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ABSTRACT

The degeneracy in the noncanonical Hamiltonian Poisson bracket structure of the Vlasov equation is removed by obtaining a symplectic leaf equation of motion; i.e. an equation of motion written in terms of a variable that manifestly preserves the Casimir constraints of the system. A nondegenerate Poisson bracket in terms of this variable is presented.

It is by now well-known that many non-dissipative continuous systems possess a Hamiltonian structure, which when viewed in terms of Eulerian variables has a noncanonical form. Examples from plasma physics include ideal magnetohydrodynamics (MHD) [1], the Vlasov equation [2], the two-fluid equations [3], and the BBGKY hierarchy [4]. A common feature of all these systems is that they posssess Casimir invariants due to the degeneracy of their Poisson structure. These invariants foliate the phase space into submanifolds, called the symplectic leaves, which are invariant under the dynamics. It is of interest to study the evolution equations restricted to a single leaf. Recently Crawford and Hislop found such a restriction for the Vlasov equation in one dimension [5]. In this paper we generalize their result to the multi-dimensional case, and also derive explicit expressions for the Poisson bracket for the Vlasov equation on a symplectic leaf. First, we briefly review the Hamiltonian structure of the Vlasov equation in order to establish our notation.

The Vlasov equation is usually written as a partial differential equation on the particle phase space $\mathbf{z}$:

$$\frac{\partial f}{\partial t} + [f, H] = 0 \,, \tag{1}$$

where $f(\mathbf{z}, t)$ is the particle distribution function, $H(\mathbf{z}, t)$ is the single particle Hamiltonian, and $[\cdot\,,\cdot]$ is the Poisson bracket. In terms of the canonical variables $\mathbf{z} = (\mathbf{q}, \mathbf{p})$ the Poisson bracket takes the familiar form:

$$[f, g] = \frac{\partial f}{\partial \mathbf{q}} \cdot \frac{\partial g}{\partial \mathbf{p}} - \frac{\partial f}{\partial \mathbf{p}} \cdot \frac{\partial g}{\partial \mathbf{q}} \,. \tag{2}$$

On the other hand, if one considers the physical observables $\mathcal{F}[f]$ which are functionals of the distribution function, one can show that their evolution obeys a Hamiltonian equation [2]

$$\frac{d\mathcal{F}}{dt} = \{\mathcal{F}, \mathcal{H}\} \,, \tag{3}$$

which is equivalent to the Vlasov equation (1). Here $\mathcal{H}$ is the Hamiltonian functional which satisfies $\delta\mathcal{H}/\delta f = H$, and $\{\,\cdot\,,\cdot\,\}$ is a Lie-Poisson bracket, defined by

$$\{\mathcal{F}_1, \mathcal{F}_2\} = \int d\Gamma\, f \left[\frac{\delta \mathcal{F}_1}{\delta f}, \frac{\delta \mathcal{F}_2}{\delta f}\right] \,, \tag{4}$$

where $d\Gamma$ denotes the Liouville measure on the particle phase space, e.g., $d\Gamma = d^3\mathbf{q}d^3\mathbf{p}$ in canonical coordinates.

A striking feature of the Lie-Poisson bracket (4) is its infinite degeneracy: if we consider observables of the form

$$\mathcal{C}[f] = \int d\Gamma\, C(f) \,, \tag{5}$$

where $C(f)$ is an arbitrary smooth function, then it is obvious that $\mathcal{C}$ commutes with any functional of f. Therefore such observables are conserved for any Hamiltonian $\mathcal{H}$:

$$\frac{d\mathcal{C}}{dt} = [\mathcal{C}, \mathcal{H}] = \int d\Gamma\, f\{C'(f), H\} = 0\,. \tag{6}$$

These conserved quantities are known as the *Casimirs.* They define a foliation of the space of distribution functions into invariant submanifolds, which are symplectic by the Kirillov-Kostant-Souriau theorem [6]. Each of these submanifolds or symplectic leaves can be characterized as a group orbit. Given a distribution $f_0(\mathbf{z})$, there is a unique symplectic leaf that passes through it. Let Λ denote a canonical transformation of the particle phase space, then the points f on this leaf have the form $f = f_0 \circ \Lambda$. We can say that the group of canonical transformations generates the leaf that passes through f_0. In order that the group action and the leaf have a one-one correspondence, we must exclude the isotropy subgroup that leaves $f = f_0$. Following Ref. 5, we represent a group element by a Lie series: $\Lambda = e^{L_W}$, where $L_W = [W, \cdot]$ is the Hamiltonian vector field of W. Thus a point on the leaf near f_0 can be written as

$$f = e^{L_W} f_0\,. \tag{7}$$

The generating function W should satisfy the condition $L_W f_0 \neq 0$, so that it can be interpreted as a local coordinate system on the leaf.

Our goal is to express the Vlasov equation (1) in terms of W. Our method is based on the following identity [7]:

$$\frac{\partial}{\partial t} e^{L_W} = L_{\alpha(L_W)\partial_t W}\, e^{L_W} \tag{8}$$

where,

$$\alpha(z) \equiv \int_0^1 d\theta\, e^{\theta z} = \frac{e^z - 1}{z} \tag{9}$$

is an entire function. We can derive (9) from the simpler identity

$$\frac{d}{d\theta} e^{\theta L_W} = e^{\theta L_W} L_W\,. \tag{10}$$

Differentiating (10) with respect to time yields

$$\frac{d}{d\theta}\left(\frac{\partial}{\partial t}e^{\theta L_W}\right) = \left(\frac{\partial}{\partial t}e^{\theta L_W}\right)L_W + e^{\theta L_W}L_{\partial_t W} , \tag{11}$$

which can be rearranged into

$$\frac{d}{d\theta}\left[\left(\frac{\partial}{\partial t}e^{\theta L_W}\right)e^{-\theta L_W}\right] = e^{\theta L_W}L_{\partial_t W}e^{-\theta L_W} = L_{e^{\theta L_W}(\partial_t W)} . \tag{12}$$

The last equality follows from the Campbell-Baker-Hausdorff formula. Integrating (12) with respect to θ from 0 to 1, we then obtain Eq. (8). We also note that this identity remains true if ∂_t is replaced by a general variation δ, a fact that will be used later in deriving the leaf Poisson bracket.

Applying (8) to (7) we immediately obtain

$$\frac{\partial f}{\partial t} = [\alpha(L_W)\partial_t W, f] , \tag{13}$$

which clearly is tangent to the leaf. (Generally a vector tangent to the leaf at point f has the form $\{G, f\}$, where G can be any smooth function. See [5,8].) Substituting (13) into the Vlasov equation (1) and using the invariance of the Poisson bracket under a canonical transformation yields

$$e^{L_W}\{e^{-L_W}(\alpha(L_W)\partial_t W - H), f_0\} = 0 . \tag{14}$$

We see that the first factor in the bracket must commute with f_0; denote this factor by C, an arbitrary function which satisfies $\{C, f_0\} = 0$. In the general case when f_0 has no special symmetry, we have $C = C(f_0)$. Thus we arrive at the Vlasov equation in terms of W:

$$\frac{\partial W}{\partial t} = \beta(L_W)H + \beta(-L_W)C , \tag{15}$$

where $\beta(z) \equiv 1/\alpha(z)$, or more explicitly:

$$\beta(z) \equiv \frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} , \tag{16}$$

where B_n are the Bernoulli numbers. Clearly $\beta(z)$ is analytic near the real axis. The role of function C in (15) is to keep W away from the isotropy kernel; it should be chosen so that $\{\partial_t W, f_0\} \neq 0$.

For the one-dimensional case considered in Ref. 5, $f_0 = f_0(v)$ was assumed to be an equilibrium, and $H = v^2/2 - \phi(x)$. So we have $C = C(v)$. Expanding (15) in a power series, and using $C(v)$ to remove the x-independent part of the right-hand side order by order, we find that (15) gives the same result as Eq. (31) in Ref. 5.

Now let us turn to the Poisson bracket for the leaf equation (15). By (7) we can regard any functional of f, $\mathcal{F}[f]$, also as a functional of W: $\hat{\mathcal{F}}[W] = \mathcal{F}[f(W)]$. Using (8) (see the comment at the end of that paragraph) we obtain, similar to (13),

$$\delta f = \{\alpha(L_W)\delta W, f\} \; ; \tag{17}$$

then by the chain rule, i.e., upon equating $\delta\hat{\mathcal{F}}[W;\delta W] = \delta\mathcal{F}[f;\delta f]$, we find

$$\frac{\delta\hat{\mathcal{F}}}{\delta W} = \alpha(L_W) L_{f_0} e^{-L_W} \left(\frac{\delta\mathcal{F}}{\delta f}\right) \, . \tag{18}$$

Note that for a Casimir $\mathcal{C}$ we have $\delta\hat{\mathcal{C}}/\delta W = 0$. Since f_0 is known, we can solve this equation for $\delta\mathcal{F}/\delta f$. Formally, we denote the inverse of L_{f_0} by $L_{f_0}^{-1}$ (after taking out the isotropy kernel C). Then

$$\frac{\delta\mathcal{F}}{\delta f} = e^{L_W} L_{f_0}^{-1} \beta(L_W) \frac{\delta\hat{\mathcal{F}}}{\delta W} + e^{L_W} C. \tag{19}$$

Note that $\{C, f_0\} = 0$ implies $\{e^{L_W} C, f\} = 0$, meaning that the second term in the above equation is the component transverse to the leaf, so it does not contribute to the Lie-Poisson bracket (4). Upon substituting (19) into (4) we obtain

$$[\mathcal{F}_1, \mathcal{F}_2] = \int d\Gamma \, \frac{\delta\hat{\mathcal{F}}_1}{\delta W} \left(\beta(-L_W) L_{f_0}^{-1} \beta(L_W)\right) \frac{\delta\hat{\mathcal{F}}_2}{\delta W} \, . \tag{20}$$

This Poisson bracket is non-degenerate in the sense that, if $[\mathcal{F}_1, \mathcal{F}_2] = 0$ for all $\mathcal{F}_1$ such that $\delta\hat{\mathcal{F}}_1/\delta W \neq 0$, then we have $\delta\hat{\mathcal{F}}_2/\delta W = 0$. Any invariants of Eq. (15) must therefore come from the symmetry in the Hamiltonian.

The Lie series representation (7) has the advantage of being coordinate independent; in particular it does not require canonical variables. However, the formal power series can be cumbersome in practice. In the following we develop another version of our results by using a mixed-variable generating function, which requires canonical variables but can be easier to manipulate.

Let the canonical transformation Λ be given by $S(\mathbf{q},\mathbf{P},t)$, with

$$\mathbf{p} = \frac{\partial S}{\partial \mathbf{q}}, \qquad \mathbf{Q} = \frac{\partial S}{\partial \mathbf{P}} , \tag{21}$$

where it is assumed that the Jacobian matrix of the transformation is non-degenerate:

$$\det(\omega_{ij}) \equiv \left| \frac{\partial^2 S}{\partial q_i \partial P_j} \right| \neq 0 . \tag{22}$$

Here to be explicit we use the F_2-type generating function, but our method below can be adapted without difficulty to other types of generating functions. Locally one can always find a generating function that satisfies the non-degeneracy condition similar to (22) [9]. It is natural to work in the mixed-variable space $(\mathbf{q},\mathbf{P})$, where the particle Poisson bracket (2) (denoted by subscript m) becomes

$$\{f,g\}_m = J_{ij} \left(\frac{\partial f}{\partial q_j} \frac{\partial g}{\partial P_i} - \frac{\partial f}{\partial P_i} \frac{\partial g}{\partial q_j} \right) , \tag{23}$$

where the convention of summing over repeated indices is used, and $\mathbf{J}$ satisfies $J_{ij}\omega_{jk} = \delta_{ik}$. The distribution functions in various spaces are related to each other through Eq. (21):

$$f(\mathbf{q},\mathbf{p},t) \equiv f_0(\mathbf{Q},\mathbf{P}) \equiv f_m(\mathbf{q},\mathbf{P},t) . \tag{24}$$

Now we calculate $\partial f/\partial t$ with $(\mathbf{q},\mathbf{p})$ held fixed. From (21) we have

$$\begin{aligned} \left(\frac{\partial \mathbf{P}}{\partial t} \right)_{(\mathbf{q},\mathbf{p})} &= -\mathbf{J} \cdot \frac{\partial^2 S}{\partial \mathbf{q} \partial t} , \\ \left(\frac{\partial \mathbf{Q}}{\partial t} \right)_{(\mathbf{q},\mathbf{p})} &= \frac{\partial^2 S}{\partial \mathbf{P} \partial t} + \frac{\partial^2 S}{\partial \mathbf{P} \partial \mathbf{P}} \cdot \left(\frac{\partial \mathbf{P}}{\partial t} \right)_{(\mathbf{q},\mathbf{p})} ; \end{aligned} \tag{25}$$

also from (24) we have

$$\frac{\partial f_0}{\partial \mathbf{Q}} = \mathbf{J} \cdot \frac{\partial f_m}{\partial \mathbf{q}} ,$$
$$\frac{\partial f_0}{\partial \mathbf{P}} = \frac{\partial f_m}{\partial \mathbf{P}} - \frac{\partial^2 S}{\partial \mathbf{P} \partial \mathbf{P}} \cdot \frac{\partial f_0}{\partial \mathbf{Q}} . \tag{26}$$

Together they lead to the following equation which is analogous to (13):

$$\left(\frac{\partial f}{\partial t}\right)_{(\mathbf{q},\mathbf{p})} = \frac{\partial f_0}{\partial \mathbf{Q}} \cdot \left(\frac{\partial \mathbf{Q}}{\partial t}\right)_{(\mathbf{q},\mathbf{p})} + \frac{\partial f_0}{\partial \mathbf{P}} \cdot \left(\frac{\partial \mathbf{P}}{\partial t}\right)_{(\mathbf{q},\mathbf{p})} = \left\{ f_m, \frac{\partial S}{\partial t} \right\}_m . \tag{27}$$

We remark again that this relation still holds if we replace the time derivative by a generic variation: this tells us the direction of the isotropy kernel. On the other hand it is obvious that

$$\{f, H\} = \left\{ f_m, H\left(\mathbf{q}, \frac{\partial S}{\partial \mathbf{q}}, t\right) \right\}_m . \tag{28}$$

Hence the Vlasov equation in the mixed-variable space reads

$$\left\{ f_m, \frac{\partial S}{\partial t} + H\left(\mathbf{q}, \frac{\partial S}{\partial \mathbf{q}}, t\right) \right\}_m = 0 . \tag{29}$$

The first factor in the bracket must commute with f_m. Let $C(\mathbf{Q}, \mathbf{P})$ be an arbitrary function that commutes with $f_0(\mathbf{Q}, \mathbf{P})$, then we arrive at the leaf equation in terms of S:

$$\frac{\partial S}{\partial t} + H\left(\mathbf{q}, \frac{\partial S}{\partial \mathbf{q}}, t\right) = C\left(\frac{\partial S}{\partial \mathbf{P}}, \mathbf{P}\right) . \tag{30}$$

This modified Hamilton-Jacobi equation was first introduced by Pfirsch and Morrison [10]. It can also be derived directly from an action principle [11].

Employing the same procedure as before we can derive the leaf Poisson bracket in terms of S. Here we only display the result:

$$\{\mathcal{F}_1, \mathcal{F}_2\} = \int d\Gamma_m \frac{\delta \mathcal{F}_1}{\delta S} \left(L_{f_m}^{-1}\right) \frac{\delta \mathcal{F}_2}{\delta S}, \tag{31}$$

where $d\Gamma_m = d^3\mathbf{q} d^3\mathbf{P} \det(\omega)$ is the Liouville measure in the mixed-variable space.

In conclusion, we have derived the Vlasov equation on a symplectic leaf, where all points are now presumbly dynamically accessible. We also found explicit expressions for the cosymplectic form of the Kirillov-Kostant-Souriau symplectic structure. Similar methods are expected to apply to other non-dissipative models that describe fluids and plasmas.

REFERENCES

[1] P.J. Morrison and J.M. Greene, Phys. Rev. Lett. **45**, 790 (1980)

[2] P.J. Morrison, Phys. Lett. **80A**, 383 (1980); J.E. Marsden and A. Weinstein, Physica **4D**, 394 (1982).

[3] R.G. Spencer and A.N. Kaufman, Phys. Rev. A **25**, 2437 (1982).

[4] J.E. Marsden, P.J. Morrison, and A. Weinstein, Contempary Mathematics **28**, 115 (1984).

[5] J.D. Crawford and P.D. Hislop, Phys. Lett. **134A**, 19 (1988).

[6] R. Abraham and J.E. Marsden, "Foundations of mechanics" 2nd Ed. (Benjamin, New York, 1978) p.302.

[7] D.H. Sattinger and O.L. Weaver, "Lie groups and algebras with applications to physics, geometry, and mechanics" (Springer-Verlag, New York, 1986) p.35

[8] P.J. Morrison and D. Pfirsch, Phys. Rev. A **40**, 3898 (1989).

[9] V.I. Arnold, "Mathematical methods of classical mechanics" (Springer-Verlag, New York, 1978) p.267.

[10] P.J. Morrison and D. Pfirsch, Phys. Fluids B, (1991, to appear).

[11] H. Ye and P.J. Morrison, submitted to Phys. Fluids (1991).

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