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M.D. Calvin, R.D. Hazeltine, P.M. Valanju,
and E.R. Solano
Institute for Fusion Studies
The University of Texas at Austin
Austin, Texas 78712

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R.D. Hazeltine, M.D. Calvin, P.M. Valanju, and E.R. Solano
Institute for Fusion Studies
The University of Texas at Austin
Austin, Texas 78712
and
Fusion Research Center
The University of Texas at Austin
Austin, Texas 78712

We analytically calculate the neutral particle distribution and its effects on ion heat and momentum transport in three dimensional plasmas with arbitrary temperature and density profiles. A general variational principle taking advantage of the simplicity of the charge-exchange (CX) operator is derived to solve self-consistently the neutral-plasma interaction problem. To facilitate an extremal solution, we use the short CX mean-free-path (λ_x) ordering. Further, a non-variational, analytical solution providing a full set of transport coefficients is derived by making the realistic assumption that the product of the CX cross section with relative velocity is constant. The effects of neutrals on plasma energy loss and rotation appear in simple, sensible forms. We find that neutral viscosity dominates ion viscosity everywhere, and in the edge region by a large factor.

1. Introduction

Conventional transport theory of high-temperature plasmas treats only completely ionized plasmas with no significant numbers of neutrals. However, the bulk transport is significantly affected by the edge, where the plasma is not completely ionized. Edge neutrals, whose motion is not affected by the magnetic field, interact with ions through charge-exchange (CX) and impact ionization. During a CX collision a neutral and ion exchange identities. It thus appears as if the neutral has scattered, suddenly changing its speed and direction. A neutral may continue this CX "scattering" many times before it is finally ionized due to impact with an ion. As Sacharov [1] first pointed out, it is in this way that low energy neutrals near the wall can gain energy through frequent CX scatterings and penetrate into the plasma interior, where they may affect plasma transport and rotation.

Although it would be desirable to solve the full plasma-neutral problem analytically, the near equality of the plasma and hot neutral density scale lengths makes it very hard to do so. Most analytic solutions available [1-7] work only in a one dimensional slab geometry. They also assume prespecified, fixed, uniform plasma density and temperature profiles, as well as a uniform external neutral source. Neither the effects of the neutrals on the plasma nor the changes in the external neutral source due to changes in plasma flux are calculated. These restrictions severely limit their use in experiments.

In this paper, we show that the scope of the analytical solutions can be broadened considerably by making use of variational techniques. We can solve the neutral-plasma interaction problem in a self-consistent manner without assuming any spatial symmetry, and without neglecting the effects of neutrals on the plasma. This is achieved in two steps. First. the simplicity of the charge exchange collision operator is used to set up a general variational principle. The process of finding a minimal solution is simplified by using the smallness of the ratio of the ionization to the charge exchange mean-free-path. To each order in this small parameter, variationally accurate transport coefficients for neutral particle, energy and momentum fluxes can be read off from the minimized entropy production rate.

Furthermore, upon making the realistic assumption that the product of CX cross section with relative velocity $\sigma_x|\mathbf{v}-\mathbf{v}'|$ is constant, we calculate the neutral particle, heat and momentum fluxes and present the full set of neutral transport coefficients. A similar calculation is given by Vekshtein and Ryutov [8].

We then discuss the effects of CX on ion fluid behavior and transport. Momentum and energy moments of the ion and neutral kinetic equations are analyzed in the short CX mean-free-path regime. The effects of neutrals on plasma energy loss and rotation appear in particularly simple, intuitively sensible forms. After examining the contribution of neutrals to ion viscosity, we find that neutral viscosity dominates ion viscosity everywhere, and in the edge region by a large factor. The influence of neutrals on the isotopic dependence of energy confinement is also discussed.

2. Kinetic equations

To begin we briefly review neutral kinetic theory [9]. Neutral particles are subject to three inelastic processes:

1. Charge exchange (CX) collisions locally conserve both ions and neutrals; in effect, only energy and momentum are exchanged. We write the CX operator as

$$X(f,g) \equiv \int d^3v'\,\sigma_x|\mathbf{v}'-\mathbf{v}|\{f(\mathbf{v})g(\mathbf{v}')-f(\mathbf{v}')g(\mathbf{v})\}\ , \tag{1}$$

where f and g are the ion and neutral particle distribution functions respectively. Charge exchange occurs with frequency ν_x. σ_x is of course the CX cross section. Notice that

$$X(g,f) = -X(f,g)\ . \tag{2}$$

2. Impact ionization, a neutral sink, is proportional to g and to n_e, the electron density (impact ionization due to ions is negligible). We suppress dependence on the electron

distribution and write the impact ionization rate as $\nu_z g$.

3. Recombination, a neutral source, is proportional to the product of f and n_e; we denote it by $\nu_r f$. Other neutral sources, such as from gas puffing, are distinctly local.

Neutrals are not subject to mean forces, and their elastic collision rate is negligible. Hence the kinetic equation for g is

$$\frac{\partial g}{\partial t} + \mathbf{v} \cdot \nabla g = -X(g, f) - \nu_z g + \nu_r f \ . \tag{3}$$

This is instructively compared to the corresponding ion kinetic equation,

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \mathbf{a} \cdot \frac{\partial f}{\partial \mathbf{v}} - C(f) = -X(f, g) + \nu_z g - \nu_r f \ . \tag{4}$$

Here $\mathbf{a}$ is the acceleration due to electric and magnetic fields and C the Coulomb collision operator. The point is that ion population changes due to recombination or impact ionization precisely balance neutral changes.

In discussing charge exchange, it is advantageous to track the charge flow, rather than individual particles. Thus a CX event is viewed as an exchange of momentum and energy between a neutral and an ion, each particle maintaining its species identity. Because the initial momenta of the colliders are not correlated, CX yields large-angle scattering events. From this point of view a neutral will survive any number of CX "scatterings;" it disappears only upon impact ionization (or wall interaction), so that the neutral lifetime is $1/\nu_z$.

Section 4 of the present work provides a variational principle for the general solution of Eq. (4). However, most of our results, including the explicit transport formulae, pertain only in the special case

$$\nu_z \ll \nu_x \ , \tag{5}$$

in which each neutral suffers many charge exchanges in its lifetime. In typical tokamak experiments ionization is indeed slower than CX although not always by a large factor. For example, in TEXT [10] ν_x exceeds ν_z by factors of 3 to 5. (The recombination rate ν_r is typically somewhat smaller still.)

The rates ν_x and ν_z are conveniently measured in terms of the corresponding mean-free-paths, $\lambda = v_n/\nu$, where v_n is the thermal speed of the neutral population. We shall refer to $\lambda_x = v_n/\nu_x$ as the CX mean-free-path; it is measured in centimeters in most tokamak discharges. The impact ionization length λ_z is a measure of the total path travelled by a neutral; notice that in our case this path is far from linear since Eq. (5) describes a neutral that changes direction several times before ionization.

The third length of interest is the scale length L for neutral density variation. A random walk argument [9] using (5) shows that

$$L \sim (\lambda_x \lambda_z)^{1/2} \tag{6}$$

and therefore that

$$\lambda_x \ll L \ll \lambda_z \ . \tag{7}$$

In other words, the ordering (5) is consistent with the short CX mean-free-path regime. In Sec. 5 we focus attention on the short mean-free-path limit. Our argument will show that these are self-consistent orderings.

3. H-theorem

Here we adopt classical arguments to establish some important properties of the CX operator.

Consider the bilinear form

$$\Theta[G_1, G_2] \equiv \int d^3v \left[\frac{G_1}{(n_n \hat{f})}\right] X(G_2, f) \ , \tag{8}$$

where G_1 and G_2 are any two neutral distributions, n_n is the neutral density and $\hat{f}$ is an ion distribution normalized to have unit particle density:

$$\hat{f} = \frac{f}{n_i} \ .$$

It will appear presently that if g is the solution to Eq. (3), then $\Theta[g,g]$ is the rate of neutral entropy production. [Our notation is to use g for the solution — if only approximate — to (3); upper-case G will refer to an arbitrary neutral distribution function or trial function.] We refer to $\Theta[G_1, G_2]$ as the "CX bilinear form."

First we show that

$$\Theta[G_1, G_2] = \Theta[G_2, G_1] \; . \tag{9}$$

This symmetry forces the neutral transport coefficients to have Onsager symmetry.

The demonstration follows a conventional pattern. We use "Boltzmann notation," $f' \equiv f(v')$, so that (1) becomes

$$X(f,g) \equiv \int d^3v \, \sigma_x |\mathbf{v}' - \mathbf{v}| (fg' - f'g) \; .$$

From (1) and (2) we have

$$\Theta[G_1, G_2] = -\int d^3v \, d^3v' \left(\frac{\sigma_x}{n_n}\right) |\mathbf{v}' - \mathbf{v}| \left(\frac{G_1}{\hat{f}}\right) (f\, G_2' - f' G_2) \; ,$$

or, after relabeling integration variables,

$$\Theta[G_1, G_2] = -\int d^3v \, d^3v' \left(\frac{\sigma_x}{n_n}\right) |\mathbf{v}' - \mathbf{v}| \left(\frac{G_1'}{\hat{f}'}\right) (f'\, G_2 - f G_2') \; ,$$

and symmetrizing,

$$\Theta[G_1, G_2] = \left(\frac{1}{2}\right) n_i \int d^3v \, d^3v' \left(\frac{\sigma_x}{n_n}\right) |\mathbf{v}' - \mathbf{v}| \hat{f} \hat{f}' \left(\frac{G_1}{\hat{f}} - \frac{G_1'}{\hat{f}'}\right) \left(\frac{G_2}{\hat{f}} - \frac{G_2'}{\hat{f}'}\right) \tag{10}$$

from which (9) follows.

We also use (10) to express the quadratic form $\Theta[G,G]$ as

$$\Theta[G, G] = \left(\frac{1}{2}\right) n_i \int d^3v \, d^3v' \left(\frac{\sigma_x}{n_n}\right) |\mathbf{v}' - \mathbf{v}| \hat{f} \hat{f}' \left(\frac{G}{\hat{f}} - \frac{G'}{\hat{f}'}\right)^2 \tag{11}$$

showing that it can vanish only if $g/\hat{f}$ is independent of velocity. Thus we have

$$\Theta[g, g] = 0 \Longrightarrow G(\mathbf{x}, \mathbf{v}) = N(\mathbf{x}) \hat{f}(\mathbf{x}, \mathbf{v}) \tag{12}$$

for an arbitrary spatial function N.

It is obvious from (1) that the CX operator vanishes when g= $N\hat{f}$; the interesting feature of (12) is that $g = N\hat{f}$ is the only solution to $X(g, f) = 0$.

Finally (11) yields the inequality

$$\Theta[G, G] > 0 \qquad \text{for any } G \ . \tag{13}$$

This with (12) completes the entropy theorem ("H-theorem") for CX. We see that G relaxes, through charge exchange, to have the velocity dependence of the (not necessarily Maxwellian) ion distribution. Θ is evidently the entropy production rate.

4. General variational principle

With some simplification (for example, the assumption of constant CX cross-section), the isothermal, one-dimensional version of Eq. (3) can be solved analytically for any mean-free-path ordering by singular eigenfunction techniques [6]. It is likely that a generalization of these methods could treat three-dimensional cases [11]. However, singular eigenfunctions are not easily adapted to allow for temperature variation.

We present here a complementary approach to neutral particle physics, based on a variational principle. The variational approach allows for arbitrary temperature variation — an important improvement in realism since temperature variation can be steep near the tokamak edge. It also allows for arbitrary energy dependence of the CX cross section as well as three-dimensional geometry. More importantly it provides relatively simple asymptotic formulas for various quantities of interest in limiting parameter regimes.

On the other hand, it should be emphasized that the variational principle has its own complications, especially at long mean-free-path, where global trial functions are called for.

We derive the general variational principle in this section. Its specialization to the short-λ_x case — a much simpler and more obviously practical formalism — is considered in Sec. 5.

We begin by introducing the scalar product,

$$\{G_1, G_2\} \equiv \int d^3x\, d^3v \frac{G_1 G_2}{(n_n \widehat{f})} = \{G_2, G_1\} , \tag{14}$$

where f is the ion distribution as in Sec. 2. We also introduce the spatially integrated CX bilinear form

$$\int d^3x\ \Theta[G_1, G_2] \equiv \overline{\Theta} = \{G_1, X(G_2, f)\} .$$

Now consider the steady-state version of (3):

$$\mathbf{v} \cdot \nabla g + X(g, f) + \nu_z g = \nu_r f . \tag{15}$$

It is combined with its adjoint

$$-f\mathbf{v} \cdot \nabla \left(\frac{g^\dagger}{f}\right) + X(g^\dagger, f) + \nu_z g^\dagger = \nu_r f , \tag{16}$$

in a conventional way

$$\{g^\dagger, \mathbf{v} \cdot \nabla g\} - \left\{g, f\mathbf{v} \cdot \nabla \left(\frac{g^\dagger}{f}\right)\right\} + 2\overline{\Theta}[g^\dagger, g] + 2\nu_z \{g^\dagger, g\} = \nu_r \{f, g + g^\dagger\} ,$$

to construct the form

$$S[G^\dagger, G] = V[G^\dagger, G] + \overline{\Theta}[G^\dagger, G] + \nu_z \{G^\dagger, G\} - \nu_r \{f, G + G^\dagger\} , \tag{17}$$

where

$$V[G^\dagger, G] \equiv \left\{\left(\frac{1}{2}\right) G^\dagger, \mathbf{v} \cdot \nabla G\right\} - \left\{\left(\frac{1}{2}\right) G, f\mathbf{v} \cdot \nabla \left(\frac{G^\dagger}{f}\right)\right\} . \tag{18}$$

Here G and $G^\dagger$ are to be viewed as trial functions for the solutions to (15) and (16) respectively. We observe that evaluation at the exact solutions g and $g^\dagger$ gives

$$S[g, g^\dagger] \equiv S^* = V[g^\dagger, g] + \overline{\Theta}[g^\dagger, g] + \nu_z \{g^\dagger, g\} = -\left(\frac{1}{2}\right) \nu_r \{f, g + g^\dagger\} . \tag{19}$$

This relation is essentially an entropy production law; the terms involving $\overline{\Theta}, \nu_z$ and ν_r measure entropy production by CX, ionization and recombination respectively. The first term, V, describes entropy flow.

After recalling (14) we find that

$$\frac{\delta V}{\delta G^\dagger} = \int d^3x\, d^3v \left(\frac{1}{\bar{f}}\right) \mathbf{v}\cdot\nabla G = \{1, \mathbf{v}\cdot\nabla G\} \ ,$$

and similarly

$$\frac{\delta V}{\delta G} = \left\{1, f\mathbf{v}\cdot\nabla\left(\frac{G^\dagger}{f}\right)\right\} \ .$$

Moreover, because of (9),

$$\frac{\delta\overline{\Theta}}{\delta G^\dagger} = \{1, X(G,f)\} \ , \frac{\delta\overline{\Theta}}{\delta G} = \{1, X(G^\dagger, f)\} \ ,$$

whence

$$\frac{\delta S}{\delta G^\dagger} = \{1, \mathbf{v}\cdot\nabla G + X(G,f) + \nu_z G - \nu_r f\} \ ,$$

$$\frac{\delta S}{\delta G} = \left\{1, -fv\cdot\nabla\left(\frac{G^\dagger}{f}\right) + X(G^\dagger, f) + \nu_z G^\dagger - \nu_r f\right\} \ .$$

Thus the variational principle

$$\frac{\delta S}{\delta G^\dagger} = 0 \tag{20}$$

reproduces (15), while the principle

$$\frac{\delta S}{\delta G} = 0 \tag{21}$$

yields (16). We summarize these facts by writing

$$\delta S = 0 \ . \tag{22}$$

A normalized version of (22) is easily constructed. One finds that

$$\delta\overline{H} = 0 \ ,$$

where

$$\overline{H}(G, G^\dagger) \equiv \nu_r^2 \frac{\{f, G+G^\dagger\}^2}{(V[G^\dagger, G] + \overline{\Theta}[G^\dagger, G] + \nu_z\{G^\dagger, G\})} \ . \tag{23}$$

This functional has the extremal value

$$\overline{H}^* = 4S^* .$$

Whichever form is used, the general variational theory has two disadvantages. First, both forms are spatially nonlocal so that trial functions must include both **x**- and **v**-dependence. Second, because the operator in (15) is not self-adjoint, trial functions for both g and $g^\dagger$ must be provided. Of course one prefers a variational principle involving only velocity integration, and using only symmetric (self-adjoint) bilinear forms.

Nonetheless, in any parameter regime that allows approximate analytic solution to (15) and (16), the variational principles (22) and (23) become directly useful. For then we can obtain higher-order information from quite simple integrals. For example, suppose that the CX rate is small: $\nu_x \ll v_n/L$. Then (15) and (16) are, asymptotically, first-order partial differential equations with well-understood Green's functions. Substitution of the solutions into H or S would provide an entropy balance law including CX effects through first order in ν_x.

5. Short mean-free-path theory

5.1 Ordered kinetic equations

We now turn attention to a case where the variational theory is both local and self-adjoint. Here we adopt the ordering (7). We define the small parameter

$$\Delta \equiv \frac{\lambda_x}{L} \approx \frac{L}{\lambda_z} , \tag{24}$$

and expand the solution to (15) as

$$g = g_0 + g_1 \cdots , g_k = \mathcal{O}(\Delta^k) .$$

We also assume

$$\nu_\tau \sim \Delta^2 \nu_x \tag{25}$$

consistently with (5). Then, precisely as in Chapman-Enskog theory, we obtain a sequence of ordered equations for the g_k.

After writing (15) as

$$X(g,f) + \mathbf{v}\cdot\nabla g + \nu_z g = f\nu_r$$

we see that the first three orders are given by

$$\Delta^0{:}\; X(g_0,f) = 0\ ; \tag{26}$$

$$\Delta^1{:}\; X(g_1,f) + \mathbf{v}\cdot\nabla g_0 = 0\ ; \tag{27}$$

$$\Delta^2{:}\; X(g_2,f) + \mathbf{v}\cdot\nabla g_1 + \nu_z g_0 = f\nu_r\ . \tag{28}$$

The CX conservation law,

$$\int d^3v\, X(G,f) = 0\ , \tag{29}$$

for any G, provides a solubility condition in each order. Thus we must have, from (27),

$$\nabla\cdot\int d^3v\, \mathbf{v} g_0 = 0\ , \tag{30}$$

and from (28),

$$\nabla\cdot\int d^3v \mathbf{v} g_1 = \nu_r n_i - \nu_z n_n\ . \tag{31}$$

In view of (12), Eq. (26) has the unique solution

$$g_0 = n_n(\mathbf{x})\hat{f}\ , \tag{32}$$

where n_n is the (lowest order) neutral density.

5.2 Variational principle for short λ_x

To make further progress one must specify the ion distribution. We choose $\hat{f}$ to correspond to a Maxwellian moving with velocity $\mathbf{V}$. Then

$$\begin{aligned}\hat{f} &= \pi^{-3/2} v_{ti}^{-3} \exp\left[-\tfrac{(\mathbf{v}-\mathbf{V})^2}{v_{ti}^2}\right]\ ,\\ v_{ti} &= \left(\frac{2T_i}{m_i}\right)^{1/2}\ ,\end{aligned} \tag{33}$$

with

$$\nabla \cdot (n_n \mathbf{V}) = 0 \tag{34}$$

in order to satisfy (30). Thus $\hat{f}$ depends upon position through $\mathbf{V}(\mathbf{x})$ and $T_i(\mathbf{x})$. While (33) is not exact, it closely approximates the observed ion distribution in most confinement experiments. The simple velocity shift, although strictly consistent with neoclassical theory only in the isothermal case, is not far from theoretical predictions and allows for the rapid rotation observed in the edge regions of some experiments. More elaborate non-Maxwellian corrections could be included in the present formalism with some loss of simplicity.

Since we will measure the neutral fluxes in a frame moving with velocity $\mathbf{V}$, we will drop all $\mathbf{V}$-dependent terms except those involving its gradient. Our first order equation has become

$$\mathbf{X}(g_1, f) = Q \equiv -\mathbf{v} \cdot \nabla g_0 \ , \tag{35}$$

where Q is found to be

$$Q = -n_n \hat{f} \mathbf{v} \cdot \left[\nabla \ln p_n + \left(\frac{v^2}{v_{ti}^2} - \frac{5}{2} \right) \nabla \ln T_i + \left(\frac{2}{v_{ti}^2} \right) \mathbf{v} \cdot \nabla \mathbf{V} \right] \tag{36}$$

and where we have used $p_n = n_n T_i$. We solve (35) via a straightforward specialization of the general variational method developed in Sec. 4. We recall the entropy production rate,

$$\Theta[G_1, G_2] = \int d^3v \left[\frac{G_1}{(n_n \hat{f})} \right] X(G_2, f) \tag{8}$$

and introduce the linear form

$$P[G] \equiv \int d^3v \, \frac{GQ}{(n_n \hat{f})} \ , \tag{37}$$

to find that the functional

$$H[G, G] \equiv \frac{P^2[G]}{\Theta[G, G]} \tag{38}$$

is variational:

$$\delta H = 0 \ , \tag{39}$$

at $G = g_1$. Since (35) implies $\Theta[g_1, g_1] = P[g_1]$, the extremal value of H is

$$H[g_1, g_1] = P[g_1] = \Theta[g_1, g_1] \ , \tag{40}$$

the entropy production rate. In terms of the first order neutral particle, heat, and momentum fluxes defined by

$$\mathbf{\Gamma}_n = \int d^3v \, g_1 \mathbf{v} \tag{41}$$

$$\mathbf{q}_n = \int d^3v \, g_1 \mathbf{v} \left(\frac{v^2}{v_{ti}^2} - \frac{5}{2} \right) \tag{42}$$

$$\mathbf{P}_n = v_{ti}^{-1} \int d^3v \, g_1 \mathbf{v}\mathbf{v} \tag{43}$$

respectively; (40), (37) and (36) give rise to

$$H[g_1, g_1] = -\mathbf{\Gamma}_n \cdot \nabla \ln p_n - \mathbf{q}_n \cdot \nabla \ln T_i - \mathbf{P}_n : v_{ti}^{-1} \nabla \mathbf{V} \ . \tag{44}$$

We see that the neutral entropy production rate H is given by a "canonical" product of forces and fluxes.

The neutral fluxes are related to the neutral transport coefficients by the relations

$$\begin{aligned} \mathbf{\Gamma}_n &= -L_{11} \nabla \ln p_n - L_{12} \nabla \ln T_i - \mathbf{L}_{13} \cdot v_{ti}^{-1} \nabla \mathbf{V} \ , \\ \mathbf{q}_n &= -L_{12} \nabla \ln p_n - L_{22} \nabla \ln T_i - \mathbf{L}_{23} \cdot v_{ti}^{-1} \nabla \mathbf{V} \ , \\ \mathbf{P}_n &= \left(-\mathbf{L}_{31} \nabla \ln p_n - \mathbf{L}_{32} \nabla \ln T_i - L_{33} v_{ti}^{-1} \nabla \mathbf{V} \right) + \text{ transpose} \ . \end{aligned} \tag{45}$$

Using (45) we compute the entropy production rate (44) as

$$\begin{aligned} H[g_1, g_1] &= L_{11} |\nabla \ln p_n|^2 + 2L_{12} \nabla \ln p_n \cdot \nabla \ln T_i + L_{22} |\nabla \ln T_i|^2 \\ &+ 2\mathbf{L}_{13} \nabla \ln p_n : v_{ti}^{-1} \nabla \mathbf{V} + 2\mathbf{L}_{23} \nabla \ln T_i : v_{ti}^{-1} \nabla \mathbf{V} + L_{33} \, v_{ti}^{-2} (\nabla \mathbf{V})^2 \ , \end{aligned} \tag{46}$$

a quadratic form in the gradients ("thermodynamic forces"). The point is that once an expression of the form (46) is found for H, the transport coefficients can be read off by inspection. This formalism, like (44), follows the pattern of conventional transport theory.

In second order we need only the solubility condition, (31). Since the neutral particle flux is defined to be

$$\mathbf{\Gamma}_n = \int d^3v\, g_1 \mathbf{v} \ ,$$

in view of (34), (31) can be expressed as

$$\nabla \cdot \mathbf{\Gamma}_n = \nu_r n_i - \nu_z n_n \ . \tag{47}$$

Since the ion parameters n_i and T_i are presumed given, (47) should be viewed as a constraint on the neutral density profile. We note that the time derivative term can be included straightforwardly in (47),

$$\frac{\partial n_n}{\partial t} + \nabla \cdot \mathbf{\Gamma}_n = \nu_r n_i - \nu_z n_n \ , \tag{48}$$

provided we use the self-consistent ordering $\partial/\partial t = \mathcal{O}(\Delta^2)$.

Our results (54) will show that

$$\mathbf{\Gamma}_n \approx \left(\frac{n_n}{\nu_x}\right) \frac{v_n^2}{L_n} \approx n_n \Delta\, v_n \ .$$

We substitute this estimate into (44) to obtain the ordering

$$\Delta \frac{v_n}{L} \approx \nu_x \Delta^2 \approx \nu_z \ ,$$

as anticipated in (25). Thus our orderings are internally consistent.

We conclude this discussion by showing that the extremum of H is in fact a maximum. Let $G = g + \delta g$ and define

$$\delta\Theta_1 \equiv \Theta(\delta g, g) + \Theta(g, \delta g), \delta\Theta_2 \equiv \Theta(\delta g, \delta g) \ ,$$
$$\delta P \equiv P[\delta g] \ .$$

Then we have, without approximation,

$$\begin{aligned} \Theta[G, G] &= \Theta^* + \delta\Theta_1 + \delta\Theta_2 \ , \\ P[G] &= P^* + \delta P \ . \end{aligned} \tag{49}$$

Also, Eq. (40) implies $\Theta^* = P^*$ where the asterisks denote extremal values, and (39) implies

$$\delta\Theta_1 = 2\delta P \ .$$

We use these facts to evaluate H[G,G], with the result

$$H = H^* \left\{ 1 - \frac{\delta\Theta_2}{\Theta^*} + \left(\frac{1}{4}\right)\left(\frac{\delta\Theta_1}{\Theta^*}\right)^2 \right\} , \tag{50}$$

correct through second order. Of course there is no first order term because of (39).

Now consider the quantity, $\Theta[xg + \delta g, xg + \delta g]$, where x is an arbitrary constant. For this choice (49) becomes

$$\Theta[xg + \delta g, xg + \delta g] = \Theta^* x^2 + \delta\Theta_1 x + \delta\Theta_2 \ .$$

Since (13) does not allow the value of Θ to cross the real axis, the quadratic equation

$$\Theta^* x^2 + \delta\Theta_1 x + \delta\Theta_2 = 0$$

cannot have two real roots. Therefore its discriminant cannot be positive:

$$4\Theta^* \delta\Theta_2 \geq (\delta\Theta_1)^2 ,$$

and (50) implies

$$H \leq H^* ; \tag{51}$$

the extremum is indeed a maximum.

5.3 Constant $\sigma_x|\mathbf{v} - \mathbf{v}'|$ approximation

In addition to the short λ_x approximation, the realistic assumption that $\sigma_x|\mathbf{v} - \mathbf{v}'|$ is nearly independent of velocity allows us to avoid the variational approach and to calculate the neutral fluxes directly. A similar calculation is given by Vekshtein and Ryutov [8]. We will solve the problem by expansion in Δ. Thus we first solve the first-order equation

$$X(g_1, f) = Q , \tag{35}$$

where

$$Q = -n_n \hat{f} \mathbf{v} \cdot \left[\nabla \ln p_n + \left(\frac{v^2}{v_{ti}^2} - \frac{5}{2} \right) \nabla \ln T_i + \left(\frac{2}{v_{ti}^2} \right) \mathbf{v} \cdot \nabla \mathbf{V} \right] . \tag{36}$$

Assuming now that $\sigma_x |\mathbf{v} - \mathbf{v}'|$ is constant, we evaluate the CX operator (1) to find

$$X(g_1, f) = \nu_x \int d^3v' (\hat{f} g_1 - \hat{f} g_1') = \nu_x \left(g_1 - \hat{f} \int d^3v' g_1' \right) \tag{52}$$

where we have taken $\sigma_x |\mathbf{v} - \mathbf{v}'| = \int d^3v \, \hat{f} \sigma_x |\mathbf{v} - \mathbf{v}'| = \nu_x / n_i$. We now observe that since

$$\int d^3v Q = 0 ,$$

it is clear from (52) that (35) has the exact solution

$$g_1 = \frac{Q}{\nu_x} . \tag{53}$$

Thus we find the first order correction to the neutral distribution

$$g_1 = -\left(\frac{n_n f}{\nu_x} \right) \mathbf{v} \cdot \left[\nabla \ln p_n + \left(\frac{v^2}{v_{ti}^2} - \frac{5}{2} \right) \nabla \ln T_i + \left(\frac{2}{v_{ti}^2} \right) \mathbf{v} \cdot \nabla \mathbf{V} \right] . \tag{54}$$

Now, using the definitions (41)–(43), we may directly calculate the neutral fluxes to obtain

$$\mathbf{\Gamma}_n = -\frac{1}{2} n_n \nu_x \lambda_x^2 \nabla \ln p_n , \tag{55}$$

$$\mathbf{q}_n = -\frac{5}{4} n_n \nu_x \lambda_x^2 \nabla \ln T_i , \tag{56}$$

$$\mathbf{P}_n = -\frac{1}{2} n_n \nu_x \lambda_x^2 v_{ti}^{-1} (\nabla \mathbf{V} + \text{ transpose}) . \tag{57}$$

As indicated by (45), the neutral transport coefficients can be written

$$L_{lm} = \alpha_{lm} n_n \nu_x \lambda_x^2 \tag{58}$$

where

$$\alpha_{11} = 0.5 , \quad \alpha_{22} = 1.25 , \quad \alpha_{33} = 0.5 , \tag{59}$$

and

$$\alpha_{lm} = 0 \quad \text{for} \quad l \neq m .$$

The form of these neutral transport coefficients is consistent with our physical picture. In any realistic plasma there will always be a population of neutral particles that are not affected by the magnetic field. These neutrals may undergo many CX collisions before being ionized due to impact with an ion. Each CX collision results in a random change of neutral momentum. The result is that neutrals execute a random walk of step-size λ_x and frequency ν_x.

6. Charge exchange and ion transport

6.1 Moment equations

Our discussion of ion fluid behavior and transport is based on two moments of Eqs. (3) and (4): the momentum and energy moments. The momentum conservation law for ions has the form

$$\frac{\partial}{\partial t}(m_i n_i \mathbf{V}_i) + \nabla \cdot \mathbf{P}_i - en_i(\mathbf{E} + c^{-1}\mathbf{V}_i \times \mathbf{B}) = \mathbf{F}_e - \mathbf{F}_x + \nu_z m_i n_n \mathbf{V}_n - \nu_r m_i n_i \mathbf{V}_i \ .$$

Here

$$n_i \mathbf{V}_i \equiv \int d^3v\, \mathbf{v} f \tag{60}$$

is the ion flow,

$$\mathbf{P}_i \equiv \int d^3v\, m_i \mathbf{v}\mathbf{v} f \tag{61}$$

is the ion stress tensor,

$$\mathbf{F}_e \equiv \int d^3v\, m_i \mathbf{v} C$$

is the collisional friction force on ions due to Coulomb collisions with electrons, and

$$\mathbf{F}_x \equiv \int d^3v\, m_i \mathbf{v}\, X(f,g) \tag{62}$$

measures the effective friction due to charge exchange. It is convenient to define

$$\mathbf{F}_n \equiv -\mathbf{F}_x + \nu_z m_i n_n \mathbf{V}_n - \nu_r m_i n_i \mathbf{V}_i \ . \tag{63}$$

We refer to F_n as the "neutral friction." Then we have

$$\frac{\partial}{\partial t}(m_i n_i \mathbf{V}_i) + \nabla \cdot \mathbf{P}_i - e n_i(\mathbf{E} + c^{-1}\mathbf{V}_i \times \mathbf{B}) = \mathbf{F}_e + \mathbf{F}_n \ . \tag{64}$$

The corresponding neutral force law,

$$\frac{\partial}{\partial t}(m_i n_i \mathbf{V}_i) + \nabla \cdot \mathbf{P}_i = -\mathbf{F}_n \ , \tag{65}$$

is obtained by changing subscripts, assuming $m_n = m_i$ and noting that $e_n = 0$.

Next consider pressure evolution. After multiplying (2) by $m_i v^2/2$ and integrating over velocity we find that ion pressure,

$$p_i \equiv \int d^3v\, f m_i (\mathbf{v} - \mathbf{V}_i)^2/2 \ , \tag{66}$$

evolves according to

$$\left(\frac{3}{2}\right)\frac{\partial}{\partial t}\left[p_i + \left(\frac{1}{2}\right) m_i n_i V_i^2\right] + \nabla \cdot Q_i = \mathbf{V}_i \cdot (\mathbf{F}_e + e n_i \mathbf{E}) + W_e + W_n \tag{67}$$

where

$$\mathbf{Q}_i \equiv \int d^3v\, f \mathbf{v}\, \frac{m_i v^2}{2} \tag{68}$$

is the ion energy flux,

$$W_e \equiv \int d^3v\, C\, \frac{m_i v^2}{2} \tag{69}$$

is the Coulomb energy exchange with electrons, and

$$\begin{aligned} W_n \equiv & \int d^3v\, X(f,g) \frac{m_i v^2}{2} + \mathbf{V}_i \cdot \mathbf{F}_n \\ & + \nu_z \left[p_n + \left(\frac{1}{2}\right) m_i n_n V_n^2\right] - \nu_r \left[p_i + \left(\frac{1}{2}\right) m_i n_i V_i^2\right] \ , \end{aligned} \tag{70}$$

is the energy gained by ions due to inelastic ion-neutral interaction. We denote the first term in (70) by

$$W_x \equiv \int d^3v\, X(f,g)\, \frac{m_i v^2}{2} \ ; \tag{71}$$

it usually dominates the sum. The neutral counterpart to (67) is

$$\left(\frac{3}{2}\right)\frac{\partial}{\partial t}\left[p_n + \left(\frac{1}{2}\right)m_n n_n V_n^2\right] + \nabla\cdot\mathbf{Q}_n = -W_n \ . \tag{72}$$

Here $\mathbf{Q}_n$ is related to the neutral heat flux by

$$\mathbf{Q}_n = \mathbf{q}_n - \left(\frac{5}{2}\right)\mathbf{\Gamma}_n \ . \tag{73}$$

6.2 Perpendicular ion flow

Any change in ion momentum due to neutral interactions is balanced by a corresponding change in neutral momentum. How do the neutrals dispose of their changed momentum? In general, they might accelerate, or they might propagate the momentum change to the walls, by viscous dissipation. In the short mean-free-path regime considered here, the net effect of momentum exchange is to allow the neutral pressure gradient to act on ions:

$$p_i \rightarrow p_i + p_n \ . \tag{74}$$

To see this conclusion explicitly, we add (64) and (65). Since $\mathbf{V}_i \cong \mathbf{V}_n$ while $n_n \ll n_i$, the general result is

$$\frac{\partial}{\partial t}(m_i n_i \mathbf{V}_i) + \nabla\cdot(\mathbf{P}_i + \mathbf{P}_n) - e n_i(\mathbf{E} + c^{-1}\mathbf{V}_i\times\mathbf{B}) = \mathbf{F}_e \ . \tag{75}$$

A sharper result pertains in the short CX mean-free-path regime, where both stresses are approximately isotropic. Note isotropy of the neutral stress results from short CX-mean-free-path,

$$\mathbf{P}_n = I p_n + \mathcal{O}\left(\frac{\lambda_x}{L}\right) \ ,$$

while that of the ion stress is an artifact of small gyroradius ρ:

$$\mathbf{P}_i = I p_i + \mathcal{O}\left(\frac{\rho}{L}\right) \ .$$

By a conventional argument the acceleration and friction terms in (75) are also $\mathcal{O}\left(\frac{\rho}{L}\right)$. Hence, after solving (75) for $\mathbf{V}_i$ we have

$$\mathbf{V}_i = \mathbf{b}V_{\|} + \left(\frac{c}{eBn_i}\right)\mathbf{b}\times\left[en_i\mathbf{E} + \nabla(p_i + p_n)\right] \tag{76}$$

showing that neutrals, although obviously unmagnetized, contribute like a magnetized species to the diamagnetic drift. Since the neutral and ion pressure gradients are opposed in much of the edge region, the observed effect of (76) is diminished ion diamagnetic rotation. It should be emphasized that the effect of CX on $V_{\|}$ has not been considered here.

Since there is no corresponding effect on electron diamagnetism, the perpendicular plasma current is affected by neutrals in the obvious way:

$$\mathbf{J}_{\perp} = \left(\frac{c}{B}\right)\mathbf{b}\times\nabla(p_e + p_i + p_n)\ . \tag{77}$$

One implication of (77) is that experimental estimates of plasma beta must be performed with care whenever neutrals may be present.

A final conclusion from (76) is that neutrals are unlikely to affect ion particle transport, in contradiction to some previous work. The radial particle flux in an axisymmetric system is proportional to the toroidal component of the friction force, F_T. Since $F_{nT} = -(\nabla p_n)_T = -(1/R)\partial p/\partial\zeta$, where R is the major radius and ζ the toroidal angle, the axisymmetric effect of neutrals on Γ_i vanishes exactly. But even with asymmetry the effect appears small, because the flux-surface average will annihilate, or nearly annihilate, $\mathbf{b}\times\nabla p_n$.

Observe next that, according to (75), not just the scalar pressure but the entire stress tensors of neutrals and ions act additively in ion dynamics. This circumstance is significant because measurements of ion viscosity in the tokamak are anomalously high. Thus the question arises as to whether CX can account for anomalous viscosity. From (57) and

$$\mathbf{P}_n = -2\eta_n v_{ti}^{-1}(\nabla\mathbf{V} + \text{ transpose})$$

neutral viscosity is found to be

$$\eta_n = \left(\frac{1}{4}\right)\left(\frac{n_n T_i}{\nu_x}\right)$$

ion viscosity [12] is

$$\eta_i = \left(\frac{3}{10}\right)\left(\frac{n_i T_i}{\Omega_i^2 \tau_i}\right) .$$

Here Ω_i is the ion gyrofrequency and τ_i is the ion-ion collision time as defined by Braginskii [12]. Using typical profiles from TEXT, we find that the quantity

$$\frac{\eta_n}{\eta_i} \approx \left(\frac{n_n}{n_i}\right)\left(\frac{\Omega_i^2 \tau_i}{\nu_x}\right) \tag{78}$$

varies from 10^2 at the center to 10^6 at the wall 30 cm away. Since λ_x varies from 8 cm at the center to 13 cm at the wall, we are still within the window of validity for short λ_x theory. We therefore find that neutral viscosity dominates ion viscosity everywhere, and in the edge region by a large factor.

6.3 Ion energy transport

Similar physics applies to ion energy transport, except that here, in the absence of a conservation law analogous to (29), the effect is large. After CX delivers ion energy to the neutrals, it diffuses rapidly by neutral heat conduction, as described by (56). Since part of the neutral flux is proportional to the ion temperature gradient, the effect of ion-neutral energy exchange will appear as enhanced ion heat conduction together with convection.

To estimate the importance of this heat conduction process, we compare it to the neoclassical ion heat loss,

$$Q_{NC} \sim \nu_i \rho^2 \left(\frac{B}{B_p}\right)^2 \nabla p_i \tag{79}$$

where ν_i is the Coulomb collision frequency for ion-ion collisions, B_p is the poloidal magnetic field and a factor of $(r/R)^{1/2} \sim 1$ is suppressed. The corresponding measure of the new process is Q_n; Eq. (56) provides

$$\frac{Q_n}{Q_{NC}} \sim \left(\frac{n_n}{n_i}\right)\left(\frac{\nu_x}{\nu_i}\right)\left(\frac{\lambda_x}{\rho_p}\right)^2 . \tag{80}$$

This ratio exceeds unity in typical circumstances because its last factor is large.

The explicit calculation begins with the sum of (67) and (72). Since neutral pressure changes no faster than ion pressure,

$$\frac{\partial p_n}{\partial t} \sim \left(\frac{n_n}{n_i}\right) \frac{\partial p_i}{\partial t} \ll \frac{\partial p_i}{\partial t} ,$$

(at least after some relaxation time) and since the speeds V_i and V_n are smaller than either thermal speed, the general result is

$$\left(\frac{3}{2}\right) \frac{\partial}{\partial t} \left[p_i + \left(\frac{1}{2}\right) m_i n_i V_i^2\right] + \nabla \cdot (\mathbf{Q}_i + \mathbf{Q}_n) = \mathbf{V}_i \cdot (\mathbf{F}_e + e n_i \mathbf{E}) + W_e . \tag{81}$$

Here all the terms in (81) are conventional — pertinent to a neutral-free plasma — except Q_n Thus, as anticipated, neutral energy transport simply adds, in the ion energy balance equation, to ion energy transport.

An observation in tokamaks is the isotopic dependence of energy confinement. Global energy confinement time τ_E is known [13] to scale with average isotopic mass number M as

$$\tau_E \approx M^{0.5} .$$

We obtain from (56) a neutral thermal conductivity χ_n that scales as

$$\chi_n \approx M^{-0.5} .$$

We find that χ_n and local τ_E are inversely related anywhere the quantity (80) is large. Again using typical TEXT profiles, we find that Q_n/Q_{NC} ranges from 10 to 10^3 within the outermost 13 cm in TEXT. We therefore find locally that $\tau_E \approx M^{0.5}$ in the edge region. The significance of this local agreement, with regard to global confinement, remains to be investigated.

7. Summary

Taking advantage of the simplicity of the charge-exchange (CX) operator, we have developed a general variational principle for finding the effects of neutrals on plasma transport and rotation. This general variational method has the following advantages: it treats three dimensional plasmas with arbitrary temperature and density profiles; it includes the effects of neutrals on the plasma; it allows for arbitrary CX cross section and mean-free-path; and it provides relatively simple asymptotic formulas for various quantities of interest in limiting parameter regimes. However, this general variational method also has the following disadvantages: it is spatially nonlocal so that the trial functions must include both **x**- and **v**-dependence; and it involves operators that are not self-adjoint, therefore requiring dual trial functions. Nonetheless, in any parameter regime that allows an approximate analytic solution, the general variational principle becomes directly useful in obtaining higher-order information about CX effects from quite simple integrals.

The special case of short CX mean-free-path (λ_x) is found to provide an ordering in which the variational theory is both local and self-adjoint. We find that this short λ_x specialization of the general variational method may be used to obtain expressions for the neutral entropy production from which variationally accurate neutral transport coefficients can be read off by inspection.

The realistic simplification that the product of CX cross section with relative velocity ($\sigma_x|\mathbf{v} - \mathbf{v}'|$) is constant has allowed us, in the case of short λ_x, to avoid the variational approach and to find the neutral distribution function directly. We have thus calculated the neutral particle, heat and momentum fluxes and have presented a full set of neutral transport coefficients. Their form confirms our physical picture of neutrals executing a random walk with step size λ_x.

Our findings about neutral transport combined with analysis of the momentum and energy moments of the ion and neutral kinetic equations leads us to several simple, sensible conclusions about the effects of neutrals on ion fluid behavior and transport.

We find that the neutral stress simply adds to the ion stress in the ion momentum balance equation. Since the ion and neutral pressure gradients oppose each other in much of the edge region, the observed effect on ion perpendicular flow is diminished ion diamagnetic rotation.

Furthermore, because of the effect of neutrals on the perpendicular plasma current (77), experimental estimates of plasma beta must be performed with care whenever neutrals may be present.

CX causes neutral viscosity to contribute directly to ion viscosity. Neutral viscosity is compared to classical perpendicular ion viscosity and is found to dominate everywhere, and in the edge region by a large factor. Thus CX appears to be related to measurements of anomalously high ion viscosity in the tokamak.

Similarly, neutral energy flux simply adds to ion energy flux in the ion energy balance equation. Although ion particle transport is shown to be unaffected by CX, the effect on ion energy transport is enhanced ion heat conduction. The neutral heat flux is compared to the neoclassical ion heat flux (80) and is found to be larger in typical circumstances.

Global energy confinement time τ_E is known [13] to scale with average isotopic mass number M as $\tau_E \approx M^{0.5}$. The present theory predicts this dependence in any region where neutral energy conduction due to CX dominates energy transport.

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