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Abstract

We study analytically the low frequency stability of a long thin annular layer of energetic ions in a background plasma with finite axial and zero azimuthal magnetic field. We find that although the equilibrium is susceptible to the kink instability low mode number perturbations can be stabilized in the limit of $\frac{N_i}{N_b} \rightarrow 0$ when the current layer is close to the maximum field reversal parameter. We also present a brief discussion of the tearing mode stability criteria of such strong current layers with respect to the placement of conducting walls.

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I. Introduction

A long field-reversed configuration of energetic large orbit ions embedded in a charged neutralizing background plasma has been proposed as the basis of a fusion reactor.[1] The energetic ion component consists of a mixture of deuterium and tritium ions having similar velocities and thus directed energies of $\sim 400\,\text{keV}$ and $\sim 600\,\text{keV}$, respectively. The energetic ions move in roughly circular orbits with radial betatron oscillations within an annular layer about the axis of symmetry. The azimuthal current of the energetic ions is large enough to produce reversal of the axial magnetic field $B(r)$ on the inside of the annulus. Due to the rapid variation of B within the annulus the deuterium and tritium orbits overlap. The "temperature" associated with the radial betatron oscillations is of the order of $50\,\text{keV}$. The fusion energy cross section for d-t fusion is near its maximum value at these energies. The energetic ion density, $N_b \sim 10^{14}/\text{cm}^3$, is assumed to be several orders of magnitude larger than the background ion density, $N_i \sim 10^{12}/\text{cm}^3$, in order to reduce background plasma drag of the energetic ions. The background electrons provide charge neutralization and are also hot in order to reduce electron drag. For the mentioned parameters, the fusion time is of the order of 10 s. If the energetic ion "liftime" exceeds the fusion time, an energy multiplication factor $F \equiv \frac{\text{(Fusion probability)(Fusion energy per reaction)}}{\text{Initial ion beam energy}}$ of about 20 is achievable. For such values of F, it is possible to have a modest size fusion reactor without ignition. The energetic ion layer can be maintained by repetitive injection pulses from a series of ion diodes.

A critical issue for the proposed system is the low frequency stability of the equilibrium configuration. Annular equilibria produced by relativistic electrons embedded in a dense plasma, the well-known Astron system, have been generated and observed to exhibit a low-frequency precessional instability predicted by Furth.[2] This instability is found to be stabilized by conducting walls.[3] A number of further studies has been made of the low-frequency stability of kink modes of high-energy, high-current particle rings embedded in a dense, low-temperature background plasma.[4–9] In the case of large aspect ratio, circular poloidal cross-section rings, the kink mode of azimuthal mode number $\ell \geq 1$ is found to be stable for $\Omega_\beta < \Omega_0$, where Ω_β is the self-magnetic-field betatron frequency and Ω_0 is the circulation frequency of the ring ions.[4] For circular cross section rings the radial (Ω_r) and axial (Ω_z) betatron frequencies are equal ($\Omega_r = \Omega_z = \Omega_\beta$). The kink mode $\ell = 1$ corresponds to a rigid tilting of the ring. For noncircular cross section rings the condition for tilt stability is $\Omega_z < \Omega_0$ (Ref. 9). Because Ω_z decreases with increasing axial length of the ring, while Ω_0 changes relatively little, the ring tends to be stabilized by axial elongation. However, even for elongated rings or layers there may a kink instability with $\ell \geq 2$. The condition for stability to these kink modes is $\eta_s < 3$, where $\eta_s = \left|\frac{r_0}{B(r_0)} \frac{\partial B}{\partial r}\right|$ is the self-field index[10] (r_0 is the layer radius). This prediction has been verified in numerical simulation studies.[11,12] The above kink stability criterion is necessary but not sufficient, and thus it is not clear whether a violation of this criterion implies instability. It may be noted that earlier investigations[13] of the stability of equilibria containing a small component of energetic particles have found stability "bands" in parameter space despite violation of the magnetohydrodynamic (MHD) stability criterion. Further, an essential aspect of the system proposed by Rostoker[1] is the very low ion density of the background plasma relative to the energetic ion density, which is necessary to decrease the drag on the energetic ions. In this limit, the background ion inertia (which has an important role in the unstable "kink" perturbations) is negligible, and therefore the nature of the kink instability is likely to be changed.

In this paper, we study analytically the low frequency stability of a long thin annular layer of energetic ions in a background plasma with finite axial and zero azimuthal magnetic field. We consider only flute perturbations in which there is no variation along the magnetic field. We focus primarily on kink modes with azimuthal mode numbers $\ell \geq 2$. We find that the equilibrium is susceptible to the kink instability although low mode number perturbations can be stabilized in the limit of $\frac{N_i}{N_b} \to 0$ and a strong current layer where almost complete field reversal is achieved. However, with a strong current layer the system is susceptible to tearing instability and we therefore also present (in the Appendix) a brief discussion of tearing mode ($\ell = 0$) stabilization by the placement of conducting walls in close proximity to the boundaries of the annular layer.

In Sec. II, we derive the approximate Hamiltonian of the energetic ion beam motion. In Sec. III, we discuss the energetic ion beam response to low frequency flute perturbations, and we construct a quadratic variational form of the eigenmode equations. Parallel electric field perturbations can be neglected, while the parallel magnetic field perturbations within the annular layer are considered to be proportional to the gradient of the equilibrium magnetic field (the "rigid" displacement approximation) since such perturbations tend to minimize the compressional magnetic energy. We view the quadratic form as a dispersion functional, and in Sec. IV we obtain approximate dispersion relations by substituting appropriate trial functions in the quadratic form. We find unstable modes in the limit of high and low background ion densities. In Sec. V, we discuss our results and suggest modifications of the equilibrium which may lead to more stable configurations.

II. Equilibrium

We consider an equilibrium configuration consisting of a long cylindrical annulus of energetic ions encircling the axis of symmetry and undergoing radial betatron oscillations in a field reversed magnetic field (Fig. 1). The energetic ion component is assumed to be charged neutralized by a cold background plasma. The annular region is bounded by conducting walls at $r = r_w$ and $r = r_c$ in order to stabilize $l = 0$ tearing modes.

A. Ion orbits

The equilibrium Hamiltonian of the energetic ion beam is

$$H_0 = \frac{p_r^2}{2m_b} + \frac{p_z^2}{2m_b} + V(p_\theta, r) \ , \tag{1}$$

where

$$V(p_\theta, r) = \frac{\left(p_\theta - \frac{e}{c}\psi(r)\right)^2}{2m_b r^2}$$

is the effective potential. The energy (H_0) as well as the axial (p_z) and azimuthal (p_θ) canonical momenta are constants of the motion. The equilibrium vector potential is $\mathbf{A}_0 = \frac{1}{r}\psi(r)\widehat{\theta}$, and the magnetic field is $\mathbf{B} = B(r)\widehat{z}$ where $B(r) = \frac{1}{r}\frac{\partial\psi}{\partial r}$.

Let $\frac{\partial V(p_\theta, r)}{\partial r} = 0$ at $r = r_\beta$. We consider only the "betatron" root

$$\left(p_\theta - \frac{e}{c}\psi(r_\beta)\right) + \frac{e}{c}r_\beta^2 B(r_\beta) = 0 \ .$$

This equation defines

$$r_\beta = r_\beta(p_\theta) \ .$$

Thus, those ions with energy H_0 and canonical momenta p_θ, p_z such that

$$p_r^2 = 2m_b\left(H_0 - V(p_\theta, r_\beta)\right) - p_z^2 \ ,$$

$$= 0$$

will describe circular orbits of radius r_β about the axis of symmetry.

If we expand $V(p_\theta, r)$ about $r = r_\beta$, we obtain the following approximate Hamiltonian for ions with small radial excursions about the betatron radius

$$H_0 = \frac{p_r^2}{2m_b} + \frac{p_z^2}{2m_b} + \frac{\left(p_\theta - \frac{e}{c}\psi(r_\beta)\right)^2}{2m_b r_\beta^2} + \frac{(r - r_\beta)^2}{2} m_b \Omega_\beta^2(r_\beta) \ , \tag{2}$$

where

$$\Omega_\beta^2(r_\beta) = \left[\Omega(r)\frac{\partial}{\partial r} r\Omega(r)\right]_{r=r_\beta}$$

$$\Omega(r) = \frac{eB(r)}{m_b c}$$

and we have used the identity $\frac{\partial r_\beta}{\partial p_\theta} = -\frac{\Omega(r_\beta)}{m_b r_\beta \Omega_\beta^2(r_\beta)}$.

The radial motion is simple harmonic with frequency equal to the betatron frequency $\Omega_\beta(r_\beta)$. The equilibrium orbits may therefore be approximated by

$$r - r_\beta = \delta r_\beta \sin\phi \tag{3a}$$

$$p_r = \delta r_\beta m \Omega_\beta \cos\phi \tag{3b}$$

$$\frac{d\phi}{dt} = \Omega_\beta \tag{3c}$$

$$\begin{aligned} \frac{d\theta}{dt} &= \frac{\partial H}{\partial p_\theta} \\ &= -\Omega(r_\beta) - \frac{(\delta r_\beta)^2 \Omega}{2 r_\beta \Omega_\beta} \frac{\partial \Omega_\beta}{\partial r_\beta} + \frac{\delta r_\beta}{r_\beta} \Omega \sin\phi + \cdots \ , \end{aligned} \tag{3d}$$

where the radial oscillation amplitude δr_β is given by

$$m_b^2 \Omega_\beta^2 (\delta r_\beta)^2 \equiv 2 m_b H_0 - p_z^2 - \frac{\left(p_\theta - \frac{e}{c}\psi(r_\beta)\right)^2}{r_\beta^2}$$

and we assume $(\delta r_\beta / r_\beta)^2 \ll 1$.

B. Distribution function

We consider the energetic ion beam to be described by the distribution function

$$F_b(H_0, p_\theta) = \frac{r_0 N_b}{2\pi m_b} \delta(H_0 - \varepsilon_0)\, \delta(p_\theta - p_0 - \delta p) \ , \tag{4}$$

where the beam energy ε_0 is related to p_0 by

$$\varepsilon_0 = V(p_0, r_0)$$

and

$$r_0 = r_\beta(p_0) \ .$$

For this distribution function, the equilibrium ion beam density is

$$N = \int d^3p\, F_b = \frac{N_b r_0}{2\pi} \sum_{p_r} \int dp_\theta\, dp_z \int_{\widehat{H}_0}^{\infty} \frac{dH_0}{r\,|p_r|} \delta(H_0 - \varepsilon_0)\, \delta(p_\theta - p_0 - \delta p)$$

$$= \frac{N_b r_0}{r} \int dp_\theta \delta\left(p_\theta - p_0 - \delta p\right) \Theta\left(p_{0z}^2\right) , \tag{5}$$

where

$$|p_r| = \left[2m_b H_0 - p_z^2 - 2m_b V\left(p_\theta, r\right)\right]^{1/2}$$

$$\widehat{H}_0 = \frac{p_z^2}{2m_b} + V\left(p_\theta, r\right)$$

$$p_{0z}^2 = 2m_b \varepsilon_0 - 2m_b V\left(p_\theta, r\right)$$

$$\Theta\left(p_{0z}^2\right) = \begin{cases} 1 & p_{0z}^2 \geq 0 \\ 0 & p_{0z}^2 < 0 \ . \end{cases}$$

Expanding about $p_\theta = p_0$, $r = r_0$, we obtain for p_{0z}^2

$$p_{0z}^2 = 2m_b \Omega\left(r_0\right)\left(p_\theta - p_0\right) - \left(r - r_0\right)^2 m_b^2 \Omega_\beta^2\left(r_0\right) + \cdots \ .$$

Thus, the ion beam density is finite within an annulus of thickness Δ

$$N = \begin{cases} \dfrac{N_b r_0}{r}, & \left(r - r_0\right)^2 < (\Delta/2)^2 \\ 0, & \text{otherwise} , \end{cases} \tag{6}$$

where

$$(\Delta/2)^2 = \frac{2\delta p \Omega\left(r_0\right)}{m_b \Omega_\beta^2\left(r_0\right)} \ll r_0^2 \ .$$

The ion beam current $J_{0\theta}$ is

$$J_{0\theta} = \int d^3p\, F_b \frac{e\left(p_\theta - \frac{e}{c}\psi\right)}{m_b r} \approx -\int d^3p\, F_b e r_\beta \Omega\left(r_\beta\right)\left\{1 + \frac{\left(r - r_\beta\right)^2}{2r_\beta^2} \frac{\Omega_\beta^2}{\Omega^2\left(r_\beta\right)}\right\}$$
$$\approx -N e r_0 \Omega\left(r_0\right) \ . \tag{7}$$

The ion beam is assumed to carry the total current and thus the magnetic field inside the annulus may be approximated by a linear variation

$$B(r) = B\left(r^-\right) + B\left(r_0\right) \frac{\omega_b^2 v_{0\theta}}{\Omega(r_0) c^2}\left(r - r^-\right) , \qquad r^+ > r > r^- , \tag{8}$$

where $\omega_b^2 = \frac{4\pi N_b e^2}{m_b}$, $\Omega(r_0) = \frac{eB(r_0)}{m_b c}$, $v_{0\theta} = r_0\Omega(r_0)$, and r^+, r^- are the inner and outer radius of the annulus ($\Delta = r^+ - r^-$).

We define the field reversal parameter $\delta\widehat{\Omega}$ to be

$$\delta\widehat{\Omega} \equiv \frac{B\left(r^+\right) - B\left(r^-\right)}{B\left(r_0\right)} = \frac{\omega_b^2 r_0 \Delta}{c^2} , \tag{9}$$

where $B\left(r_0\right) = \left(B\left(r^+\right) + B\left(r^-\right)\right)/2$. The field reversal factor is

$$\zeta = \frac{B\left(r^+\right) - B\left(r^-\right)}{B\left(r^+\right)} = \frac{2\delta\widehat{\Omega}}{2 + \delta\widehat{\Omega}} .$$

With field reversal, $B\left(r^-\right) < 0$ (we adopt the convention that $B\left(r^+\right) > 0$) and $\delta\widehat{\Omega} > 2$, $\zeta > 1$.

The equilibrium density of the charge neutralizing background electrons and ions (assumed to be cold) are N_e, N_i where

$$N_i + N_b = N_e .$$

III. Linear Dispersion Relation

A. Perturbed fields

We are interested in the stability of the equilibrium to low frequency (ω) perturbations with $\omega \ll c/r_0$. We consider flute perturbations with time and azimuthal angular dependence given by $\sim e^{il\theta - i\omega t}$, and we neglect parallel electric field perturbations.

We find it convenient to choose a gauge in which the scalar potential $\widetilde{\phi} = 0$ and the vector potential $\widetilde{\mathbf{A}}$ is given by

$$\begin{aligned} \widetilde{\mathbf{A}} &= \nabla\xi(r)e^{il\theta - i\omega t} + \chi(r)e^{il\theta - i\omega t}\nabla r \\ &= \left[\left(\frac{\partial\xi}{\partial r} + \chi\right)\widehat{r} + il\frac{\xi}{r}\widehat{\theta}\right] e^{il\theta - i\omega t} , \end{aligned}$$

where we express $\widetilde{\mathbf{A}}$ in terms of field variables $\xi(r)$ and $\chi(r)$. $\widehat{r}$ and $\widehat{\theta}$ are unit vectors in the radial and azimuthal directions respectively.

The perturbed electromagnetic fields are

$$\widetilde{E}_r = \frac{i\omega}{c}\left(\frac{\partial \xi}{\partial r} + \chi\right) e^{il\theta - i\omega t}$$

$$\widetilde{E}_\theta = -\frac{\omega l \xi}{cr} e^{il\theta - i\omega t}$$

$$\widetilde{B}_z = -\frac{il\chi}{r} e^{il\theta - i\omega t} .$$

The magnetic field perturbation is proportional to χ while the curl-free part of the electric field perturbation is due to ξ.

In the vacuum region, $r^- > r > 0$ and $r_w > r > r^+$, outside the annulus of energetic ion beams, the perturbed magnetic field (in the limit of low frequency $\left|\frac{\omega r}{lc}\right| \ll 1$) is negligibly small $\left(\chi \approx \frac{\omega^2 r^2}{c^2 \ell^2}\frac{\partial \xi}{\partial r} \ll \frac{\partial \xi}{\partial r}\right)$, and the field variable $\xi(r)$ is approximately determined by

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \xi}{\partial r} - \frac{\ell^2}{r^2}\xi = 0 . \tag{10}$$

The solution of this equation is

$$\xi(r) = \xi\left(r^-\right) \frac{\left[\left(\frac{r}{r^-}\right)^l - \left(\frac{r^-}{r}\right)^l \left(\frac{r_c}{r^-}\right)^{2l}\right]}{\left[1 - \left(\frac{r_c}{r^-}\right)^{2l}\right]} , \qquad r^- > r > r_c \tag{11a}$$

$$\xi(r) = \xi\left(r^+\right) \frac{\left[\left(\frac{r^+}{r}\right)^l - \left(\frac{r}{r^+}\right)^l \left(\frac{r^+}{r_w}\right)^{2l}\right]}{\left[1 - \left(\frac{r^+}{r_w}\right)^{2l}\right]} , \qquad r_w > r > r^+ , \tag{11b}$$

where $\xi(r)$ is continuous at $r = r^+$, r^- and $\xi\left(r_w\right) = 0$, $\xi\left(r_c\right) = 0$, due to conducting walls at $r = r_w$, $r = r_c$.

Inside the annulus, $r^+ > r > r^-$, we assume $\chi(r)$ to have the form

$$\chi = \chi_0 r \frac{\partial B}{\partial r} , \tag{12}$$

where χ_0 is constant. Thus the magnetic field perturbation $\tilde{B}_z$ is proportional to $\partial B/\partial r$

$$\tilde{B}_z = -il\chi_0 \frac{\partial B}{\partial r} e^{il\theta - i\omega t} .$$

This perturbation is exactly a "rigid" displacement of the annulus for $l = 1$, and corresponds to the equilibrium magnetic field moving with the displacement of the layer. It is hereafter referred to as the rigid mode approximation.[2,14]

The rigid mode approximation can be justified by an examination of the quadratic variational form of the eigenmode equations. It can be shown that in the limit of $\Omega_\beta^4 > \Omega^2(r_\beta)\left(\omega^2 + \ell^2\Omega^2(r_\beta)\right)$, the quadratic form is dominated by a "large" term proportional to $\left(\partial\chi_0/\partial r\right)^2$, or more physically a "large" term proportional to the magnetic compressional energy. Thus, to minimize the magnetic compressional energy, we need to take to lowest order (in the layer) χ_0 constant independent of r. Because of its complicated structure, we will not write down the complete quadratic form including terms proportional to $\left(\partial\chi_0/\partial r\right)^2$. Instead we take χ_0 to be constant at the outset, and in Sec. IIIC, we construct a simplified quadratic variational form valid in the limit where the rigid mode approximation is applicable.

We also find it convenient to introduce a new field variable $C_0(r)$ inside the annulus in terms of which $\xi(r)$ may be expressed

$$\xi(r) = C_0(r) - \chi_0 r B(r) . \tag{13}$$

Thus, inside the annulus

$$\frac{\partial \xi(r)}{\partial r} = \frac{\partial C_0}{\partial r} - \chi_0 B - \chi_0 r \frac{\partial B}{\partial r} \tag{14}$$

and the perturbed radial electric field is

$$\tilde{E}_r = \frac{i\omega}{c}\left(\frac{\partial C_0}{\partial r} - \chi_0 B\right) e^{il\theta - i\omega t} .$$

At the boundaries of the annulus

$$\xi\left(r^\pm\right) = C_0\left(r^\pm\right) - \chi_0 r^\pm B\left(r^\pm\right) . \tag{15}$$

We assume (and justify later) that $C_0(r)$ varies slowly inside the annulus: $\partial C_0/\partial r \ll C_0/\Delta$. If this inequality is not satisfied, the perturbation of the background plasma would result in large radial electric fields, and the energy in the radial electric field would not be minimized.

We now proceed to evaluate the perturbed currents of the beam and background plasma induced by these perturbed fields.

B. Perturbed currents

The perturbed ion beam distribution function is determined by the linearized Liouville equation

$$\frac{\partial f}{\partial t} + [f, H_0] + [F, H_1] = 0 \ , \tag{16}$$

where the Poisson brackets are defined in the usual way with respect to the canonical variables $(p_i \equiv p_r, p_\theta, p_z,\ q_i \equiv r, \theta, z)$, and F is the beam equilibrium distribution function (we delete the subscript 'b' for convenience).

$$[f, g] = \sum_i \left\{ \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right\} \ ,$$

and the perturbed Hamiltonian H_1 can be written as follows:

$$H_1 = -\frac{e}{c}\mathbf{v}\cdot\mathbf{A}_\perp = -\frac{e}{mc}\left[p_r\frac{\partial \xi}{\partial r} + \frac{\left(p_\theta - \frac{e}{c}\psi\right)}{r}\frac{il\xi}{r} + p_r \chi_0 r \frac{\partial B}{\partial r} \right] e^{il\theta - i\omega t}$$

$$= -\frac{e}{c}\frac{d}{dt}\xi e^{il\theta - i\omega t} - i\frac{\omega e}{c}\xi e^{il\theta - i\omega t} - \frac{e}{mc}p_r \chi_0 r \frac{\partial B}{\partial r} e^{il\theta - i\omega t} \ . \tag{17}$$

The solution of this equation is

$$f = [F, W] \ , \tag{18}$$

where $W \equiv -\int^t dt' H_1 \left(p_r', r', \theta', t'\right)$ and the integration in time is along the equilibrium phase space trajectory

$$r' - r = \delta r_\beta \left[\sin\left(\phi + \Omega_\beta \left(t' - t\right)\right) - \sin\phi\right]$$

$$\theta' - \theta = -\Omega(r_\beta)(t'-t) - \frac{\delta r_\beta}{r_\beta}\frac{\Omega(r_\beta)}{\Omega_\beta}\left[\cos\left(\phi + \Omega_\beta(t'-t)\right) - \cos\phi\right]$$

$$p'_r = m\Omega_\beta \delta r_\beta \cos\left(\phi + \Omega_\beta(t'-t)\right) .$$

Since $C_0(r)$ is assumed to vary slowly through the annular layer and $B(r)$ varies linearly, we can make the following expansions to evaluate the time integration of H_1

$$C_0(r') = C_0(r) + (r'-r)\frac{\partial C_0}{\partial r} + \cdots$$

$$r'B(r') = rB(r) + (r'-r)\frac{\partial}{\partial r}rB(r) + \cdots .$$

Hence

$$\xi(r') = \xi(r) + (r'-r)\frac{\partial \xi}{\partial r} + \cdots .$$

We then obtain for W

$$W(p_r, p_\theta, r, \theta, t) = \widehat{W}(p_r, p_\theta, r)\, e^{il\theta - i\omega t} , \tag{19}$$

where

$$\begin{aligned}\widehat{W} &= g_1(p_r, r_\beta(p_\theta), r)\left[C_0(r) - \chi_0 rB(r)\right] \\ &+ g_2(p_r, r_\beta(p_\theta), r)\left[l\Omega(r_\beta) r\chi_0 \frac{\partial B}{\partial r} + \omega\left(\frac{\partial C_0}{\partial r} - \chi_0 B\right)\right]\end{aligned} \tag{20}$$

and

$$g_1(p_1, r_\beta(p_\theta), r) \equiv \frac{e}{c}\frac{l\Omega(r_\beta)}{(\omega + l\Omega(r_\beta))} + \frac{e}{c}\frac{\left[(\omega + l\Omega(r_\beta))(r - r_\beta) - \frac{ip_r}{m}\right]\omega l\Omega(r_\beta)}{r_\beta(\omega + l\Omega(r_\beta))\left(\Omega_\beta^2 - (\omega + l\Omega(r_\beta))^2\right)}$$

$$g_2(p_r, r_\beta(p_\theta), r) \equiv \frac{e}{c}\frac{\left[\Omega_\beta^2(r - r_\beta) - \frac{ip_r}{m}(\omega + l\Omega(r_\beta))\right]}{(\omega + l\Omega(r_\beta))\left(\Omega_\beta^2 - (\omega + l\Omega(r_\beta))^2\right)} .$$

The perturbed ion beam currents are determined by

$$\widetilde{\mathbf{J}}_b = \int d^3p \frac{e}{m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}_0\right) f - \int d^3p \frac{e^2}{mc}\widetilde{\mathbf{A}} F , \tag{21}$$

where $d^3p = \frac{1}{r} dp_r\, dp_\theta\, dp_z$.

Substituting for f, we obtain for the radial perturbed current

$$\begin{aligned}
\widetilde{J}_{br} &= \int d^3p \frac{e}{m} p_r [F, W] - \int d^3p \frac{e^2}{mc} \widetilde{A}_r F \\
&= \int d^3p \frac{e}{m} p_r \left\{ \frac{\partial}{\partial r} F \frac{\partial W}{\partial p_r} - \frac{\partial}{\partial p_r} F \frac{\partial W}{\partial r} + \frac{\partial}{\partial \theta} F \frac{\partial W}{\partial p_\theta} - \frac{\partial}{\partial p_\theta} F \frac{\partial W}{\partial \theta} \right\} \\
&- \int d^3p \frac{e^2}{mc} \widetilde{A}_r F \ .
\end{aligned} \tag{22}$$

Since F is even in p_r and $\partial W / \partial p_r$ is independent of p_r, the first term is zero. The fourth term yields zero on integration. The third term is smaller than the second term by $\Delta / r_0 \ll 1$. Thus, $\widetilde{J}_{br}$ may be approximated by

$$\begin{aligned}
\widetilde{J}_{br} &= \int d^3p\, F \frac{\partial W}{\partial r} - \int d^3p \frac{e^2}{mc} \widetilde{A}_r F = e^{il\theta - i\omega t} \int d^3p \frac{e^2}{mc} F \frac{(\omega + l\Omega(r_\beta))\, \Psi(C_0, \chi_0, r, r_\beta)}{\left(\Omega_\beta^2 - (\omega + l\Omega(r_\beta))^2\right)} \\
&+ \text{higher order terms in } \frac{\Delta}{r_0} ,
\end{aligned} \tag{23}$$

where

$$\begin{aligned}
\Psi(C_0, \chi_0, r, r_\beta) &\equiv l\Omega(r_\beta)\, r \frac{\partial B}{\partial r} \chi_0 + \omega \left(\frac{\partial C_0}{\partial r} - \chi_0 B \right) \\
&+ \frac{\omega l \Omega(r_\beta)}{r_\beta (\omega + l\Omega(r_\beta))} (C_0 - \chi_0 r B) \ .
\end{aligned} \tag{24}$$

Similarly, we obtain for the azimuthal perturbed current

$$\begin{aligned}
\widetilde{J}_{b\theta} &= \int d^3p \frac{e}{mr} \left(p_\theta - \frac{e}{c}\psi \right) [F, W] - \int d^3p \frac{e^2}{mc} \widetilde{A}_\theta F \\
&= \int d^3p \frac{e}{mr} \left(p_\theta - \frac{e}{c}\psi \right) \frac{\partial}{\partial r} F \frac{\partial W}{\partial p_r} + \int d^3p \frac{e}{mr} \left(p_\theta - \frac{e}{c}\psi \right) ilF \frac{\partial W}{\partial p_\theta} \\
&+ \int d^3p \frac{e}{m} \frac{il}{r} FW - \int d^3p \frac{e^2}{mc} \widetilde{A}_\theta F
\end{aligned}$$

$$= -e^{il\theta - i\omega t} \int d^3p \frac{e^2}{mc} F \frac{il\omega^2}{(\omega + l\Omega(r_\beta))^2} \frac{(C_0 - \chi_0 r B)}{r}$$

$$+ e^{il\theta - i\omega t} \int d^3p \frac{e^2}{mc} \frac{\left(p_\theta - \frac{e}{c}\psi\right)}{mr} \left[\frac{il\Omega(r_\beta)}{r_\beta(\omega + l\Omega(r_\beta))} - i\frac{\partial}{\partial r} \right] \frac{F\Psi(C_0, \chi_0, r, r_\beta)}{\Omega_\beta^2 - (\omega + l\Omega(r_\beta))^2}$$

$$+ \text{higher order terms in } \frac{\Delta}{r_0}, \tag{25}$$

where we have made use of the relations

$$\frac{\partial r_\beta}{\partial p_\theta} = -\frac{\Omega}{m r_\beta \Omega_\beta^2}$$

$$\frac{\partial \Omega}{\partial p_\theta} = -\frac{\Omega}{m r_\beta \Omega_\beta^2} \frac{\partial \Omega}{\partial r_\beta} = -\frac{1}{m r_\beta^2}\left(1 - \frac{\Omega^2}{\Omega_\beta^2}\right)$$

$$\frac{\left(p_\theta - \frac{e}{c}\psi(r)\right)}{mr} = -r_\beta \Omega(r_\beta) + \cdots .$$

The perturbation of the cold background plasma produces perturbed currents given by

$$\tilde{\mathbf{J}}_p = i\frac{\omega}{c} \sum_j \frac{N_j e_j^2}{m_j\left(\omega^2 - \Omega_j^2\right)} \left\{ \Omega_j \hat{z}_\times \widetilde{\mathbf{A}} + i\omega \widetilde{\mathbf{A}} \right\}$$

$$= \sum_j \frac{N_j e_j^2}{m_j c \left(\omega^2 - \Omega_j^2\right)} \left\{ \hat{r} \left[-\omega^2 \left(\frac{\partial C_0}{\partial r} - \chi_0 B \right) + \frac{\omega l \Omega_j}{r} (C_0 - \chi_0 r B) \right] \right.$$

$$\left. + \hat{\theta} \left[i\omega\Omega_j \left(\frac{\partial C_0}{\partial r} - \chi_0 B \right) - \frac{il\omega^2}{r} (C_0 - \chi_0 r B) \right] \right\}, \tag{26}$$

where the sum is over the electron ($j = e$) and ion ($j = i$) species.

C. Quadratic form of eigenmode equations

Substituting the perturbed currents in Maxwell's equation, we obtain the eigenmode equations

$$\nabla_\times \left(\nabla_\times \widetilde{\mathbf{A}} \right) + \frac{1}{c^2} \frac{\partial^2 \widetilde{\mathbf{A}}}{\partial t^2} = \frac{4\pi}{c} \left(\tilde{\mathbf{J}}_b + \tilde{\mathbf{J}}_p \right) . \tag{27}$$

These eigenmode equations can be written more compactly as a quadratic variational form. If we multiply Eq. (27) by the adjoint function $\widetilde{\mathbf{A}}_{\perp}^{+} = \widehat{\mathbf{A}}_{\perp}^{+} e^{-il\theta + i\omega t}$, and integrate over space, we obtain

$$
\begin{aligned}
L\left(C_0^+, \chi_0^+, C_0, \chi_0\right) &= \frac{1}{4\pi}\int d^3r \left\{\widetilde{B}_z \widetilde{B}_z^+ - \frac{\omega^2}{c^2}\widetilde{\mathbf{A}}\cdot\widetilde{\mathbf{A}}^+\right\} - \frac{1}{c}\int d^3r \widetilde{\mathbf{J}}_p \cdot \widetilde{\mathbf{A}}^+ - \frac{1}{c}\int d^3r \widetilde{\mathbf{J}}_b \cdot \widetilde{\mathbf{A}}^+ \\
&= \frac{1}{4\pi}\int d^3r\, \ell^2 \chi_0^2 \left(\frac{\partial B}{\partial r}\right)^2 - \frac{\omega^2}{4\pi c^2}\int d^3r \left[\left(\frac{\partial C_0}{\partial r} - \chi_0 B\right)^2 + \frac{\ell^2}{r^2}\left(C_0 - \chi_0 r B\right)^2\right] \\
&+ L_p\left(C_0, \chi_0\right) + L_b\left(C_0, \chi_0\right) \ ,
\end{aligned} \tag{28}
$$

where

$$
\begin{aligned}
L_p\left(C_0, \chi_0\right) &\equiv -\frac{1}{c}\int d^3r \widetilde{\mathbf{J}}_p \cdot \widetilde{\mathbf{A}}^+ \\
&= \int d^3r \sum_j \frac{N_j e_j^2}{m_j c^2\left(\omega^2 - \Omega_j^2\right)}\left\{\left[\omega\left(\frac{\partial C_0}{\partial r} - \chi_0 B\right) - \frac{l\Omega_j}{r}\left(C_0 - \chi_0 r B\right)\right]^2\right. \\
&\left. + \frac{\ell^2\left(\omega^2 - \Omega_j^2\right)}{r^2}\left(C_0 - \chi_0 r B\right)^2\right\}
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
L_b\left(C_0, \chi_0\right) &\equiv -\frac{1}{c}\int d^3r\, \widetilde{\mathbf{J}}_p \cdot \widetilde{\mathbf{A}}^+ = \left\langle \frac{e^2}{mc^2} F \frac{\omega^2}{\left(\omega + l\Omega\left(r_\beta\right)\right)^2} \frac{\ell^2}{r^2}\left(C_0 - \chi_0 r B\right)^2\right\rangle \\
&+ \left\langle \frac{e^2}{mc^2} \frac{F}{\left(\Omega_\beta^2 - \left(\omega + l\Omega\left(r_\beta\right)\right)^2\right)} \frac{\left(p_\theta - \frac{e}{c}\psi(r)\right)}{mr^2\Omega\left(r_\beta\right)} \Psi^2\left(C_0, \chi_0, r, r_\beta\right)\right\rangle \\
&+ \text{higher order terms in} \frac{\Delta}{r_0} \ .
\end{aligned} \tag{30}
$$

The angular brackets denote integration over the phase space variables $\langle ( ) \rangle \equiv \int d^3p\, d^3r( )$. We have deleted the superscript $+$ on the functions $\widehat{\mathbf{A}}^+$ since $\widehat{\mathbf{A}}^+ = \widehat{\mathbf{A}}$. This is due to the symmetry of the quadratic form: $\widehat{\mathbf{A}}^+$ are solutions of the same eigenmode equations with the same boundary conditions as $\widehat{\mathbf{A}}$. The eigenmode equations are the Euler-Lagrange equations obtained from first variations of the quadratic form with respect to C_0 and X_0 and are identical to Eq. (27).

We do not attempt to solve these equations exactly. Instead, we consider our variational quadratic form to be a dispersion functional, and for $C_0(r)$ we substitute an approximate solution based on the thinness of the plasma layer and thereby obtain the dispersion relation for the eigenvalue ω.

Furthermore, we simplify the analysis by considering the limit of large betatron frequency $\Omega_\beta^4 > \Omega^2(r_\beta)(\omega^2 + \ell^2\Omega^2(r_\beta))$.

Since

$$\frac{1}{\left(\Omega_\beta^2 - (\omega + l\Omega(r_\beta))^2\right)} = \frac{1}{\left(\Omega_\beta^2 - \Omega^2(r_\beta)\right)}\left\{1 + \frac{(\omega + l\Omega(r_\beta))^2 - \Omega^2(r_\beta)}{\left(\Omega_\beta^2 - (\omega + l\Omega(r_\beta))^2\right)}\right\}$$

and

$$\frac{\frac{\partial B}{\partial r}}{\Omega_\beta^2 - \Omega^2(r_\beta)} = \frac{B(r_\beta)}{r_\beta \Omega^2(r_\beta)},$$

we can approximate the beam contribution to the quadratic form in the limit of large betatron frequency as follows:

$$L_b(C_0, X_0) = \left\langle F\frac{e^2}{mc^2}\frac{\omega^2}{(\omega + l\Omega(r_\beta))^2}\frac{\ell^2}{r^2}(C_0 - X_0 rB)^2\right\rangle$$

$$+ \left\langle \frac{e}{c}F\frac{\left(p_\theta - \frac{e}{c}\psi\right)}{mr}\ell^2\frac{\partial B}{\partial r}X_0^2\right\rangle - \left\langle Fm\ell^2X_0^2\left[(\omega + l\Omega(r_\beta))^2 - \Omega^2(r_\beta)\right]\right\rangle$$

$$- \left\langle F\frac{e}{c}2lX_0\left[\omega\left(\frac{\partial C_0}{\partial r} - X_0B\right) + \frac{\omega l\Omega(r_\beta)}{r_\beta(\omega + l\Omega(r_\beta))}(C_0 - X_0 rB)\right]\right\rangle$$

$$\approx -\frac{1}{4\pi}\int d^3r\,\ell^2\left(\frac{\partial B}{\partial r}\right)^2 X_0^2$$

$$+\int d^3r \left\{ \frac{N_b e^2}{m_b c^2} \frac{\omega^2}{(\omega + l\Omega_0)^2} \frac{\ell^2}{r^2} (C_0 - \chi_0 r B)^2 - N_b m_b \ell^2 \left[(\omega + l\Omega_0)^2 - \Omega_0^2 \right] \chi_0^2 \right.$$

$$\left. - N_b \frac{e}{c} 2l\chi_0 \left[\omega \left(\frac{\partial C_0}{\partial r} - \chi_0 B \right) + \frac{\omega l \Omega_0}{(\omega + l\Omega_0) r_0} (C_0 - \chi_0 r B) \right] \right\} , \qquad (31)$$

where

$$\Omega_0 = \Omega(r_\beta)_{p_\theta = p_0 + \delta_p} = \Omega(r_0) - \frac{\Delta^2}{8r_0} \frac{\partial}{\partial r_0} \Omega(r_0) + \cdots .$$

Thus the quadratic form [Eq. (28)] may now be approximated by

$$r_0^4 L (C_0, \chi_0) = \frac{r_0^2}{\Delta^2} L_0 (C_0, \chi_0) + \frac{r_0}{\Delta} L_1 (C_0, \chi_0) + L_2 (C_0, \chi_0)$$

$$- \frac{\omega^2 r_0^4}{4\pi c^2} \int d^3r \left[\left(\frac{\partial \xi}{\partial r} \right)^2 + \frac{\ell^2}{r^2} \xi^2 \right] , \qquad (32)$$

where $\int d^3r$ denotes integration over the vacuum region

$$L_0 (C_0, \chi_0) \equiv \int d^3r \, a_1(r) \left(\frac{\partial C_0}{\partial x} \right)^2 \qquad (33)$$

$$L_1 (C_0, \chi_0) \equiv - \int d^3r \, 2a_2(r) C_0 \frac{\partial C_0}{\partial x} + \int d^3r \, 2a_3(r) \widehat{\chi}_0 \frac{\partial C_0}{\partial x} \qquad (34)$$

$$L_2 (C_0, \chi_0) = \int d^3r \, \alpha_0 C_0^2 + \int d^3r \, 2\alpha_1 C_0 \widehat{\chi}_0 + \int d^3r \, \alpha_2 \widehat{\chi}_0^2 \qquad (35)$$

$$x \equiv \frac{(r - r_0)}{\Delta}$$

$$\widehat{\chi}_0 \equiv \chi_0 B_0 r_0$$

$$B_0 = B(r_0)$$

$$a_1(r) = \sum_j \frac{N_j e_j^2 r_0^2}{m_j c^2} \frac{\omega^2}{\left(\omega^2 - \Omega_j^2 \right)} = - \frac{N_i e^2 r_0^2}{m_i c^2} \frac{\omega^2 \left(\Omega_i^2 - \omega^2 / b \right)}{\Omega_i^2 \left(\Omega_i^2 - \omega^2 \right)} + O \left(\frac{m_e}{m_i} \right)$$

$$a_2(r) = \sum_j \frac{N_j e_j^2 r_0^2}{m_j c^2} \frac{\omega l \Omega_j}{\left(\omega^2 - \Omega_j^2 \right)} = - \frac{N_e e^2 r_0^2 l}{m_i c^2} \frac{\omega \left(\Omega_i^2 \left(\frac{N_i}{N_e} - 1 \right) + \omega^2 \right)}{\Omega_i \left(\Omega_i^2 - \omega^2 \right)} + O \left(\frac{m_e}{m_i} \right)$$

$$a_3(r) = (-a_1(r) + a_2(r))\frac{B(r)}{B_0} - \frac{N_b e^2 r_0^2}{m_b c^2}\frac{l\omega}{\Omega_0}$$

$$= \frac{N_i e^2 r_0^2}{m_i c^2}\frac{B(r)}{B_0}\frac{\omega^2}{(\Omega_i^2 - \omega^2)}\left(1 - \frac{l\omega}{\Omega_i}\right) + O\left(\frac{m_e}{m_i}\right)$$

$$\alpha_0(r) = \ell^2\left[a_1(r) + \frac{N_b e^2 r_0^2}{m_b c^2}\frac{\omega^2}{(\omega + l\Omega_0)^2}\right]$$

$$\alpha_1(r) = -\alpha_0(r)\frac{B(r)}{B_0} - \frac{N_b e^2 r_0^2 \ell^2}{m_b c^2}\frac{\omega}{(\omega + l\Omega_0)} + a_2(r)\frac{B}{B_0}$$

$$\alpha_2(r) = \alpha_0(r)\frac{B^2(r)}{B_0^2} - \frac{N_b e^2 r_0^2 \ell^2}{m_b c^2}\left\{\frac{(\omega + l\Omega_0)^2}{\Omega_0^2} - 1 - \frac{2\omega}{(\omega + l\Omega_0)}\frac{B(r)}{B_0}\right\}$$

$$+ (a_1(r) - 2a_2(r))\frac{B^2(r)}{B_0^2} + \frac{N_b e^2 r_0^2}{m_b c^2}\frac{2l\omega}{\Omega_0}\frac{B(r)}{B_0} ,$$

where the summation over j does not include the energetic beam ions. We note that typically L_0, L_1, L_2 are of the same order, while the contribution from the vacuum is smaller (for completeness, we retain this contribution).

In the last term of Eq. (32), $\xi(r)$ is determined in the vacuum region by Eq. (11) with $\xi(r^{\pm})$ related to $\widehat{C}_0$ and $\widehat{X}_0$ by Eq. (15).

IV. Dispersion Relation

We first define a "standard ordering" given by

$$\frac{N_i}{N_e} = \left(1 - \frac{N_b}{N_e}\right) > \left|\frac{l\Delta}{r_0}\right|\left(\left|\frac{\omega}{\Omega_i}\right| + \left|\frac{\Omega_i}{\omega}\right|\frac{N_b}{N_e}\right) ,$$

where $|a_1| > \frac{\Delta}{r_a}|a_2|$. Furthermore, we assume that the conducting walls are not in contact with the boundaries of the plasma annulus so that

$$\frac{4\pi N_i e^2}{m_i(\Omega_i^2 + \omega^2)} > \left|\frac{l\Delta}{r_0}\right|\left\{Z_l\left(\frac{r_c}{r^-}\right) + Z_l\left(\frac{r^+}{r_w}\right)\right\} ,$$

where

$$Z_l(s) \equiv \frac{1+s^{2l}}{1-s^{2l}} \, .$$

The largest term in the quadratic form in a Δ/r_0 ordering is $L_0\left(C_0, X_0\right)$, and it is due to the inertial response of the background plasma ions. In order to minimize this response, it is necessary to minimize the magnitude of $\left|\widetilde{E}_r\right| \sim \left|\frac{\omega}{c}\frac{\partial C_0}{\partial r}\right|$ by taking C_0 to be constant to lowest order in Δ/r_0. Thus C_0 equal to a constant is the lowest order solution of the eigenmode equation. This is consistent with the assumption made earlier that C_0 varies slowly inside the annulus.

Let

$$C_0 = \widehat{C}_0 + \frac{\Delta}{r_0} C_0^{(1)} + \cdots \, , \qquad r^+ > r > r^- \, , \tag{36}$$

where $\widehat{C}_0$ is equal to a constant. Substituting for C_0 in $L\left(C_0, X_0\right)$ and extremizing with respect to $\partial C_0^{(1)}/\partial x$, we obtain

$$\frac{\partial C_0^{(1)}}{\partial x} = \frac{a_2(r)}{a_1(r)}\widehat{C}_0 - \frac{a_3(r)}{a_1(r)}\widehat{X}_0 \, , \tag{37}$$

where

$$\frac{a_2}{a_1} \approx -\frac{N_e}{N_i}\frac{l\Omega_i}{\omega}\left[\left(1-\frac{N_i}{N_e}\right) - \frac{\omega^2}{\left(\Omega_i^2 - \frac{\omega^2}{b}\right)}\right]$$

$$\frac{a_3}{a_1} \approx -\frac{B(r)}{B_0} + \frac{\omega l \Omega_i}{\left(\Omega_i^2 - \frac{\omega^2}{b}\right)}\frac{B(r)}{B_0}$$

and

$$b \equiv \left(1 + \frac{N_i}{N_e}\frac{m_i}{m_e}\right) \gg 1 \, .$$

The perturbed radial electric field $\frac{i\omega}{c}\left(\frac{\partial C_0^{(1)}}{\partial r} - \frac{\widehat{X}_0 B}{r_0 B_0}\right)$ induced by the perturbed azimuthal field $-\frac{\omega l}{cr}\left(\widehat{C}_0 - \widehat{X}_0\frac{B}{B_0}\right)$ is peaked at the location of the lower hybrid resonance $\omega^2 = b\Omega_i^2(r)$.

Thus, within the annulus

$$\frac{r_0^2}{\Delta^2} L_0\left(C_0, \chi_0\right) + \frac{r_0}{\Delta} L_1\left(C_0, \chi_0\right) + L_2\left(C_0, \chi_0\right) = \int d^3r \left[-\frac{a_2^2}{a_1} + \alpha_0\right] \widehat{C}_0^2$$

$$+ \int d^3r \left[\frac{a_2 a_3}{a_1} + \alpha_1\right] 2\widehat{\chi}_0 \widehat{C}_0 + \int d^3r \left[-\frac{a_3^2}{a_1} + \alpha_2\right] \widehat{\chi}_0^2 \,, \tag{38}$$

where

$$-\frac{a_2^2}{a_1} + \alpha_0 \approx \frac{N_e e^2 \ell^2}{m_i c^2} \left\{ \frac{N_e}{N_i} \left(\frac{N_i}{N_e} - 1\right)^2 - \frac{b\omega^2}{\left(b\Omega_i^2 - \omega^2\right)} + \frac{N_b m_i}{N_e m_b} \frac{\omega^2}{\left(\omega + l\Omega_0\right)^2} \right\}$$

$$\frac{a_2 a_3}{a_1} + \alpha_1 \approx \frac{N_e e^2 \ell^2}{m_i c^2} \left\{ \frac{B}{B_0} \frac{b\omega^2}{\left(b\Omega_i^2 - \omega^2\right)} - \frac{N_b m_i B}{N_e m_b B_0} \frac{\omega^2}{\left(\omega + l\Omega_0\right)^2} - \frac{N_b m_i}{N_e m_b} \frac{\omega}{\left(\omega + l\Omega_0\right)} \right\}$$

$$-\frac{a_3^2}{a_1} + \alpha_2 \approx \frac{N_e e^2 \ell^2}{m_i c^2} \left\{ - \frac{N_i B^2}{N_e B_0^2} \frac{b\omega^2}{\left(b\Omega_i^2 - \omega^2\right)} + \frac{N_b m_i B^2}{N_e m_b B_0^2} \frac{\omega^2}{\left(\omega + l\Omega_0\right)^2} \right.$$

$$\left. - \frac{N_b m_i}{N_e m_b} \left[\frac{\left(\omega + l\Omega_0\right)^2}{\Omega_0^2} - 1 - \frac{2\omega B}{\left(\omega + l\Omega_0\right) B_0} \right] \right\} .$$

Since $B(r)$ varies linearly inside the annulus, the spatial integrals can readily be evaluated. The quadratic form may then be written as follows:

$$r_0^2 L\left(\widehat{C}_0, \widehat{\chi}_0\right) = \int dz \frac{2\pi N_e r_0 \Delta e^2 \ell^2}{m_i c^2} \left[\widehat{C}_0^2 \mathcal{A}_0 + 2\widehat{C}_0 \widehat{\chi}_0 \mathcal{A}_1 + \widehat{\chi}_0^2 \mathcal{A}_2\right] \,, \tag{39}$$

where

$$\mathcal{A}_0 = \frac{N_e}{N_i} \left(1 - \frac{N_i}{N_e}\right)^2 - \frac{N_e m_i}{N_i m_b} \frac{\omega b^{1/2}}{2\Omega_0 \delta\widehat{\Omega}} \left(\log \frac{\Lambda_1^+}{\Lambda_1^-} - \log \frac{\Lambda_2^+}{\Lambda_2^-}\right)$$

$$+ \frac{N_b}{N_e} \frac{m_i}{m_b} \frac{\omega^2}{\left(\omega + l\Omega_0\right)^2} - \frac{N_b m_i}{N_e m_b} \frac{\omega^2 r_0^2}{|l| c^2 \delta\widehat{\Omega}} \left(Z_l\left(\frac{r_c}{r^-}\right) + Z_l\left(\frac{r^+}{r_w}\right)\right) \tag{40}$$

$$\mathcal{A}_1 = \frac{m_i^2}{m_b^2} \frac{\omega^2}{2\Omega_0^2 \delta\widehat{\Omega}} \log \frac{\Lambda_1^+ \Lambda_2^+}{\Lambda_1^- \Lambda_2^-} - \frac{N_b}{N_e} \frac{m_i}{m_b} \frac{\omega\left(2\omega + l\Omega_0\right)}{\left(\omega + l\Omega_0\right)^2}$$

$$+ \frac{N_b m_i}{N_e m_b} \frac{\omega^2 r_0^2}{|l| c^2 \delta\widehat{\Omega}} \left(\frac{B\left(r^-\right)}{B_0} Z_l\left(\frac{r_c}{r^-}\right) + \frac{B\left(r^+\right)}{B_0} Z_l\left(\frac{r^+}{r_w}\right)\right) \tag{41}$$

$$\mathcal{A}_2 = -\frac{N_i}{N_e} \frac{m_i^2}{m_b^2} \frac{\omega^2}{\Omega_0^2} \left[1 + \frac{m_i}{m_b} \frac{\omega}{2\Omega_0 b^{1/2} \delta\widehat{\Omega}} \left(\log \frac{\Lambda_1^+}{\Lambda_1^-} - \log \frac{\Lambda_2^+}{\Lambda_2^-}\right)\right]$$

$$+\frac{N_b}{N_e}\frac{m_i}{m_b}\left[1-\frac{(\omega+l\Omega_0)^2}{\Omega_0^2}+\frac{\omega^2}{(\omega+l\Omega_0)^2}\left(1+\frac{\Delta^2}{12}\left(\frac{1}{B_0}\frac{\partial B}{\partial r}\right)^2\right)+\frac{2\omega}{(\omega+l\Omega_0)}\right]$$

$$-\frac{N_b m_i}{N_e m_b}\frac{\omega^2 r_0^2}{|l|c^2\delta\widehat{\Omega}}\left(\frac{B^2(r^-)}{B_0^2}Z_l\left(\frac{r_c}{r^-}\right)+\frac{B^2(r^+)}{B_0^2}Z_l\left(\frac{r^+}{r_w}\right)\right) \tag{42}$$

$$\Lambda_1^{\pm}\equiv\Omega_i\left(r^{\pm}\right)-\frac{\omega}{b^{1/2}}$$

$$\Lambda_2^{\pm}\equiv\Omega_i\left(r^{\pm}\right)+\frac{\omega}{b^{1/2}}$$

$$Z_l(s)\equiv\frac{\left(1+s^{2l}\right)}{\left(1-s^{2l}\right)}\ .$$

We have extremized our quadratic form with respect to $\partial C_0/\partial x$. We still have the freedom to extremize Eq. (39) with respect to the overall constant $\widehat{C}_0$ in order to determine the unknown constant $\widehat{C}_0$ in terms of $\widehat{X}_0$. Thus by extremizing with respect to $\widehat{C}_0$, we obtain

$$\widehat{C}_0=-\frac{\mathcal{A}_1}{\mathcal{A}_0}\widehat{X}_0\ . \tag{43}$$

Substituting for $\widehat{C}_0$, we obtain the dispersion relation

$$\mathcal{A}_2-\frac{\mathcal{A}_1^2}{\mathcal{A}_0}=0\ . \tag{44}$$

For equilibria in which there is no field reversal ($1>\delta\widehat{\Omega}$ but $\delta\widehat{\Omega}>\frac{\ell\Delta}{r_0}$ to satisfy the large betatron frequency assumption), we obtain an approximate solution of the dispersion relation in the limit $1>\left|\frac{\omega}{\ell\Omega_0}\right|^2>\frac{m_b}{m_i}\frac{N_b}{N_i},\ \frac{(\delta\widehat{\Omega})^2}{12}>\frac{m_b N_b}{m_i N_e}\frac{(\ell^2-1)}{\ell^2}$, where

$$\mathcal{A}_0\approx-\frac{N_e m_i^2}{N_i m_b^2}\ell^2\widehat{\omega}^2\left(1+\frac{(\delta\widehat{\Omega})^2}{4}\right)$$

$$\mathcal{A}_1\approx\frac{m_i^2}{m_b^2}\ell^2\widehat{\omega}^2\left(1+\frac{(\delta\widehat{\Omega})^2}{12}\right)$$

$$\mathcal{A}_2\approx-\frac{N_i m_i^2}{N_e m_b^2}\ell^2\widehat{\omega}^2\frac{N_b}{N_e}\frac{m_i}{m_b}(1-\ell^2)$$

$$\widehat{\omega}\equiv\frac{\omega}{\ell\Omega_0}\ .$$

Substituting in Eq. (44), we obtain the unstable root:

$$\widehat{\omega} = i\left[\frac{N_b m_b(\ell^2-1)}{N_i m_i \ell^2}\frac{12}{(\delta\widehat{\Omega})^2}\right]^{1/2}$$

and the perturbations are unstable with growth rate

$$\mathrm{Im}(\omega) \approx \Omega_0\left[\frac{N_b}{N_i}\frac{m_b}{m_i}\left(\ell^2-1\right)\frac{12}{(\delta\widehat{\Omega})^2}\right]^{1/2} \quad . \tag{45}$$

For equilibria in which there is field reversal ($\delta\widehat{\Omega} > 2$) so that a resonance at the lower hybrid frequency $\omega = \pm b^{1/2}\Omega(r)$ is possible at some radius r, we obtain an approximate solution in the limit $\widehat{\omega}^2 \sim \frac{N_b}{N_e} \ll 1$, where

$$\begin{aligned}
\log \Lambda_1^- &= -i\pi \ , \\
\log \Lambda_2^- &\to i\pi \ , \\
\log \frac{\Lambda_1^+\,\Lambda_2^+}{\Lambda_1^-\,\Lambda_2^-} &\to \log\frac{\Omega_i^2(r^+)}{\Omega_i^2(r^-)} \ , \\
\mathcal{A}_0 &\approx i\pi \tfrac{N_e}{N_i}\tfrac{m_b}{m_i}\tfrac{\omega b^{1/2}}{\delta\widehat{\Omega}} \\
\mathcal{A}_1 &\approx \frac{m_i^2}{m_b^2}\frac{\omega^2}{2\Omega_0^2\,\delta\widehat{\Omega}}\log\frac{\Omega_i^2(r^+)}{\Omega_i^2(r^-)} \ , \\
\mathcal{A}_2 &\approx -\frac{N_i}{N_e}\frac{m_i^2}{m_b^2}\frac{\omega^2}{\Omega_0^2} + \frac{N_b}{N_e}\frac{m_i}{m_b}(1-\ell^2) \\
\mathcal{A}_2 &\gg \frac{\mathcal{A}_1^2}{\mathcal{A}_0} \ .
\end{aligned}$$

The dispersion relation may then be approximated by

$$\frac{\omega^2}{\Omega_0^2} = -\frac{N_b\,m_b}{N_i\,m_i}(\ell^2-1)$$

and the perturbations are unstable with growth rate

$$\mathrm{Im}(\omega) = \Omega_0\left[\frac{N_b\,m_b}{N_i\,m_i}(\ell^2-1)\right]^{1/2} \quad . \tag{46}$$

We solve Eq. (44) numerically, neglecting the typically small contributions from the vacuum region surrounding the annulus. In Figs. 2 and 3 we plot the frequency $\hat{\omega}$ as a function of $\frac{N_b}{N_e}$ for several values of the azimuthal mode number ℓ and the field reversal parameter $\delta\widehat{\Omega}$. We have included solutions for values of $\frac{N_b}{N_e} \to 1$ since Eq. (44) remains valid in this limit for reasons discussed below. We predict instability for all non-zero values of $\frac{N_b}{N_e}$. However, for equilibria with large field reversal where $\delta\widehat{\Omega} > 25.6$, $\ell = 2$ can be stable (see Eq. (50)).

We note that if $N_i/N_e \to 0$ or there is field reversal (then the pole contribution from resonance at the lower hybrid frequency yields $\mathcal{A}_0 \sim b^{1/2} \gg 1$); $\mathcal{A}_0 \gg \mathcal{A}_1$ and hence $\widehat{C}_0 \ll \widehat{X}_0$. Strictly, the "standard ordering" is not valid in the limit $N_i/N_e \to 0$. However, as we demonstrate below, the quadratic form is extremized by $C_0 = 0$ when $N_i/N_e \to 0$. As this is also the prediction from our "standard ordering" analysis we can validly use the dispersion relation given by Eq. (44) for arbitrary N_i/N_e.

We now consider the low background ion density limit where the "standard ordering" formally fails and the following inequality applies:

$$\left[\frac{N_b}{N_e}\left|\frac{\Omega_i}{\omega}\right| + \left|\frac{\omega}{\Omega_i}\right|\right] > \left(1 - \frac{N_b}{N_e}\right)\left|\frac{r_0}{l\Delta}\right| .$$

Further, we assume that

$$\frac{N_e}{N_b}\frac{m_b}{m_i}\left|\frac{\omega_b^2}{\omega\Omega_i}\right| > \left(Z_l\left(\frac{r_c}{r^-}\right) + Z_l\left(\frac{r^+}{r_w}\right)\right) .$$

In this case, $\left|\frac{\Delta}{r_0}a_2\right| > \left\{|a_1|, \left|\frac{\Delta}{r_0}a_2\right|\right\}$, and the dominant term in the quadratic form [Eq. (32)] is

$$r_0^4 L\left(C_0, X_0\right) = \frac{r_0}{\Delta}\int d^3r\frac{1}{r}\left(\frac{\partial}{\partial x}a_2 r\right)C_0^2 .$$

This term is due to the "electric drift" of the plasma electrons, and since the energetic beam ions do not respond similarly, large charge density perturbations would arise unless the field variable C_0 is zero. Thus the quadratic form is extremized to lowest order in Δ/r_0 by choosing $C_0 = 0$. The dispersion relation is therefore $\mathcal{A}_2 = 0$ with $N_i/N_e = 0$.

If we neglect the term proportional to $\frac{\omega^2 r_0^2}{|l| c^2 \delta\hat{\Omega}} \ll 1$ which is the contribution of the vacuum region between the plasma annulus and the conducting walls, we obtain

$$\frac{(\omega + l\Omega_0)^4}{\Omega_0^2} - (2\omega + l\Omega_0)^2 - \omega^2 \frac{\left(\delta\hat{\Omega}\right)^2}{12} = 0 \; . \tag{47}$$

For $\left(\delta\hat{\Omega}\right)^2/12 < 1$, the solutions are

$$\frac{\omega}{\Omega_0} = \begin{cases} -(l-1) \pm i(l-1)^{1/2} \\ -(l-1) \pm (l+1)^{1/2} \end{cases}$$

and the growth rate of the unstable mode is

$$\mathrm{Im}\,\omega = (l-1)^{1/2}\Omega_0 \; . \tag{48}$$

For $\ell > \frac{\delta\hat{\Omega}}{4(3)^{1/2}} \gg 1$. The solution of the unstable mode is

$$\omega = -\ell\Omega_0 + \delta\hat{\Omega}\,\frac{\Omega_0}{4(3)^{1/2}} + i\Omega_0 \left(\frac{\ell\delta\hat{\Omega}}{2(3)^{1/2}}\right)^{1/2} \; . \tag{49}$$

For a given value of l, Eq. (47) predicts stability for sufficiently large $\left(\delta\hat{\Omega}\right)^2/12$. Let $x = \frac{\omega}{l\Omega_0}$. We can rewrite Eq. (47) as follows:

$$Y_1(x) = Y_2(x)$$

where

$$Y_1(x) \equiv \ell^2 (x+1)^4$$

$$Y_2(x) = x^2 \frac{\left(\delta\hat{\Omega}\right)^2}{12} + (2x+1)^2 \; .$$

We note that $(l > 1)$

$$Y_1 > Y_2, \qquad x = 0$$

$$Y_2 > Y_1, \qquad x = -1$$

$$Y_1 > Y_2, \qquad x \to \pm\infty \; .$$

Hence there are two real roots of $Y_1(x) = Y_2(x)$ for $x < 0$, and a sufficient condition for the occurrence of two real roots of $Y_1(x) = Y_2(x)$ for $x > 0$ is the existence of a finite value of $x = x_0 > 0$ such that

$$Y_2(x_0) > Y_1(x_0)$$

that is

$$\frac{\left(\delta\widehat{\Omega}\right)^2}{12} > \frac{\ell^2 (x_0+1)^4}{x_0^2} - \frac{(2x_0+1)^2}{x_0^2} .$$

The minimum value of $(x_0+1)^4/x_0^2$ for $x_0 > 0$ occurs at $x_0 = 1$. Thus a sufficient condition for the occurrence of four real roots of $Y_1(x) = Y_2(x)$ (and therefore stability) is

$$\left(\delta\widehat{\Omega}\right)^2 > 192\ell^2 - 108 . \tag{50}$$

For $l = 2$,

$$\delta\widehat{\Omega} > 25.7 .$$

We solve Eq. (47) numerically, and in Fig. 4 we plot the frequency of the unstable modes as a function of the field reversal factor $\delta\widehat{\Omega}$. For $l = 2$ stability, we estimate numerically that $\delta\widehat{\Omega} > 25.6$.

V. Discussion

We have investigated the low frequency stability of an equilibrium configuration consisting of a thin annular layer of energetic large orbit ions in a neutralizing background plasma. The energetic ion motion exhibits two characteristic frequencies, the frequency of radial betatron oscillations $\Omega_\beta = \Omega_0 \left(1 + \frac{r_0}{\Delta}\,\delta\widehat{\Omega}\right)^{1/2}$ and the revolution about the axis of symmetry Ω_0. Here $\frac{r_0}{\Delta} \gg 1$, and field reversal requires $\delta\widehat{\Omega} = \frac{\Delta}{\Omega_0}\frac{\partial\Omega_0}{\partial r_0} > 2$. $\delta\widehat{\Omega}$ is proportional to the current per unit axial length and is related to the field reversal factor ζ by $\zeta = \frac{2\delta\widehat{\Omega}}{2+\delta\widehat{\Omega}}$.

In order to simplify the analysis:

1) We neglected parallel electric field perturbations and we considered only flute perturbations with no variations along the magnetic field.

2) We considered the limit of a very thin annular layer where $\frac{\Delta}{r_0}|\ell| < 1$. In this limit, the betatron frequency Ω_β is much larger than the beam circulation frequency Ω_0, and for low frequency perturbations $\omega^2 \lesssim \ell^2\Omega_0^2$,

$$\Omega_\beta^2 > \Omega_0(\omega^2 + \ell^2\Omega_0^2)^{1/2} .$$

This inequality also allows us to make the rigid mode approximation.[2,14]

3) We neglected the temperature of the background plasma. We are therefore assuming that the thermal velocities of the background species are smaller than the phase velocity of the unstable perturbations.

4) We assume that the dielectric properties of the system are dominant so that perturbations can be considered quasineutral:

$$\frac{1}{(\Omega_i^2 + \omega^2)}\left[\omega_{pi}^2 \frac{r_0}{|\ell|\Delta} + \frac{m_b}{m_i}\omega_b^2\left\{\left|\frac{\Omega_i}{\omega}\right| + \left|\frac{\omega}{\Omega_i}\right|\frac{N_e}{N_b}\right\}\right] > \left\{Z_\ell\left(\frac{r_c}{r^-}\right) + Z_\ell\left(\frac{r^+}{r+w}\right)\right\} .$$

The electric field energy within the beam equilibrium is then large compared to that in the vacuum.

The beam equilibrium distribution function was taken to be monoenergetic with no spread in the particle canonical momentum p_θ. In reality, a small spread in p_θ is likely. However, we expect our analysis to be valid for small spreads in p_θ provided the spread introduced in the circulation frequency $\Omega(r_\beta(p_\theta))$ is small. This requires: $(\omega + \ell\Omega_0) > |\ell|\frac{\Delta}{r_0}\Omega_0$.

We constructed a quadratic variational form, ordered the individual terms in the smallness parameter $\frac{\Delta}{r_0} \ll 1$, and by successive extremization of the lowest order terms, we obtained the dispersion relation (equation (44)).

For $\frac{N_b}{N_e} \ll 1$, the equilibrium is unstable for azimuthal mode numbers $\ell \geq 2$ with growth rates

$$\mathrm{Im}\,\omega = \Omega_0\left[(\ell^2 - 1)\frac{m_b N_b}{m_i N_i}\frac{12}{(\delta\hat{\Omega})^2}\right]^{1/2} \quad , \; 2 > \delta\hat{\Omega} > \left[\frac{12(\ell^2-1)m_b N_b}{\ell^2 m_i N_i}\right]^{1/2}$$

$$\mathrm{Im}\,\omega = \Omega_0 \left[(\ell^2 - 1) \frac{m_b N_b}{m_i N_i} \right]^{1/2} , \quad \delta\widehat{\Omega} > 2 \, .$$

Instability persists for finite values of $\frac{N_b}{N_e}$. When $\frac{N_b}{N_e} = 1$, the growth rate of the unstable mode is

$$\mathrm{Im}\,\omega = \Omega_0 (\ell - 1)^{1/2} , \quad \frac{\delta\widehat{\Omega}}{2(3)^{1/2}} < 1$$

$$\mathrm{Im}\,\omega = \Omega_0 \left(\frac{\ell \delta\widehat{\Omega}}{2(3)^{1/2}} \right)^{1/2} , \quad \ell > \frac{\delta\widehat{\Omega}}{2(3)^{1/2}} > 1 \, .$$

However, stability can be achieved for mode numbers $2 \leq \ell < \ell_0$ if the following sufficient condition $\left(\frac{N_b}{N_e} = 1\right)$ is satisfied, $\delta\widehat{\Omega} > (192\ell_0^2 - 108)^{1/2}$. Thus, $\delta\widehat{\Omega}$ must be greater than 25.6 in order to stabilize the $\ell = 2$ mode. Higher ℓ modes are more difficult to stabilize but their effect on containment should be less detrimental.

Residual instabilities due to the coupling of ion beam modes to the background plasma at frequencies equal to the local lower hybrid frequency may still persist but they have small growth rates when $\frac{N_b}{N_e} \sim 1$. Such ion beam-plasma interactions have previously been investigated by Gerver and Sudan.[14]

This geometry has $\ell = 1$ marginally stable. However, when field line curvature is taken into account, the $\ell = 1$ mode becomes the well-known precessional mode which can be stabilized by quadrupole fields, walls, or toroidal fields.[2]

We note that the stabilization of kink modes requires strong current layers ($\delta\widehat{\Omega}$ large). This raises the question of whether the axially extended annular equilibrium is stable to "tearing" $\ell = 0$ modes. This topic has previously been discussed by many authors,[15,16,17,18] but for convenience, we reproduce in Appendix A the stability analysis for an annular layer in the slab approximation.

We find that stability to $\ell = 0$ modes can only be achieved for a flat current profile if *both* outer and inner conducting walls at $r = r_w$ and $r = r_c$ respectively are placed close to the boundaries of the annulus. The general stability criteria is of the form $(r_w - r_c) < 2g\Delta$ where the numerical factor g is $g = 1$ for flat current profiles. The magnitude of g exceeds unity with hollow current profiles. However such equilibria can be produced with distribution functions (where $\langle v_r^2 \rangle = \langle v_z^2 \rangle$) only if the perpendicular energy of betatron oscillations approaches and exceeds the directed energy of the ion beams, $\langle v_r^2 \rangle \gtrsim v_{\theta 0}^2$. For this situation, the assumptions used in our analysis is violated.

We note that smooth current profiles with a maximum inside the annulus (which can be produced by smooth distribution functions $F(H, p_\theta, p_z)$) requires conducting wall nearly touching the layer if $\langle v_r^2 \rangle = \langle v_z^2 \rangle$. Less stringent tearing instability can be attained by having $\langle v_z^2 \rangle > \langle v_r^2 \rangle$ as shown in Refs. 16 and 17. We also note that attaining a hollow or flat beam current profile rather than a peaked one requires high beam quality for which the spread of p_θ is less than $p_\theta - p_0$, where p_0 is the angular momentum of the betatron orbit. Thus if self-collisions force $\langle v_r^2 \rangle = \langle v_z^2 \rangle$ a system without a toroidal magnetic field is susceptible to tearing mode instability. The effect of a toroidal magnetic field is clearly important and needs further study.

In addition, we expect that the presence of an azimuthal magnetic field will also be effective in enhancing stability to tearing $\ell = 0$ mode instabilities.

However, the stability to kink $\ell \geq 2$ modes of hollow or peaked current profiles or equilibria with azimuthal magnetic fields is not covered by our analysis and remains to be investigated.

The result of this analysis indicates the existence of a possible stability window for moderate values of the azimuthal mode number $\ell (\ell \geq 2)$ when the reversal limit (that is as $\delta\widehat{\Omega} \to \infty$, $\zeta \equiv \frac{2\delta\widehat{\Omega}}{2+\delta\widehat{\Omega}} \to 2$) is approached. It could be difficult to achieve experimentally, since $\ell = 2$ stability requires the field reversal factor $\zeta \geq 1.86$, whereas ζ has an upper limit of 2.

However, the model equilibrium of a thin field-reversed layer is somewhat idealized. More favorable stability criteria may be expected for thick layers. It is noteworthy that θ-pinch experiments have already established the occurrence of relatively long-lived field-reversed equilibria. It is interesting to note that on the basis of MHD theory (without rotation), $\zeta = 2$ for systems in such an equilibrium.

We observe that Lovelace's criterion[10] for stability requires

$$I \equiv \frac{r_0}{B(r_0)} \frac{\partial B}{\partial r_0} < 3 \ . \tag{51}$$

Our analysis does not indicate any transition to stability for small $\delta\widehat{\Omega}$. However as we have assumed $\frac{r_0}{\Delta}\delta\widehat{\Omega} \gg 3$, and it is not surprising that we have not recovered Lovelace's criterion. Lovelace suggested that $I > 3$ would be unstable. This is generally confirmed in our analysis although at very large I values we find surprisingly that low azimuthal mode number perturbations can be stable when $\frac{N_b}{N_e} \to 1$. We note that with thick layers Lovelace's stability criterion can be satisfied even for a field reversed layer, and this suggests an alternative limit for stable operation of a long layer of energetic ion beams where the geometry resembles a field reversed θ-pinch.

Further stability studies need to consider equilibria configurations with a layer of finite thickness, finite axial length, and a finite azimuthal magnetic field.

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Appendix A

In this Appendix, we discuss the stability of long annular field-reversed equilibria to $\ell = 0$ perturbation of the form $\widetilde{\phi} = 0$, $\widetilde{\mathbf{A}} = A_\theta(r)e^{ikz}\widehat{\theta}$, and $\widetilde{f} = f(r)e^{ikz}$. These perturbations, if unstable, can lead to the "break-up" of the cylindrical annulus into "rings" of finite axial length. We consider time independent perturbations and we analyze the eigenmode equations for the occurrence of "neighboring stationary states." The existence of such states implies instability.[18,15] However, if they do not occur, the equilibrium is stable to these perturbations.

The perturbed ion distribution function $\widetilde{f}$ is:

$$[\widetilde{f}, H_0] = -[F(H_0, p_\theta), H_1] = -\frac{\partial F_0}{\partial H_0}[H_0, H_1]$$

where

$$H_1 = -\frac{e}{m_b r c}\left(p_\theta - \frac{e\psi}{c}\right)\widetilde{A}_\theta = r\widetilde{A}_\theta\,\frac{\partial H_0}{\partial \psi}\ .$$

Thus

$$f(r) = rA_\theta(r)\,\frac{\partial F}{\partial \psi}$$

and the perturbed ion beam current $\delta J_{b\theta}$ is:

$$\begin{aligned}
\frac{4\pi}{c}\,\delta J_{b\theta} &= -\frac{4\pi N e^2}{m_b c^2}\,A_\theta(r) \\
&\quad + \frac{4\pi e}{m_b c}\,A_\theta(r)\int d^3p\left(p_\theta - \frac{e}{c}\psi\right)\frac{\partial F}{\partial \psi} \\
&= rA_\theta(r)\,\frac{\partial}{\partial \psi}\,\frac{4\pi}{c}\,J_{0\theta}(\psi) = \frac{A_\theta(r)}{B(r)}\,\frac{\partial}{\partial r}\,\frac{4\pi}{c}\,J_{0\theta} \\
&= -\frac{A_\theta(r)}{B(r)}\,\frac{\partial^2 B}{\partial r^2}\ .
\end{aligned}$$

The perturbed currents of the cold background plasma is zero. The eigenmode equation (with k the eigenvalue) is therefore given by

$$\frac{\partial^2 A_\theta}{\partial x^2} - k^2 A_\theta = -\frac{4\pi}{c}\,\delta J_{b\theta} \tag{A1}$$

where $x = r - r_0$ and we neglect the curvature of the annulus $\left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} \rightarrow \frac{\partial^2}{\partial x^2}\right)$. It is sufficient to discuss only the $k \rightarrow 0$ limit since this determines the condition for marginal stability.[18,15]

In the case of a thin annulus with uniform current where we assume exact field reversal:

$$B = \begin{cases} \hat{B} & x > \frac{\Delta}{2} \\ \hat{B}\,\frac{2x}{\Delta} & \frac{\Delta}{2} > x > -\frac{\Delta}{2} \\ -\hat{B} & x < -\frac{\Delta}{2} \end{cases}$$

we obtain ($k = 0$):

$$\frac{\partial^2 A_\theta}{\partial x^2} = -\frac{2A_\theta}{\Delta}\left[\delta\left(x + \frac{\Delta}{2}\right) + \delta\left(x - \frac{\Delta}{2}\right)\right] \tag{A2}$$

where $\delta(x)$ is the Dirac delta function.

We assume the annulus is bounded by conducting walls at $x = x_2$ and at $x = -x_1$ ($A_\theta = 0$ at $x = x_2, -x_1$).

The solutions of Eq. (A2), inside and outside the annulus, are:

$$A_\theta = \begin{cases} A_\theta\left(\frac{\Delta}{2}\right)\left(\frac{1}{2} + \frac{x}{\Delta}\right) + A_\theta\left(-\frac{\Delta}{2}\right)\left(\frac{1}{2} - \frac{x}{\Delta}\right), & \frac{\Delta}{2} > x > -\frac{\Delta}{2} \\ A_\theta\left(\frac{\Delta}{2}\right)\frac{(x_2 - x)}{\left(x_2 - \frac{\Delta}{2}\right)}, & x_2 > x > \frac{\Delta}{2} \\ A_\theta\left(-\frac{\Delta}{2}\right)\frac{(x_1 + x)}{\left(x_1 - \frac{\Delta}{2}\right)}, & -\frac{\Delta}{2} > x > -x_1 \end{cases}$$

By imposing the matching conditions at $x = \pm\frac{\Delta}{2}$:

$$\left[\frac{\partial A_\theta}{\partial x}\right]_{\frac{\Delta}{2}-\epsilon}^{\frac{\Delta}{2}+\epsilon} = -\frac{2}{\Delta}A_\theta\left(\frac{\Delta}{2}\right)$$

$$\left[\frac{\partial A_\theta}{\partial x}\right]_{-\frac{\Delta}{2}-\epsilon}^{-\frac{\Delta}{2}+\epsilon} = -\frac{2}{\Delta}A_\theta\left(-\frac{\Delta}{2}\right)$$

where ϵ is infinitesimally small, we find that $x_2 + x_1 = 2\Delta$ at marginal stability ($k = 0$). No solutions for real values of k exist and hence the equilibrium is stable to $\ell = 0$ perturbations if

$$x_2 + x_1 < 2\Delta\ . \tag{A3}$$

This result implies that the $\ell = 0$ mode can be stabilized only by placing two conducting walls relatively close to the boundaries of the annulus. Hollow equilibrium current maxima at the boundaries tend to be more stable than those with a maximum at the center of the annulus. In order to explore the sensitivity of the stability criterion to the current profile, we consider the following model of the magnetic field variation inside the annulus which is in fact an exact solution for the equilibrium distribution function discussed in Section IIB:

$$B = \begin{cases} \hat{B} & x > \frac{\Delta}{2} \\ \hat{B}\frac{\sinh \alpha x}{\sinh \frac{\alpha\Delta}{2}} & \frac{\Delta}{2} > x > -\frac{\Delta}{2} \\ -\hat{B} & x < -\frac{\Delta}{2} \end{cases}$$

where $\frac{\alpha^2\Delta^2}{4} = \frac{\omega_b^2\Delta^2}{4c^2} \approx \frac{\omega_b^2 p_r^2}{m_b^2 c^2 \Omega_\beta^2} \approx \frac{v_T^2}{v_{0\theta}^2}$. The corresponding eigenmode equation ($k = 0$) is:

$$\frac{\partial^2 A_\theta}{\partial x^2} = \alpha^2 A_\theta \Theta\left(\frac{\Delta^2}{4} - x^2\right) - A_\theta \alpha \coth \alpha x \left[\delta\left(x - \frac{\Delta}{2}\right) + \delta\left(x + \frac{\Delta}{2}\right)\right] \tag{A4}$$

where $\Theta(x)$ is the step function.

The solution inside the annulus is

$$A_\theta = \left[A_\theta\left(\frac{\Delta}{2}\right) - A_\theta\left(-\frac{\Delta}{2}\right)\right]\frac{\sinh \alpha x}{2\sinh \frac{\alpha\Delta}{2}}$$

$$+\left[A_\theta\left(\frac{\Delta}{2}\right) + A_0\left(-\frac{\Delta}{2}\right)\right]\frac{\cosh \alpha x}{2\cosh \frac{\alpha\Delta}{2}}, \quad \frac{\Delta}{2} > x > -\frac{\Delta}{2}.$$

Proceeding as before, we find that the $\ell = 0$ mode can be stabilized if

$$x_2 + x_1 < \Delta\left[1 + \frac{2}{\alpha\Delta}\sinh\frac{\alpha\Delta}{2}\cosh\frac{\alpha\Delta}{2}\right].$$

Thus, as $\frac{\alpha\Delta}{2}$ increases (more hollow current profiles), stability can be achieved with the conducting walls at larger distances from the boundaries of the annulus.

As an example of a smooth current profile with a maximum at the mid-point of the annulus, we consider

$$B = \begin{cases} \widehat{B} & x > \dfrac{\Delta}{2} \\ \widehat{B}\,\sin\dfrac{\pi x}{\Delta} & \dfrac{\Delta}{2} > x > -\dfrac{\Delta}{2} \\ -\widehat{B} & x < -\dfrac{\Delta}{2} \end{cases}$$

For this case, the current at the boundaries is zero, and the eigenmode equation ($k = 0$) is

$$\frac{\partial^2 A_\theta}{\partial x^2} = \frac{\pi^2}{\Delta^2} A_\theta \Theta \left(\frac{\Delta^2}{4} - x^2 \right) . \tag{A5}$$

The marginal stability condition is $x_2 + x_1 = \Delta$ and the conducting walls must be at the boundaries of the annulus to obtain stability to $\ell = 0$ perturbations.

We note that the eigenmode equation (Eq. (A1)) was derived assuming an equilibrium distribution function $F_0(H_0, p_\theta)$ which depends only on H_0 and p_θ. Thus the average "temperatures of the radial $\langle v_r^2 \rangle$ and axial $\langle v_z^2 \rangle$ motion are equal. If $F_0 = F_0(H_0, p_\theta, p_z)$, we can have $\langle v_r^2 \rangle \neq \langle v_z^2 \rangle$. It has been shown by Berk and Pearlstein,[16] Uhm and Davidson,[17] that when

$$\left\langle v_z^2 \right\rangle > \left\langle v_r^2 \right\rangle ,$$

significant improvement in the conduction wall requirements needed for tearing $\ell = 0$ mode stability can be achieved.

References

1. N. Rostoker and A. Fisher, 6th Int. Conf. on High Power Particle Beams, Kobe, Japan, June 9-12 (1986).
2. H.P. Furth, Phys. Fluids **8**, 2020 (1965).
3. J.W. Beal, M. Brettschneider, N.C. Christofilos, R.E. Hester, W.A.S. Lamb, W.A. Sherwood, R.L. Spoerlin, P.B. Weiss, and R.E. Wright, in Plasma Physics and Controlled Nuclear Fusion Research (IAEA, Vienna, 1969) Vol. I, p. 967.
4. R.V. Lovelace, Phys. Rev. Lett. **35**, 162 (1975); and Phys. Fluids **19**, 723 (1976).
5. R.V. Lovelace, Phys. Fluids **21**, 1389 (1978).
6. R.V. Lovelace, Phys. Rev. Lett. **41**, 1801 (1978).
7. R.N. Sudan and M.N. Rosenbluth, Phys. Fluids **22**, 282 (1979).
8. J.M. Finn and R.N. Sudan, Phys. Fluids **22**, 1198 (1975), J.M. Finn, Phys. Fluids **22**, 1770 (1979); J.M. Finn and R.N. Sudan, Nucl. Fusion **22**, 1443 (1982).
9. C.B. Ruchti and R.V. Lovelace, Phys. Fluids **27**, 1789 (1984).
10. R.V. Lovelace, Phys. Fluids **22**, 708 (1979).
11. D.S. Harned, Phys. Fluids **25**, 1915 (1982).
12. Z. Mikic and E.C. Morse, Phys. Fluids **30**, 2806 (1987).
13. H.L. Berk and H.V. Wong, and K.T. Tsang, Phys. Fluids **30**, 2681 (1987).
14. R.V. Lovelace and R.N. Sudan, Phys. Fluids **15**, 1842 (1972).
15. M.J. Gerver and R.N. Sudan, Phys. Fluids **22**, 686 (1979).

16. K.D Marx, Phys. Fluids **11**, 357 (1968).

17. H.L. Berk and C.D. Pearlstein, Phys. Fluids **15**, 2396 (1972).

18. H. Uhm and R.C. Davidson, Phys. Fluids **23**, 348 (1980).

19. D. Pfirsch, Z. Naturforschg. **17a**, 861 (1962).

Figure Captions

1. Annular equilibrium of beam ions.

   a) Cylindrical annulus of energetic ions of radius r_0 and thickness Δ bounded by conducting walls at $r = r_w$ and $r = r_c$.

   b) Cross-section of annulus — betatron orbit of energetic ions.

2. Real and imaginary part of the frequency $\hat{\omega}$ for equilibria with no field reversal ($\delta\hat{\Omega} = 0.2$) as a function of $\frac{N_b}{N_e} \cdot \left(\frac{m_b}{m_i} = 2\right)$.

   a) $\ell = 2$

   b) $\ell = 3$

   c) $\ell = 4$

3. Real and imaginary part of the frequency $\hat{\omega}$ of the $\ell = 2$ mode for field-reversed equilibria ($\delta\hat{\Omega} = 3, 10, 30$) as a function of $\frac{N_b}{N_e} \cdot \left(\frac{m_b}{m_i} = 2\right)$.

4. Real and imaginary part of the frequency $\hat{\omega}$ of the $\ell = 2$ mode for $\frac{N_b}{N_e} = 1$, $\frac{m_b}{m_i} = 2$, as a function of the field-reversed parameter $\delta\hat{\Omega}$.

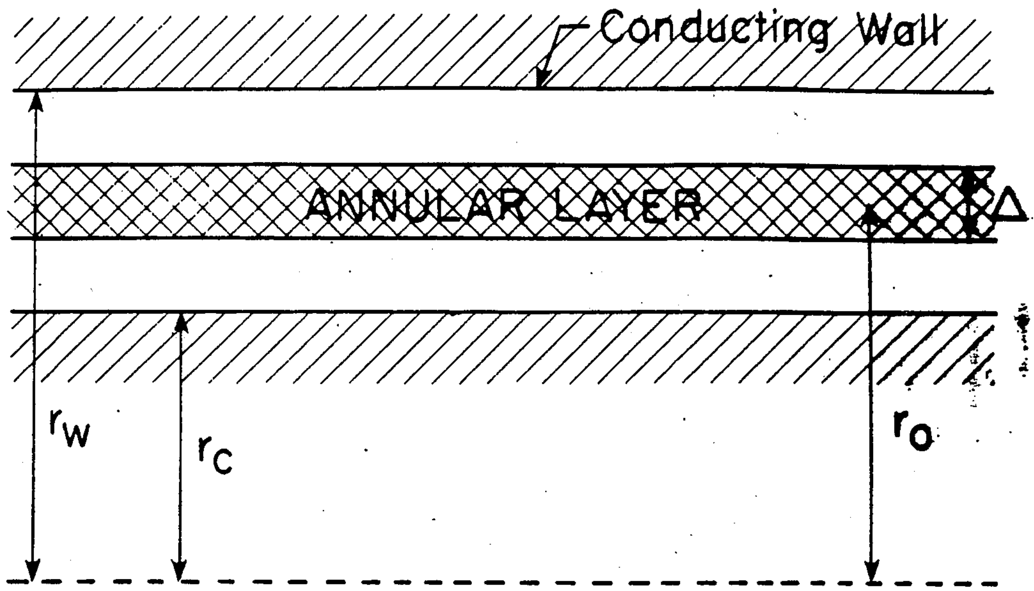


Fig. 1(a)

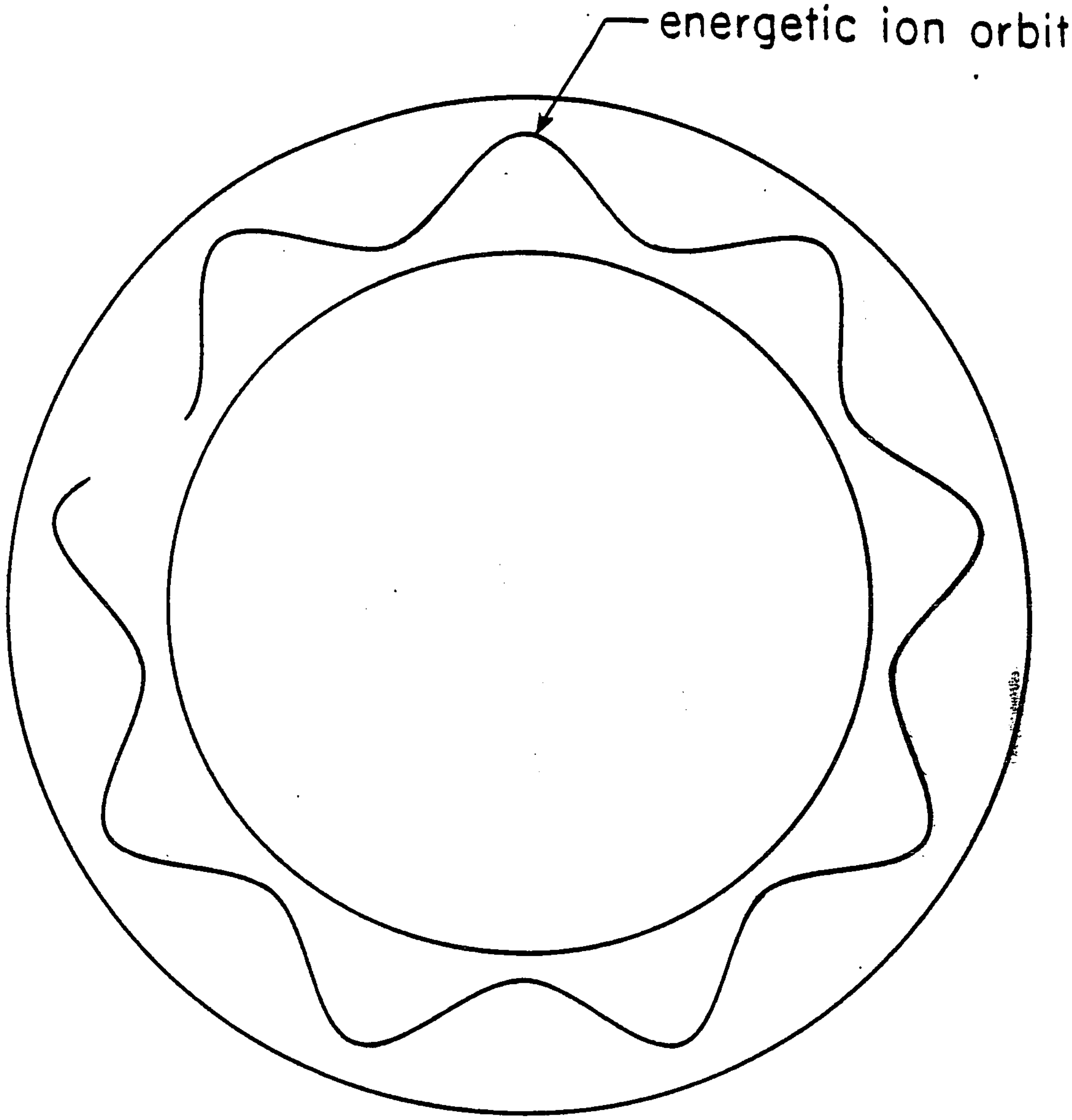
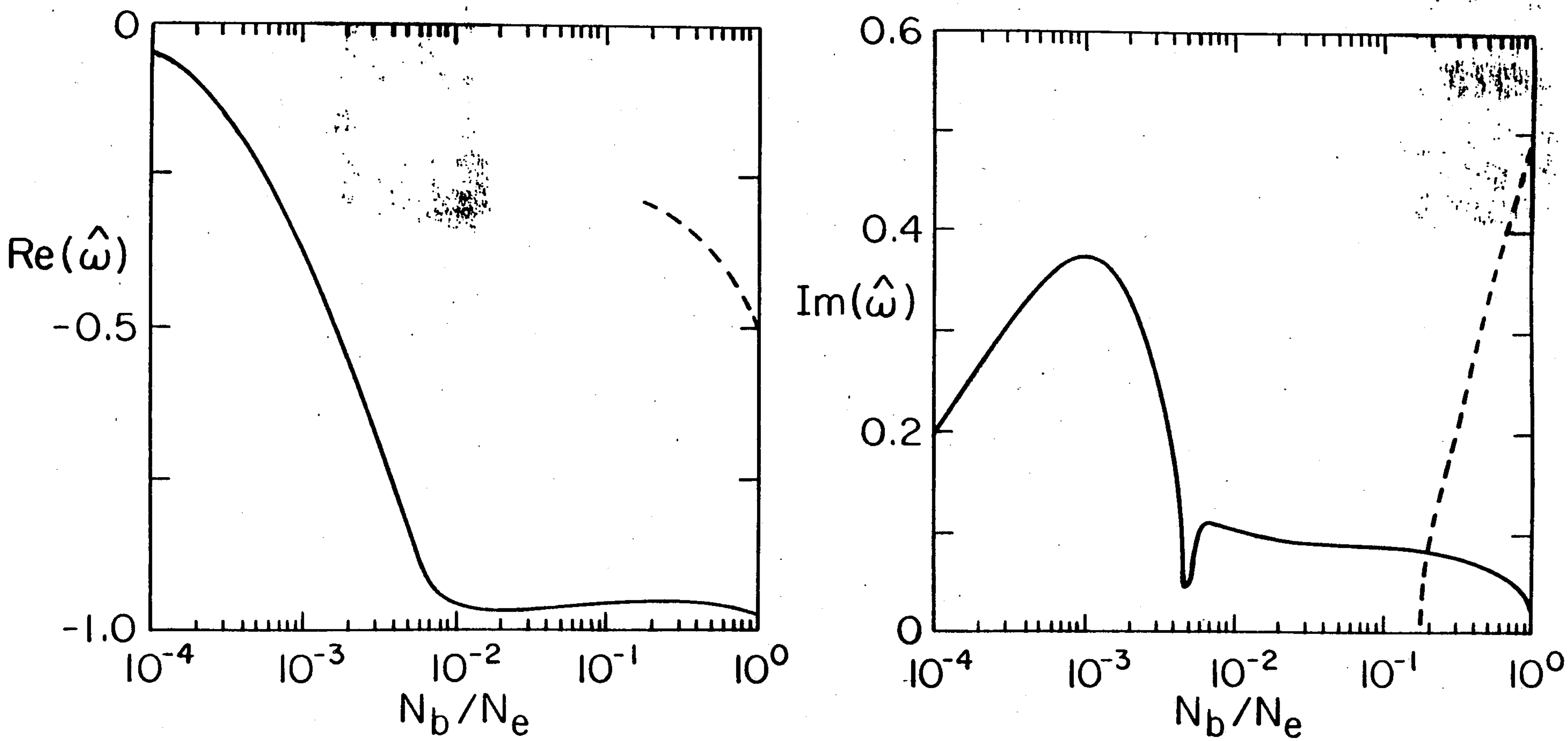
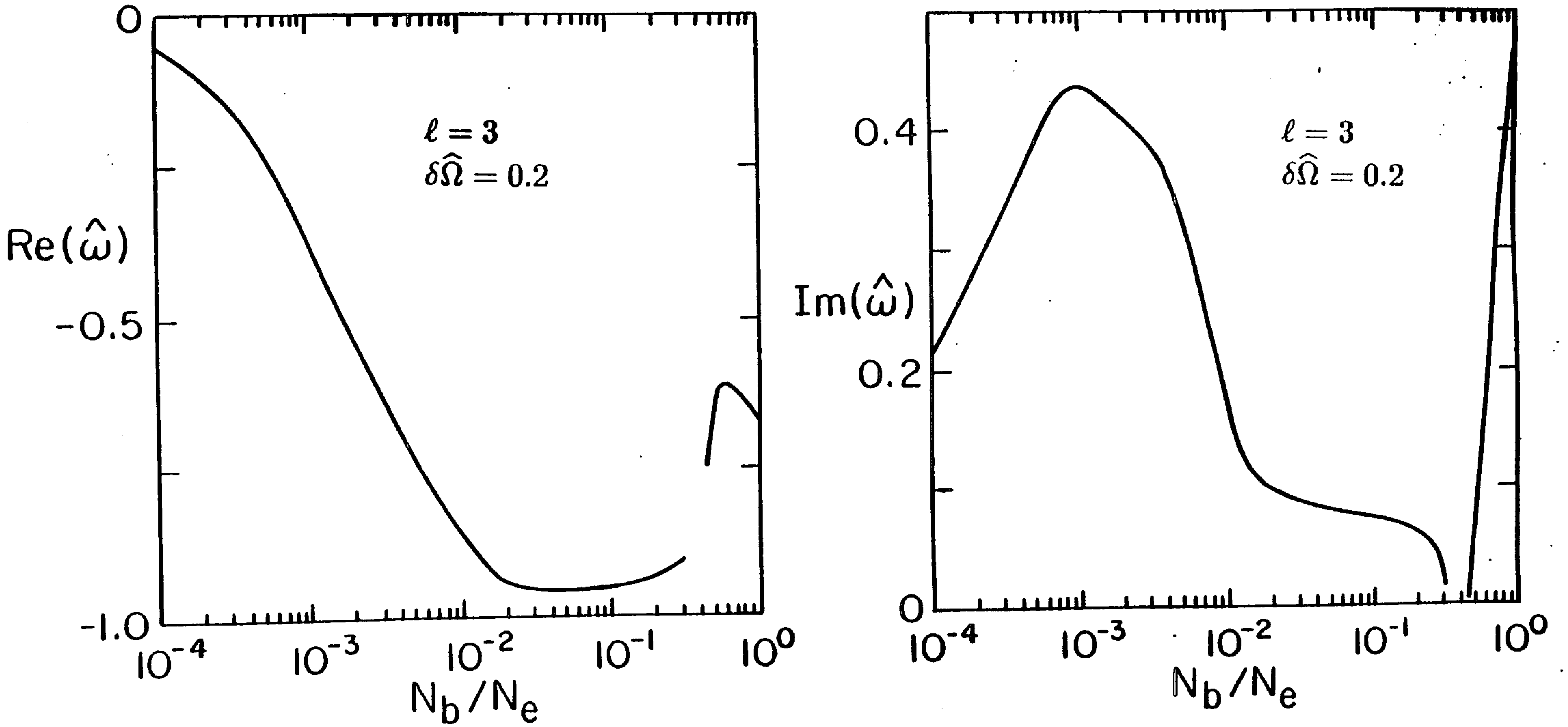


Fig. 1(b)



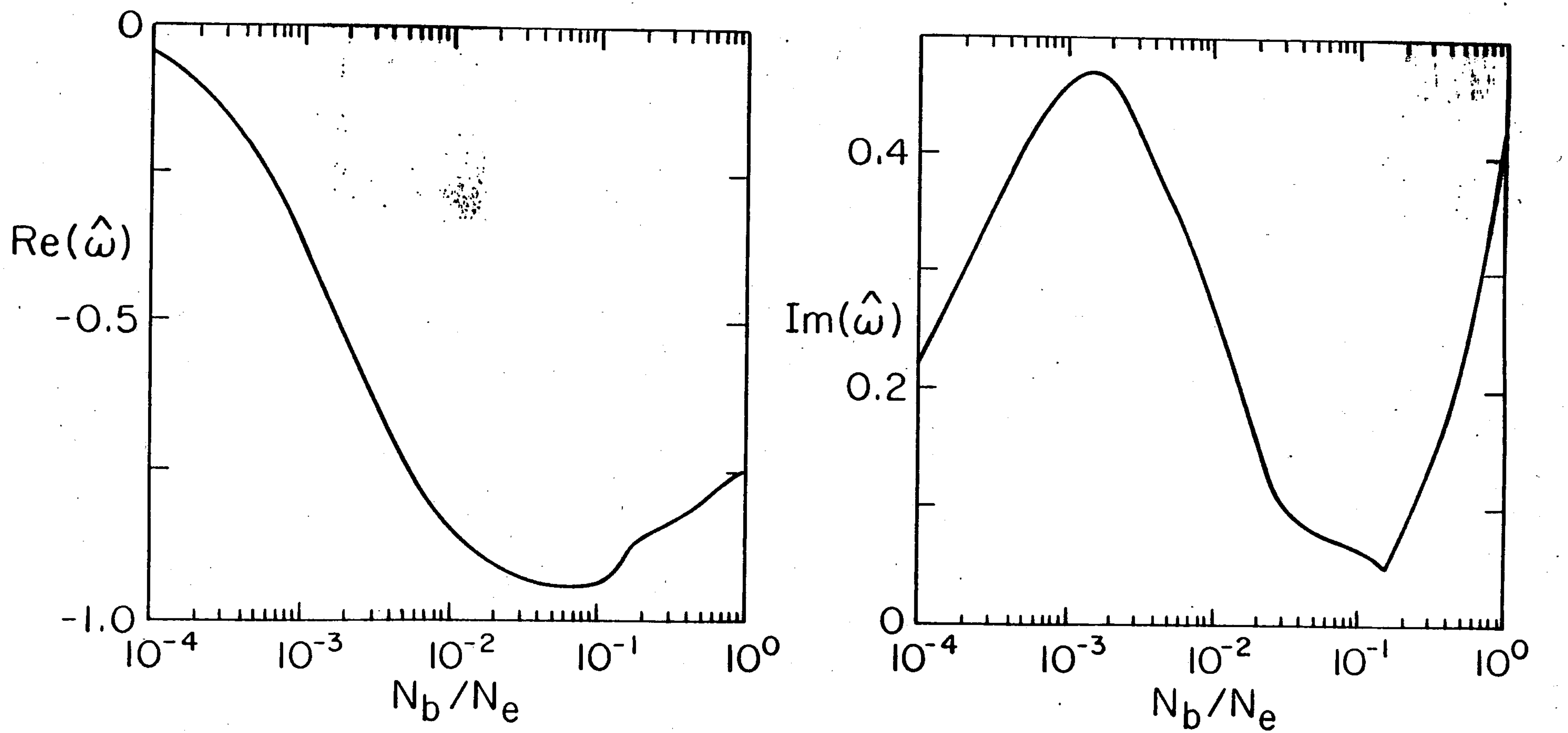
0
Re($\hat{\omega}$)
-0.5
-1.0
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}
N_b/N_e
0.6
0.4
Im($\hat{\omega}$)
0.2
0
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}
N_b/N_e

Fig. 2(a)



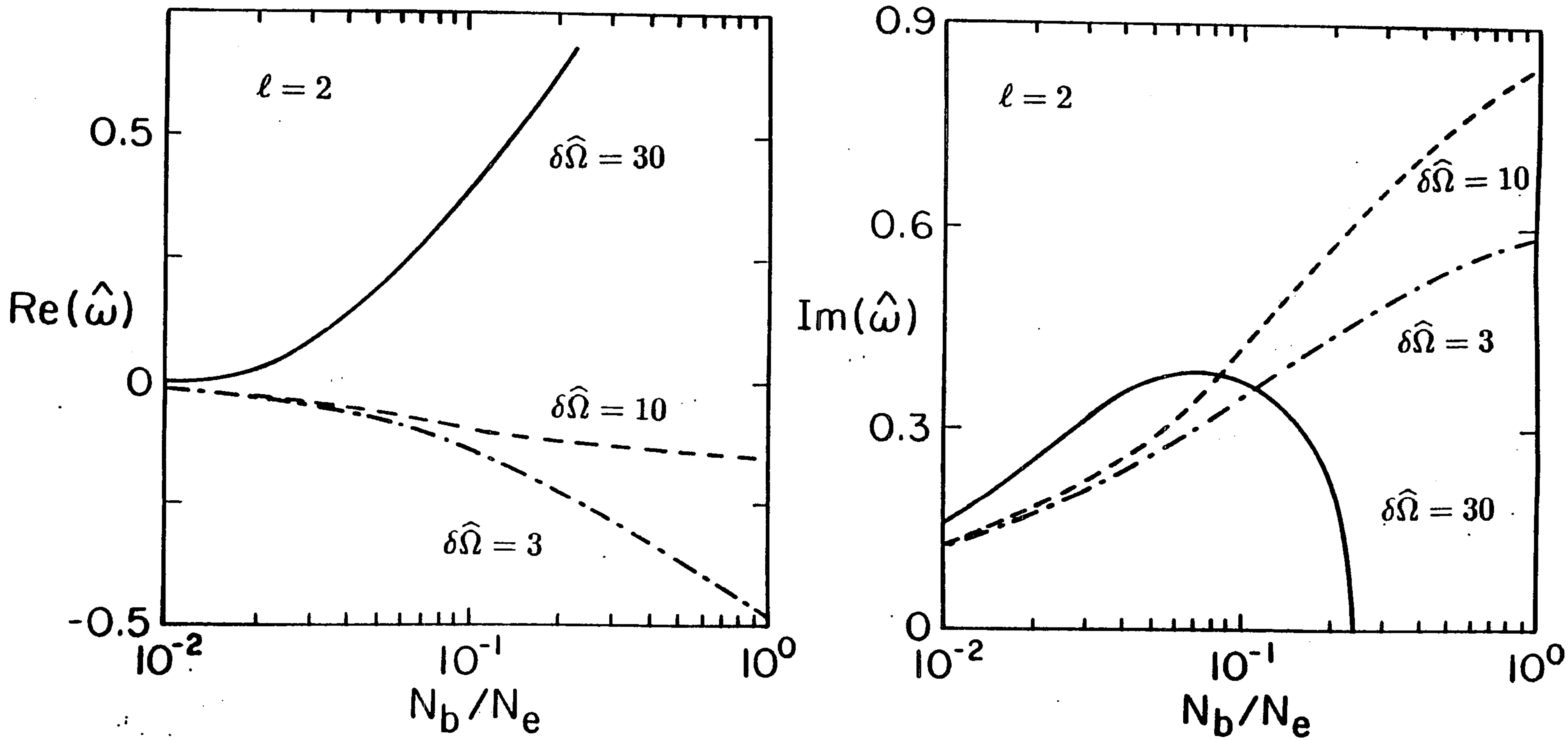
Re($\hat{\omega}$)
0
-0.5
-1.0
$\ell = 3$
$\delta\hat{\Omega} = 0.2$
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}
N_b/N_e
Im($\hat{\omega}$)
0.4
0.2
0
$\ell = 3$
$\delta\hat{\Omega} = 0.2$
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}
N_b/N_e

Fig. 2 (b)



0
Re($\hat{\omega}$)
-0.5
-1.0
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}
N_b/N_e
0.4
Im($\hat{\omega}$)
0.2
0
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}
N_b/N_e

Fig. 2(c)



Re($\hat{\omega}$)
0.5
0
-0.5
$\ell = 2$
$\delta\hat{\Omega} = 30$
$\delta\hat{\Omega} = 10$
$\delta\hat{\Omega} = 3$
10^{-2}
10^{-1}
10^{0}
N_b/N_e
Im($\hat{\omega}$)
0.9
0.6
0.3
0
$\ell = 2$
$\delta\hat{\Omega} = 10$
$\delta\hat{\Omega} = 3$
$\delta\hat{\Omega} = 30$
10^{-2}
10^{-1}
10^{0}
N_b/N_e

Fig. 3

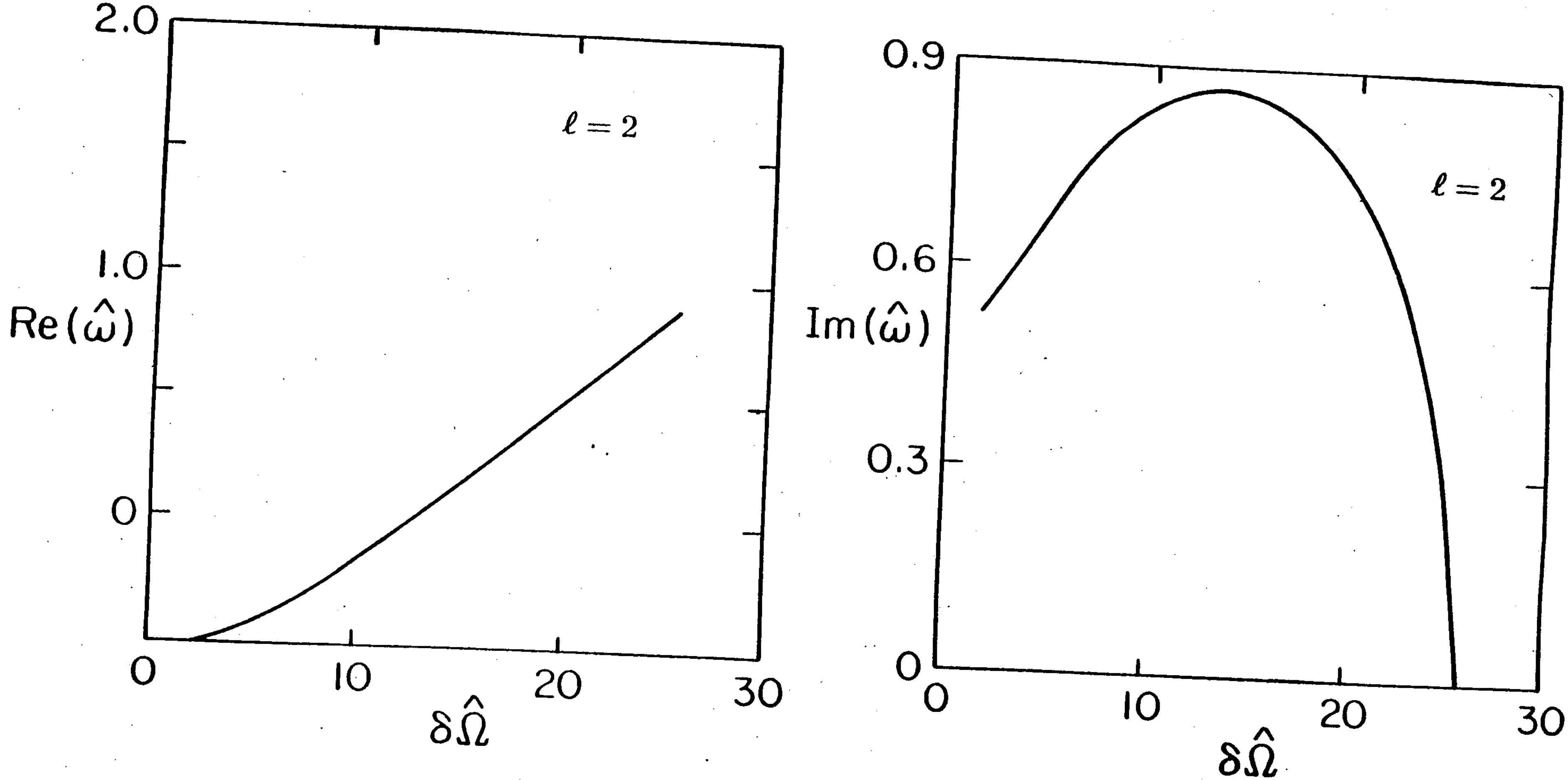

2.0
1.0
0
Re($\hat{\omega}$)
$\ell = 2$
0
10
20
30
$\delta\hat{\Omega}$
0.9
0.6
0.3
0
Im($\hat{\omega}$)
$\ell = 2$
0
10
20
30
$\delta\hat{\Omega}$

Fig. 4