INSTITUTE FOR FUSION STUDIES

DOE/ET-53088-474 IFSR #474

Toroidal Plasma Reactor with Low External Magnetic Field

A.D. Beklemishev,[a] V.A. Gordin, R.R. Khayrutdinov,[a]
V.I. Petviashvili,[a] and T. Tajima
Institute for Fusion Studies
The University of Texas at Austin
Austin, Texas 78712

January 1991

[a]*permanent address: Kurchatov Institute of Atomic Energy, Moscow 123182, U.S.S.R.*

THE UNIVERSITY OF TEXAS

AUSTIN

Toroidal Plasma Reactor with Low External Magnetic Field

A.D. Beklemishev,[a] V.A. Gordin, R.R. Khayrutdinov,[a]
V.I. Petviashvili,[a] and T. Tajima
Institute for Fusion Studies
The University of Texas at Austin
Austin, Texas 78712

Abstract

A toroidal pinch configuration with safety factor $q < 0.5$ decreasing from the center to periphery without field reversal is proposed. This is capable of containing high pressure plasma with only small toroidal external magnetic field. Sufficient conditions for magnetohydrodynamic stability are fulfilled in this configuration. The stability is studied by constructing the Lyapunov functional and investigating its extrema both analytically and numerically. Comparison of the Lyapunov stability conditions with the conventional linear theory is carried out. Stable configurations are found with average β near 15%, with magnetic field associated mainly with plasma current. The β value calculated with the external magnetic field can be over 100%. Fast charged particles produced by fusion reactions are asymmetrically confined by the poloidal magnetic field (and due to the lack of strong toroidal field). They thus generate a current in the noncentral part of plasma to reinforce the poloidal field. This current drive can sustain the monotonic decrease of q with radius.

[a]Permanent address: Kurchatov Institute of Atomic Energy, Moscow 123182, U.S.S.R.

I. Introduction

It is desirable to explore stable high beta plasma configurations. One approach is to explore a high beta tokamak ($q > 1$) equilibrium, in particular, that of the second stability.[1] Another approach[2] is to look for a reversed field pinch (RFP) ($q < 1$), spheromak, or field reversed configuration with no toroidal field (FRC). The present investigation tries to explore a high beta toroidal confinement configuration without field reversal with a very small q. This configuration can be supported by very small external toroidal fields and is ideal MHD stable, satisfying sufficient conditions for stability. Such a configuration has many potential strengths such as less stringent magnet conditions, charged fusion product driven current, possibility for adoption of advanced fuels, etc.

The stability of the stationary state of ordinary differential equations was investigated by Lyapunov.[3] A sufficient condition for such stability is based upon the properties of the extrema of a function, the so-called Lyapunov function, defined in the phase space of the system. The idea of applying such a theory to partial differential evolution equations of infinite-dimensional phase space began to evolve earlier, but they were extended to applications to the hydrodynamic equations only in the Seventies. At that time linear magnetohydrodynamic (MHD) theory, which studies the necessary conditions for stability, had already been well developed (see, e.g., Ref. 4). In the case of ideal MHD the linear theory indicates a set of equilibrium for which all frequencies of the linear oscillations are real. It is well known that this is only a necessary, but not a sufficient condition for the stability of the original nonlinear system. For instance, it does not take into account the possibility of a thresholdless instability caused by the interaction of three or more linear waves in an inhomogeneous medium. Such a nonlinear instability is possibly observed in linearly stable flows of water, for example. It is well known[5] that according to the linear theory the instabilities of laminar flow must start with the Reynolds number Re larger than 5800. Experimentally, the instability starts with Re $\sim$ 2000. This is apparently related to a nonlinear instability,[5] which is not found in the linear theory.

We are interested in distinguishing a narrower class that satisfies the sufficient conditions for stability among the set of linearly stable plasma states. The main point of Lyapunov's theory is finding the appropriate Lyapunov function (functional). A functional series of first integrals connected with helicity[6] is useful for this purpose. The stationary solutions of the MHD equations which are checked for Lyapunov stability are obtained as extremals i.e., as solutions of variational Euler's equations for the appropriate Lyapunov functionals. The class of equilibrium solutions is thereby restricted and we cannot select the necessary functional for any arbitrary solutions of the Grad-Shafranov equation.[4]

The next step consists in checking the positive-definiteness of the second variation of the Lyapunov functional in the vicinity of the extremal. There appear difficulties connected with the infinite-dimensionality of the phase space. These are the possibility of a continuous spectrum for the Jacobi equations (as discussed in Secs. III and IV) with coefficients which have poles on resonance surfaces, and the diverse definitions of positive definiteness and of strict positive definiteness. We show that these problems are the same as are met in linear theory. So the only difference in the Lyapunov method with linear theory is the restrictions on the class of equilibrium states: not every equilibrium is extremum of integral of motion.

Earlier, Taylor[7] proposed to find stable states by minimizing the functional

$$\int (B^2 - k\mathbf{A}\cdot\mathbf{B})d^3x \quad , \quad \mathbf{B} = \nabla \times \mathbf{A} \quad , \quad k = \text{const.} \tag{1}$$

where $\mathbf{A}\cdot\mathbf{B}$ is the helicity density and $\mathbf{A}$ the vector potential of the magnetic field. Minimization leads in this case to stable force-free (pressure p = const.) configurations. Variations in the pressure are neglected, which is valid provided there exists in the plasma a mechanism for a fast equilization of the pressure. At the extremals of (1), in a cylindrical or toroidal geometry, the longitudinal magnetic field can reverse. Using the more general helicity integrals[6] makes it possible to construct a Lyapunov functional which has an extremal with non-vanishing pressure gradients.

II. Helicity Integrals

The ideal MHD equations describe our system:

$$\begin{gathered} \partial_t \mathbf{B} = \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B}) \ , \quad \boldsymbol{\nabla} \cdot \mathbf{B} = 0 \ , \\ d_t p + \gamma p \, \boldsymbol{\nabla} \cdot \mathbf{v} = 0 \ , \quad \partial_t \rho + \boldsymbol{\nabla} \cdot \rho \mathbf{v} = 0 \ , \\ \rho d_t \mathbf{v} = -\boldsymbol{\nabla} p + (\boldsymbol{\nabla} \times \mathbf{B}) \times \mathbf{B}/4\pi \ , \end{gathered} \tag{2}$$

(γ is the adiabatic exponent, $\gamma = 5/3$), where ∂_t is the Eulerian time derivative while d_t the Lagrangian. They possess the following first integrals (the components of the momentum, the energy, the cross helicity, and the entropy series, respectively):

$$\begin{gathered} \int \rho \mathbf{v} \, d^3x \quad , \quad W = \int \left(\frac{B^2}{8\pi} + \frac{p}{\gamma - 1} + \frac{\rho v^2}{2} \right) d^3x \ , \\ \int \rho \mathbf{v} \cdot \mathbf{B} \, d^3x \quad , \quad \int_D \rho f \left(\frac{p}{\rho^\gamma} \right) d^3x \ , \end{gathered} \tag{3}$$

where f is an arbitrary function, D is an arbitrary fluid volume; when the integration domain is not specified, integration should be carried out over the whole volume V inside the conducting sheath. On the boundary ∂V we assume that the conditions of impenetrability and infinite conductivity are satisfied:

$$\mathbf{n} \cdot \mathbf{v} \Big|_{\partial V} = 0 \quad , \quad \mathbf{n} \cdot \mathbf{B} \Big|_{\partial V} = 0 \ , \tag{4}$$

where $\mathbf{n}$ is the normal vector to the boundary.

From the first equation of (2) it follows that

$$\partial_t \mathbf{A} = \mathbf{v} \times \mathbf{B} - \boldsymbol{\nabla} \varphi(\mathbf{x}, t) \ , \tag{5}$$

where φ is an arbitrary function. It is shown in Ref. 8 that for any gauge, or any choice of the function φ, the functional $\int \mathbf{A} \cdot \mathbf{B} \, d^3x$ is a first integral of the set (2).

To generalize this result, we put $\varphi = \mathbf{v}\cdot\mathbf{A}$ and define the local helicity density $h = \mathbf{A}\cdot\mathbf{B}$. It satisfies a continuity equation:

$$\partial_t h + \nabla \cdot h\mathbf{v} = 0 \ . \tag{6}$$

Hence it follows that the quantity $\mu = p^{1/\gamma}/h$ is conserved along the particle orbit:

$$\partial_t \mu + \mathbf{v}\cdot\nabla\mu = 0 \ .$$

For the enlarged set of Eqs. (2) and (6) there exists a functional series of first integrals

$$K_F = \int hF(\mu) d^3x \ , \tag{7}$$

where F is an arbitrary function.[6]

III. Lyapunov Functional

As the Lyapunov functional we use a first integral in the form of the sum of the energy and the helicity

$$L = W + K_F \ . \tag{8}$$

Variations are carried out in terms of the independent variables $\mathbf{A}$, $p, \mathbf{v}$. It follows from (3) that by first varying with respect to p one can eliminate $p, \delta p$ algebraically and then reduce the problem to one of studying the simplified functional Y, which depends on $\mathbf{A}$ alone:

$$Y = \int \left[B^2/2 + U(h)\right] d^3x \ . \tag{9}$$

Here U is the helicity function, which determines the structure of the equilibrium in accordance with the formula $\delta_A Y = 0$; we thus have

$$\nabla\times\mathbf{B} + 2\theta\mathbf{B} + \nabla\theta\times\mathbf{A} = 0 \quad , \quad \theta \equiv d_h U \ . \tag{10}$$

The pressure is expressed in terms of the integral

$$p = p(h) = \int_0^h h d_h^2 U \, dh/4\pi \ . \tag{11}$$

The Taylor states [7] can be obtained from Eq. (10) assuming $\theta \equiv d_h U = \text{const.}$ and hence $d_h p = 0$. The general equilibrium state (10) and (11) is stable[6] under the conditions

$$0 \leq h d_h p \leq \gamma p \quad , \quad \delta^2_{\mathbf{A}} Y \geq 0 \; . \tag{12}$$

For plasma inside a highly conducting metal chamber the following boundary conditions apply

$$\mathbf{n} \cdot \mathbf{B} = 0 \; , \; \mathbf{n} \cdot \delta\mathbf{B} = 0 \; , \; p = 0 \; , \; \nabla p = 0 \; , \; h = 0 \; , \; \nabla^2 p = 0 \; . \tag{13}$$

The first two of these conditions are a consequence of the conservation of the circulation of $\mathbf{A}$ along contours on the boundary; the third is the condition that the plasma is thermally insulated and the last three conditions follow from (10) and from the first inequality in (12). Note also that (10) and (11) are equivalent to the system of equations

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = 4\pi \, \nabla p \; , \qquad \mathbf{A} \cdot \nabla \times \mathbf{B} + 2\theta h = 0 \; . \tag{14}$$

Let us examine the second variation of functional (9):

$$\delta^2_{\mathbf{A}} Y = \int (\delta B^2/2 + U' h_2 + U'' h_1^2/2) d^3\mathbf{x} \; , \tag{15}$$

where $h_1 = \mathbf{A} \cdot \delta\mathbf{B} + \mathbf{B} \cdot \delta\mathbf{A}$, $h_2 = \delta\mathbf{A} \cdot \delta\mathbf{B}$, $\delta\mathbf{B} = \nabla \times \delta\mathbf{A}$. Here the prime denotes the derivative with respect to the argument. Let us minimize (15) by a variational method. Varying (15), we find the Jacobi equation

$$\nabla \times \nabla \times \delta\mathbf{A} + U' \nabla \times \delta\mathbf{A} + \nabla \times U' \delta\mathbf{A} + U'' \mathbf{B} h_1 + \nabla \times U'' \mathbf{A} h_1 = 0 \; . \tag{16}$$

IV. Lyapunov Stability and the Linear MHD Analysis

The sufficient condition of stability is the positive definiteness of the second variation of Lyapunov functional. This is always satisfied if there are no solutions to the Jacobi equation

(16). Any solution of Eq. (16) is also a solution to all equations which can be derived from it, so that the absence of solutions to any one of such "derived" equations is sufficient for stability. In this section we show that the system of linearized MHD equations does follow from Eq. (16) and thus is able to describe sufficient conditions of stability.

The divergence operator applied to Eq. (16) yields

$$\delta\mathbf{B}\cdot\nabla U' + \mathbf{B}\cdot\nabla(U''h_1) = 0\ . \tag{17}$$

Noting that $U''h_1 = \delta U'$, we find Eq. (17) to be the linearized form of equation which describes plasma pressure as a function of magnetic field line

$$\delta(\mathbf{B}\cdot\nabla p) = 0\ . \tag{18}$$

The linearized MHD equation of balance between the Lorentz force and the pressure gradient

$$\delta(\mathbf{J}\times\mathbf{B} - \nabla p) = \delta\mathbf{J}\times\mathbf{B} + \mathbf{J}\times\delta\mathbf{B} - \nabla\delta p = 0 \tag{19}$$

can also be obtained from Eq. (16). Indeed, let us substitute $\delta\mathbf{J} = \frac{1}{4\pi}\nabla\times(\nabla\times\delta\mathbf{A})$ from Eq. (16) into the combination $S \equiv \delta\mathbf{J}\times\mathbf{B} + \mathbf{J}\times\delta\mathbf{B}$. This yields the chain of equalities:

$$S = \frac{1}{4\pi}\left(\mathbf{B}\times[\nabla U'\times\delta\mathbf{A}] + \mathbf{B}\times[\nabla\times(U''h_1\mathbf{A})] + \mathbf{B}\times[\nabla U'\times\mathbf{A}]\right) =$$

$$\frac{1}{4\pi}\left(h_1\nabla U' + h\nabla(U''h_1) - \mathbf{A}\,(\delta\mathbf{B}\cdot\nabla U' + \mathbf{B}\cdot\nabla(U''h_1)\right) = \frac{1}{4\pi}\nabla(U''hh_1) = \nabla\delta p\ . \tag{20}$$

This proves that the closed set of MHD equations (18) and (19) follows from (16) and thus describes sufficient conditions of stability. Algebraic transformations in (20) are made using condition (17) and the equilibrium equation (10).

This system was often studied in papers on linear MHD stability of magnetic confinement systems and useful stability condition such as the Mercier criterion[9] were obtained. These conditions are usually only necessary but not sufficient for the stability of the system, since they describe the certain existence of some solution of system (18) and (19) and not the absence of any one. Thus, in order to ensure the relevance of these criteria to our case, we have to show that the class of solutions to Eq. (16) is not much narrower than that of the MHD system, or that Eq. (16) follows from (18) and (19) with reasonable additional constraints.

From the above derivation procedure it follows that the component of Eq. (16) perpendicular to $\mathbf{B}$ can be reconstructed from Eq. (19), but the parallel component is defined only up to some arbitrary solution of equation $\mathbf{B} \cdot \nabla F = 0$ so that

$$\nabla \times \nabla \times \delta\mathbf{A} + U'\nabla \times \delta\mathbf{A} + \nabla \times U'\delta\mathbf{A} + U''\mathbf{B}h_1 + \nabla \times U''\mathbf{A}h_1 = \mathbf{B}F \ . \tag{21}$$

It is easy to see that the system (18) and (19) can be derived from (21) as well as from (16). Fortunately, the class of nontrivial solutions of $\mathbf{B} \cdot \nabla F = 0$ is relatively narrow and consists of the $m = 0$ perturbation $F = F(\psi)$ and of a series of the δ-functional spikes on rational flux surfaces. Both types of resulting perturbations are related to resistive reconnection or diffusion of magnetic fields and thus can be excluded from consideration by using ideal MHD constraints on solutions of (18) and (19) around resonant flux surfaces. In this limit the MHD system is equivalent to Jacobi equation (16).

V. Axisymmetric Toroidal Configurations

We now study extremals which depend on r and z but not on angle ϕ. We express the magnetic field and the current in the cylindrical system of coordinates (r, ϕ, z) in terms of the Stokes potential. We write $\psi = -r\mathbf{A} \cdot \mathbf{e}_\phi$. Hence

$$r\mathbf{B} = \mathbf{e}_\phi \times \nabla\psi + \mathbf{e}_\phi I \quad , \quad \mathbf{B} \cdot \nabla\psi = 0 \ , \tag{22}$$

where $\mathbf{e}_\phi$ is the unit vector in the azimuthal direction, and ψ and I are arbitrary functions.

It follows from (22) that

$$r\nabla \times \mathbf{B} = -\mathbf{e}_\phi \times \nabla I + \mathbf{e}_\phi \Delta^* \psi \ , \tag{23}$$

$$\Delta^* \equiv r\partial_r r^{-1}\partial_r + \partial_z^2 \ . \tag{24}$$

Substituting (22) and (23) into (14), we find that I, h, and p are locally single-valued functions of ψ. It follows from (18) that the function ψ satisfies the Grad-Shafranov equation which in our case has the form

$$\Delta^*\psi = -4\pi r^2 \dot{p} - I\dot{I} \quad , \quad 4\pi\dot{p} = h\dot{\theta} \ , \tag{25}$$

and the easily integrable equation also follows:

$$\dot{I} = 2\theta + \psi\dot{\theta} \ , \tag{26}$$

where the dots henceforth denote the derivative with respect to ψ. Thus in Eq. (25) there is one arbitrary function, $\theta(\psi)$, rather than two as usual.[4]

In view of the axial symmetry of the problem, the boundary condition (4) is equivalent to the condition

$$\psi\Big|_{\partial V} = \text{const.} \ . \tag{27}$$

We determine the vector potential $\mathbf{A}$. According to the Lyapunov conditions this must be a single-valued vector field, continuous in the region V, such that the current $\mathbf{j}$ is finite everywhere in V.

We change to an orthogonal set of coordinates (ψ, ϑ, ϕ), where ϑ and ϕ are the poloidal and toroidal angles, respectively. The components of the metric tensor are

$$l_1 = 1/|\nabla\psi| \ , \ l_2 = 1/|\nabla\vartheta| \ , \ l_3 = r$$

and

$$B_1 = 0 \ , \ B_2 = \nabla_1\psi/r \ , \ B_3 = I/r \ , \ A_3 = -\psi/r \ .$$

It follows from (11) and (22) that $h = h(\psi)$ and that by definition we have

$$h = A_2 \, \nabla_1\psi/r + A_3 \, \dot{I}/r \ , \tag{28}$$

where $\nabla_i = l_i^{-1}\,\partial_i$ is the ith component of the gradient in the (ψ, ϑ, ϕ) coordinate system. The third component of the magnetic field is

$$B_3 = I/r = (\partial_1 l_2 A_2 - \partial_2 l_1 A_1)/l_1\, l_2 \ . \tag{29}$$

As $\mathbf{A}$ is single-valued, we have

$$\oint \partial_2(l_2\, A_1)\partial\vartheta = 0$$

and, in view of (29), we find

$$\partial_1(hM_1 + \psi I M_2) = IM_2 \ , \tag{30}$$

where

$$M_1 \equiv \oint H d\vartheta \ , \quad M_2 \equiv \oint H r^{-2} d\vartheta \ , \quad H \equiv l_1 l_2 l_3 \ . \tag{31}$$

We note that $2\pi \int M_1\, d\psi = V$ is the volume inside the $\psi =$ const. surface, $\Phi = \int M_2\, d\psi$ the toroidal magnetic flux, $\chi = 2\pi\psi$ the poloidal flux, and ψ_c the value of the function ψ on the magnetic axis; $q = IM_2/2\pi = d_\chi \Phi$. Equation (28) now gives a relation between the helicity density h and the safety factor q:

$$h = [\Phi(\psi) - 2\pi\psi q]M_1^{-1} = \Phi d_V \chi - \chi d_V \Phi \ . \tag{32}$$

From (28), (29), and (32) we get a unique expression for the components of the vector potential

$$A_1 = l_1^{-1} \int_0 (\partial_1 l_2 A_2 - l_1 l_2 I/r) d\vartheta \ ,$$

$$A_2 = \psi I(1 - r^2 M_2/M_1) l_1 + \Phi l_1 r/M_1 \ .$$

As we assume the current on the magnetic axis to be finite, it follows that the component A_2 must be a differentiable function. As $l_1 = 1/|\boldsymbol{\nabla}\psi|$ has a pole on the magnetic axis, while M_1 is finite, we must have on the axis, where $\psi = \psi_c$

$$\Phi(\psi_c) = 0 \ . \tag{33}$$

It also follows from (11) that

$$p = p(\psi) = \int_{\psi_b}^{\psi} h\,\dot{\theta}\,d\psi \ . \tag{34}$$

The above expressions will be used to find the equilibrium defined by $\delta Y = 0$. The check of stability of this equilibrium also includes condition $\delta^2 Y > 0$, which involves solving the linearized MHD system. This can be done by standard methods only for simple equilibria, such as the cylindrical pinch,[6] or by restricting the class of admissible perturbations. In the examples described in the following section the study is restricted to localized modes, so that the Mercier criterion[9] is used instead of the full system of partial differential equations. Convenient form of Mercier criterion was published in Ref. 10:

$$\frac{(2\pi\dot{q})^2}{4\dot{p}} > \left[\dot{p}\oint\frac{H d\vartheta}{B_2^2} - \dot{M}_1\right]\oint\frac{\nu B^2}{B_2^2}d\vartheta + 2\pi\dot{q}I\oint\frac{\nu d\vartheta}{B_2^2} - \dot{p}I\left(\oint\frac{\nu d\vartheta}{B_2^2}\right)^2 \ , \tag{35}$$

where $\nu = IHr^{-2}$.

VI. Numerical Examples

We first pick a set of the safety factor q and θ as a function of the poloidal flux ψ. This generates an equilibrium so constructed to satisfy the Grad-Shafranov equation (25). However, the arbitrary choice of q and θ does not guarantee the Lyapunov relation (26). We have to look for the set of q and θ that satisfy the condition. To accomplish this, we adapt θ iteratively by minimizing the difference between a functional (called M_2) that is generated from the Lyapunov satisfying Grad-Shafranov equation and a functional M_2 that is simply from the Grad-Shafranov solution.

We solve the equilibrium equation (25) by the "inverse variable" technique, using the code POLAR.[11] We use the normalized poloidal flux coordinate ρ that is determined by equation:

$$\psi = (\psi_c - \psi_b)(1-\rho) + \psi_b \quad , \quad (0 \le \rho \le 1) \tag{36}$$

where ψ_c and ψ_b are poloidal fluxes on magnetic axis and boundary, respectively. We specify the profiles $\frac{dp}{d\psi}(\rho)$ and $q(\rho)$ as input parameters for the equilibrium problem. From the solution of Grad-Shafranov equations we obtain the coordinates $R(\rho,\vartheta)$ and $z(\rho,\vartheta)$ of the magnetic surfaces. Here we solve equilibrium problem with fixed plasma boundary. The coordinates of plasma boundary are given by equations:

$$\begin{cases} R = R_0 + a(\cos\vartheta - g\sin^2\vartheta) \ , \\ z = \varepsilon a \sin\vartheta \ , \end{cases}$$

where R_0 is the geometrical plasma center, a is the minor radius, ε is ellipticity, g is triangularity, ϑ is poloidal angle. We represent the profile of the safety factor q by polynomial[9]

$$q = q_c + \sum q_k x^k \ , \tag{37}$$

where $k = 1, \ldots, 5$; $x = \psi - \psi_c$, $q_c = -\frac{1}{R}$, $q_1 = \frac{1}{I_c R}\left(1 - \frac{29}{16R^2}\right)$, and I and I_c are the poloidal current and the value of I on the magnetic axis. q_2 is a free parameter and the values of q_3, q_4, and q_5 are determined by the following conditions: $\dot{q}_{bd} = 0$, $h_{bd} = 0$ where h is the helicity density,

$$h = \left(\int_{\psi_c}^{\psi} q d\psi - \psi q\right) \frac{2\pi}{M_1} \ . \tag{38}$$

Here the subscript bd denotes values on the plasma boundary.

The pressure is determined by Eq. (34). For given θ, I can be obtained from (26). When derivatives are calculated numerically, it produces large numerical error and during iterations the value of errors increases. To avoid these numerical errors, we use analytical expression for θ variable:

$$\theta = \theta_0 + \sum_{k=2}^{5} C_k x^k \ , \tag{39}$$

where $C_1 = 0$ from the Mercier condition at $x = 0$. From Eq. (26) we find

$$I = I_c + \int_{\psi_c}^{\psi} (2\theta + \dot{\theta}\psi) d\psi \tag{40}$$

and calculate $M_2^* = \frac{2\pi q}{I}$. Then we find the minimum of functional $F(M_2, M_2^*) = \int_{\psi}^{\psi_b} (M_2 - M_2^*)^2 \, x^2 d\psi$ with respect to parameters C_k. Using the obtained parameters C_k, we recalculate $\dot{\theta}$ and substituting into expression for pressure equation (40), we solve new equilibrium. All this procedure is repeated until required accuracy is obtained.

We solve Eqs. (25)–(26) in dimensionless units. For unit length we take x_u, for unit magnetic field B_u. They are to be chosen for convenience sake.

Figure 1 shows an example of the magnetic surfaces of our configuration. In this particular example we show several quantities of interest as a function of the radius r. The $r = 0$ is the toroidal axis position and $r = 5$ (in arbitrary unit) is the magnetic axis (the center of the poloidal radius or the poloidal axis). In the following curves are terminated at the plasma edge. Figure 2 shows the profiles of toroidal and poloidal magnetic fields. Note that the poloidal field is zero on the poloidal axis, as it should, and the toroidal field on the plasma boundary is extremely low, about 4% of the maximum toroidal field at the magnetic axis. In Fig. 3 we show the profiles of the toroidal current and poloidal current in the plasma (and on the surface). The profile of the safety factor is displayed in Fig. 4. The q value varies from ~ 0.2 at the center to less than 0.1 (about 0.08) in the present example. Figure 5 shows the profiles of the plasma pressure and the helicity density. Displayed in Fig. 6 is the profile of the quantity M_2, which is related to the toroidal magnetic flux, given by Eq. (31). This profile is computed from given $q(\psi)$ and $\theta(\psi)$. After adapting these θ functions numerically to the condition of sufficient stability (26) iteratively, the two profiles of M_2 ultimately coincide (converge) for the solution. Figure 7 shows the functional dependence of U, U', and P as a function of the helicity density h. If the obtained (Liapunov-stable) equilibrium were the Taylor state,[7] $U(h) = \text{const.} \times h$ and thus $U'(h) = \text{const.}$ The Taylor state corresponds to a constant pressure equilibrium. In fact, our relaxed state has a pressure profile and in particular $U'(h) \approx \text{const.} + \text{const.} \times h^{1/2}$, corresponding to $U(h) \propto h + \text{const.} \times h^{3/2}$. The second term in U' and U represents the non-Taylor like profile. The physical reason why the second term in $U(h)$ goes like $h^{3/2}$ is unclear at this moment. However, it is noted that in a separate investigation with a driven boundary condition[12] we witnessed an approximate equilibrium. In this equilibrium the global Beltrami condition $\nabla \times \mathbf{B} = \alpha \mathbf{B}$ with α being the global condition constant is replaced by the local Beltrami condition $\nabla \times \mathbf{B} = \alpha(\psi)\mathbf{B}$, where α is a function of the flux function and approximately proportional to $\psi^{1/2}$. It should also be noted that in a numerical simulation[13] the pressure profile tends to relax to be uniform. Thus we need to investigate how long the pressure profile is maintained and if necessary how the profile is maintained either by drive (or fueling) or by fusion products. To obtain dimensional quantities from Figs. 1 to 6, the following transformations should be made

$$\begin{array}{ll} \mathbf{B} \longleftrightarrow B_u \mathbf{B} \; ; & \mathbf{A} \longleftrightarrow x_u \, B_u \mathbf{A} \\ q \longleftrightarrow q & p \longleftrightarrow B_u^2 \, p \\ h \longleftrightarrow x_u \, B_u^2 \, h & \text{and so on .} \end{array}$$

For instance it might be convenient to take $x_u = 1\,\mathrm{m}$, $B_u = 5 \times 10^4$ Gauss.

VII. Operation Considerations and Current Drive

For a confinement system it is crucial to operate it as a reactor mode. This means that (i) the energy density is high enough to be attractive, (ii) the (near) steady-state operation is possible, (iii) the energy conversion and fueling is efficient, (iv) the wall load of heat and neutron fluence is low enough, and inexpensive enough, among other requirements. These conditions point to our desire to achieve: (i) the highest possible plasma beta or the lowest possible external magnetic fields required, (ii) possible advanced fuel to reduce neutron fluence, (iii) a self-sustained current due to fusion products, (iv) simple wall/magnets requirements, (v) possible direct energy conversion, etc.

Let us consider the current drive[14–17] by the charged fusion products such as energetic alpha particles and protons. It is crucial that our configuration has relatively weak toroidal magnetic fields. In the absence of strong toroidal fields the changed fusion products can have a large orbit, not far smaller than the system's dimension. This leads to the possibility of asymmetric (or momentum selective) confinement. For simplicity, for the moment, let us ignore the toroidal fields entirely. Imagine a charged fusion product has a momentum into the board. If the poloidal field at the location is upward, this particle (positively charged, executes a Larmor motion toward the outside of the torus. Either by the enlarged Larmor radius outside of the plasma or by some design to snare such particles, it is possible to lose particles that come out of the plasma region. On the other hand, suppose that a charged particle has a momentum out of the board. This particle executes a Larmor motion toward the interior of the torus and thus keeps confined. The net result is the overall toroidal momentum out of the board. As has been discussed in Ref. 14, this net momentum gives rise to a drag of electrons that yields a radial spinout of electrons, and thus a radial electric field. This radial electric field is responsible for the toroidal $\mathbf{E} \times \mathbf{B}$ rotation of the plasma. With the Ohkawa effect[15] due to the different atomic numbers per charge, however, this net momentum flow accompanies the toroidal current: The toroidal (ϕ) Ohkawa current is given by

$$J_\phi = Z_f \, n_f \, e \, \langle v_{f\phi} \rangle \left(1 - \frac{Z_f}{Z_{\text{eff}}} \right) \ , \tag{41}$$

where Z_f and Z_{eff} are the charge of fusion products and the background plasma ions, n_f and v_f are the density and the toroidal flow velocity of the fusion products. This current, if $(1 - Z_f/Z_{\text{eff}}) > 0$, is in the direction to reinforce the poloidal magnetic field.[16] Thus it can be said that the charged fusion products can self-sustain (or enhance) the poloidal fields by the selective confinement.

Consider the Fokker-Planck equation

$$\mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \frac{q}{m_c} \mathbf{v} \times \mathbf{B} \cdot \frac{\partial f}{\partial \mathbf{v}} = C(f) + S\delta(v - v_0) \ , \tag{42}$$

where the collision term is

$$C(f) = v \frac{\partial}{\partial \mathbf{v}} \cdot \left[\frac{\mathbf{v}}{|v|} (v^3 + v_i^3) f \right] , \tag{43}$$

with v_0 the initial speed of the fusion product and v_i being the threshold speed below which the scattering collisionless begin to become important relative to slow down collisions and the source term is the fusion product source

$$S = n_1 n_2 \langle \sigma v \rangle . \tag{44}$$

If the constants of motion are p_ϕ and v $(\frac{1}{2} mv^2 =$ energy) during the period shorter than the slow-down time ν_s^{-1} but longer than the bounce time, then we have $f = f(vp_\phi)$, but f not dependent on v_z.

As treated in Ref. 14, we can integrate Eq. (43) along the characteristics. The resultant solution of the distribution function is

$$f(v, p_\phi) = \frac{\widehat{G}\left[v_0(p_\phi, v), p_{\phi 0}(p_\phi, v)\right]}{v^3 + v_i^3} \cdot \Theta\left[p_\phi - p_{\phi \min}(v)\right] \cdot \Theta\left[p_{\phi \max}(v) - p_\phi\right] , \tag{45}$$

where

$$\widehat{G}(v_0, p_{\phi 0}) = \frac{\widehat{S}(v_0, p_{\phi 0}) v_0^2}{\widehat{\nu}(v_0, p_{\phi 0})} , \tag{46}$$

and the caret quantities mean, for example,

$$\widehat{Q}(p_\phi, v) = \frac{\int_s dr\, dz\, Q(r, z)}{\int_s dr\, dz} , \tag{47}$$

where the s refers to the curve that bounds the accessible phase space region with given p_ϕ and v at the present time (as permitted by the maximum and minimum possible toroidal momenta $p_{\psi \max}$ and $p_{\psi \min}$). Here the subscript 0 refers to the initial (at-birth) value and Θ is the Heavyside step function.

The current due to the fusion product is calculated as

$$J_\phi(r, z) = q\rho \int d^3v\, f\, v_\phi . \tag{48}$$

We can show that the current $(J_\phi)_{en}$ induced by the energetic particles in comparison with the background plasma current $(J_\phi)_p$ that had produced the equilibrium poloidal fields is given[14] as

$$\frac{J_{en}}{J_p} = \frac{E}{T_p} \frac{\nu_f}{\nu_s} \frac{\beta}{10} \alpha I(\alpha) \; , \tag{49}$$

where E is the energetic particle energy, T_p the plasma temperature, β the core plasma beta, $\nu_f = n \langle \sigma v \rangle$ the fusion reaction rate, and α the ratio of the minor radius to the energetic particle Larmor radius. Here the dimensionless function $I(\alpha)$ is shown for cases where α is relatively small (5), and relatively large (20) in Fig. 7. It is clear in this case that a substantial amount of current can be generated by the fusion charged particles.

If we revive weak toroidal magnetic fields, we expect that the above result holds in an essential way. For example, the asymmetric confinement should remain operative. This is particularly the case toward the outside of the minor radial direction. We thus expect that our reactor configuration can, on the order of magnitude, sustain the poloidal magnetic fields by the charged fusion products.

In short, the current profile generated by the charged fusion products of the system tends to be hollow. In addition, the parallel or antiparallelism of the toroidal current and the toroidal magnetic fields makes a significant difference in the induced current. We, however, need more thorough work with the toroidal field effects. If the Larmor radius of energetic particles is substantially smaller than the poloidal radius of the plasma, the particle dynamics is approximately that of drift kinetic. In this case energetic particles execute the vertical drift motion due to the toroidal field gradient, transporting interior energetic particles toward the periphery. Thereby they can then be asymmetrically lost. In this case the net amount of toroidal flow should be reduced. It should be noted that the generated current near the surface can be transported toward interior by resistive or anomalous relaxation. Additional external current drive may be needed for support of the q profile while heating the plasma without loss of stability until fusion conditions are achieved.

Certain comments are in order. First, in order to maintain the equilibrium discussed, we need to make the boundary conditions with a metallic wall. A solid wall may be problematic, as energetic particles tend to sputter the surface of such a metal. Perhaps we may employ a net of metallic mesh that is essentially conducting and can be given a definite electric potential, yet is permeable for particles. Once charged particles come out of this boundary, it is important to capture them. This is for the purpose of the current drive described above as well as for that of the efficient and direct energy conversion of charged particles.

VIII. Discussion

Fusion energy through magnetic confinement has progressed by a large stride recently, culminating to reach a nearly "scientific breakeven" within a factor two or so in the world's largest tokamak experiments, as reported in the latest IAEA meeting in Baltimore, U.S.A,, this year (1990). Among fusion concepts the tokamak concept has come furthest. Nevertheless, there exists a criticism that a tokamak reactor has some undesirable features.[18] It points out that the capital cost of constructing a tokamak fusion reactor is much higher than that of a typical fission reactor, based on the low reactor power density, high system complexity, and other factors. Therefore, it is desirable to explore fusion concepts to circumvent these characteristics. One of the major motivations of the present paper has been to squarely address how to ameliorate the technical complexity of the external magnetic system.

In magnetic fusion there are two fundamental reactor requirements. One is the ignition condition and the other is the ignited energy gain. We assume here that an attractive magnetic fusion reactor should be ignited. The ignition condition may be expressed as the ratio of the fusion power to the energy loss rate being greater than unity:

$$r \equiv \frac{P_{\text{fus}}}{P_E} = \frac{n^2 \langle \sigma v \rangle \varepsilon_{\text{fus}} f_\alpha}{nT/\tau_E} > 1 \ , \tag{50}$$

where ε_{fus} is the fusion energy (e.g. 14.7 MeV for the DT process), f_α is the fraction of alpha particle containment before slowdown, n and T are the plasma density and temperature, and τ_E the energy confinement time. Here we assumed the radiative loss time τ_{rad} is sufficiently long enough to be ignored (but could be important for high temperature advanced fuel reactor). As $\langle\sigma v\rangle \propto T^2$ in typical fusion operation regimes, the quantity r is proportional to $nT\,\tau_E\,f_\alpha$. Of course, all the physics is in determination of τ_E. The condition (50) is a function of n, T, τ_E, f_α, and τ_{rad}, all basically the properties of the fusion plasma itself. The condition of ignited energy gain may be stated that the auxiliary power usage compared with the available fusion power is sufficiently small:

$$\frac{P_{\mathrm{aux}}}{\eta\,P_{\mathrm{fus}} - P_{\mathrm{aux}}} < \frac{1}{\nu}\,, \tag{51}$$

where P_{aux} is the power necessary to sustain the auxiliary processes of the reactor such as the plasma heating, the magnet current power supply, and the vacuum and coolant power supply, η is the overall energy conversion efficiency $\eta = \eta_{dc}\,f_c + \eta_{th}\,M\,f_n$ with f_c and f_n being the fraction of charged particles and neutrons out of fusion products, η_{dc} and η_{th} the energy conversion efficiency by the direct convertor and the thermal converter, M being the multicative factor for fissile materials through neutron breeding. The number ν is the figure of merit of the reactor power utilization and the larger it is, the more power is available for use and typically taken to be in the neighborhood of 7. It can be shown that the condition (51) depends on n, T, the plasma volume V_p, the magnet volume V_m among others.

The neutron fluence $\Gamma_n = \varepsilon_n\,\phi$, where ε_n is the neutron energy and ϕ the neutron flux, is related to the fusion power by

$$\frac{\Gamma_n}{P_{\mathrm{fus}}} = \frac{\varepsilon_n}{\varepsilon_{\mathrm{fus}}}\,a\,, \tag{52}$$

where a is an appropriate (linear) dimension of the fuel (plasma). For a DT tokamak reactor a is of the order of 1m, while for a light water fission reactor (LWR) and a fast breeding reactor (FBR) typically a on the order of 1cm. The fusion power goes like

$$P_{\mathrm{fus}} \propto n^2\,T^2 \sim \beta^2\,B^4\,, \tag{53}$$

and for a tokamak this becomes

$$P_{\text{fus}} \propto (\epsilon/q)^2 B_t^2 , \tag{54}$$

where B is the (strongest or toroidal B_t) magnetic field, ϵ and q are the inverse aspect ratio and the safety factor and are approximately fixed constant for a tokamak.

A schematic reactor is shown in Fig. 9. In a DT reactor the first wall (FW) has to withstand the high neutron fluence. Perhaps the maximum it can stand is $\sim 20\,\text{MWyr/m}^2$, from which the average lifetime should be determined. The blanket (either liquid or solid) has to transport heat and act as breeding, if necessary. The shield (S) and the superconducting magnets (SC) have to sit outside of all these. The shield has to be sufficiently thick (enough for the stopping power of neutrons). The superconducting system has to withstand the neutron penetration and heat to keep the superconductor from heating, stress, and material deterioration. The current J_s and magnetic field B_s on the surface of superconductors should be less than the critical current J_c and magnetic field B_c, and yet the magnetic field in the plasma, say, nearly 2m away from the surface of superconductors has to be sufficiently strong toroidal field. These are formidable technical requirements.

In light of these technical demands, the present machine has several (potential) benefits as a reactor. First, the toroidal magnetic field, which is the only external field, near the surface of the plasma and thus near the surface of magnets is much smaller than the typical poloidal field. Therefore, the magnetic field in Eq. (53) is the poloidal field and the coil magnetic field condition is a couple of magnitude less severe than that of an equivalent tokamak. This will relax the severity of Eq. (51), as well, because the auxiliary power needed for magnets reduces accordingly. Secondly, the lack of strong toroidal field and thus the fairly large value of overall plasma beta mean that the synchrotron radiation power loss is not as severe as in a tokamak even for very high temperature, as the synchrotron loss goes like

$$P_{\text{syn}} \propto n\,T_e\,B^2 \sim \beta\,B^4 , \tag{55}$$

where B is the total magnetic field. As we already pointed out in Sec. VII, the relative smallness of the toroidal field allows one to yield the fusion product sustained current drive of the poloidal field, opening the possibility for a steady-state reactor.

The reduced energy loss due to synchrotron radiation permits us to operate at a much higher temperature, thus permits us to adopt an advanced fuel such as D^3He instead of the DT fuel. The advanced fuel operation pertains to a number of reactor benefits. First, it reduces the neutron fluence Γ_n by two orders of magnitude, thus enhancing the effective energy density according to Eq. (52). This further lessens the load on the magnet. Since a major portion of orbits of charged fusion products have a large Larmor radius, there exists an option of direct energy conversion of these energetic charged particle energy.

One intriguing magnetic configuration more extreme than the present ultra-low q configuration is the isolated magnetic configuration.[19,20] This configuration is spheromak-like, except that it does not have an external field outside the separatrix, thus allowing a configuration without external magnets. However, either a swirl of flows should be present or the pressure profile is concave (its interior is lower than the exterior) to reconcile the virial theorem. The swirl of plasma flows might be maintained by the swirl of liquid metal flows as a part of the blanket/shield system. If the interior temperature is higher than the exterior, such a pressure profile may be thermodynamically unstable, although radiative plasmas often satisfy such a profile. Such a configuration is, however, absolutely stable in terms of MHD. The investigation of isolated magnetless magnetic configurations, however, is left for future research.

The work was supported by the U.S. Department of Energy DE-FG05-80ET-53088. We are grateful for discussions with Prof. J.B. Taylor and Prof. T. Sato.

References

1. B. Coppi, A. Ferreira, J.W. Mark, and J.J. Ramos, Nucl. Fusion **19**, 715 (1979).

2. E. Teller, ed. *Fusion*, Vol. 1.B (Academic, New York, 1981).

3. A.M. Lyapunov, *The General Problem of the Stability of Motion* (Gostekhizdat, Moscow, 1950).

4. G. Bateman, *MHD Instabilities* (MIT Press, Cambridge, Mass, 1978); B.B. Kadomtsev, in *Reviews of Plasma Physics*, ed. M.A. Leontovich (Consultant Bureau, New York, 1966), Vol. 2, p. 153.

5. D.D. Joseph, *Stability of Fluid Motion* (Springer, Berlin, 1976); L.D. Landau and E.M. Lifshitz, *Fluid Mechanics* (Pergamon, Oxford, 1959) Ch. 3.

6. V.A. Gordin and V.I. Petviashvili, Zh. Eksp. Teor. Fiz. **95**, 1711 (1989) [Sov. Phys. JETP **68**, 988 (1989)].

7. J.B. Taylor, Rev. Mod. Phys. **58**, 741 (1986).

8. L. Woltjer, Proc. Nat. Acad. Sci. **44**, 489 (1958).

9. C. Mercier, Nucl. Fusion **1**, 47 (1960).

10. J.W. Connor, R.J. Hastie, and J.B. Taylor, Nucl. Fusion **4**, 365, 1 (1979).

11. L.M. Degtyarev, V.V. Drozdov, Comp. Phys. Reports, **2**, 343 (1985).

12. N. Bekki, T. Tajima, and J.W. Van Dam, Institute for Fusion Studies preprint #327 (1988) (unpublished).

13. R. Horiuchi and T. Sato, Phys. Fluids **29**, 4174 (1986) and also private communication with them (1990).

14. H.L. Berk, H. Momota, and T. Tajima, Phys. Fluids **30**, 3548 (1987).

15. T. Ohkawa, Nucl. Fusion **10**, 185 (1970).

16. M. Okamoto, Nucl. Fusion **27**, 833 (1987).

17. D.E. Baldwin and M.E. Rensink, Comments on Plasma Phys. and Contr. Fusion **4**, 55 (1978).

18. D. Pfirsch and K.H. Schmitter, IEEE Conf. Publ. No. 233 "Fourth International on Energy Options — The Role of Alternatives in the World Energy Scene", p. 350 (1984).

19. V.I. Petviashvili and V.V.Yankov, in *Reviews of Plasma Physics*, ed. B.B. Kadomtsev (Consultant Bureau, New York, 198) Vol. 14, p. 1.

20. S. Oliveira and T. Tajima, to be published.

Figure Captions

1. Sequence of magnetic surfaces from $\psi = 0.2$ at magnetic axis ($r = 5.8; z = 0$) till $\psi = 0.8$ at the conducting wall. Units of normalization are given in Sec. VI.

2. Profiles of toroidal (1) and poloidal (2) magnetic fields at $z = 0$.

3. Profiles of $(\nabla \times B)_t$ (1) and $(\nabla \times B)_p$ (2) at $z = 0$. To obtain current densities in (ampere/cm^2) multiply value on Fig. 3 by B_u/x_u (B_u in gauss, x_u in c.

4. Profile of safety factor q at $z = 0$. Note high value of shear $\nabla q/q$ except the central and boundary regions.

5. Profiles of plasma pressure (1) and 0.1 of helicity density (2). At the plasma boundary $p = \nabla p = \nabla^2 p = h = \nabla h = 0$ (force-free state).

6. Profiles of M_2 calculated at given $q(\psi)$ and $\theta(\psi)$. After adapting these functions numerically to additional condition on equilibrium (26), the two profiles coincide.

7. The functional dependence of quantities U, U', and P as a function of the helicity density h.

8. The normalized toroidal current induced by the fusion product ($B_\phi = 0$ assumed). (a) For $\alpha = 5$ (b) For $\alpha = 20$.

9. Schematic reactor configuration: P, the plasma; FW, the first wall; B, the blanket, and S and SC are the neutron shield and the superconductor.

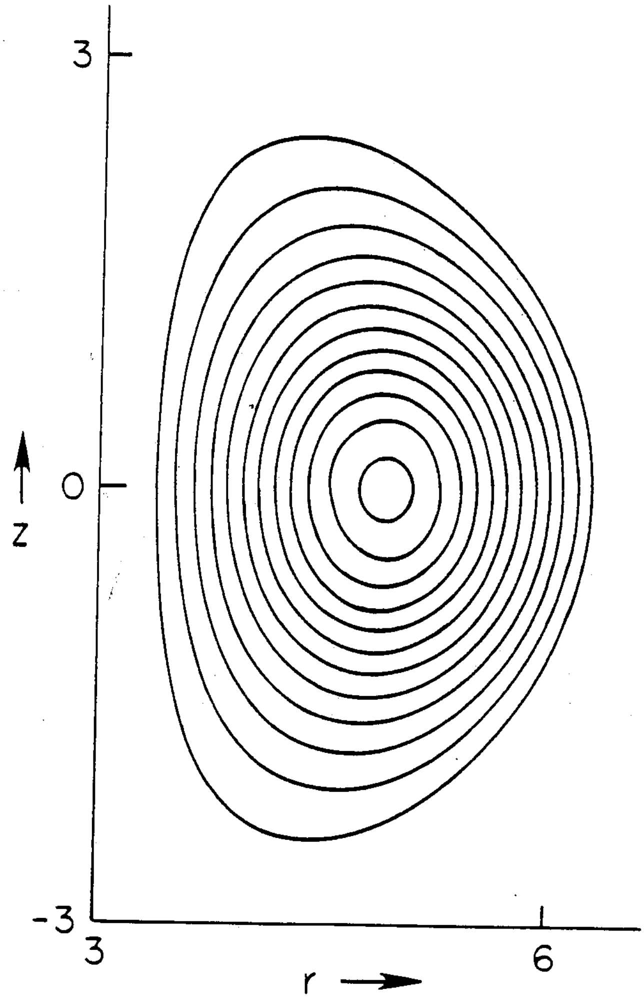

Fig. 1

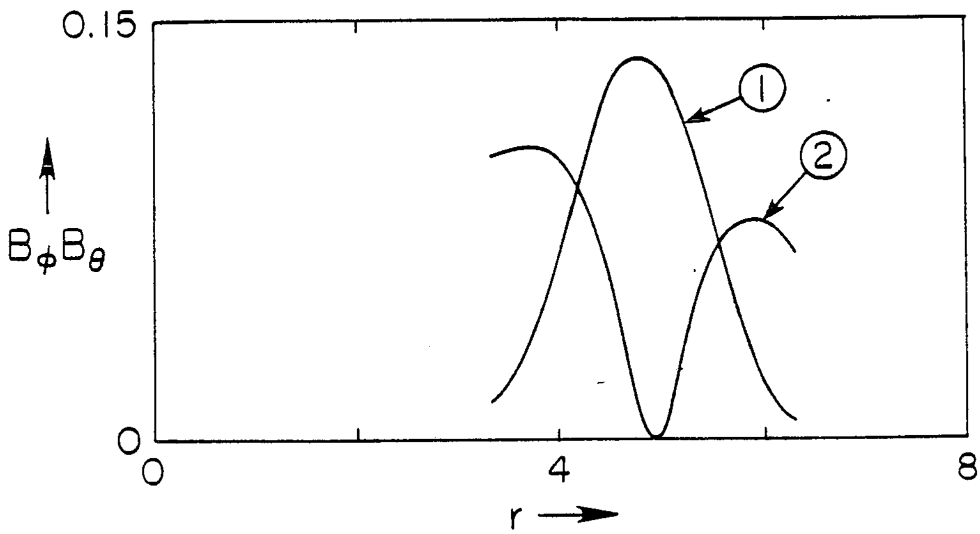

Fig. 2

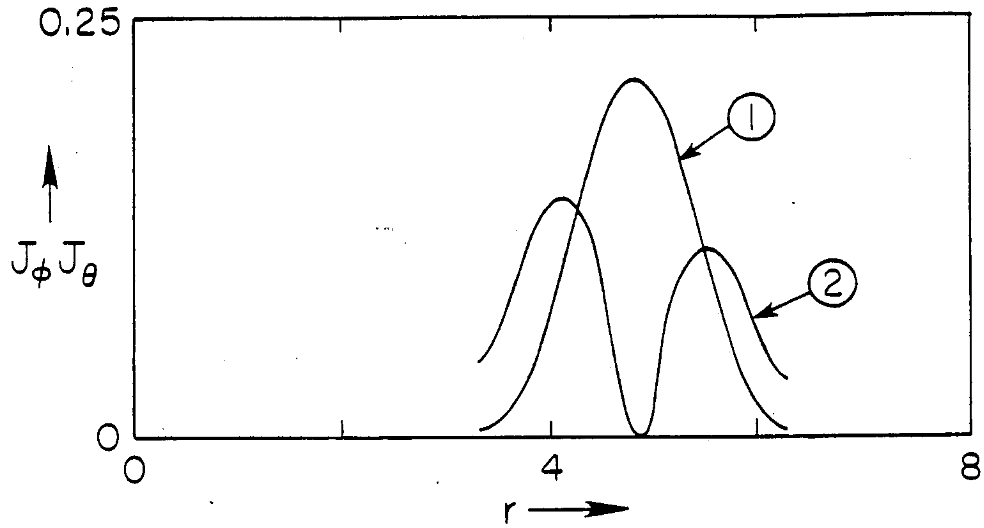

Fig. 3

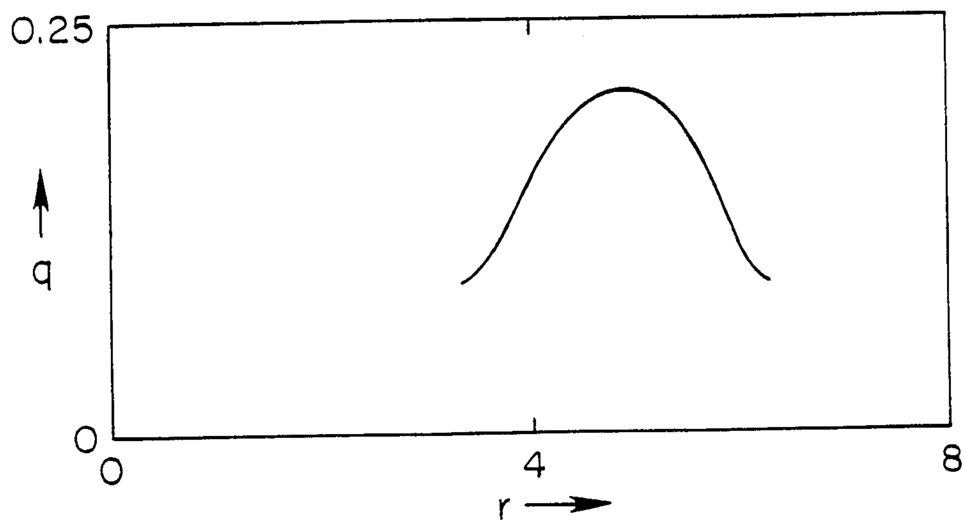

Fig. 4

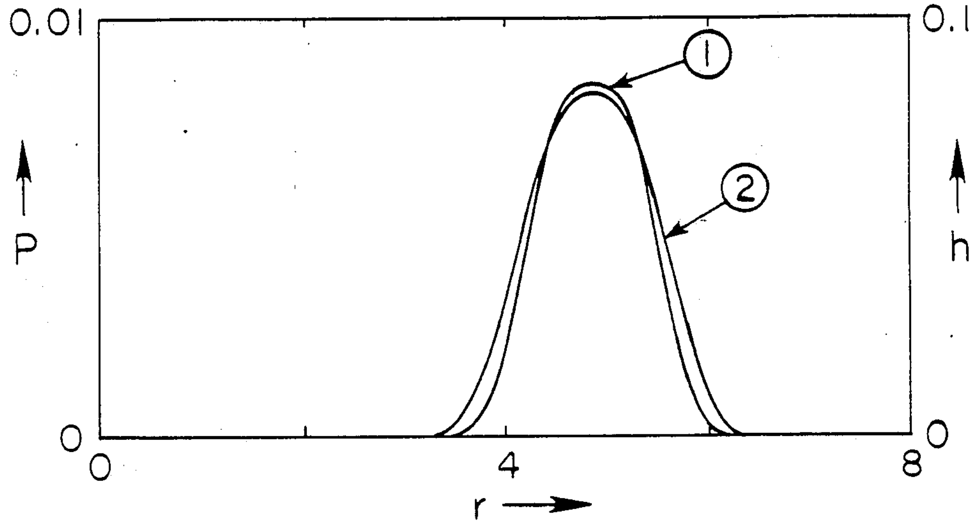

Fig. 5

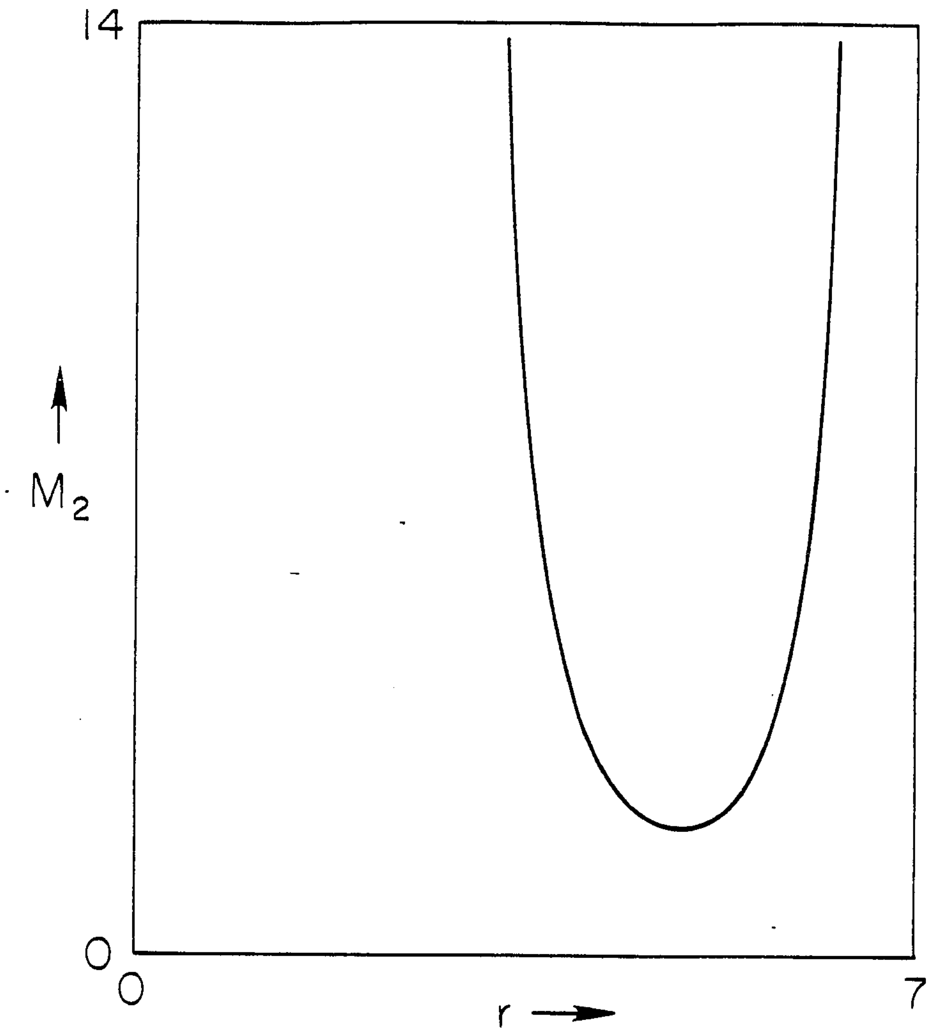

Fig. 6

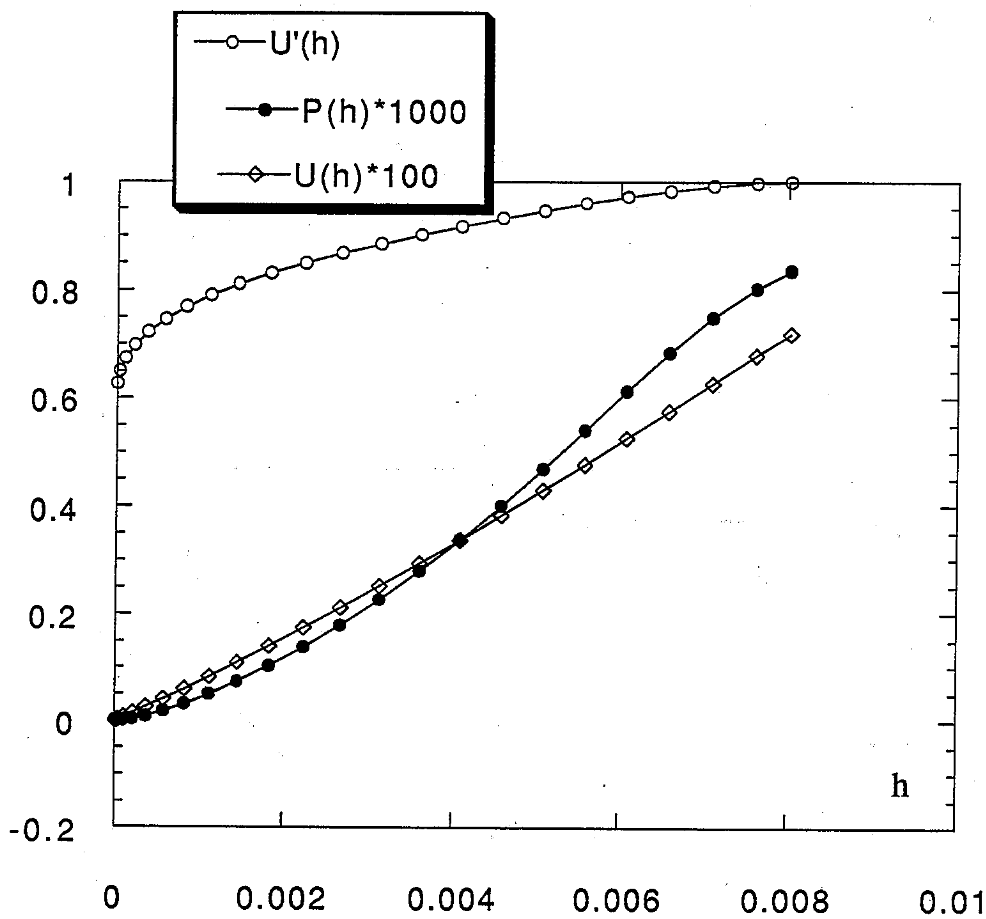

Fig. 7

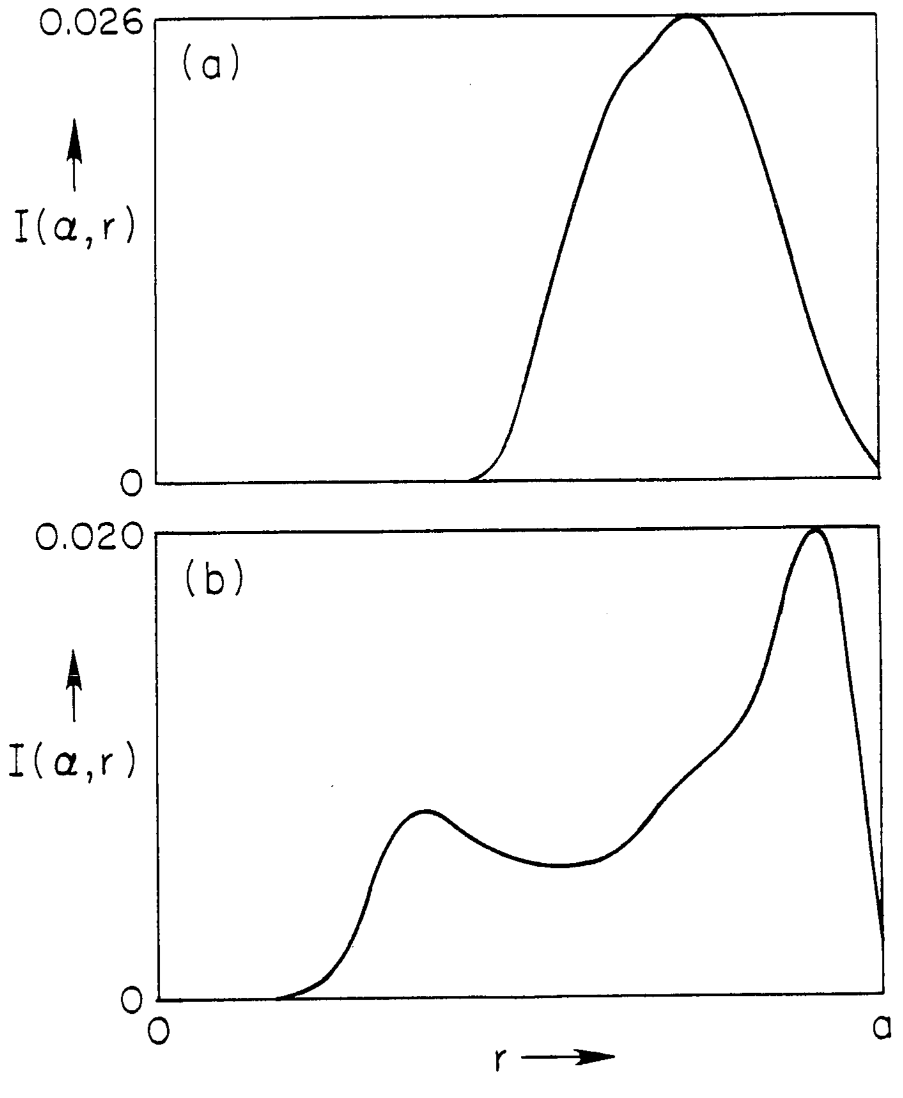

Fig. 8

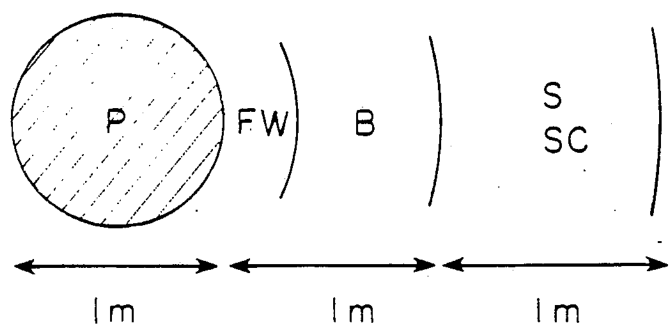

Fig. 9