η and consequently cannot yield a dynamo effect when $\eta \to 0$. In accordance with this picture, our calculations lead to an average dynamo field over a magnetic surface, which, outside Alfvén resonant layers, is proportional to resistivity.

The conservation of helicity, e.g. Eq. (69), might suggest that the relation between helicity flux and current drive at an Alfvén resonant layer should resemble that between energy flux and plasma heating. However this is not the case. There is a flux of *energy* across magnetic surfaces independent of resistivity which is absorbed at the Alfvén resonant layer. As already noted, there is no such transverse helicity flux in the absence of resistivity. Nevertheless, within the Alfvén resonant layer, the current drive is enhanced relative to its value in the bulk plasma by a factor $(\omega a^2/(\eta + \nu))$, ($\nu = \mu/\rho_0$ the kinematic viscosity). However this contribution averages to zero across the layer. The average dynamo field in the layer is enhanced by a factor $(\omega a^2/(\eta + \nu))^{2/3}$ and depends on the magnetic shear and the gradient of the Alfvén speed. In terms of helicity balance, the enhanced current drive in the

$$\mathbf{q}' = \frac{2i}{\omega}\,\mathbf{e}_\omega \times \mathbf{e}_\omega^*, \tag{75}$$

which has the same divergence as $\mathbf{q}$ but is manifestly gauge invariant. This form also brings out the connection between wave helicity flux and wave polarization; in particular, plane polarized waves carry no helicity. Furthermore, the helicity flux transverse to the equilibrium magnetic field is

$$\mathbf{q}'_\perp = \frac{-2i}{\omega B_0^2}(\mathbf{e}_\omega\cdot\mathbf{B}_0)(\mathbf{e}_\omega^* \times \mathbf{B}_0) + \text{c.c.}, \tag{76}$$

showing that a flux of helicity perpendicular to $\mathbf{B}_0$ requires a non-zero component of the wave electric field parallel to $\mathbf{B}_0$.

In the presence of resistivity (introducing Ohm's law), we have

$$\nabla\cdot\mathbf{q}' + 2\eta\mu_0(\mathbf{b}_\omega\cdot\mathbf{j}_\omega^* + \text{c.c.}) + 2\mathbf{B}_0\cdot\mathbf{E} = 0, \tag{77}$$

($\mathbf{j}_\omega = \mu_0^{-1}\nabla\times\mathbf{b}_\omega$) which relates the flux of wave helicity to the dynamo field. For small resistivity, Eq. (76) also shows that $\nabla\cdot\mathbf{q}'_\perp$ is proportional to resistivity. Consequently, as indicated by the general discussion of Sec. III, it is only by injecting helicity *parallel* to the equilibrium magnetic field (or by an initial condition) that a dynamo effect independent of η can be produced in a resistive fluid plasma. Furthermore, by integrating Eq. (77) over a resonant layer it is apparent that the enhanced dynamo field within the layer, discussed in the preceding section, does *not* arise from the flux of helicity $\mathbf{q}'_\perp$. Instead it is related to the helicity $\eta\mu_0\tilde{\mathbf{j}}\cdot\tilde{\mathbf{b}}$ "dissipated" by the fluctuations *within* the layer. Consequently this enhanced dynamo effect cannot be deduced from a global helicity balance.

$$\frac{\partial H}{\partial t} + \nabla\cdot\mathbf{Q} + 2\mathbf{B}\cdot\mathbf{E} = 0, \tag{69}$$

where

$$H = \mathbf{A}\cdot\mathbf{B}, \tag{70}$$

$$\mathbf{Q} = \Phi\mathbf{B} + \mathbf{E}\times\mathbf{A}. \tag{71}$$

Here **A** is the magnetic vector potential and Φ the electric potential. The integral of H within a magnetic surface is the magnetic helicity

$$K = \int_{V_0} dV\, \mathbf{A}\cdot\mathbf{B}\,. \tag{72}$$

This quantity is gauge invariant and also a dynamic invariant in weakly resistive plasma.[17,18] The quantities H and **Q** are not individually gauge invariant, but one may regard them as helicity density and flux.

When the time-independent equilibrium and fluctuating quantities $\mathbf{B}_0$, $\mathbf{b}_\omega$, etc. are introduced, as in Sec. III, Eqs. (70) and (71) lead to a time-average wave helicity

$$h = \mathbf{a}_\omega\cdot\mathbf{b}_\omega^* + \text{c.c.} \tag{73}$$

and a wave helicity flux

$$\mathbf{q} = \phi_\omega\mathbf{b}_\omega^* + \mathbf{e}_\omega\times\mathbf{a}_\omega^* + \text{c.c.} \tag{74}$$

(where $\mathbf{b}_\omega = \nabla\times\mathbf{a}_\omega$, $\mathbf{e}_\omega = -\nabla\phi_\omega + i\omega\mathbf{a}_\omega$, and $\nabla\times\mathbf{e}_\omega = i\omega\mathbf{b}_\omega$). However, we may introduce a more convenient form for the wave helicity flux,

$$P = (\omega a B_0^2(x_o)\mu_0^{-1}) \frac{1}{k^2} \int_{-\infty}^{\infty} d\sigma \left| \frac{\partial^2 b_{x0}}{\partial \sigma^2} \right|^2, \tag{65}$$

which, with Eq. (56), becomes

$$P = 2\pi |C|^2 \frac{\rho_0 \omega^2 a^2}{k^2} \left| \frac{\partial \omega_A}{\partial x} \right|_{x_o}. \tag{66}$$

The current driven by the dynamo effect in the resonant layer can be estimated from Eq. (63) as

$$J \sim \frac{B_0}{\mu_0 a} \frac{a}{(\varepsilon + \gamma)^{1/3}} \frac{a\mathbf{x}}{kB_0} \cdot (\mathbf{k} \times \frac{\partial \mathbf{B}_0}{\partial x}) \frac{h}{kB_0} \left| \frac{a}{\omega_A} \frac{\partial \omega_A}{\partial x} \right|_{x_o}^{1/3}, \tag{67}$$

so that the current drive "efficiency" is

$$\frac{J}{P} \sim \frac{a}{\eta B_0} \frac{a\mathbf{x}}{h} \cdot (\mathbf{k} \times \frac{\partial \mathbf{B}_0}{\partial x}) \frac{\varepsilon}{(\varepsilon + \gamma)^{1/3}} \left| \frac{a}{\omega_A} \frac{\partial \omega_A}{\partial x} \right|_{x_o}^{-2/3}. \tag{68}$$

For a purely Ohmically driven discharge the corresponding efficiency would be $a/(\eta B_0)$, so that Eq. (68) indicates that the present efficiency is $\sim \varepsilon/(\varepsilon + \gamma)^{1/3}$ of the Ohmic value.

VI. HELICITY

Previous studies of dynamo effect current drive[1-7] made use of the concept of helicity conservation. For any magnetic field,[15,16]

$$\mathcal{L}(\mathbf{b}_{s1} - \frac{i\mathbf{k}}{k^2}\frac{\partial b_{x1}}{\partial \sigma} + \frac{i b_{x0}}{f_0 k^2}\mathbf{k}\times(\mathbf{k}\times\mathbf{B}_1)) = 0. \tag{60}$$

Again, matching to the outer solution excludes the homogeneous contribution and

$$\mathbf{b}_{s1} = \frac{i\mathbf{k}}{k^2}\frac{\partial b_{x1}}{\partial \sigma} - \frac{i b_{x0}}{f_0 k^2}\mathbf{k}\times(\mathbf{k}\times\mathbf{B}_1). \tag{61}$$

Then the integrated dynamo field is

$$\int_{layer} dx\, \mathbf{E}\cdot\mathbf{B}_0 = \frac{\eta B_0^2(x_o)}{(\varepsilon+\gamma)^{1/3}}\frac{4}{f_0 k^2}\,\mathbf{x}\cdot(\mathbf{k}\times\mathbf{B}_1)\int_{-\infty}^{\infty} d\sigma \left|\frac{\partial b_{x0}}{\partial \sigma}\right|^2. \tag{62}$$

Since the layer width ~ $(\varepsilon+\gamma)^{1/3}a$, this indicates that within the resonance layer the *average* dynamo field is enhanced by a factor $(\varepsilon+\gamma)^{-2/3}$. Using Eq. (56) for b_{x0}, Eq. (62) becomes

$$\int_{layer} dx\, \mathbf{E}\cdot\mathbf{B}_0 = \frac{8\pi}{3}\Gamma(\tfrac{1}{3})\frac{\eta B_0^2 |C|^2}{a}\frac{a}{(\varepsilon+\gamma)^{1/3}}\frac{a\mathbf{x}}{kB_0}\cdot(\mathbf{k}\times\frac{\partial \mathbf{B}_0}{\partial x})\frac{h}{kB_0}\left|\frac{3a}{\omega_A}\frac{\partial \omega_A}{\partial x}\right|_{x_o}^{1/3}, \tag{63}$$

where ω_A is the Alfvén frequency $h/(\mu_0\rho_0)^{1/2}$. This shows that the enhanced field is proportional to the magnetic shear and to the cube root of the gradient of the Alfvén frequency.

The power per unit area dissipated in the layer

$$P = \int_{layer} dx\,(\mu|\nabla\times\mathbf{v}|^2 + \eta|\nabla\times\mathbf{b}|^2) \tag{64}$$

is independent of the viscosity and resistivity.[11-14] It is given by

$$\frac{b_{x0}}{f_0} = C\{im \int_0^\infty dp\, p^{-1} exp(-\tfrac{1}{3}p^3|K|)\,[1 - exp(impo)] + ln\,\lambda\}, \qquad (56)$$

where $m = K/|K|$ and $K = f_0^2/\mathcal{A}_1$.

The lowest order component $\mathbf{b}_{s0}$ in the inner region is given by

$$\mathcal{L}(\mathbf{b}_{s0} - \frac{i\mathbf{k}}{k^2}\frac{\partial b_{x0}}{\partial \sigma}) = 0. \qquad (57)$$

Matching to the outer solution excludes the homogeneous contribution to Eq. (57) (as in Sec. IV) and

$$\mathbf{b}_{s0} = \frac{i\mathbf{k}}{k^2}\frac{\partial b_{x0}}{\partial \sigma}. \qquad (58)$$

Then the leading order contribution to the dynamo field in the resonant layer becomes

$$\mathbf{E}\cdot\mathbf{B}_0 = \frac{\eta B_0^2(x_o)}{a(\varepsilon + \gamma)}\frac{1}{f_0 k^2}\,\mathbf{x}\cdot(\mathbf{k}\times\mathbf{B}_0)\frac{\partial}{\partial \sigma}(b_{x0}^* \frac{\partial^2 b_{x0}}{\partial \sigma^2} + \text{c.c.})\,. \qquad (59)$$

Equation (59) indicates that, relative to the fluctuations outside the layer, the field within the layer is increased by a factor $(\varepsilon + \gamma)^{-1}$. However, this contribution vanishes when integrated across the layer. To find the integrated dynamo field in the layer, we must determine the higher order fluctuations.

At next order, after substituting for $\mathbf{b}_{s0}$ from Eq. (58), the component $\mathbf{b}_{s1}$ is given by

It may be seen from Eq. (44) that the width of the resonant layer (i.e. the region where resistivity and viscosity are important) is $\tau \sim (\varepsilon + \gamma)^{1/3}$. We therefore introduce a scale parameter $\lambda = (\varepsilon + \gamma)^{1/3}$ and a stretched coordinate $\sigma = \tau/\lambda$, expand the equilibrium quantities in powers of λ around $\sigma = 0$,

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1 \lambda\sigma + \dots, \tag{49}$$

$$f = f_0 + f_1 \lambda\sigma + \dots, \tag{50}$$

$$\mathcal{A} = \mathcal{A}_1 \lambda \sigma + \mathcal{A}_2 \lambda^2 \sigma^2 + \dots, \tag{51}$$

and write

$$b_x = b_{x0} + \lambda b_{x1} + \dots, \tag{52}$$

$$\mathbf{b}_s = \lambda^{-1} \mathbf{b}_{s0} + \mathbf{b}_{s1} + \dots. \tag{53}$$

Then in lowest order, the inner solution b_{x0} satisfies

$$\frac{\partial}{\partial\sigma} \mathcal{L} \left(\frac{\partial b_{x0}}{\partial\sigma} \right) = 0, \tag{54}$$

where

$$\mathcal{L} \equiv \left(\frac{\partial^2}{\partial\sigma^2} + \frac{i\mathcal{A}_1 \sigma}{f_0^2} \right). \tag{55}$$

The solution of Eq. (54) that asymptotically matches to the outer solution, Eq. (47), is[11] while the variables v_x and $\mathbf{v}_s$ are given by Eqs. (31) and (32).

The resonant layer has been discussed in connection with Alfvén wave heating, in the absence of magnetic shear, by Kappraff and Tataronis[11] and by Davila.[12] As we will see, there are considerable differences between the calculations of the heating effect and the dynamo effect. The latter depends on magnetic shear and requires a solution to higher order. Accordingly we first reconsider the resonant layer solutions.

As usual, we treat both ε and γ as small and seek an "inner" solution of Eqs. (44) and (45) valid at the resonant layer where resistivity and viscosity are important. This inner solution is then matched to "outer" solutions for which the resistivity and viscosity are negligible ($\varepsilon = \gamma = 0$). The outer solutions are governed by

$$\frac{\partial}{\partial \tau}[\mathcal{A}\frac{\partial}{\partial \tau}(\frac{b_x}{f})] - k^2\mathcal{A}\frac{b_x}{f} = 0, \tag{46}$$

and near the resonant layer they have the form

$$\frac{b_x}{f} = C(ln|\tau| + im\pi), \tag{47}$$

where $m = 0, \pm 1$. The outer solution for $\mathbf{b}_s$ is given by

$$\mathbf{b}_s = \frac{i}{k^2}[\mathbf{k}\frac{\partial b_x}{\partial \tau} - \frac{b_x}{f}\mathbf{k}\times(\mathbf{k}\times\frac{\partial \mathbf{B}}{\partial \tau})]. \tag{48}$$

where $\mathbf{x}$ is a unit vector and v_x and b_x are given by Eqs. (37) and (39). Note that for wavelengths $ka \sim 1$ the dynamo field is proportional to $(\eta k^2/\omega)vb$ and not vb as Eq. (9) might imply. It can therefore drive only a fraction $(b/B_0)^2$ of the equilibrium current, in agreement with the conclusion of Bellan and Schalit for a uniform, non-resonant plasma.

V. THE ALFVÉN RESONANT LAYER

We now turn to the situation at the Alfvén resonant layer ($A(x) = 0$), where $\mathbf{v}$ and $\mathbf{b}$ are singular when η and μ are zero. We introduce the normalized quantities $\tau = (x - x_o)/a$, $\mathbf{k} = a\mathbf{k}$, $\mathbf{b} = \mathbf{b}/B_0(x_o)$, $\mathbf{B} = \mathbf{B}_0/B_0(x_o)$, $f = ah(x)/B_0(x_o)$, $g = a\omega(\mu_0\rho_0(x))^{1/2}/B_0(x_o)$, $\mathcal{A} = g^2 - f^2$, $\mathcal{D} = a^2D$ $(= (\partial^2/\partial\tau^2 - k^2))$, and write $\varepsilon = \eta/\omega a^2$, $\gamma = \mu/(\omega a^2\rho_0(x_o))$. Then from Eqs. (31) - (35) we obtain

$$\frac{\partial}{\partial\tau}[\mathcal{A}\frac{\partial}{\partial\tau}(\frac{b_x}{f})] - k^2\mathcal{A}\frac{b_x}{f} = i\varepsilon\{\frac{\partial}{\partial\tau}[g^2\frac{\partial}{\partial\tau}(\frac{\mathcal{D}b_x}{f})] - k^2g^2\frac{\mathcal{D}b_x}{f}\} + i\gamma g^2(x_o)\mathcal{D}^2(\frac{b_x}{f}) + \varepsilon\gamma g^2(x_o)\mathcal{D}^2(\frac{\mathcal{D}b_x}{f}), \tag{44}$$

$$\begin{aligned}\mathcal{A}\mathbf{b}_s - i\varepsilon g^2\mathcal{D}\mathbf{b}_s - i\gamma g^2(x_o)f\,\mathcal{D}(\frac{\mathbf{b}_s}{f}) - \varepsilon\gamma g^2(x_o)f\,\mathcal{D}(\frac{\mathcal{D}\mathbf{b}_s}{f}) &= \frac{i}{k^2}\mathcal{A}[\mathbf{k}\frac{\partial b_x}{\partial\tau} - \frac{b_x}{f}\mathbf{k}\times(\mathbf{k}\times\frac{\partial\mathbf{B}}{\partial\tau})] \\ &+ \varepsilon\frac{g^2}{k^2}[\mathbf{k}\frac{\partial}{\partial\tau}(\mathcal{D}b_x) - \frac{\mathcal{D}b_x}{f}\mathbf{k}\times(\mathbf{k}\times\frac{\partial\mathbf{B}}{\partial\tau})] \\ &+ \gamma\frac{g^2(x_o)f}{k^2}\mathcal{D}[\frac{\mathbf{k}}{f}\frac{\partial b_x}{\partial\tau} - \frac{b_x}{f^2}\mathbf{k}\times(\mathbf{k}\times\frac{\partial\mathbf{B}}{\partial\tau})] \\ &- i\varepsilon\gamma\frac{g^2(x_o)f}{k^2}\mathcal{D}[\frac{\mathbf{k}}{f}\frac{\partial}{\partial\tau}(\mathcal{D}b_x) - \frac{\mathcal{D}b_x}{f^2}\mathbf{k}\times(\mathbf{k}\times\frac{\partial\mathbf{B}}{\partial\tau})],\end{aligned} \tag{45}$$

where $A(x) \equiv \omega^2\mu_0\rho_0 - h^2$. Note that $A(x) = 0$ defines the Alfvén resonant layer, at which Eq. (39) is singular when $\mu = 0$. The component $\mathbf{v}_s$ is given by

$$\mathcal{L}(\mathbf{v}_s - \frac{i\mathbf{k}}{k^2}\frac{\partial v_x}{\partial x}) = 0, \tag{40}$$

where $\mathcal{L} = A(x) - i\mu\omega\mu_0 D$, so that

$$\mathbf{v}_s = \frac{i\mathbf{k}}{k^2}\frac{\partial v_x}{\partial x} + \hat{\mathbf{v}}_s, \tag{41}$$

with $\mathcal{L}\hat{\mathbf{v}}_s = 0$. The homogeneous solution $\hat{\mathbf{v}}_s$, which decays exponentially as $exp[i^{3/2}(A/\omega\mu)^{1/2}x]$, represents an evanescent wave at the plasma boundary and is not relevant to the current drive problem. Therefore

$$\mathbf{v}_s = \frac{i\mathbf{k}}{k^2}\frac{\partial v_x}{\partial x} \tag{42}$$

and $\mathbf{k} \times \mathbf{v}_s = 0$. [We remark, however, that in a uniform, unbounded plasma, the homogeneous solution $\hat{\mathbf{v}}_s$ represents an independent oscillation with $\omega^2 = h^2/(\mu_0\rho_0)$ and $\mathbf{k} \times \mathbf{v}_s \neq 0$ which propagates parallel to $\mathbf{B}_0$. This allows one to construct waves of arbitrary polarization and helicity and to exploit the first term in Eq. (22).]

Thus, the dynamo field in the bulk plasma (i.e. outside the resonant layer) can be expressed as

$$\begin{aligned} \mathbf{E}\cdot\mathbf{B}_0 = \frac{\eta}{\omega}\{ & \frac{\mathbf{x}\cdot(\mathbf{B}_0 \times \mathbf{k})}{k^2}[\frac{\partial v_x}{\partial x} D b_x^* + \frac{b_x^*}{h} D(h\frac{\partial v_x}{\partial x})] \\ & + \frac{b_x^*}{h}(\mathbf{x} \times \mathbf{B}_0)\cdot D(v_x \frac{\partial \mathbf{B}_0}{\partial x})\} + \text{c.c.}, \end{aligned} \tag{43}$$

$$\omega\rho_0 v_x + \frac{h}{\mu_0} b_x + i\frac{\partial p}{\partial x} = i\mu D v_x , \quad (33)$$

$$\omega\rho_0 \mathbf{v}_s + \frac{h}{\mu_0}\mathbf{b}_s - \frac{i}{\mu_0} b_x \frac{\partial \mathbf{B}_0}{\partial x} - \mathbf{k}p = i\mu D \mathbf{v}_s , \quad (34)$$

$$\frac{\partial v_x}{\partial x} + i\mathbf{k}\cdot\mathbf{v}_s = 0, \quad (35)$$

where $D \equiv (\partial^2/\partial x^2 - k^2)$ and $k^2 = k_y^{\,2} + k_z^{\,2}$. The dynamo field is

$$\mathbf{E}\cdot\mathbf{B}_0 = \frac{\eta}{i\omega}\mathbf{B}_0\cdot(\mathbf{v}_x \times D\,\mathbf{b}_s^{*} + \mathbf{v}_s \times D\mathbf{b}_x^{*}) + \text{c.c.} . \quad (36)$$

It is clear from Eq. (36) that, except at resonant layers, $\mathbf{E}\cdot\mathbf{B}_0$ may be calculated to first order in resistivity from the $\eta = 0$ form of the fluctuations **v**, **b**. Accordingly, we first consider the fluctuations in this limit. The situation at a resonant layer is discussed in the next section.

When $\eta = 0$,

$$b_x = -\frac{h}{\omega} v_x , \quad (37)$$

$$\mathbf{b}_s = -\frac{h}{\omega}\mathbf{v}_s - \frac{i}{\omega}\frac{\partial \mathbf{B}_0}{\partial x} v_x , \quad (38)$$

and v_x is given by

$$\frac{\partial}{\partial x}\left(A(x)\frac{\partial v_x}{\partial x}\right) - k^2 A(x) v_x + i\mu\omega\mu_0 D^2 v_x = 0, \quad (39)$$

model, we consider the sheet pinch in which equilibrium quantities vary only in the x-direction and

$$\mathbf{B}_0 = (0, B_y(x), B_z(x)), \tag{24}$$

$$\rho_0 = \rho_0(x). \tag{25}$$

The fluctuations are considered to be produced by an external antenna, represented by boundary conditions at $x = \pm a$, and we Fourier analyze the fluctuations into components

$$\tilde{\mathbf{b}}(\mathbf{x}, t) = \mathbf{b}(x)e^{i\psi} + \text{c.c.}, \tag{26}$$

$$\tilde{\mathbf{v}}(\mathbf{x}, t) = \mathbf{v}(x)e^{i\psi} + \text{c.c.}, \tag{27}$$

$$\tilde{P}(\mathbf{x}, t) = p(x)e^{i\psi} + \text{c.c.}, \tag{28}$$

where $\psi = k_y y + k_z z - \omega t$. It is convenient to write $\mathbf{k} = (0, k_y, k_z)$, $h = \mathbf{k}\cdot\mathbf{B}_0$, and to resolve $\mathbf{b}$ and $\mathbf{v}$ into components perpendicular and parallel to the magnetic surface (denoted by subscript x and s, respectively),

$$\mathbf{b} = \mathbf{b}_x + \mathbf{b}_s, \tag{29}$$

$$\mathbf{v} = \mathbf{v}_x + \mathbf{v}_s. \tag{30}$$

Then from Eqs. (5) - (7),

$$\omega b_x + h v_x = i\eta D b_x, \tag{31}$$

$$\omega \mathbf{b}_s + h \mathbf{v}_s + i v_x \frac{\partial \mathbf{B}_0}{\partial x} = i\eta D \mathbf{b}_s, \tag{32}$$