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S.V. Bazdenkov[a)]
Institute for Fusion Studies
The University of Texas at Austin
Austin, Texas 78712

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permanent add.: [a)]I.V. Kurchatov Institute of Atomic Energy, Moscow 123 182, U.S.S.R.

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S.V. Bazdenkov[a]
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Abstract

The cyclone-anticyclone asymmetry; i.e., the predominant generation of anticyclones in rotating shallow water, is considered from the viewpoint of flow relaxation toward vortices with minimal energy and fixed enstrophy ("selective decay" process). Three invariants of the set of equations for rotating shallow water are taken into account: total energy, enstrophy, and "mass." A nonlinear second order differential equation is obtained that describes the relaxed flows. It is shown that the anticyclone-like solution corresponds to a minimal energy value, in comparison with the cyclone-like solution for the same generalized enstrophy and "mass.".

[a] I.V. Kurchatov Institute of Atomic Energy, Moscow 123 182, U.S.S.R.

I. Introduction

Experiments with rotating shallow water and also numerical simulations[1–3] reveal the effect of cyclone-anticyclone asymmetry in which anticyclone-like vortices are predominantly observed whereas cyclone-like ones occur only occasionally. In these experiments, a thin liquid layer that is much thinner than the characteristic length of a flow inhomogeneity rotates with the characteristic frequency Ω in the presence of a gravitational force, $\mathbf{g}$. The paraboloid-shape bottom provides an equilibrium layer with a spatially uniform depth H. Any flow with nonzero velocity in the rotating coordinate system leads to the occurrence of depth perturbations. In the case of anticyclones, these perturbations look like hills and the vorticity vector is in the direction opposite to that of the rotation $\mathbf{\Omega}$. Inversely, cyclones correspond to valley-like perturbations of the depth H, with the vorticity vector and $\mathbf{\Omega}$ in the same direction. Thus, during the process of flow evolution, especially in the case of a turbulent flow, anticyclones are predominantly observed. As for cyclones, they are hardly created.

Various reasons for this asymmetry can be considered. It could be due to a peculiarity of the experimental device, for example, the way in which the velocity shear is maintained. Another explanation, more general and device-independent, for this asymmetry can be related with the linear stability analysis result[4] that an anticyclonic profile for sheared flow of the rotating shallow water is more stable than a cyclonic profile.

In the present paper we propose another explanation, namely, that the cyclone-anticyclone asymmetry is a manifestation of the self-organization process in the two-dimensional flow of the rotating shallow water. This process is closely connected with "selective decay"[5,6] of the ideal invariants under the dissipation action. In accordance with the "selective decay" mechanism, the realistic dissipative system (flow) tends to the state corresponding to extremum value of the most rapidly decaying integral of motion, under the condition that any other integrals are conserved. The choice and the number of integrals of motion which are important from the viewpoint of the "selective decay" mechanism depends on the specifics of the system. Here we will consider 2D flow relaxation under the "guidance" of the main three integrals of motion for rotating shallow water: namely, the total energy, the potential enstrophy, and the total fluid amount (mass). In this case, the relaxed flow to which the realistic dissipative flow tends during the turbulent evolution corresponds to the state with minimum total energy, under the condition that enstrophy and mass are conserved. We will show that the relaxed flow corresponds to the anticyclone-like vorticity distribution, and hence anticyclones are the preferable structures.

This paper is organized as follows: In Sec. II the rotating shallow water equations and the three main integrals of motion are presented. In Sec. III the "selective decay" approach is considered and the corresponding 2D nonlinear differential equation describing the relaxed flow is obtained. In Sec. IV the case of plane-tangential flows is analyzed and the existence of only the anticyclone-like profiles for the relaxed flow is shown. In Sec. V axially symmetric relaxed flows are considered. It is shown that anticyclones are the most preferable (with minimal total energy) relaxed flows. Our conclusions are presented in Sec. VI.

II. Basic Equations

The simple model of rotating shallow water with constant Coriolis parameter Ω can be described by the following set of equations

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} = -g \cdot \boldsymbol{\nabla} H - \boldsymbol{\Omega} \cdot \hat{\mathbf{z}} \times \mathbf{v} \ , \tag{1}$$

$$\frac{\partial H}{\partial t} + \boldsymbol{\nabla}(H \cdot \mathbf{v}) = 0 \ . \tag{2}$$

Here "H" is the depth of the shallow water; "g" is the gravitational acceleration; $\mathbf{v}$ is the flow velocity vector in the (x, y)-plane of a rotating system of coordinates; $\hat{\mathbf{z}}$ is the unit vector in the z-direction perpendicular to the (x, y)-plane; $\boldsymbol{\nabla} = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y}$; and the other notations are standard. The geometry for the problem is illustrated in Fig. 1. We do not take into account here the spatial inhomogeneity of the Coriolis parameter Ω (so called "β-effect") i.e., we neglect the difference between $\hat{\mathbf{z}}$ and the unit vector $\mathbf{n}$ normal to the bottom at any point. Thus, the radius of curvature of the bottom, R, is considered to be much larger than the characteristic length $L = (\frac{1}{H} \cdot |\boldsymbol{\nabla} H|)^{-1}$; i.e., $R \gg L \gg H$. As will be shown, the cyclone-anticyclone asymmetry is a sufficiently strong effect, which occurs even without the β-effect being taken into account. The term "cyclone" means that the z-component of the vorticity vector, $\boldsymbol{\omega}(\equiv \boldsymbol{\nabla} \times \mathbf{v})$ has the same sign as Ω, and the spatial depth profile looks like a valley. Inversely, an anticyclone corresponds to $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ having opposite signs and the spatial profile of the depth looks like a hill. Of course, one does not always have a clear understanding whether the flow can be considered as cyclone or, perhaps anticyclone, since solutions of Eqs. (1) and (2) with spatially oscillating signs of ω and $\boldsymbol{\nabla}^2 H$ can exist. This problem arises, for example, in Sec. V where axially-symmetric "oscillating" solutions are considered. Nevertheless, it is possible to classify any solution as a cyclone-like vortex or as an anticyclone-like one in the case when these oscillations are only a disappearing "tail" around a well-distinguished "kernel."

The ideal Eqs. (1) and (2) have three well-known integrals of motion: the total energy E, which is the sum of the kinetic and potential energies of a fluid in a gravitational field,

$$E = \frac{1}{2} \int_D \left(H \cdot v^2 + g \cdot H^2 \right) \, dx \, dy - E_\infty \; ; \tag{3}$$

the enstrophy S, which is closely related to the potential vorticity, $\xi = \frac{1}{H}(\omega + \Omega)$, being frozen-in with the flow,

$$S = \int_D \frac{1}{H} (\omega + \Omega)^2 dx \, dy - S_\infty \; ; \tag{4}$$

and the mass M of the fluid,

$$M = \int_D H \, dx \, dy - M_\infty \; . \tag{5}$$

Here integration over the flow domain, D, (which may be infinite) is assumed, and renormalization terms E_∞, S_∞, and M_∞ are introduced to avoid divergence of the integrals in the case of an infinite domain D. If D is a finite domain, then E_∞, S_∞, and M_∞ are equal to zero.

III. "Selective Decay" Relaxation

Let us now consider dissipative flows that are more realistic. In the shallow water approximation, at least two kinds of dissipation mechanism can be considered: viscosity effects, which are described by the $(\nu \cdot \nabla^2 \mathrm{v})$-term on the right-hand side of Eq. (1); and friction between the flow and the bottom, corresponding to the term $-\frac{1}{\tau}\mathrm{v}$ on the right-hand side of Eq. (1). Here ν is the kinematic viscosity coefficient, and τ is the characteristic momentum relaxation time. If dissipation effects are taken into account, then the "integrals of motion" E and S are not exactly conserved:

$$\frac{dE}{dt} = \nu \cdot \int_D H (\mathrm{v} \cdot \nabla^2 \mathrm{v}) dx \, dy - \frac{1}{\tau} \int_D H \cdot v^2 dx \, dy \; , \tag{6}$$

$$\frac{dS}{dt} = 2\nu \cdot \int_D \frac{(\omega + \Omega)}{H} \cdot \nabla^2 \omega \, dx \, dy - \frac{2}{\tau} \int_D \frac{\omega(\omega + \Omega)}{H} dx \, dy \; . \tag{7}$$

The dissipation, of course, does not affect the continuity equation, Eq. (2), and the corresponding conservation law $\frac{dM}{dt} = 0$, which is valid even for dissipative flow.

The decay rates for energy and enstrophy given in Eqs. (6) and (7) are different from each other, but in the general case of spatially nonuniform depth H, it is not clear which of these two invariants is more rapidly decaying. It seems that the enstrophy S can be considered as a long-lived invariant in comparison with the energy E in the case when friction between the flow and the bottom is the main dissipative process (this is the case for most of the experiments with rotating shallow water). Indeed, if $\nu \to 0$ in Eqs. (6) and (7), then $(\frac{1}{E}\frac{dE}{dt})$ and $(\frac{1}{S}\frac{dS}{dt}$ are of the same order of magnitude, but $\frac{dE}{dt}$ is strictly negative whereas the sign of $\frac{dS}{dt})$ is not definite. So, during the same characteristic time τ, the energy E is monotonically decreasing in value, whereas the enstrophy S can oscillate around a slowly decreasing average value. This conclusion appears to be in contradiction with the well-known result that for 2D incompressible viscous flows, the enstrophy is a rapidly decaying integral of motion in comparison with the energy. But in reality there is no contradiction because two different dissipation mechanisms are considered.

In accordance with the "selective decay" approach,[5,6] a system (i.e., flow) tends to the state corresponding to the minimum value of the rapidly decaying invariants, under the condition that other long-lived invariants are approximately constant. Let us consider the total energy E, given in Eq. (3), as the integral of motion that must be minimized under the condition that enstrophy S and mass M are conserved. The corresponding variational problem is as follows:

$$\delta \mathcal{L} = 0 \quad , \quad \text{where}$$

$$\mathcal{L} = E + \lambda \cdot S + \mu \cdot M \ . \tag{8}$$

Here λ and μ are undefined Lagrange multipliers, which can be calculated as usual, from the conditions that enstrophy S and mass M are equal to their corresponding initial values.

Varying the functional $\mathcal{L}$, one can obtain the following set of Euler's equations:

$$\mathrm{v} = \frac{2\lambda}{H}\hat{\mathbf{z}} \times \nabla\left(\frac{\omega+\Omega}{H}\right) , \tag{9}$$

$$\frac{v^2}{2} + gH + \mu - \lambda\left(\frac{\omega+\Omega}{H}\right)^2 = 0 . \tag{10}$$

Here the velocity variation, $\delta\mathrm{v}$, and the depth variation, δH, are considered to be independent, and Eqs. (9) and (10) correspond to $\delta\mathrm{v}$ and δH, respectively. Multiplying Eq. (9) by $\left(\frac{\omega+\Omega}{H}\right)$ and using Eq. (10), one can obtain the equilibrium equation

$$g \cdot \nabla H = -\nabla\left(\frac{v^2}{2}\right) - (\omega+\Omega)[\hat{\mathbf{z}} \times \mathrm{v}] , \tag{11}$$

which is the same as Eq. (1) with $\frac{\partial}{\partial t} \equiv 0$. Let us notice that Eq. (11) describes all possible equilibrium flows and does not select any special one. The selection is made by the use of Eq. (9), which depends on the peculiarities of the functional $\mathcal{L}$. Therefore, the most important (and informative) equation from the viewpoint of the description of relaxed flows is Eq. (9), while Eq. (11) describes the relationship between the velocity, v, and the depth, H, in any arbitrary equilibrium flow. By solving Eqs. (9) and (10), one can obtain the solutions in the form of $\mathbf{v}[\mathbf{x};\lambda,\mu]$ and $H[\mathbf{x};\lambda,\mu]$. The undefined Lagrange multipliers λ, and μ can be determined from the nonlinear problem $S[\lambda,\mu] = S_0$ and $M[\lambda,\mu] = M_0$, where S_0 and M_0 are the initial preset values of enstrophy and mass. Finally, the total energy E can be calculated for the solution $\mathrm{v}[\mathrm{x};\lambda,\mu]$ and $H[\mathbf{x};\lambda,\mu]$, and the solution with the minimal energy can be selected.

Instead of this complicated procedure, it is useful to consider the approximate solution of Eq. (11) that corresponds to the well-known "geostrophic" flow

$$\mathrm{v} = \frac{g}{\Omega}\hat{\mathbf{z}} \cdot \nabla H , \tag{12}$$

and describes the balance between the Coriolis force and the "pressure" gradient term, $g \cdot \nabla H$, in Eq. (11). It can be obtained in the limit that the Obukhov-Rossby radius,

$r_0 = \frac{1}{\Omega}\sqrt{gH_\infty}$, is small in comparison with the characteristic depth spatial inhomogeneity length, $L = (\frac{1}{H}|\nabla H|)^{-1}$. Formally, $\mathbf{v}$ in Eq. (12) is obtained to the lowest order in a $(\frac{1}{\Omega})$ power expansion at $\Omega \to \infty$. To the same order in a $(\frac{1}{\Omega})$ power expansion, the expression for the z-component of the vorticity vector is as follows:

$$\omega = \frac{g}{\Omega}\nabla^2 H \, . \tag{13}$$

By using Eqs. (12) and (13), together with Eq. (9), one can obtain the nonlinear differential equation that describes relaxed geostrophic flows;

$$\frac{2\lambda}{H}\left(\nabla^2 H + \frac{\Omega^2}{g}\right) - \frac{1}{2}H^2 = \text{const.} \tag{14}$$

We will consider only the localized solutions of Eq. (14), for which the boundary conditions are

$$H(|\mathbf{x}| = \infty) = H_\infty \, , \tag{15a}$$

$$\mathbf{v}(|\mathbf{x}| = \infty) = 0 \, , \tag{15b}$$

By the use of Eq. (10) at $|\mathbf{x}| \to \infty$, it is possible to express the value H_∞ in terms of λ and μ:

$$H_\infty^3 + \frac{\mu}{g}H_\infty^2 - \lambda \cdot \frac{\Omega^2}{g} = 0 \, . \tag{16}$$

If the boundary conditions in Eqs. (15a) and (15b) are taken into account in Eq. (14), then it is possible to write Eq. (14) in the following form

$$\nabla^2 H - (H - H_\infty) \cdot \left\{\frac{1}{4\lambda}\left(H + \frac{h_\infty}{2}\right)^2 \cdot \left(1 - \frac{gH_\infty^3}{16\Omega^2\lambda}\right)\right\} = 0 \, . \tag{17}$$

Now it is useful to introduce the dimensionless variables

$$H \equiv \widetilde{H} \cdot H_\infty \, , \; \mathbf{x} \equiv \tilde{\mathbf{x}} \cdot \frac{\sqrt{gH_\infty}}{\Omega} \, , \; \widetilde{\nabla}^2 \equiv \partial^2/\partial\tilde{x}^2 + \partial^2/\partial\tilde{y}^2 \, .$$

Then the relaxed (i.e., most preferred) geostrophic flows are described by the dimensionless equation,

$$\widetilde{\nabla}^2 \widetilde{H} - (\widetilde{H} - 1) \cdot \left[1 + A \cdot \widetilde{H} \cdot (\widetilde{H} + 1)\right] = 0 \, , \tag{18}$$

where $A \equiv \frac{gH_\infty^3}{4\lambda\cdot\Omega^2}$. The corresponding boundary condition is

$$\widetilde{H}(|\tilde{\mathrm{x}}| = \infty) = 1 \ . \tag{19}$$

We can write the expressions for the integrals of motion, Eqs. (3), (4), and (5), in terms of the dimensionless variables as follows:

$$E = \frac{g^2 \cdot H_\infty^3}{2\Omega^2} \cdot \int_D \left\{ \widetilde{H}^2 - 1 + \widetilde{H} \cdot (\widetilde{\nabla} \widetilde{H})^2 \right\} d\tilde{x}\, d\tilde{y} \ , \tag{20}$$

$$S = g \cdot \int_D \left\{ \frac{1}{\widetilde{H}} (1 + \widetilde{\nabla}^2 \widetilde{H})^2 - 1 \right\} d\tilde{x}\, d\tilde{y} \ , \tag{21}$$

$$M = \frac{gH_\infty^2}{\Omega^2} \cdot \int_D (\widetilde{H} - 1)\, d\tilde{x}\, d\tilde{y} \ . \tag{22}$$

Here the integration over the infinite domain D is assumed, and the renormalization constants E_∞, S_∞, and M_∞ are properly chosen.

Thus, if at the initial moment $t = 0$ the values of enstrophy, S_0, and mass, M_0, are known, then the most preferred (self-organized) geostrophic flow is fully described by Eqs. (18) and (19) and Eqs. (21) and (22). Indeed, in accordance with Eqs. (18) and (19), the solution $\widetilde{H}(\tilde{\mathrm{x}}; A)$ depends only on the parameter A. The value of this parameter can be calculated from the equality $S[A] = S_0$, where the functional $S[A]$ is defined by Eq. (21). Once the value of A is known, the parameters H_∞ and λ can be determined from the equality $M[A, H_\infty] = M_0$ and from the definition of A. That is to say, we have the equalities

$$\begin{cases} H_\infty = \left[\dfrac{g}{M_0 \cdot \Omega^2} \displaystyle\int_D (\widetilde{H} - 1)\, d\tilde{x}\, d\tilde{y} \right]^{-1/2} \ , \\[2ex] \lambda = \dfrac{gH_\infty^3}{4A\Omega^2} \ , \end{cases} \tag{23}$$

where the parameter A obeys the equation

$$\frac{S_0}{g} = \int_D \left\{ \frac{(1 + \widetilde{\nabla}^2 \widetilde{H})^2}{\widetilde{H}} - 1 \right\} d\tilde{x}\, \tilde{y} \tag{24}$$

and $\widetilde{H}$ is the solution of Eqs. (18) and (19); i.e., it depends only on the spatial coordinates, $\tilde{\mathrm{x}}$, and on the parameter A.

IV. Plane-Tangential Anticyclone-Like Relaxed Flows

Let us now consider the solution of Eqs. (18) and (19) in the form of a plane-tangential flow when $\widetilde{H}$ depends on only one of the $(\tilde{x}, \tilde{y})$-coordinates, say $\tilde{x}$: $\widetilde{H} \equiv \widetilde{H}(\tilde{x})$. In this case Eq. (18) can be transformed to the following form:

$$\frac{1}{2}\left(\frac{d\widetilde{H}}{d\tilde{x}}\right)^2 - U(\widetilde{H}) = 0 \ , \tag{25a}$$

where

$$U(\widetilde{H}) = \frac{1}{2}(\widetilde{H}-1)^2 \cdot \left[1 + \frac{A}{2}(\widetilde{H}+1)^2\right] \ . \tag{25b}$$

The boundary condition of Eq. (19) is used in the definition of $U(\widetilde{H})$. The dependence of the potential energy, U, on the dimensionless depth, $\widetilde{H}$, at different values of the parameter A is shown in Fig. 2. In accordance with Eq. (25a), only positive or zero values of $U(\widetilde{H})$ are accessible; i.e., the solution of Eqs. (25a) and (25b) does not exist when $A < -2$ where $A > 0$, only solutions with a "dry bottom," viz. $\widetilde{H} = 0$, or an infinitely increasing depth, viz., $\widetilde{H} \to \infty$, are possible. Indeed, the only extremal $\widetilde{H}$ value with $\frac{d\widetilde{H}}{d\tilde{x}} = 0$ is $\widetilde{H} = 1$, if the case of $A > 0$ is considered. These solutions are "bad" from the physical point of view, and we exclude them. When $-2 < A < -\frac{1}{2}$, only "bad" solutions with a dry bottom can exist, as well as when $-\frac{1}{2} < A < 0$ and $\widetilde{H} < 1$. Hence, the only region of A values in which nontrivial "good" solutions exist is $-\frac{1}{2} < A < 0$ and $\widetilde{H} < 1$ (see Fig. 2b). This corresponds to the existence of the root $U(\widetilde{H}) = 0$ at $\widetilde{H} > 2$. However, the inequality $\widetilde{H} > 1$ means that the most preferred self-organized vorticity distribution in the case of a plane-tangential flow is anticyclone-like vorticity. No regular cyclone-like solutions of Eqs. (25) exist.

V. Axially Symmetric Vortices

The axially symmetric solutions of Eq. (18) corresponds to a radially varying $\widetilde{H}(\widetilde{r})$ in the cylindrical system of coordinates. Such solutions are described by the following equation

$$\frac{1}{\widetilde{r}}\frac{d}{d\widetilde{r}}\left(\widetilde{r}\,\frac{d\widetilde{H}}{d\widetilde{r}}\right)-(\widetilde{H}-1)\cdot\left[1+A\cdot(\widetilde{H}+1)\cdot\widetilde{H}\right]=0\ . \tag{26}$$

Because of the explicit $\widetilde{r}$-dependence, the analysis of the solutions of Eq. (26) is complicated. Nevertheless, it is possible to obtain useful qualitative information about these solutions. Multiplying Eq. (26) by $\left(\frac{d\widetilde{H}}{d\widetilde{r}}\right)$, one can obtain

$$\frac{d}{d\widetilde{r}}\left[\frac{1}{2}\left(\frac{d\widetilde{H}}{d\widetilde{r}}\right)^{2}-U(\widetilde{H})\right]+\frac{1}{\widetilde{r}}\left(\frac{d\widetilde{H}}{d\widetilde{r}}\right)^{2}=0\ ,$$

where $U(\widetilde{H})$ is the same as in Eq. (25b). By integrating this equation, one obtains

$$\frac{1}{2}\left(\frac{d\widetilde{H}}{d\widetilde{r}}\right)^{2}=U(\widetilde{H})+\int_{\widetilde{r}}^{\infty}\frac{d\widetilde{r}}{\widetilde{r}}\left(\frac{d\widetilde{H}}{d\widetilde{r}}\right)^{2}\ . \tag{27}$$

The last term in Eq. (27) is positive definite, so the boundary condition (which is assumed), namely,

$$\left.\frac{d\widetilde{H}}{d\widetilde{r}}\right|_{\widetilde{r}=0}=0\ ,$$

can be valid only if $U(\widetilde{H})\Big|_{\widetilde{r}=0}<0$. This means that solutions of Eq. (27) exist only in the case when $A<0$. Most of these solutions are oscillating ones, and it is not a well-defined problem to distinguish which of them are anticyclones or cyclones. For example, at large enough distances from the center of a cylindrical system of coordinates, the linearized version of Eq. (26) (i.e., with $|\widetilde{H}-1|\to 0$) has solutions in the form of Bessel functions. We will not discuss the oscillating solutions with "flakey-paste"-like structures for the sheared flow, for which only a few oscillations of the depth $\widetilde{H}$ occur, with large amplitude. These solutions do not contribute to an understanding of the cyclone-anticyclone asymmetry, due to the impossibility to classifying them. Hence, we have to shrink the solutions under consideration and investigate more representative solutions from the viewpoint of cyclone-anticyclone classification. Let us consider the solutions that consist of two parts: a vortex-like kernel with monotonically varying $\widetilde{H}$ at $\widetilde{r} < \widetilde{r}_*$, and a disappearing, oscillating "tail" at $\widetilde{r} > \widetilde{r}_*$. Another parameter (besides $\widetilde{r}_*$) is the value of $\widetilde{H}$ at the origin location of the coordinates, $\widetilde{H}_0 \equiv \widetilde{H}(\widetilde{r} = 0)$. In order to have a "tail" with small amplitude, which is matched with the kernel at $\widetilde{r} = \widetilde{r}_*$, we assume also the "boundary" condition $\widetilde{H}(\widetilde{r} = \widetilde{r}_*) = 1$. Of course, the probability of analytically finding an exact solution of Eq. (26) with these properties is very small. Therefore it is more convenient first to consider the variational problem of Eq. (8), with the following trial function

$$\widetilde{H} = \begin{cases} \widetilde{H}_0 + (1 - \widetilde{H}_0) \cdot \left(\dfrac{\widetilde{r}}{\widetilde{r}_*}\right)^2 , & \widetilde{r} < \widetilde{r}_* \\ 0 , & \widetilde{r} > \widetilde{r}_* \, . \end{cases} \tag{28}$$

Equation (28) represents a three-parameter $(H_\infty, \widetilde{H}_0, \widetilde{r}_*)$ family of vortices, where $\widetilde{r}_*$ describes the vortex size, and where H_0 and H_∞ are the central (dimensionless) and peripheral depth, so that $\widetilde{H}_0 < 1$ corresponds to cyclones and $\widetilde{H}_0 > 1$ corresponds to anticyclones. Two of these parameters are fixed by the condition that the enstrophy and mass are known (being preset):

$$\frac{S_0}{\pi g} = \frac{\widetilde{r}_*^2 \cdot \ell n \, \widetilde{H}_0}{(\widetilde{H}_0 - 1)} \cdot \left[1 + \frac{4(1 - \widetilde{H}_0)}{\widetilde{r}_*^2}\right]^2 , \tag{29a}$$

$$\frac{M_0}{\pi g} = \frac{H_\infty^2}{2\Omega^2} \cdot \widetilde{r}_*^2 \cdot (1 + \widetilde{H}_0) \, . \tag{29b}$$

These equations are equivalent to the inclusion of S and M into the functional $\mathcal{L}$ of Eq. (8). By means of Eqs. (29a) and (29b), it is possible to express $\widetilde{r}_*$ and H_∞ in terms of $\widetilde{H}_0$:

$$\widetilde{r}_*^2 = 4 \cdot (1 - \widetilde{H}_0) \cdot \left\{ - \left(1 + \frac{S_0}{\pi g} \cdot \frac{1}{4 \ell n \, \widetilde{H}_0}\right) \pm \sqrt{\left(1 + \frac{S_0}{2\pi g} \cdot \frac{1}{4 \cdot \ell n \, \widetilde{H}_0}\right)^2 - 1} \right\}^{-1} , \tag{30}$$

$$H_\infty = \left\{ \frac{2 M_0 \Omega^2}{\pi g} \cdot \frac{1}{\widetilde{r}_*^2 \cdot (1 + \widetilde{H}_0)} \right\}^{1/2} . \tag{31}$$

As a result, we obtain an expression for the total energy E as an implicit function of $\widetilde{H}_0$:

$$\frac{E}{\pi g} = \frac{gH_\infty^3}{2\Omega^2} \cdot \widetilde{r}_*^2 \left\{ \widetilde{H}_0^2 + \widetilde{H}_0 \cdot (1-\widetilde{H}_0) \cdot \left(1 + \frac{2(1-\widetilde{H}_0)}{\widetilde{r}_*^2}\right) + \frac{(1-\widetilde{H}_0)^2}{3} \left(1 + \frac{4(1-\widetilde{H}_0)}{\widetilde{r}_*^2}\right) \right\} . \tag{32}$$

The last step is to analyze the $E(\widetilde{H}_0)$ dependence and to find at what value of $\widetilde{H}_0$ the energy E is minimal. In Fig. 3, the $E(\widetilde{H}_0)$ dependence is presented qualitatively. The two branches, labeled "1" and "2," corresponds to different signs of the root in Eq. (30) ("$-$" and "+", respectively). When $\widetilde{H}_0 < \widetilde{H}_{0c}(S_0)$, only complex (unphysical) values $\widetilde{r}_*^2$ are possible. Here the critical value $\widetilde{H}_{0c}(S_0)$ decreases when S_0 increases and $\widetilde{H}_{0c}(S_0) = 1$. Because we are interested in the solutions of the form in Eq. (28) with minimal energy E, the branch "1" is the most attractive for our purpose. In Fig. 4, the branch "1" for the $E(\widetilde{H}_0)$ dependence is presented for various values of enstrophy S_0. As can be seen, the anticyclones ($\widetilde{H}_0 > 1$) are the most preferred vortices with minimal energy. The amplitude ($\widetilde{H}_0$) of these relaxed vortices decreases along with the growth in the enstrophy.

VI. Conclusions

The experimentally observed cyclone-anticyclone asymmetry is a manifestation of the self-organization process that takes place in rotating shallow water. In the present paper it has been shown that the anticyclone-like vortices are the most preferred relaxed states to which a realistic dissipative flow tends during the "selective decay" self-organization process. The anticyclones are characterized by minimal energy E, under the condition that the enstrophy S and the mass M are conserved. The corresponding variational problem was formulated and the nonlinear two-dimensional differential equation for shallow water of depth π, was obtained in the geostrophic limit where $L = (\frac{1}{H}|\nabla H|)^{-1} \gg r_0 = \frac{1}{\Omega}\sqrt{gH}$ [see Eqs. (18) and (19)]. The solutions of this equation were analyzed in the cases of both plane-tangential and also axially-symmetric vortex-like relaxed flow. In both cases, the anticyclone-like solutions are shown to have minimal energy.

Another interesting problem that was not considered in the present paper is the existence of axially non-uniform solutions of Eq. (18), say, in the form of coupled vortices.

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Figure Captions

1. Schematic of the geometry: "L" indicates the thin liquid layer (shallow water), "B" indicates the bottom, and "H" is the depth of the shallow water. Other notations are explained in the text.

2. Dependence of the "potential energy" U on the dimensionless depth $\widetilde{H}$

   a) Curve 1: $A > 0$; Curve 2: $A < -2$.

   b) Curve 1: $-2 < A < -\frac{1}{2}$; Curve 2: $-\frac{1}{2} < A < 0$.

   The shaded area corresponds to the existence of a "good" solution.

3. Dependence of the total energy E layer on the depth of a shallow water at the center of a trial vortex [Eq. (28)]. Branches labeled "1" and "2" corresponds to different signs of the root in the expression for the vortex width r_*.

4. Branch "1" from Fig. 3 for three different values of the enstrophy, $\widetilde{S} = \frac{S}{\pi g}$:

   a) Curve 1: $\widetilde{S} = 1$; Curve 2: $\widetilde{S} = 5$; Curve 3: $\widetilde{S} = 25$.

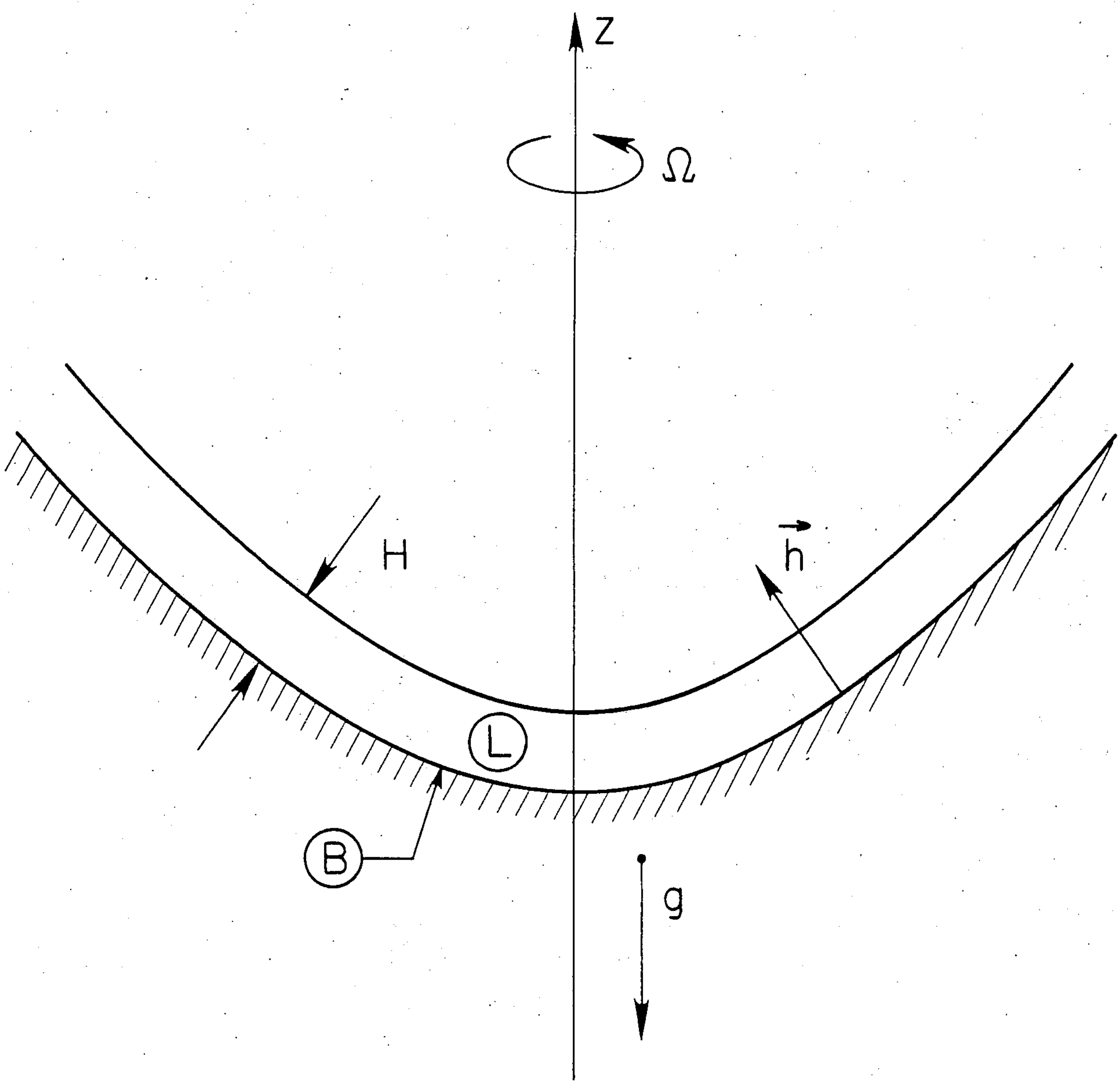

Fig. 1

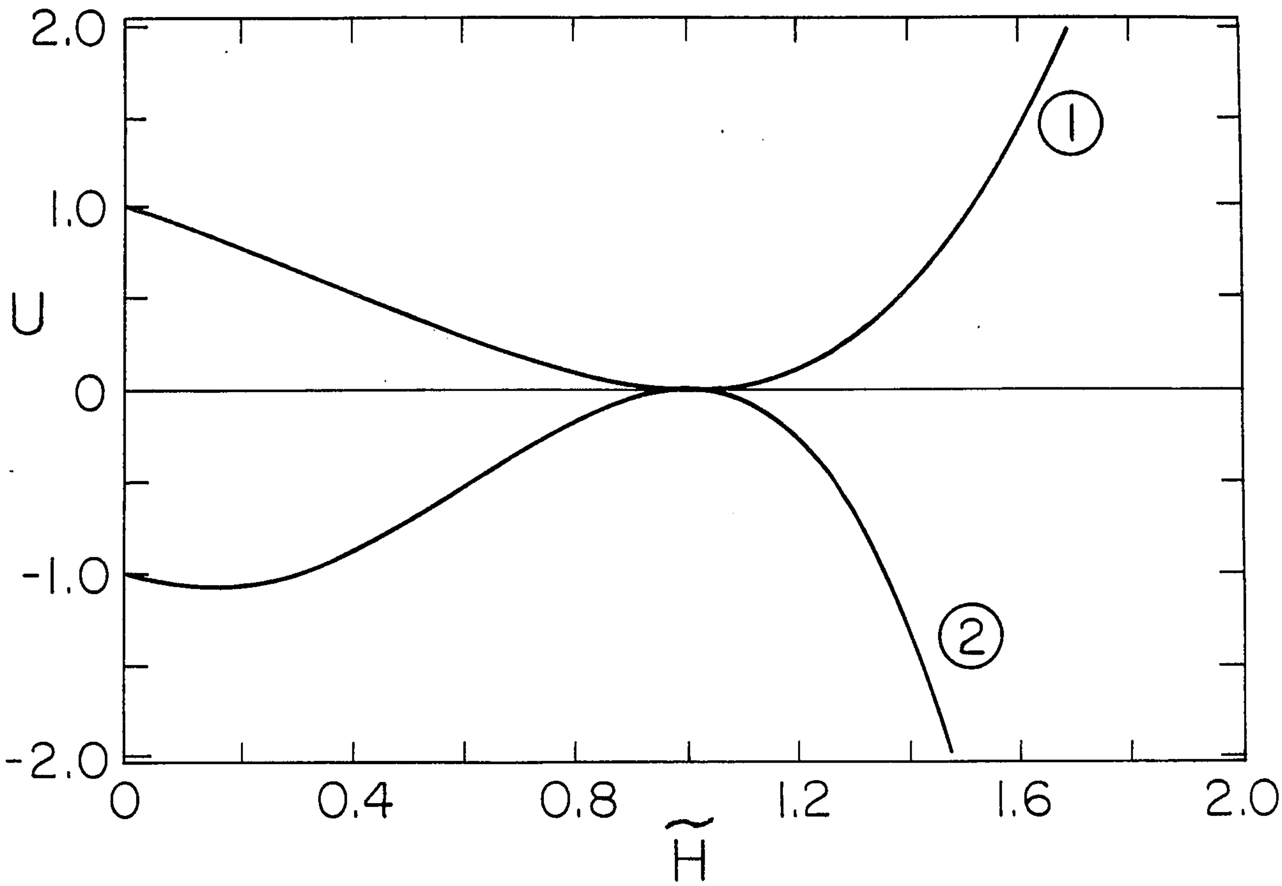

Fig. 2a

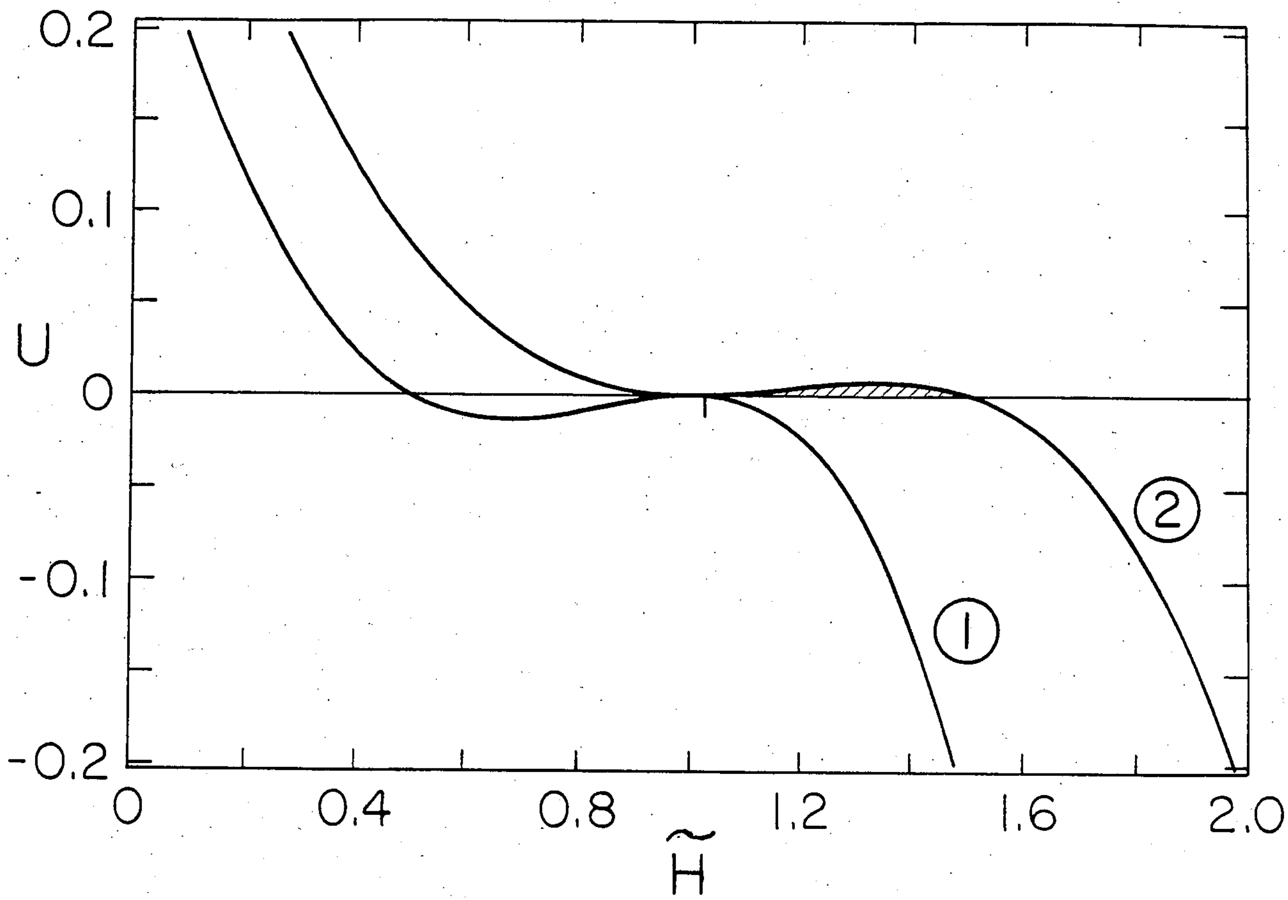

Fig. 2b

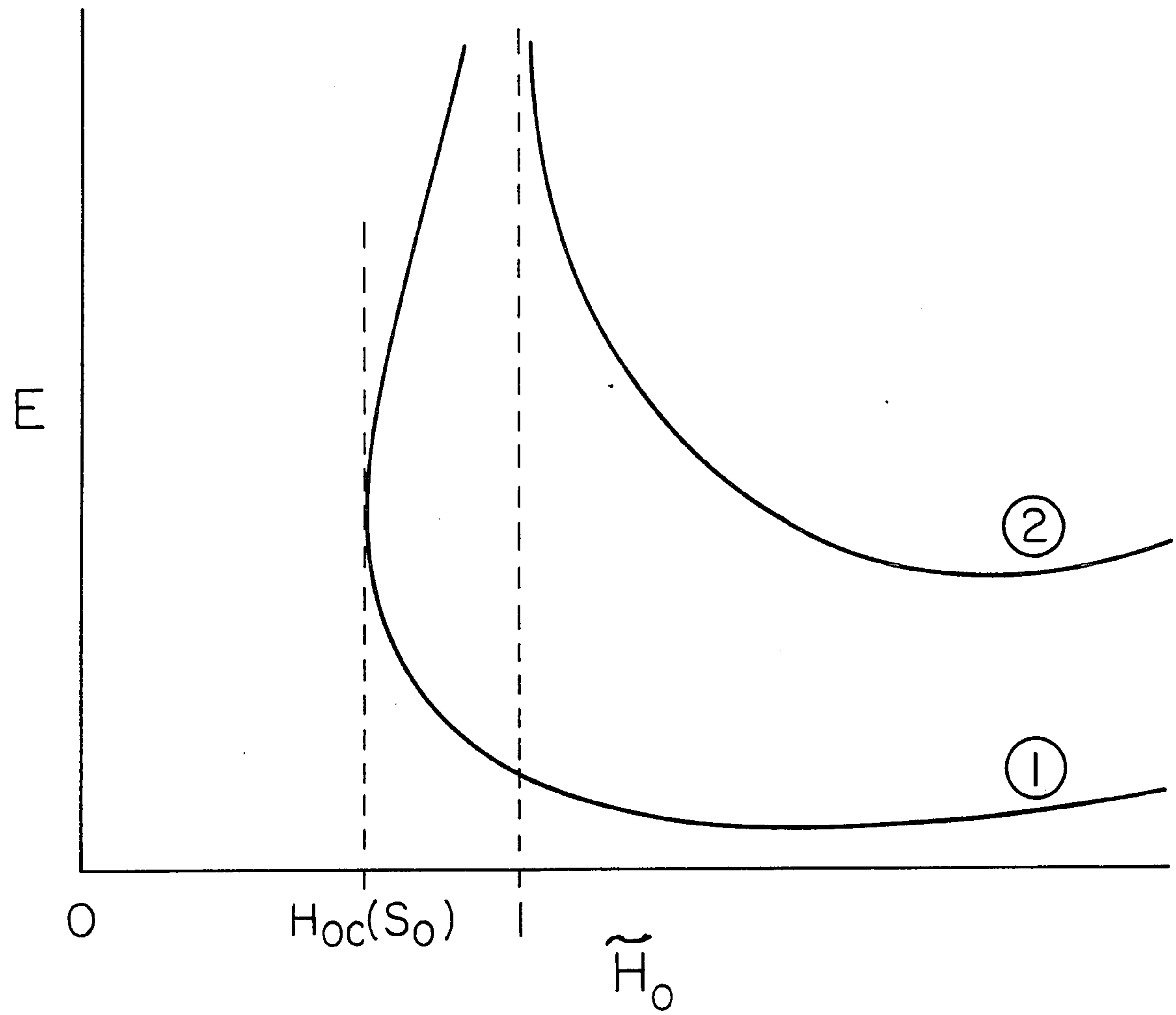

Fig. 3

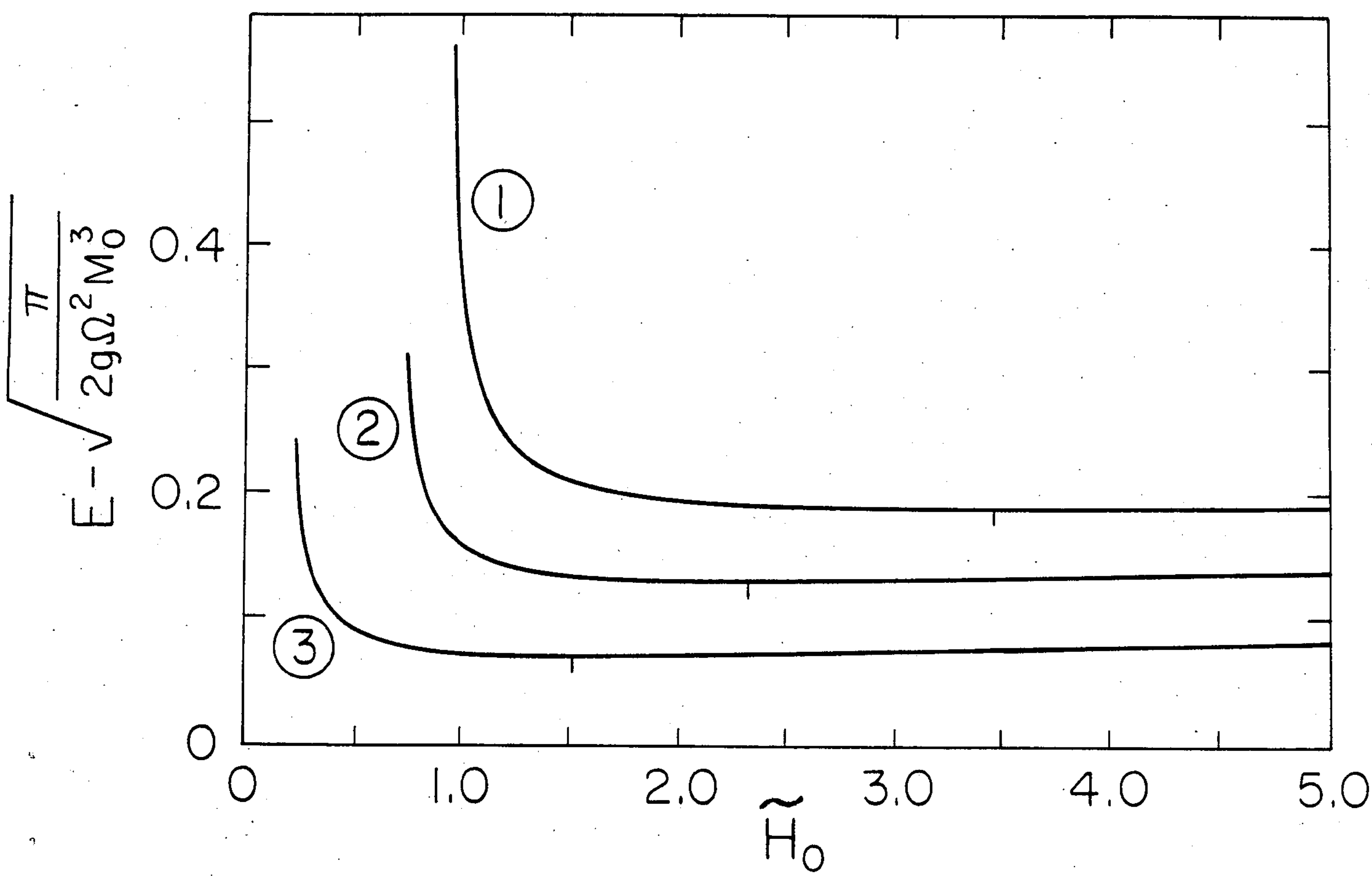

Fig. 4