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INTEGRAL EQUATION FORMULATION OF ELECTROMAGNETIC MODE EQUATIONS

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The coupled set of differential equations describing the electromagnetic perturbations in Tokamak plasmas is reduced to a single simple integral equation with a symmetric kernel. Obvious analytical and computational advantages are discussed.

Slab model analyses of various electromagnetic instabilities in Tokamaks are generally carried out within the framework of the coupled differential equations[1].

$$\psi'' - k^2\psi = \frac{\sigma}{x^2}(\psi - x\phi) \quad , \tag{1}$$

$$\phi'' - k^2\phi = \frac{\sigma}{x x_A^2}(\psi - x\phi) \quad , \tag{2}$$

where ϕ is the normalized electrostatic potential, $\psi = (\omega A_\parallel / c k_\parallel')$ is proportional to the parallel component of the vector potential, $x_A^2 = \omega(\omega + \omega_{ix})/(k_\parallel' v_A)^2$, $k_\parallel' = k/L_S$, k is the poloidal mode number, L_S is the shear length, and σ, the generalized conductivity, can be a very complicated function of x. Equations (1) and (2) are the statements of the parallel component of Ampere's law $J_\parallel = \sigma_\parallel E_\parallel$, and charge neutrality respectively.

Conventionally this system is solved under two distinct approximations. Most drift wave analyses ignore the vector potential ψ, and most tearing mode analyses ignore terms proportional to k^2. Thus proper analytical methods to deal with finite β drift waves or finite k tearing modes for a general conductivity have been almost nonexistent. Below we show that the above set of coupled equations is reducible to a comparatively simple single integral equation with a symmetric kernel. In addition the integral equation is in the localized variable $E_\parallel$ or $J_\parallel$ as distinct from ψ and ϕ which are not localized for tearing modes. The integral equation is easily amenable to a variational principle for an analytical treatment. It must also be stressed that even for numerical work, this method seems to be considerably better because we are dealing with one equation in a localized variable; and by the very nature of the integral equation, the boundary conditions are built in. In the following analyses, we assume that $\sigma(x) = \sigma(-x)$, which is, in general, true. The method, however, can be extended even if σ was not an even or odd function of x.

With the definitions

$$J(x) = \sigma/x^2(\psi - x\phi) \equiv \frac{\sigma}{x^2} E \quad , \tag{3}$$

Eqs. (1) and (2) can be rewritten as

$$\psi'' - k^2\psi = J(x) \quad , \tag{4}$$

$$\phi'' - k^2 = \frac{x}{x_A^2} J(x) \quad , \tag{5}$$

which can be formally solved to obtain the particular integrals

$$\psi_p = -\frac{1}{2k} \int_{-\infty}^{+\infty} dx' e^{-k|x-x'|} J(x') \quad , \tag{6}$$

$$\phi_p = -\frac{1}{2k} \int_{-\infty}^{+\infty} dx' e^{-k|x-x'|} J(x') \frac{x'}{x_A^2} \quad . \tag{7}$$

Notice that the solutions ψ_p and ϕ_p are localized, i.e., $\psi_p \to 0$, $\phi_p \to 0$ as $|x| \to \infty$. These are, in general, the required solutions for all twisting modes including drift waves, and large poloidal mode number modes. There is, however, one very important exception; the low poloidal mode number tearing modes. We shall deal with this case later.

For localized solutions, we simply equate

$$\psi \equiv \psi_p = -\frac{1}{2k}\int_{-\infty}^{+\infty} dx' e^{-k|x-x'|} J(x') \quad , \tag{8}$$

$$\phi \equiv \phi_p = -\frac{1}{2k}\int_{-\infty}^{+\infty} dx' e^{-k|x-x'|} J(x') \frac{x'}{x_A^2} \quad . \tag{9}$$

From Eqs. (8) and (9), we construct

$$\psi - x\phi = \frac{x^2}{\sigma} J = -\frac{1}{2k}\int_{-\infty}^{+\infty} dx' e^{-k|x-x'|}\left(1 - \frac{xx'}{x_A^2}\right) J(x') \quad , \tag{10}$$

which can also be written as

$$\frac{x^2}{\sigma} J = \int_{-\infty}^{+\infty} dx' \; K(x,x') J(x')$$

where (11)

$$K(x,x') = K(x',x) = -\frac{1}{2k}\left(1 - \frac{xx'}{x_A^2}\right) e^{-k|x-x'|}$$

is a symmetric kernel. Equation (11) readily allows a variational principle

$$S = \left\langle \frac{x^2}{\sigma} J^2 \right\rangle - \iint_{-\infty}^{+\infty} dx\, dx'\, K(x,x')J(x)J(x'), \tag{12}$$

which can be exploited using standard methods for stability analysis[1-2]. Equation (11) could be written in an alternative way

$$\frac{\sigma}{x^2} E = \int_{-\infty}^{+\infty} dx'\, \bar{K}(x,x')E(x') \quad , \tag{13}$$

with a

$$\bar{K}(x,x') = \bar{K}(x',x) = -\frac{1}{2k} e^{-k|x-x'|} \frac{\sigma(x)\sigma(x')}{x^2 x'^2} (1 - \frac{xx'}{x_A^2})$$

which is also a symmetric kernel. Thus one can use either Eq. (11) or Eq. (13) for an analytical stability analysis because both allow a variational treatment. For numerical work Eq. (11) is clearly much better because of its simplicity. Since either Eq. (11) or Eq. (13) does not have a meaningful k=0 limit, we now treat this special case. Since it applies only to tearing modes, we have $\psi(x) = \psi(-x)$, $\phi(x) = -\phi(-x)$, and $J(x) = J(-x)$.

For this limit (k=0), Eqs. (4) and (5) can be easily solved to obtain

$$\psi = \frac{1}{2} \int_{-\infty}^{+\infty} dx' |x-x'| J(x')dx' + a \quad , \tag{14}$$

$$\phi = \frac{1}{2} \int_{-\infty}^{+\infty} dx' |x-x'| \frac{x'}{x_A^2} J(x')dx' + bx \tag{15}$$

where we have added the even homogeneous solution (a = constant) to construct ψ, and the odd homogeneous solution (constant x) to construct ϕ. We remind the reader that the homogeneous equations are $\psi'' = 0$, $\phi'' = 0$ respectively.

In the conventional tearing mode theory, the perturbation ψ has the asymptotic form[2]

$$\psi_{|x| \to \infty} \to \psi_0 + \alpha|x| \quad , \tag{16}$$

with the definition

$$\Delta' = \frac{\psi'_+ - \psi'_-}{\psi_0} = \frac{2\alpha}{\psi_0} \quad . \tag{17}$$

In addition

$$(\psi - x\phi)_{|x| \to \infty} \to 0 \quad . \tag{18}$$

Within the framework of Eq. (14)-(18), it is straightforward to see that

$$a = \frac{1}{\Delta'} \int_{-\infty}^{+\infty} dx' J(x') \ , \ b=0 \quad . \tag{19}$$

Therefore, the solutions which satisfy the required boundary conditions, are

$$\psi = \frac{1}{2}\int_{-\infty}^{+\infty} dx'|x-x'|J(x')dx' + \frac{1}{\Delta'}\int_{-\infty}^{+\infty} dx'J(x') \quad , \tag{20}$$

$$\phi = \frac{1}{2}\int_{-\infty}^{+\infty} dx'|x-x'|J(x')\frac{x'}{x_A^2} \quad , \tag{21}$$

which can be combined as before to yield the integral equation

$$\frac{x^2}{\sigma} J = \frac{1}{\Delta'}\int_{-\infty}^{+\infty} dx'J(x') + \int_{-\infty}^{+\infty} dx'|x-x'|(1 - \frac{xx'}{x_A^2})J(x') \quad , \tag{22}$$

which describes the conventional tearing modes.

Now we show that we can start with ψ_p and ϕ_p given by Eqs. (6) and (7), and derive the tearing mode equation suitable for low as well as zero k. Notice that for $k\neq0$, the solutions of the homogeneous equations $\psi'' - k^2\psi = 0$, $\phi'' - k^2\phi = 0$, are e^{kx} and e^{-kx}. Remembering that for tearing modes ψ is even and ϕ is odd, we could construct

$$\psi = \psi_p + \frac{a}{2}[e^{kx} + e^{-kx}] \quad , \tag{23}$$

$$\phi = \phi_p + \frac{b}{2}[e^{kx} - e^{-kx}] \quad . \tag{24}$$

We again demand that the asymptotic behaviour given by Eq. (10)-(18). In addition, we demand that even when $|x| \to \infty$, $k|x| << 1$ so that the exponentials can be expanded. Notice that this is essential to make contact with the conventional tearing mode theory. This limits the analysis to $kw \lesssim 1$ where w is the width of the tearing layer, because $|x| \to \infty$ really means $|x| >> w$. Under these conditions, it can be shown by straightforward algebra that the finite k version of Eq. (22) becomes

$$\frac{x^2}{\sigma} J = \frac{1}{\Delta'} \int_{-\infty}^{+\infty} dx' J(x')\left(1 + \frac{k^2(x-x')^2}{4}\right)$$
$$+ \frac{1}{2} \int_{-\infty}^{+\infty} dx' J(x')\left(1 - \frac{xx}{x_A^2}\right)|x-x'|\left[1 + \frac{k^2}{6}(x-x')^2\right], \qquad (25)$$

which for collisional and semicollisional version of σ has already been discussed[3].

It is appropriate here to remark that in the large k localized tearing mode theory, Δ' is either irrelevant, or identically equals $-2k$[3].

Thus we have shown that for any generalized conductivity, the coupled set of differential equations is reducible to a rather simple single integral equation with a symmetric kernel. The advantages of this formulation for an analytic or a computational solution are obvious. Analytically difficult problems like finite β drift waves or large k tearing modes become analytically tractable.

Applications of the formalism will be presented in future work.

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