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**Stability of Drift-Wave Modons in the Presence of Temperature Gradients**

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**Abstract**

In the homogeneous Hasegawa-Mima equation the dipole vortex or modon solution is well known to be robustly stable from both analytic and numerical studies. In the inhomogeneous plasma where $\nabla T_e \neq 0$ the corresponding vortex has an external structure extending into the high temperature region. Lyapunov stability method is used to determine the stability properties of these extended vortex structures. The overall growth rate of deformation caused by the presence of temperature inhomogeneity is shown to be bounded by $(R/L_T)^2$, where $R$ is the radius of the core of the vortex and $L_T$ is the scale length of the temperature gradient. The most important source of instability is identified as the excitation of monopolar and dipolar perturbations with short spatial scales $\lesssim R$, which are approximately independent of the presence of density and temperature gradients.

# I. Introduction

In the past decade a number of stationary travelling nonlinear solitary-vortex solutions have been obtained for a class of models in plasma and fluid dynamics.[1–5] These equations are important both in plasma physics, and in oceanic and atmospheric physics, where they are used to study the convective motion responsible for anomalous transport. Coherent structures in the form of monopolar and/or dipolar vortices in plasmas arise in the strongly nonlinear stage of various plasma instabilities driven by density and temperature inhomogeneities,[1,2] plasma flows,[6] parametric processes,[7] time-stationary electric fields (Kelvin-Helmholtz instability)[8] when the particle trapping takes place.

A particularly important equation of this type is the Charney-Hasegawa-Mima equation, which describes both nonlinear drift waves in plasmas, and nonlinear Rossby waves in shallow rotating fluid such as the Earth's atmosphere and oceans. This two-dimensional nonlinear wave equation is

$$\frac{\partial}{\partial t}(1-\lambda^2\nabla^2)\varphi + v_d\frac{\partial\varphi}{\partial y} - (\mathbf{e}_z\times\nabla\varphi)\cdot\nabla\nabla^2\varphi = 0 \ . \tag{1}$$

where $v_d$ is the macroscopic fluid drift velocity, and $\lambda$ is the so-called Rossby radius. Although Eq. (1) is not fully integrable,[9] and the corresponding dipole vortex solutions (or modons) are not solitons in the strict sense, it has been proven that they are linearly stable,[10,11] while numerical[12] and laboratory[13] experiments show that they, in most cases, "survive" even large perturbations, such as collisions with other vortices propagating in the same (or opposite) direction. This property of modons allows the turbulent plasma state to be described as the superposition of coherent vortices and weakly correlated fluctuations, and to calculate the corresponding transport coefficients.[14]

Structural stability of the Hasegawa-Mima modons is an important, unresolved issue. In the present context, structural stability corresponds to the existence and the lifetime of the modons in systems which are slightly different than the one described by Eq. (1), due to the presence of the finite temperature ($\lambda^2$-gradient) and density ($v_d$-gradient) gradients over the scale of the vortex structure.[15] In the presence of weak dissipation due to plasma viscosity[16] and Landau damping by resonant electrons,[6] an adiabatic theory of dipole vortices was developed, and it was shown that they preserve their shape, but gradually spread, and change their phase velocity.

Presence of a temperature inhomogeneity which is not included in Eq. (1), is essential for the development of the drift wave instability, which is thought to be responsible for the high level of potential fluctuations in laboratory plasmas. The drift wave coherent structures lead to the anomalous particle transport, as well as to plasma self-organization in the form of vortices. In an early work[17] the existence of a new class of solutions in the form of moving monopolar vortices was suggested, these results were strongly criticized later,[18] but authors of a recent work[19] argue that such monopoles are possible (within a certain ordering of space-time scales), and prove their structural stability in the presence of a weak vector nonlinearity. Dipole vortices in such plasmas develop a wake (or oscillating tail) of linear drift waves on the high temperature side, and some numerical simulations[14,15,19,20] indicate break-up of the vortex into two monopoles, one of which is rapidly dispersed. However, it may be argued that these results were based on an equation whose validity is questionable,[18] and that the experiments did not take into account the existence of the wake in the initial data. Thus, instability of the dipole reported in these simulations could be interpreted as the inability of the plasma to build up the wake from the initial distribution of the potential used in[14,15,19,20] the initial data.

In this work we study the linear stability of dipole vortices in the presence of a strong temperature inhomogeneity using Lyapunov's functional method. We demonstrate that adoption of proper initial data, including the presence of a connected wake, may stabilize the vortex if the temperature profile satisfies certain conditions, and that the Lyapunov instability exponent for growth is a small quantity, smaller than $\left(\frac{R}{L_T}\right)^2$, where $R$ and $L_T$ are characteristic radius of the vortex and the scale length of the temperature inhomogeneity, respectively. We show analytically that the perturbations which are critical for the development of the instability corresponds to the rotation or tilting of the dipole , which may in the next stage lead to its splitting into monopole structures. However, this process occurs on a slow time scale, being related to the excitation of monopole and dipole-like perturbations with short scales in the radial direction, rather than to the temperature gradient, and we may conclude that the dipole vortex structure in the presence of a finite temperature gradient is more stable than it was previously thought.

Stability of the dipole vortex in the presence of monopole and dipole perturbations, which are critical for the stability, is investigated numerically in the case $\nabla T_e = 0$. Vortices propagating in the direction of the electron diamagnetic drift clearly show the presence of a tilt instability, which leads to the exponential growth of the initial tilt, and to the splitting of the vortex.

However, vortices propagating in the opposite direction, that is parallel to the ion diamagnetic drift, appear to be stable to tilt perturbations. In a numerical experiment they were preserving their shape for as long as $\Delta t \sim 100\tau_E$ where $\tau_E$ is the period of the vortex rotation, $\tau_E \sim R/v_E$ where $R$ is radius of the vortex and $v_E$ the mean $\mathbf{E}\times\mathbf{B}$ flow velocity in the vortex.

# II. Basic Equations

We study low frequency perturbations ($\frac{d}{dt} \ll \Omega_i$, $\Omega_i$ being the ion gyrofrequency) in plasma with cold ions and warm electrons, in which the equilibrium plasma density $n_0$, and electron temperature $T_e$ are inhomogeneous along the $y$ axis. Using the well-known expression for the ion drift velocity in the presence of a slowly time-varying electrostatic potential $\phi(x,y,t)$:

$$\mathbf{v}_{\perp i} = \frac{1}{B_0}\mathbf{e}_z \times \nabla\phi - \frac{1}{B_0\Omega_i}\left[\frac{\partial}{\partial t} + \frac{1}{B_0}(\mathbf{e}_z \times \nabla\phi)\cdot\nabla\right]\nabla_\perp\phi \tag{2}$$

and assuming quasineutrality $n_i = n_e$, and Boltzmann distributed electrons (i.e. neglecting finite electron mass effects)

$$n_e = n_0(y) \exp\left(\frac{e\phi}{T_e(y)}\right) \tag{3}$$

we readily obtain from the ion continuity equation:

$$\left[\frac{\partial}{\partial t} + \frac{1}{B_0}(\mathbf{e}_z \times \nabla\phi)\cdot\nabla\right]\left[\log\, n_0(y) + \frac{e\phi}{T_e(y)} - \frac{1}{\Omega_i B_0}\nabla_\perp^2\phi\right] = 0 \tag{4}$$

(here $e$ is the ion charge). Equation (4) is equivalent to Eq. (45) of Ref. 15 when $x$ and $y$ are interchanged and the log $n_0(y)$ is expanded in powers of $y - y_0$.

More conveniently, in the reference frame moving with the velocity $v_x$ along the $x$ axis, we may write:

$$\left\{\frac{\partial}{\partial t} + \frac{1}{B_0}[\mathbf{e}_z \times \nabla(\phi + B_0\, v_x\, y)]\cdot\nabla\right\}[u(y) + \widehat{p}\phi] = 0 \tag{5}$$

where

$$u(y) = \frac{T_e(y_0)}{e}\log\, n_0(y)\ ,$$

$$\widehat{p} = \frac{1}{\tau(y)} - \rho_s^2\nabla_\perp^2\ , \qquad \tau(y) = \frac{T_e(y)}{T_e(y_0)}\ , \qquad \rho_s^2 = \frac{T_e(y_0)}{m_i\,\Omega_e^2} \tag{6}$$

and the parameter $y_0$ is chosen arbitrarily. Any stationary solution of Eq. (5) which propagates with the velocity $v_x$ along the $x$ axes, $\phi_s$, satisfies

$$\phi_s + B_0\, v_x y = G[u(y) + \widehat{p}\phi_s] \tag{7}$$

where $G$ is arbitrary function of its argument. Evidently, the form of the stationary solution $\phi_s$ depends on our choice of the function $G$. The standard modon is obtained in a plasma with exponential density profile and homogeneous temperature,

$$n_0(y) = N_0 \exp\left(\frac{y}{L_n}\right)$$

$$T_e(y) = \text{const} \tag{8}$$

if we adopt $G$ to be a linear function, $G(\xi) = G\cdot\xi$, allowing the coefficient $G$ to have different values $G^{\text{in}}, G^{\text{out}}$, inside and outside of a closed circle with the radius $R$, respectively. Interior of the circle, $r < R$, is called the modon core. The modon has the form:

$$\phi_s^{(0)} = B_0\, v_x R \sin\theta \cdot \begin{cases} -\left(1+\dfrac{\rho^2}{\kappa^2}\right)\dfrac{r}{R} + \dfrac{\rho^2}{\kappa^2}\dfrac{J_1(\kappa r)}{J_1(\kappa R)}\,, & r < R \\ -\dfrac{K_1(\rho r)}{K_1(\rho R)}\,, & r > R \end{cases} \tag{9}$$

where $r = (x^2+y^2)^{1/2}$, $\theta = \arctan\frac{y}{x}$, $J_1, K_1$, are the first order ordinary and modified Bessel functions, and the characteristic wavenumber $\kappa$ is determined from the dispersion relation:

$$\begin{gathered} \frac{-1}{\kappa}\frac{J_2(\kappa R)}{J_1(\kappa R)} = \frac{1}{\rho}\frac{K_2(\rho R)}{K_1(\rho R)} \\ \rho = \frac{1}{\rho_s}\sqrt{1-\frac{1}{G^{\text{out}}}}\quad,\quad \kappa = \frac{1}{\rho_s}\sqrt{\frac{1}{G^{\text{in}}}-1}\quad,\quad G^{\text{out}} = \frac{eB_0}{T_e}\,v_x\,L_n \equiv \frac{v_x}{v_d} \end{gathered} \tag{10}$$

where $v_d = T_e/eBL_n$ is the electron diamagnetic drift velocity. In plasmas with strong inhomogeneities, i.e. when the density and temperature profiles are not given by Eq. (8), adoption of a linear function $G$ in Eq. (7) in the form

$$G(\xi) = G_0^{\text{out}} + G^{\text{out}}\xi$$

allows us to separate variables in Cartesian coordinates, and to write the standard WKB solution outside of the modon core:

$$\phi_s^{\text{out}} = \frac{g(y)}{\kappa^2(y)} - \frac{1}{\kappa^2(y)}\frac{d^2}{dy^2}\frac{g(y)}{\kappa^2(y)} + \frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{ikx}\left|\kappa^2(y)-k^2\right|^{-1/4}$$
$$\cdot\left[\phi^{(1)}(k)\exp\left(i\int^y\sqrt{\kappa^2(y')-k^2}\,dy'\right) + \phi^{(2)}(k)\exp\left(-i\int^y\sqrt{\kappa^2(y')-k^2}\,dy'\right)\right] \tag{11a}$$

or, more conveniently, in the cylindrical frame $r, \theta$:

$$\phi_s^{\text{out}} = \frac{g(y)}{\kappa^2(y)} - \frac{1}{\kappa^2(y)}\cdot\frac{d^2}{dy^2}\frac{g(y)}{\kappa^2(y)} + \frac{1}{2}\sum_{n=-\infty}^{\infty} e^{in\theta}\left[a_n^{(1)}(y)\mathcal{H}_n^{(1)} + a_n^{(2)}(y)\mathcal{H}_n^{(2)}\right] \tag{11b}$$

where the following notations were used:

$$\kappa^2(y) = \frac{1}{\rho_s^2}\left[\frac{1}{G^{\rm out}} - \frac{1}{\tau(y)}\right]$$

$$g(y) = \frac{1}{\rho_s^2}\left[u(y) - \frac{1}{G^{\rm out}}\cdot B_0\, v_x y + \frac{G_0^{\rm out}}{G^{\rm out}}\right] .$$

$$\mathcal{H}_n^{(1)} = \frac{1}{\pi}\int_{i\infty-\pi/2}^{-i\infty+\pi/2} d\varphi \exp i\left[-n(\theta+\varphi) + \int^r dr'\kappa(y')\sin(\theta+\varphi')\right]$$

$$\mathcal{H}_n^{(2)} = \frac{1}{\pi}\int_{-i\infty+\pi/2}^{i\infty+\frac{3\pi}{2}} d\varphi \exp i\left[-n(\theta+\varphi) + \int^r dr'\kappa(y')\sin(\theta+\varphi')\right]$$

$$\varphi' = \arcsin\left[\frac{\kappa(y)}{\kappa(y')}\sin\varphi\right] ,$$

$$a_n^{(1),(2)}(y) = \frac{1}{2\pi}\int_0^{2\pi} d\varphi e^{in\varphi}\left|\kappa(y)\cos\varphi\right|^{1/2}\cdot\phi^{(1),(2)}(\kappa(y)\sin\varphi) \tag{12}$$

and $G^{\rm out}$ is given by Eq. (10). It can be easily seen that in the limit of homogeneous temperature, $\kappa(y) = \kappa = \text{const}$, functions $\mathcal{H}_n^{(1),(2)}$ reduce to the Hankel functions $H_n^{(1),(2)}(\kappa r)$.

The WKB solution which in the region $y \simeq y_0$ behaves like the vortex (9) is obtained if the Fourier amplitudes $\phi^{(1),(2)}(k)$ are given by:

$$\phi^{(1)}(k) = \phi^{(2)}(k) = \left|\kappa^2(y_0) - k^2\right|^{1/4} . \tag{13}$$

The Bessel function approximation in Eq. (13) is used in Ref. 15, Eqs. (32)–(36). In the following we will translate the $y$ coordinate to take $y_0 = 0$.

The solution within the modon core is obtained allowing for a weakly nonlinear function $G$,

$$G(\xi) = G_0^{\rm in} + G_1^{\rm in}\xi + G_2^{\rm in}\xi^2 ,$$

and treating the nonlinearities and inhomogeneities as small perturbations. Keeping only small terms of the order $\frac{R}{L_T}$, where $L_T = \left(\frac{d}{dy}\log T_e\right)^{-1}$, we can write the interior solution as:

$$\phi_s^{\rm in} = \phi_s^{(0)} + \frac{G_0^{\rm in}}{\kappa^2\rho_s^2 G^{\rm in}} + \delta\phi_0 - \delta\phi_2\cos 2\theta \tag{14}$$

where $\phi_s^{(0)}$ is given by Eq. (9), and:

$$\delta\phi_n(r) = J_n(\kappa r)\left[a_n - \int_R^r dr\, h(r')Y_n(\kappa r')\right] + Y_n(\kappa r)\int_0^r dr'\, h(r')J_n(\kappa r') \qquad n = 0, 2$$

$$h(r) = \frac{\pi r}{4\sin^2\theta}\left[\frac{G_2^{\rm in}}{\rho_s^2 G^{\rm in}}\left(\phi_s^{(0)} + B_0\, v_x y\right)^2 - \phi_s^{(0)} y \left.\frac{d\kappa^2(y)}{dy}\right|_{y=0}\right] \ . \tag{15}$$

We assume that the cutoff is far away from the modon core, $y_c \gg R$, and with the same accuracy $\frac{R}{L_T}$ as above we take the core to be a circle with a radius $R$, slightly shifted from the origin $x = 0, y = 0$:

$$r = R(\theta) = R\cdot(1 + \eta\sin\theta)\ , \qquad \eta = \mathcal{O}\left(\frac{R}{L_T}\right) \ . \tag{16}$$

The free parameters $R, v_x, G_0^{\rm in}, G_0^{\rm out}, G_2^{\rm in}$ appearing in (11), (14)–(15) are determined from the usual continuity conditions for the functions $\phi_s$, $\frac{\partial\phi_s}{\partial r}$, and $G$. Continuity of the first cylindrical harmonic readily recovers Eq. (10), while continuity of the zeroth and second cylindrical harmonics of the potential $\phi_s$, and of the function $G(u(y) + \widehat{p}\phi_s)$ gives:

$$-\frac{\partial Y_0(\kappa R)}{\partial R}\int_0^R dr' h(r')J_0(\kappa r') + \frac{1}{2\pi}\int_0^{2\pi} d\theta \left.\frac{\partial\phi_s^{\rm out}}{\partial r}\right|_{r=R}$$

$$= \frac{1}{J_0(\kappa R)}\frac{\partial J_0(\kappa R)}{\partial R}\left[\frac{G_0^{\rm in}}{\kappa^2\rho_s^2 G^{\rm in}} - Y_0(\kappa R)\int_0^R dr' h(r')J_0(\kappa r') + \frac{1}{2\pi}\int_0^{2\pi} d\theta\phi_s^{\rm out}(r = R)\right]$$

and

$$-\frac{\partial Y_2(\kappa R)}{\partial R}\int_0^R dr' h(r')J_2(\kappa r') + \frac{1}{\pi}\int_0^{2\pi} d\theta\cos 2\theta \left.\frac{\partial\phi_s^{\rm out}}{\partial r}\right|_{r=R}$$

$$= \frac{1}{J_2(\kappa R)}\frac{\partial J_2(\kappa R)}{\partial R}\left[-Y_2(\kappa R)\int_0^R dr' h(r')J_2(\kappa r') + \frac{1}{\pi}\int_0^{2\pi} d\theta\cos 2\theta\phi_s^{\rm out}(r = R)\right]$$

where

$$G_0^{\rm in,out} = \left(1 + \eta\sin\theta\cdot R\frac{\partial}{\partial R}\right)\left[\phi_s + B_0 v_x y - G^{\rm in,out}\cdot(u(y) + \widehat{p}\phi_s)\right]_{r=R} \ . \tag{17}$$

The system of equations (17) from which, in principle, all constants of integration $G_0^{\rm in}$, $G_0^{\rm out}$, $G_2^{\rm out}$, $a_0$, $a_2$ can be determined, is closed by the requirement that the right-hand side of Eq. (17) is independent on $\theta$. Thus, the stationary solution (11) and (14) is fully determined.

It should be noted here that the outer solution with a linear function $G$ is possible only if the density profile is such that $g(y)$ in Eq. (12) is finite for both $y = \pm\infty$. One easily verifies that solution (11) is well localized at the low temperature side (where $\kappa^2(y) < 0$), while at the high temperature side beyond the cutoff $y_c$ (i.e., where $\kappa^2(y) > 0$), the vortex is accompanied by a wake of drift waves. Energy of the wake can be very large, even infinite, with the largest contribution coming from the region $y \to -\infty$. Thus, if a vortex was made initially with an incomplete wake, its central part would radiate energy until the wake is "filled." Physically, one may envisage the vortex as being produced by a drift-wave coming from the hot electron side, and the wake is simply a standing wave, due to the reflection in the cutoff region.

# III. Stability Analysis

A particular stationary solution $\phi_s$ is said to be stable if any small perturbation $\delta\phi$ remains bounded as $t \to \infty$. Linearizing Eq. (5) around $\phi_s$, and using (7), we obtain the following equation for the perturbation:

$$\frac{\partial}{\partial t}\widehat{p}\delta\phi = \widehat{A}\,\widehat{H}\,\widehat{p}\delta\phi \tag{18}$$

where the notation was used:

$$\begin{aligned} \widehat{A} &= \frac{-1}{B_0}\,\{\mathbf{e}_z \times \nabla\,[u(y) + \widehat{p}\phi_s]\} \cdot \nabla \\ \widehat{H} &= G - \widehat{p}^{-1}\,, \qquad G = G^{\rm in}, G^{\rm out}\,. \end{aligned} \tag{19}$$

Our Eq. (18) coincides with Eq. (64) in Ref. 10 by Laedke and Spatschek.

Although Eq. (18) is linear, it is impossible to find analytically its general solution, which would, of course, give us complete information about the time evolution of the response to any initial distribution of the electrostatic potential. However, the stability can be studied using a version of the Lyapunov method.

First, multiplying Eq. (18) by $\widehat{H}\,\widehat{p\delta\phi}$, and integrating for the whole space, we obtain the following conservation law:

$$\frac{\partial L}{\partial t} = 0 \tag{20}$$

where $L$ is a quadratic integral of motion:

$$L = \int dx\, dy\, \widehat{p\delta\phi}\, \widehat{H}\, \widehat{p\delta\phi} \equiv \left\langle \widehat{p\delta\phi} | \widehat{H} | \widehat{p\delta\phi} \right\rangle \ . \tag{21}$$

Next, we note that the eigenvalue problem, corresponding to the operator $\widehat{H}$:

$$\widehat{H} h = \lambda h \tag{22a}$$

or, equivalently

$$\left[ \nabla_\perp^2 + \frac{1}{\rho_s^2} \left( \frac{1}{G - \lambda} - \frac{1}{\tau(y)} \right) \right] h(\lambda, \mathbf{r}_\perp) = 0 \tag{22b}$$

generates a set of eigenfunctions $h(\lambda_{m,n}, \mathbf{r}_\perp) \equiv h_{m,n}$, which provide a complete basis. This allows us to expand the function $\widehat{p\delta\phi}$ as

$$\widehat{p\delta\phi} = \sum_{m,n} p\delta\phi_{m,n}\, h_{m,n} \ , \tag{23}$$

and to rewrite the conservation law (20) as:

$$L = \sum_{m,n} \lambda_{m,n}\, p\delta\phi_{m,n}^2 = \text{const} \tag{24}$$

Our system is stable if all amplitudes $p\delta\phi_{m,n}$ remain finite at $t \to \infty$. This is obviously fulfilled if all the eigenvalues $\lambda_{m,n}$ have the same sign, or equivalently, if the conserved quantity $L$ has the same sign for all realizations of the perturbation $\delta\phi$. It has been shown[10] that the latter statement is both a sufficient and necessary condition, in the case of homogeneous temperature.

In the absence of the temperature gradient, $\tau(y) = 1$, Eqs. (22) is readily solved, and we can express eigenfunctions $h_{m,n}$ in the form:

$$h_{m,n}(r,\theta) = a_{m,n}\, e^{in\theta} \cdot \begin{cases} \dfrac{J_n(\kappa_{m,n} r)}{J_n(\kappa_{m,n} R)}\,, & r < R \\ \dfrac{K_n(\rho_{m,n} r)}{K_n(\rho_{m,n} R)}\,, & r > R \end{cases} \tag{25}$$

where

$$\begin{aligned} \kappa_{m,n} &= \frac{1}{\rho_s}\left(\frac{1}{G^{\text{in}} - \lambda_{m,n}} - 1\right)^{1/2} \\ \rho_{m,n} &= \frac{1}{\rho_s}\left(1 - \frac{1}{G^{\text{out}} - \lambda_{m,n}}\right)^{1/2} . \end{aligned} \tag{26}$$

Here $a_{m,n}$ is determined from the normalization $\langle h_{m,n}|h_{m,n}\rangle = 1$, coefficients $G^{\text{in}}$, $G^{\text{out}}$ are given by Eq. (10), and $\lambda_{m,n}$ is determined from the following dispersion relation, resulting from the continuity of $\nabla h_{m,n}$ at $r = R$:

$$-\,\kappa_{m,n}\,\frac{J_{n-1}(\kappa_{m,n} R)}{J_n(\kappa_{m,n} R)} = \rho_{m,n}\,\frac{K_{n-1}(\rho_{m,n} R)}{K_n(\rho_{m,n} R)}\ . \tag{27}$$

The number of negative eigenvalues $\lambda_{m,n}$ in the sum (24) can now be determined by the following argument. Since the right-hand side of Eq. (27) is positive, wavenumber $\kappa_{m,n}$ must be located between zeros of the same order of the functions $J_{n-1}, J_n$:

$$j_{m,n-1} \le \kappa_{m,n} R \le j_{m,n} \tag{28}$$

(here $j_{m,n}$ is the $m$th zero of the function $J_n$). From Eq. (28) we may see that there are *only two* wavenumbers which are smaller than the wavenumber $\kappa$ of the ground state modon[10] (see Eq. (10)). These two wavenumbers $\kappa_{1,0}, \kappa_{1,1}$, correspond to the ground states of zeroth and first cylindrical harmonics, and they possess negative eigenvalues $\lambda_{1,0}, \lambda_{1,1}$, respectively. We note also that the dispersion relation for the eigenfunction $h_{1,2}$, which gives $\lambda_{1,2} = 0$, coincides with the dispersion relation for the ground state modon, Eq. (10).

In the presence of a weak temperature gradient,

$$\varepsilon = \mathcal{O}\left(\frac{R}{L_T}\right) \ll 1$$

where $R, L_T$ are scale lengths of the vortex and of the temperature inhomogeneity, respectively, and under assumption that the cutoff is far from the modon, $y_c \gg R$, eigenvalues $\lambda_{m,n}$ will be shifted only by the small amount $\varepsilon$ relative to their $T_e = \text{const}$ values. Thus, there will still be two negative eigenvalues, $\lambda_{1,0}, \lambda_{1,1}$, and possibly the third one will arise if $\lambda_{1,2} = 0$ is shifted toward negative values (however, this one will be relatively small, $|\lambda_{1,2}| \sim \varepsilon$).

In order to complete the stability proof, it would be necessary to demonstrate that amplitudes $p\delta\phi_{1,n}$, $n = 0, 1, 2$, corresponding to negative eigenvalues, are equal to zero. Their maximum values will be estimated by the use of following linear integrals of the motion:

A. Multiplying Eq. (5) by $\phi + B_0 v_x y$, and integrating for the whole space, we obtain

$$\int dx\, dy (\phi + B_0 v_x y) \cdot \frac{\partial}{\partial t} \widehat{p}\phi = 0 \ . \tag{29}$$

Since Eq. (29) is valid for arbitrary $v_x$, we obviously must have:

$$\int dx\, dy\, y\, \widehat{p}\phi = \text{const} \tag{30}$$

which, after linearization gives:

$$\int dx\, dy\, y\, \widehat{p}\delta\phi = 0 \ . \tag{31}$$

B. Multiplying Eq. (5) by $F'[u(y) + \widehat{p}\phi]$, where $F$ is an arbitrary function of its argument, we have

$$\int dx\, dy\, F[u(y) + \widehat{p}\phi] = \text{const} \tag{32}$$

or, after linearization:

$$\int dx\, dy\, F'[u(y) + \widehat{p}\phi_s] \cdot \widehat{p}\delta\phi = 0 \ . \tag{33}$$

The second integral of motion which we use below is obtained from Eq. (33) when we choose $F' = 1$:

$$\int dx\, dy\, \widehat{p}\delta\phi = 0 \ . \tag{34}$$

C. Finally, choosing $F'(\xi) = [G(\xi)]^2 - [G^{\text{out}}(\xi)]^2$ in Eq. (33), we obtain the third constraint

$$\int_{r<R(\theta)} dx\, dy\, (\phi_s + B_0\, v_x\, y)^2 \cdot \widehat{p}\delta\phi = 0\ . \tag{35}$$

Next, we expand $\widehat{p}\delta\phi$ in (31) and (34) in terms of functions $h_{m,n}$. Using Eq. (22) and performing the partial integration, we obtain after some algebra:

$$\mathbf{e}_z \cdot \sum_{m,n} p\delta\phi_{m,n} \int_{r=R(\theta)} d\boldsymbol{\ell} \times \left\{ \varphi_j \left( \frac{1}{\kappa^2_{m,n}(y)} + \frac{1}{\rho^2_{m,n}(y)} \right) \nabla h_{m,n} - \nabla \left[ \varphi_j \left( \frac{1}{\kappa^2_{m,n}(y)} + \frac{1}{\rho^2_{m,n}(y)} \right) \right] h_{m,n} \right\}$$

$$+ \sum_{m,n} p\delta\phi_{m,n} \left[ \int_{r>R(\theta)} dxdy\, h_{m,n} \nabla^2_\perp \frac{\varphi_j}{\rho^2_{m,n}(y)} - \int_{r<R(\theta)} dxdy\, h_{m,n} \nabla^2_\perp \frac{\varphi_j}{\kappa^2_{m,n}(y)} \right] = 0\ ,\ j = 0,1 \tag{36}$$

where

$$\kappa_{m,n}(y) = \frac{1}{\rho_s} \left( \frac{1}{G^{\text{in}} - \lambda_{m,n}} - \frac{1}{\tau(y)} \right)^{1/2}$$

$$\rho_{m,n}(y) = \frac{1}{\rho_s} \left( \frac{1}{\tau(y)} - \frac{1}{G^{\text{out}} - \lambda_{m,n}} \right)^{1/2}$$

$$\varphi_0 = 1$$

$$\varphi_1 = y \tag{37}$$

The last term (volume integral) in Eq. (36) obviously vanishes in the absence of temperature inhomogeneity. It can be shown that this is true also in the presence of a weak temperature gradient, when the eigenfunctions $h_{m,n}$ are approximated by the WKB solutions similar to Eqs. (11) and (14), and provided that the characteristic wavenumber $\rho_{m,n}$ is a well-behaved function on the high temperature side, that is

$$\rho^2_{m,n}(y \to -\infty) \lesssim -\frac{1}{\rho_s^2} \cdot \frac{1}{G^{\text{out}} - \lambda_{m,n}} + \exp\left(\frac{y}{L_\infty}\right) \tag{38}$$

The above analysis, with Eqs. (35) and (36), indicate that amplitudes $p\delta\phi_{m,n}$ of the electrostatic perturbations are determined by the physical processes within the modon core

$(r < R(\theta))$ only. If the cutoff is far from the modon core, the modon is only slightly perturbed by the presence of the temperature gradient, see Eqs. (11) and (14):

$$\phi_s = \phi_s^{(0)} + \mathcal{O}\left(\frac{R}{L_T}\right) \tag{39}$$

where $\phi_s^{(0)}$ is the first cylindrical harmonic given by Eq. (9), and the small correction $\mathcal{O}\left(\frac{R}{L_T}\right)$ contains only $\theta$ independent and $\cos 2\theta$ components. Furthermore, the modon core is a circle centered at $(x, y) = \left(0, \mathcal{O}\left(\frac{R}{L_T}\right)\right)$, that is

$$r = R(\theta) = R \cdot (1 + \eta \sin\theta) \ , \qquad \eta = \mathcal{O}\left(\frac{R}{L_T}\right) \tag{40}$$

which permits us to approximate line integrals in Eq. (36) as

$$\mathbf{e}_z \cdot \int_{r=R(\theta)} d\boldsymbol{\ell} \times \mathbf{f}(r,\theta) \simeq -R \int_0^{2\pi} d\theta \cdot \left\{ f_r(R,\theta) + \eta \sin\theta \frac{\partial}{\partial R}[R f_r(R,\theta)] - \eta \cos\theta f_\theta(R,\theta) \right\} \ . \tag{41}$$

We note also that eigenfunctions $h_{1,0}$, $h_{1,1}, h_{1,2}$ within the modon core are predominantly zeroth, first, and second cylindrical harmonics, having sidebands of the order $\varepsilon$

$$\begin{aligned} &h_{m,n}(r,\theta) \simeq e^{in\theta} \cdot H_{m,n}(r) \left\{ 1 + \varepsilon \left[ a_{m,n}(r) e^{i\theta} + b_{m,n}(r) e^{-i\theta} \right] \right\} \\ &a_{m,n}(r) \sim b_{m,n}(r) \sim 1 \ , \end{aligned} \tag{42}$$

and that the above linear integrals of the motion automatically give zero for the component of the function $\widehat{p}\delta\phi$ which is antisymmetric in $x$. Then within the accuracy to the first order in $\varepsilon$ we can rewrite Eqs. (31), (34), and (35) as:

$$\begin{aligned} &\sum_{m=1}^{\infty} c_{m,0}\, p\delta\phi_{m,0}^{(+)} + \varepsilon c_{m,1}\, p\delta\phi_{m,1}^{(+)} = 0 \\ &\sum_{m=1}^{\infty} d_{m,1}\, p\delta\phi_{m,1}^{(+)} + \varepsilon \left( d_{m,0}\, p\delta\phi_{m,0}^{(+)} + d_{m,2}\, p\delta\phi_{m,2}^{(+)} \right) = 0 \\ &\sum_{m=1}^{\infty} e_{m,2}\, p\delta\phi_{m,2}^{(+)} + \varepsilon \left( e_{m,1}\, p\delta\phi_{m,1}^{(+)} + e_{m,3}\, p\delta\phi_{m,3} \right) = 0 \end{aligned} \tag{43}$$

where $p\delta\phi_{m,n}^{(+)}$ are amplitudes of eigenfunctions $h_{m,n}^{(+)}$ which are even functions of $x$, and all coefficients in Eq. (43) are of the order unity,

$$c_{m,n} \sim d_{m,n} \sim e_{m,n} \sim 1 \ . \tag{44}$$

Following the authors of Ref. 10, we assume that in the sums in Eq. (43) only the ground state modes, with $m = 1$, are present, which is equivalent to the assumption that only perturbations with the longest possible radial scales are excited. Namely, if total energy of initial perturbations is very small, it seems reasonable to expect that it is distributed mostly in the ground state modes whose energy content is the smallest. Then, from Eq. (43) we can estimate the amplitudes $p\delta\phi_{1,0}^{(+)}$, $p\delta\phi_{1,1}^{(+)}$, and $p\delta\phi_{1,2}^{(+)}$ which correspond to modes with negative eigenvalues $\lambda_{1,0}, \lambda_{1,1}, \lambda_{1,2}$:

$$\begin{aligned} p\delta\phi_{1,0}^{(+)} &\sim \varepsilon^3 \mathcal{P} \\ \phi\delta\phi_{1,1}^{(+)} &\sim \varepsilon^2 \mathcal{P} \\ \phi\delta\phi_{1,2}^{(+)} &\sim \varepsilon \mathcal{P} \end{aligned} \tag{45}$$

where $\mathcal{P}$ is some typical mode amplitude with a positive value of $\lambda$. It can also be seen from Eq. (43) that at least one mode with $\lambda > 0$ is always present when modes with negative eigenvalues are excited.

The amplitudes of the antisymmetric eigenfunctions are estimated directly from Eq. (18), which separates into odd and even functions of $x$, due to the fact that the stationary solution $\phi_s$ is even in $x$

$$\frac{\partial}{\partial t} \widehat{p}\delta\phi^{(\pm)} = \widehat{A}\widehat{H}\widehat{p}\delta\phi^{(\mp)} \tag{46a}$$

or, inversely,

$$\widehat{p}\delta\phi^{(\mp)} = \widehat{H}^{-1}\widehat{A}^{-1} \frac{\partial}{\partial t} \widehat{p}\delta\phi^{(\pm)} \tag{46b}$$

where

$$\widehat{p}\delta\phi^{(\pm)}(x,y) = \frac{1}{2}\left[\widehat{p}\delta\phi(x,y) \pm \widehat{p}\delta\phi(-x,y)\right] \ . \tag{47}$$

The inverse operator $\widehat{A}^{-1}$ is given by

$$f(x,y) \equiv \widehat{A}^{-1}h(x,y) = B_0\mathbf{e}_z \cdot \int_{(0,y_0)}^{(x,y)} \mathbf{d}\boldsymbol{\ell}' \times \frac{\nabla(u(y')+\phi_s)}{[\nabla(u(y')+\phi_s)]^2} \cdot h(x',y') \tag{48}$$

where the integration is performed along the equiline $u(y)+\widehat{p}\phi_s = \text{const}$ Noting that Eq. (48) couples the $n$-th cylindrical harmonic of the function $f$ with $n-1$ and $n+1$ harmonics of $h$, we can expand Eq. (46) into cylindrical harmonics and solve for $\delta\phi^{(-)}_{m,n}$. If $\delta\phi^{(+)}_{1,n}$, $n = 0,1,2$ are kept small at all times, (see Eq. (45)), we may conclude that their antisymmetric counterparts $\delta\phi^{(-)}_{1,0}$, $\delta\phi^{(-)}_{1,1}$, $\delta\phi^{(-)}_{1,2}$ are also small quantities:

$$p\delta\phi^{(-)}_{1,0} \sim \varepsilon^2 \frac{\partial}{\partial t}\mathcal{P}$$

$$p\delta\phi^{(-)}_{1,1} \sim \varepsilon \frac{\partial}{\partial t}\mathcal{P}$$

$$p\delta\phi^{(-)}_{1,2} \sim \frac{\partial}{\partial t}\mathcal{P} \ . \tag{49}$$

Finally, from Eqs. (24), (45), and (49) we find that the Lyapunov functional (24) has the lower limit of the order $-\varepsilon^2$

$$L = -\varepsilon\alpha\left(\varepsilon + \beta T\frac{\partial}{\partial t}\right)\mathcal{P}^2 + \sum_{\lambda_{m,n}>0} \lambda_{m,n}\, p\delta\phi^2_{m,n} \gtrsim -\varepsilon^2\mathcal{P}^2 \tag{50}$$

where $\alpha \sim \beta \sim 1$ and $T \sim R/v_x$ is a typical vortex period of rotation. The growth rate of the fastest growing mode can be estimated via the variational principle,[10] yielding

$$\gamma_{\max} \sim \frac{\varepsilon}{T} \sim \frac{R}{L_T}\cdot\frac{v_0}{R} \ . \tag{51}$$

The fastest growing modes are those with $\lambda < 0$, whose amplitudes are small ($\lesssim \varepsilon\mathcal{P}$), and thus they do not contribute significantly to the overall vortex evolution. Using Eqs. (45) and

(49) we find that the "overall growth rate," i.e. rate of change of modes with the order $\mathcal{O}(1)$ is

$$\gamma \sim \frac{\varepsilon^2}{T} \sim \left(\frac{R}{L_T}\right)^2 \frac{v_0}{R}\,. \qquad (52)$$

However, these results are correct only as long as we may neglect the presence of $m \geq 2$ modes in Eq. (43), which correspond to monopoles, dipoles, and quadrupoles having shorter radial scale lengths than the ground modes $h_{1,0}, h_{1,1}$ and $h_{1,2}$. On the long time scale these modes may develop, even if they were completely absent in the initial data, possibly leading to the tilt instability, whose existence was suggested in[20] for westward propagating Rossby-wave modons in the Earth's atmosphere. Similarly, short-scale perturbations, excited by the structural change in the equation produced by Landau damping of hot electrons, can lead to the universal instability of plasma vortices.[21] Characteristic time for the tilt instability can not be estimated by the present analytic method. It may be expected that the temperature gradient does not play a crucial role in its development, since it exists also in plasmas with homogeneous temperature, and thus we conclude that the temperature gradient should not be considered as the main source of instability.

# IV. Numerical Results and Conclusions

In the preceding section we showed that in the presence of an inhomogeneous electron temperature the oscillating tail (provided that it is complete) does not contribute significantly to the instability of the Hasegawa-Mima dipole, Eq. (9). However, the main problem of the dipole stability in plasmas with *homogeneous* temperature still remains unresolved as recently pointed out by Nycander.[22]

Perturbations which are critical for the vortex stability correspond to monopole and dipole-like perturbations, as shown by Eqs. (24) and (28). Although their amplitudes are constrained by the linear integrals of motion, Eqs. (33) and (34), the presence of higher order monopole and dipole modes still make it possible for the critical modes (those with the longest radial scale length, $h_{1,0}$ and $h_{1,1}$) to grow indefinitely, which can be seen from Eq. (43).

In order to derive a constraint of the critical modes of perturbation, ($h_{1,0}$ and $h_{1,1}$), it would be necessary to make use of an infinite number of integrals of motion, which would determine the amplitudes of all the monopole and dipole modes, $h_{m,0}, h_{m,1}, m \geq 1$. However, the Hasegawa-Mima equation is not fully integrable,[9] it has only a finite number of conserved quantities, and consequently such a general proof cannot be constructed.

However, it still may be possible that vortices within a certain range of dipole parameters are stable. In order to check this, we study numerically the evolution of a vortex which is initially perturbed in the critical way, by a small monopole and dipole, which corresponds to a small asymmetry, and tilt of the dipole. If the vortex is not destroyed under such critical perturbation, we may conclude that the negative eigenvalue modes $h_{1,0}(r)$ and $h_{1,1}(r)e^{i\theta}$ in the sums in Eqs. (23) and (24) retain finite amplitudes at all times, and that consequently the vortex remains stable.

First, we test the vortex stability to the dipole (tilt) initial perturbations. We find that vortices propagating in the direction of ion diamagnetic drift, with $v_x > v_d$, i.e. faster than the linear drift waves (which corresponds to $G^{\text{out}} = v_x/v_d > 0$) are unstable. Small initial tilt is growing exponentially, leading to the splitting of the cyclone and anticyclone parts of the vortex, which propagate independently of each other, and eventually disperse.

Conversely, vortices with $v_x/v_d < 0$ are stable for the tilt perturbations. They oscillate around the equilibrium position, and preserve the shape for a long time. In our computer simulations we were able to follow two complete cycles around the box of length $L_x = 20\pi\rho_s$ in the $t\, v_d/\rho_s \cong 100$.

Next, we perturb the vortex with a small amplitude monopole

$$\phi_{dp} \rightarrow \phi_{dp} \pm \frac{\epsilon B_0 v_x R}{1+\left(\frac{r}{R}\right)^2} . \tag{53}$$

As a result, we obtain a slightly asymmetric dipole vortex. As it can be seen from Fig. 1, such asymmetric vortex has a curved trajectory. Similarly, to the previous case of a simple tilt, for dipoles propagating in the electron diamagnetic drift direction, $v_x > v_d$, such perturbations are found to be unstable, since the trajectory is curving away from the straight line. For the case $v_x/v_d = 2$ and $R/\rho_s = 6$ the dipole veers off by $\Delta y \sim R$ in the time $\Delta t\, v_d/\rho_s = 25 - 30$. In the opposite case, $v_x/v_d < 0$, shown in Fig. 2, the trajectory is always curving back toward the straight one, and the dipole is essentially following a trajectory along $x \cong v_x t$, with small oscillations in $y(t)$.

We may conclude that the vortices with $v_x/v_d < 0$ are stable for our choice of initial monopole and dipole perturbations. Since this type of perturbation is critical for the stability, we expect that they remain stable for arbitrary (small) initial perturbations.

The stability analysis of the dipole-like vortex solution of the nonlinear drift wave-Rossby wave equation for a certain class of electron temperature profiles $T_e(y)$ is presented in detail, assuming that the vortex size $R$ is small compared with the temperature gradient scale length $L_T$. The stability analysis shows that mostly the rotational or tilt perturbation give a potentially unstable contribution to Lyapunov functional. The analysis also shows that the growth rate is no larger than the vortex rotation frequency times $(R/L_T)^2$. Other studies indicate potential destabilization independent on temperature inhomogeneity, but arising from small scale modes localized to the surface of the dipole vortex and to the presence of an external shear flow in the surrounding fluid or plasma.

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## Figure Captions

1. Unstable motion of vortex propagating parallel to the electron diamagnetic direction due to a 2% monopolar perturbation of the form in Eq. (53). The angular rotation frequency is $\Omega_E \simeq 3$ in frame (a). In frame (b) at $t\, v_d/\rho_s = 20$ the position the tilt is at 45% and the center is at $\Delta x \simeq 40$ and $\Delta y = R \sim 6$. In frame (c) at $t\, v_d/\rho_s = 40$ the anticyclone ($\phi > 0$) dominates and $\Delta y \cong 9\rho_s$. In frame (d) $t\, v_d/\rho_s = 200$, about 50 rotations of the core, dispersion is setting in.

2. Stable motion of the vortex propagating in the ion diamagnetic direction. The same 2% monopolar perturbation is applied. In frame (a) the rotation rate is $\Omega_E \cong 3$ and the initial speed is $v_x/v_d = -1$. In frame (b) $t\, v_d/\rho_s = 20$ and $\Delta x \cong -20$, $\Delta y = 0$ after ten rotations. Frames (c) and (d) show the stable structure at times $t\, v_d/\rho_s = 40$ and 120, respectively. The vortex is slowly losing speed probably due to numerical effects.

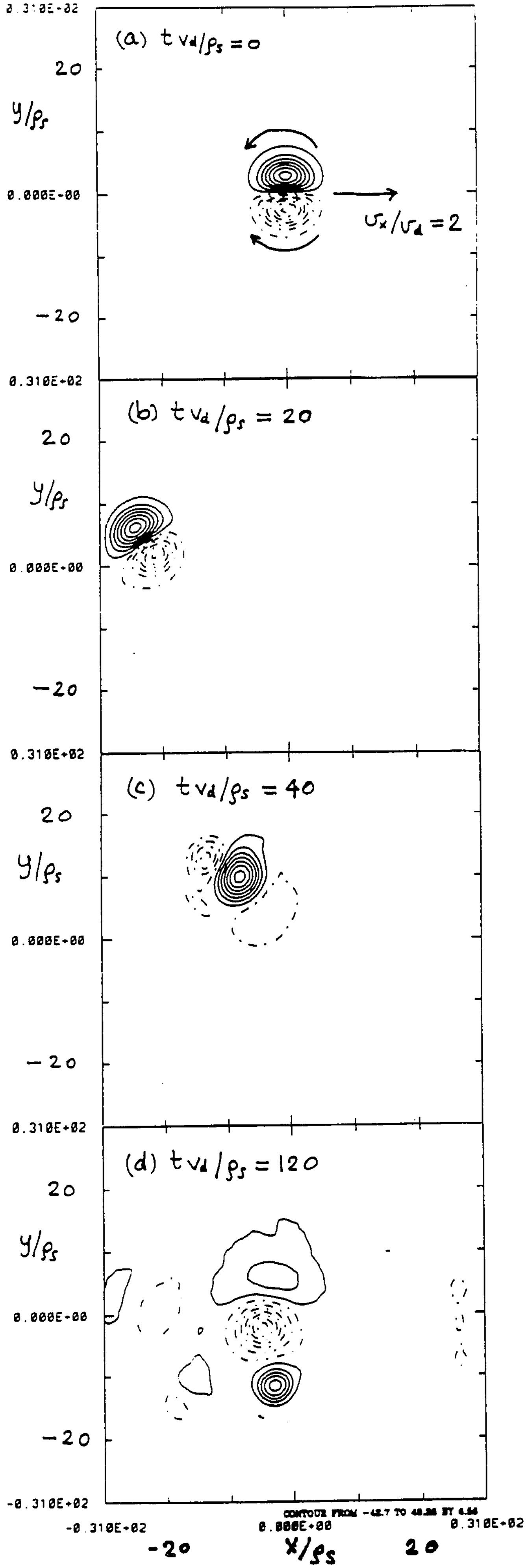
(a) $tv_d/\rho_s=0$
$v_x/v_d=2$
(b) $tv_d/\rho_s=20$
(c) $tv_d/\rho_s=40$
(d) $tv_d/\rho_s=120$
20
$y/\rho_s$
0.000E+00
−20
0.310E+02
-0.310E+02
0.000E+00
-20
$x/\rho_s$
20

Fig. 1

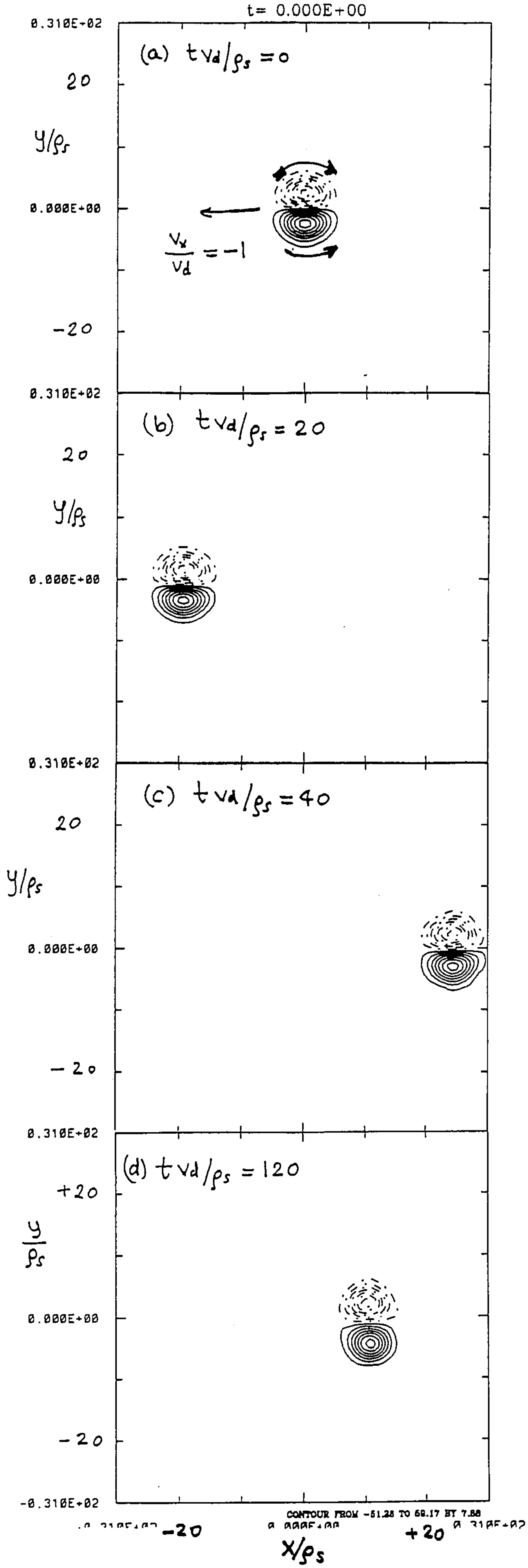
t= 0.000E+00
0.310E+02
(a) $tv_d/\rho_s = 0$
20
$y/\rho_s$
0.000E+00
$\frac{v_x}{v_d} = -1$
-20
(b) $tv_d/\rho_s = 20$
(c) $tv_d/\rho_s = 40$
(d) $tv_d/\rho_s = 120$
+20
-0.310E+02
CONTOUR FROM -51.25 TO 69.17 BY 7.88
-20
+20
$x/\rho_s$

Fig. 2